APPROXIMATION OF THE MAXWELL EIGENVALUE PROBLEM IN A LEAST-SQUARES SETTING

FLEURIANNE BERTRAND, DANIELE BOFFI, AND LUCIA GASTALDI

ABSTRACT. We discuss the approximation of the eigensolutions associated with the Maxwell eigenvalues problem in the framework of least-squares finite elements. We write the Maxwell curl curl equation as a system of two first order equation and design a novel least-squares formulation whose minimum is attained at the solution of the system. The eigensolution are then approximated by considering the eigenmodes of the underlying solution operator. We study the convergence of the finite element approximation and we show several numerical tests confirming the good behavior of the method. It turns out that nodal elements can be successfully employed for the approximation of our problem also in presence of singular solutions.

1. INTRODUCTION

The finite element approximation of the eigenmodes associated with the Maxwell system is a deeply studied and nowadays well understood topic. In particular, it is universally recognized that the natural choice for the approximation of the eigensolutions associated with the curl curl operator, is to consider Nédélec (edge) finite elements [21]. Approximations based on edge elements are optimally convergent, do not present any spurious modes, and are robust in presence of singularities due to the domain or to the presence of different materials. The interested reader is referred to the related literature; in particular to [12] for a discussion about Whitney forms in connection with this problem, to [8, 9] for the first analysis of the spectral correctness of edge elements, and to [17, 20, 7] for general surveys on this subject.

Formulations based on least-squares finite elements are widely used for the approximation of models involving partial differential equations [6]. Recent studies are investigating the behavior of the spectrum of operators associated with least-squares finite element formulations [5, 1]. These studies have their interest by themselves, and in some cases they can contribute to the design of new schemes.

In this paper we begin the study of the eigenvalues associated with the Maxwell system in the framework of least-squares finite elements. The interest of this research is twofold: on one side we discuss how to introduce a first order formulation of the Maxwell system in this framework, on the other side, we analyze rigorously the approximation of the eigensolutions with various choices of finite element spaces in two and three dimensions. Besides formulations based on edge elements, we believe that a remarkable result of our investigation is that standard Lagrangian (nodal) elements can be successfully used in two dimensions and, with some care, in three dimensions. In two dimensions, the use of nodal elements is supported by a rigorous theory, valid when the solution satisfies appropriate regularity assumptions. Our numerical investigations show that actually the approximation based on nodal

elements is performing well also in presence of strong singularities, such as those arising from reentrant corners, cracks, and material discontinuities.

In three dimensions, the theory covers the case of edge elements, while the nodal element approximation requires a more specific analysis, probably depending on the structure of the mesh, which will be the object of future investigations.

Several numerical experiments complement the theoretical results, confirming the theory and supplementing the theoretical investigations when they are not available.

Section 2 describes the problem we are dealing with and introduced the first order least-squares formulation. We continue then in Section 3 with the discussion of the two dimensional case. Indeed, the two and three dimensional case, although sharing some analogies, are intrinsically different: in two dimensions the problem is equivalent to the Laplace eigenvalue problem with Neumann boundary conditions. Section 4 is devoted to the numerical approximation of the two dimensional problem. Particular care is devoted to the definition of the solutions of our generalized eigenvalue problem and to the description of possible degenerate situations. Several two dimensional numerical results are presented in Section 5, confirming the good behavior of nodal element approximations also in presence of strong singularities. The last to sections deal with the three dimensional case: in Section 6 the theory is developed, while in Section 7 some numerical results are presented.

2. Problem setting

Let Ω be a domain in \mathbb{R}^3 . We start by the case of a contractible domain Ω where harmonic forms do not enter the characterization of our solutions. We are interested in the following eigenvalue problem associated with Maxwell's equation: find $\lambda \in \mathbb{R}$ and a non-vanishing **u** such that

(1)
$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = \lambda\varepsilon\mathbf{u} & \text{in }\Omega\\ \operatorname{div}(\varepsilon\mathbf{u}) = 0 & \text{in }\Omega\\ \mathbf{u} \times \mathbf{n} = 0 & \text{on }\partial\Omega \end{cases}$$

where **n** is the outward normal unit vector to the boundary of the domain Ω , and where μ and ε are the (possibly varying) magnetic permeability and electric permittivity, respectively.

The source problem corresponding to (1) reads: given \mathbf{f} with $\operatorname{div}(\varepsilon \mathbf{f}) = 0$, find \mathbf{u} such that

(2)
$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = \varepsilon \mathbf{f} & \text{in } \Omega\\ \operatorname{div}(\varepsilon \mathbf{u}) = 0 & \text{in } \Omega\\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

A representation of (2) as a system of first order equations could be done, in analogy to the usual procedure for the Laplace equation, as follows by introducing the auxiliary variable $\boldsymbol{\sigma} = \mu^{-1} \operatorname{curl} \mathbf{u}$, so that we have

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\mu}^{-1} \operatorname{\mathbf{curl}} \mathbf{u} & \text{in } \boldsymbol{\Omega} \\ \operatorname{\mathbf{curl}} \boldsymbol{\sigma} &= \varepsilon \mathbf{f} & \text{in } \boldsymbol{\Omega} \\ \operatorname{div}(\varepsilon \mathbf{u}) &= 0 & \text{in } \boldsymbol{\Omega} \\ \operatorname{\mathbf{u}} \times \mathbf{n} &= 0 & \text{on } \partial \boldsymbol{\Omega}. \end{aligned}$$

Unfortunately, this system is not suitable to be approximated by a Least-Squares finite element strategy on non-smooth domains when a minimization principle in $L^2(\Omega)$ is used. Indeed, the functional to be minimized would read

$$\mathcal{F}(\boldsymbol{\tau}, \mathbf{v}) = \|\boldsymbol{\tau} - \boldsymbol{\mu}^{-1} \operatorname{\mathbf{curl}} \mathbf{v}\|_{0}^{2} + \|\operatorname{\mathbf{curl}} \boldsymbol{\tau} - \varepsilon \mathbf{f}\|_{0}^{2} + \|\operatorname{div}(\varepsilon \mathbf{v})\|_{0}^{2}.$$

It is apparent that no reasonable choice of functional spaces can be made in this situation. Already in the simpler case when $\varepsilon \equiv 1$, this would imply that the variable **v** should have both divergence and curl bounded in $L^2(\Omega)$. This is a well known source of troubles for the finite element approximation when the domain has non convex corners or edges [13, 16, 14], since in that case singular solutions **u** are not in $H^1(\Omega)$.

For this reason, we make use of the first order system introduced in [8]. Namely we consider a vectorfield \mathbf{g} such that $\operatorname{curl} \mathbf{g} = \varepsilon \mathbf{f}$, $\operatorname{div}(\mu \mathbf{g}) = 0$, $(\mu \mathbf{g}) \cdot \mathbf{n} = 0$ on $\partial \Omega$, and look for the pair (\mathbf{u}, \mathbf{p}) satisfying

(3)
$$\begin{cases} \varepsilon \mathbf{u} = \operatorname{curl} \mathbf{p} & \text{in } \Omega \\ \mu^{-1} \operatorname{curl} \mathbf{u} = \mathbf{g} & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

In general ${\bf p}$ is not unique, but there is only one ${\bf p}$ that satisfies the additional conditions

(4)
$$\begin{cases} \operatorname{div}(\mu \mathbf{p}) = 0 & \text{in } \Omega\\ (\mu \mathbf{p}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

We assume ε and μ to be real scalar functions satisfying

(5)
$$0 < \underline{\varepsilon} \le \varepsilon \le \overline{\varepsilon}, \quad 0 < \mu \le \mu \le \overline{\mu}$$

for almost every ${\bf x}$ in Ω and introduce the following spaces

$$\begin{aligned} \boldsymbol{H}(\mathbf{curl}) &= \{ \mathbf{v} \in \boldsymbol{L}^2(\Omega) : \mathbf{curl} \, \mathbf{v} \in \boldsymbol{L}^2(\Omega) \} \\ \boldsymbol{H}_0(\mathbf{curl};\Omega) &= \{ \mathbf{v} \in \boldsymbol{H}(\mathbf{curl}) : \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega \} \\ \boldsymbol{H}_0(\operatorname{div}^0;\Omega;\mu) &= \{ \mathbf{q} \in \boldsymbol{L}^2(\Omega) : \operatorname{div}(\mu\mathbf{q}) = 0 \text{ in } \Omega, \ (\mu\mathbf{q}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \} \end{aligned}$$

Therefore, $\mathbf{g} \in \boldsymbol{H}_0(\operatorname{div}^0; \Omega; \mu)$ and we look for a solution $\mathbf{p} \in \boldsymbol{H}_0(\operatorname{div}^0; \Omega; \mu)$.

Proposition 1. Let us consider Problems (2) and (3) with

$$\begin{cases} \mathbf{curl}\,\mathbf{g} = \varepsilon \mathbf{f} & in \ \Omega\\ \operatorname{div}(\mu \mathbf{g}) = 0 & in \ \Omega\\ (\mu \mathbf{g}) \cdot \mathbf{n} = 0 & on \ \partial\Omega \end{cases}$$

If \mathbf{u} solves (2) then there exists \mathbf{p} so that (\mathbf{u}, \mathbf{p}) solves (3). Conversely, if \mathbf{u} solves (3) then it is also a solution of (2).

Proof. If **u** solves (2), then from div(ε **u**) = 0 we get that there exists **p** such that ε **u** = **curl p**. Such **p** is defined up to an additive gradient that can be chosen such that (4) is satisfied. Then, from the first equation in (2), we have **curl**(μ^{-1} **curl u** - **g**) = 0 which implies that μ^{-1} **curl u** - **g** = **grad** ϕ for some ϕ in $H^1(\Omega)$. On the other hand, from div(μ **g**) = 0 it follows div(μ **grad** ϕ) = 0 in Ω , and the boundary conditions on **u** and **g** imply (μ **grad** ϕ) \cdot **n** = 0 on $\partial\Omega$; hence ϕ is constant, from which we conclude that μ^{-1} **curl u** - **g** = 0.

Conversely, taking the **curl** of the second equation in (3), we get that **u** solves (2). The divergence free condition follows from $\varepsilon \mathbf{u} = \mathbf{curl p}$.

In the spirit of Least-Squares formulation, this equivalence leads to the minimization of the following functional

(6)
$$\mathcal{F}(\mathbf{v},\mathbf{q}) = \|\varepsilon^{1/2}\mathbf{v} - \varepsilon^{-1/2}\operatorname{\mathbf{curl}}\mathbf{q}\|_0^2 + \|\mu^{-1/2}\operatorname{\mathbf{curl}}\mathbf{v} - \mu^{1/2}\mathbf{g}\|_0^2$$

in the energy space $H_0(\operatorname{curl}; \Omega) \times H(\operatorname{curl})$, where we split ε and μ as the square of their square roots in order to get a symmetric system when considering the gradient of \mathcal{F} .

As mentioned above, the uniqueness of **p** requires the additional conditions stated in (4). This could be enforced by changing the energy space for the minimization of (6) to $H_0(\operatorname{curl}; \Omega) \times (H(\operatorname{curl}) \cap H_0(\operatorname{div}^0; \Omega; \mu))$. Clearly, this would lead to the same troubles described before related to the approximation of the space $H(\operatorname{curl}) \cap$ $H_0(\operatorname{div}^0; \Omega; \mu)$. We postpone the discussion about this issue and we start our investigations considering the two dimensional counterpart of (1).

3. Problem setting in two dimensions

Let Ω be a polygonal domain in \mathbb{R}^2 . The eigenvalue problem we are interested in, seeks for $\lambda \in \mathbb{R}$ and a non-vanishing **u** such that

(7)
$$\begin{cases} \mathbf{curl}(\mu^{-1}\operatorname{rot}\mathbf{u}) = \lambda\varepsilon\mathbf{u} & \text{in }\Omega\\ \operatorname{div}(\varepsilon\mathbf{u}) = 0 & \text{in }\Omega\\ \mathbf{u}\cdot\mathbf{t} = 0 & \text{on }\partial\Omega, \end{cases}$$

where \mathbf{t} is the counterclockwise tangent unit vector to the boundary of the domain Ω . We recall the two dimensional definitions of the **curl** and rot operators:

$$\operatorname{rot} \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \quad \text{with } \mathbf{v} = (v_1, v_2)^\top$$

and

$$\mathbf{curl}\,\varphi = \left(\frac{\partial\varphi}{\partial y}, -\frac{\partial\varphi}{\partial x}\right)^\top$$

For completeness, we also recall the integration by parts formula that involves these operators

$$\int_{\Omega} \mathbf{curl}(\mu^{-1} \operatorname{rot} \mathbf{u}) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mu^{-1} \operatorname{rot} \mathbf{u} \operatorname{rot} \mathbf{v} \, d\mathbf{x} - \int_{\partial \Omega} \mu^{-1} \operatorname{rot} \mathbf{u} \, \mathbf{v} \cdot \mathbf{t} \, ds$$

which is valid whenever the involved integrals are finite.

The source problem corresponding to (7) reads: given \mathbf{f} with $\operatorname{div}(\varepsilon \mathbf{f}) = 0$, find \mathbf{u} such that

(8)
$$\begin{cases} \mathbf{curl}(\mu^{-1}\operatorname{rot}\mathbf{u}) = \varepsilon \mathbf{f} & \text{in }\Omega\\ \operatorname{div}(\varepsilon \mathbf{u}) = 0 & \text{in }\Omega\\ \mathbf{u} \cdot \mathbf{t} = 0 & \text{on }\partial\Omega \end{cases}$$

and the two dimensional version of the first order system (3) is: find ${\bf u}$ and p such that

(9)
$$\begin{cases} \varepsilon \mathbf{u} = \mathbf{curl} \, p & \text{in } \Omega \\ \mu^{-1} \operatorname{rot} \mathbf{u} = g & \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{t} = 0 & \text{on } \partial \Omega \end{cases}$$

with $\mu g \in L^2_0(\Omega)$ such that $\operatorname{curl} g = \varepsilon \mathbf{f}$, where $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$ of zero mean valued functions. In this case the uniqueness of p is guaranteed by the condition

$$\int_{\Omega} \mu p \, d\mathbf{x} = 0.$$

The following proposition is the analogue of Proposition 1. We recall it here in the two dimensional setting for the reader's convenience.

Proposition 2. Let us consider problems (8) and (9) with $\varepsilon \mathbf{f} = \mathbf{curl} g$ and $\int_{\Omega} \mu g = 0$. If \mathbf{u} solves (8), then there exists p such that (\mathbf{u}, p) is solution of (9). Conversely, if (\mathbf{u}, p) is solution of (9) then \mathbf{u} solves (8).

Proof. If **u** solves (8), then from $\operatorname{div}(\varepsilon \mathbf{u}) = 0$ we get that there exists p such that $\varepsilon \mathbf{u} = \operatorname{curl} p$. Since p is defined up to an additive constant, we choose it such that the mean value of μp is zero on Ω . Then, from the first equation in (8), we have $\operatorname{curl}(\mu^{-1} \operatorname{rot} \mathbf{u} - g) = 0$ which implies that $\mu^{-1} \operatorname{rot} \mathbf{u} - g$ is constant in Ω . On the other hand, the boundary conditions on \mathbf{u} imply that the average of $\operatorname{rot} \mathbf{u} - \mu g = 0$.

Conversely, taking the **curl** of the second equation in (9), we get that **u** solves (8). The divergence free condition follows from $\varepsilon \mathbf{u} = \operatorname{curl} p$.

It is well known that in two dimensions the Maxwell system we are considering, is equivalent to a Neumann problem for the Laplace equation.

Proposition 3. The component p of the solution of (9) is the solution of the following equation

$$\begin{cases} \operatorname{rot}(\varepsilon^{-1}\operatorname{\mathbf{curl}} p) = \mu g & \text{in } \Omega\\ \frac{\partial p}{\partial \mathbf{n}} = 0 & \text{on } \partial \Omega \end{cases}$$

Remark 1. A natural question, that will be essential for the analysis of the discretization, is the regularity of the solution of (9). Thanks to Proposition 3, we can discuss first the regularity of p and then consider that $\varepsilon \mathbf{u} = \mathbf{curl} p$. Clearly, the regularity of p depends on the regularity of g, ε , and μ , and of the domain Ω . In general, it is well known that if Ω is a polygon then there exists $s \in (1/2, 1)$ such that $p \in H^{1+s}(\Omega)$ whenever $\mu g \in L^2_0(\Omega)$. Since $\mathbf{u} = \varepsilon^{-1} \mathbf{curl} p$ we have that $\mathbf{u} \in H^s(\Omega)$ and rot $\mathbf{u} = \mu \mathbf{g}$. Moreover the following a priori estimate hold true

$$\|\mathbf{u}\|_{s} + \|p\|_{1+s} \le C \|g\|_{0}.$$

Even for smoother g, ε , and μ , there are domains where the regularity of p is not higher. For instance, if Ω is the L-shaped domain, then s cannot be taken in general larger than or equal to 2/3.

For the analysis of the convergence of the eigenvalue problem, we are going to consider $g \in H^1(\Omega)$. For ε and μ smooth and nonsingular domains, we have in this case $p \in H^3(\Omega)$.

In the framework of Least-Squares finite elements, we are then led to the minimization of the functional

(10)
$$\mathcal{F}(\mathbf{v},q) = \|\varepsilon^{1/2}\mathbf{v} - \varepsilon^{-1/2}\operatorname{\mathbf{curl}} q\|_0^2 + \|\mu^{-1/2}\operatorname{rot} \mathbf{v} - \mu^{1/2}g\|_0^2$$

in the space $V \times Q$, where V and Q are defined as follows

$$V = H_0(\operatorname{rot}; \Omega)$$
$$Q = \{q \in H^1(\Omega) : \mu q \in L^2_0(\Omega)\}$$

and are equipped by the norm induced by the following scalar products

$$(\mathbf{u}, \mathbf{v})_{\boldsymbol{V}} = (\varepsilon \mathbf{u}, \mathbf{v}) + (\mu^{-1} \operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{v})$$
$$(p, q)_{Q} = (\varepsilon^{-1} \operatorname{curl} p, \operatorname{curl} q).$$

From the assumptions (5) on ε and μ , and from the Poincaré inequality, the induced norms $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_{Q}$ are equivalent to the standard ones.

A variational formulation of (10) is given by: find $\mathbf{u} \in \mathbf{V}$ and $p \in Q$ such that

(11)
$$\begin{cases} (\varepsilon \mathbf{u}, \mathbf{v}) + (\mu^{-1} \operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} p) = (g, \operatorname{rot} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ - (\mathbf{u}, \operatorname{\mathbf{curl}} q) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} p, \operatorname{\mathbf{curl}} q) = 0 & \forall q \in Q. \end{cases}$$

The next proposition states the ellipticity of the bilinear form associated with the above problem.

Proposition 4. Let

$$a: (\mathbf{V} \times Q) \times (\mathbf{V} \times Q) \to \mathbb{R}$$

be the bilinear form associated with the formulation (11), that is

$$a(\mathbf{u}, p; \mathbf{v}, q) = (\mu^{-1} \operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{v}) + (\varepsilon^{1/2} \mathbf{u} - \varepsilon^{-1/2} \operatorname{\mathbf{curl}} p, \varepsilon^{1/2} \mathbf{v} - \varepsilon^{-1/2} \operatorname{\mathbf{curl}} q).$$

Then there exists $\alpha > 0$ such that

$$a(\mathbf{v}, q; \mathbf{v}, q) \ge \alpha \left(\|\mathbf{v}\|_{\mathbf{V}}^2 + \|q\|_Q^2 \right).$$

Proof. We start observing that

$$a(\mathbf{v}, q; \mathbf{v}, q) = \|\mu^{-1/2} \operatorname{rot} \mathbf{v}\|_{0}^{2} + \|\varepsilon^{1/2} \mathbf{v} - \varepsilon^{-1/2} \operatorname{curl} q\|_{0}^{2} \ge \|\mu^{-1/2} \operatorname{rot} \mathbf{v}\|_{0}^{2}$$

For a positive β we have

$$\begin{split} a(\mathbf{v}, q; \mathbf{v}, q) &= (\mu^{-1} \operatorname{rot} \mathbf{v}, \operatorname{rot} \mathbf{v}) + (\varepsilon \mathbf{v}, \mathbf{v}) - 2(\mathbf{v}, \operatorname{\mathbf{curl}} q) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} q, \operatorname{\mathbf{curl}} q) \\ &+ 2\beta(\mathbf{v}, \operatorname{\mathbf{curl}} q) - 2\beta(\mu^{-1/2} \operatorname{rot} \mathbf{v}, \mu^{1/2} q) \pm \beta^{2}(\mu q, q) \\ &= \|\mu^{-1/2} \operatorname{rot} \mathbf{v} - \beta\mu^{1/2} q\|_{0}^{2} + (\varepsilon \mathbf{v}, \mathbf{v}) - 2(1 - \beta)(\mathbf{v}, \operatorname{\mathbf{curl}} q) \\ &+ (\varepsilon^{-1} \operatorname{\mathbf{curl}} q, \operatorname{\mathbf{curl}} q) - \beta^{2}(\mu q, q) \pm (1 - \beta)^{2}(\varepsilon^{-1} \operatorname{\mathbf{curl}} q, \operatorname{\mathbf{curl}} q) \\ &= \|\mu^{-1/2} \operatorname{rot} \mathbf{v} - \beta\mu^{1/2} q\|_{0}^{2} + \|\varepsilon^{1/2} \mathbf{v} - (1 - \beta)\varepsilon^{-1/2} \operatorname{\mathbf{curl}} q, \operatorname{\mathbf{curl}} q) \\ &= \|\mu^{-1/2} \operatorname{rot} \mathbf{v} - \beta\mu^{1/2} q\|_{0}^{2} + \|\varepsilon^{1/2} \mathbf{v} - (1 - \beta)\varepsilon^{-1/2} \operatorname{\mathbf{curl}} q\|_{0}^{2} \\ &- \beta^{2}(\mu q, q) + (2\beta - \beta^{2})\|q\|_{Q}^{2} \\ &\geq -\beta^{2}\|\mu^{-1/2} q\|_{0}^{2} + (2\beta - \beta^{2})\|q\|_{Q}^{2}. \end{split}$$

Using the Poincaré inequality $||q||_0 \le C_P ||\operatorname{curl} q||_0$ and the bounds in (5) we get

 $a(\mathbf{v}, q; \mathbf{v}, q) \ge (2\beta - \beta^2 (1 + \underline{\mu}^{-1} \overline{\varepsilon} C_P^2)) \|q\|_Q^2$

which for β small enough gives

$$a(\mathbf{v}, q; \mathbf{v}, q) \ge C_1 \|q\|_Q^2.$$

Finally, we estimate $\|\varepsilon^{1/2}\mathbf{v}\|_0$. We have

$$\varepsilon^{1/2} \mathbf{v} \|_0 \le \|\varepsilon^{1/2} \mathbf{v} - \varepsilon^{-1/2} \operatorname{\mathbf{curl}} q\|_0 + \|\varepsilon^{-1/2} \operatorname{\mathbf{curl}} q\|_0$$

from which we obtain

$$\begin{aligned} \|\varepsilon^{1/2}\mathbf{v}\|_{0}^{2} &\leq 2\left(\|\varepsilon^{1/2}\mathbf{v} - \varepsilon^{-1/2}\operatorname{\mathbf{curl}} q\|_{0}^{2} + \|q\|_{Q}^{2}\right) \\ &\leq 2(1+1/C_{1})a(\mathbf{v},q;\mathbf{v},q). \end{aligned}$$

In conclusion, we have the ellipticity result with

$$\alpha = \frac{1}{3} \min\left(1, C_1, \frac{1}{2(1+1/C_1)}\right).$$

By using the Lax–Milgram lemma, we have existence and uniqueness of the solution of (11).

Corollary 1. Given $g \in L^2_0(\Omega)$ there exists one and only one solution of (11) which satisfies the a priori stability bound

$$\|\mathbf{u}\|_{V} + \|p\|_{Q} \le C \|g\|_{0}.$$

We introduce the solution operator $T: Q \to Q$ defined as follows: given $g \in Q$,

$$Tg = p,$$

where $p \in Q$ is the second component of the solution of (11). The regularity stated in Remark 1 implies that T is compact. The following proposition states that T is self-adjoint.

Proposition 5. For all g_1 and g_2 in Q it holds

$$(Tg_1, g_2)_Q = (g_1, Tg_2)_Q.$$

Proof. Let us denote by $\mathbf{u}_i \in \mathbf{V}$ the other component of the solution associated with Tg_i (i = 1, 2). Then, from the definition of T and of the scalar products in \mathbf{V} and Q, it follows

$$(Tg_1, g_2)_Q = (\varepsilon^{-1} \operatorname{\mathbf{curl}}(Tg_1), \operatorname{\mathbf{curl}} g_2) = (\mathbf{u}_1, \operatorname{\mathbf{curl}} g_2) = (\operatorname{rot} \mathbf{u}_1, g_2)$$
$$= (\mathbf{u}_2, \mathbf{u}_1)_V - (\mathbf{u}_1, \operatorname{\mathbf{curl}}(Tg_2))$$
$$= (\mathbf{u}_2, \mathbf{u}_1)_V - (\varepsilon^{-1} \operatorname{\mathbf{curl}}(Tg_1), \operatorname{\mathbf{curl}}(Tg_2))$$
$$= (\mathbf{u}_1, \mathbf{u}_2)_V - (\varepsilon^{-1} \operatorname{\mathbf{curl}}(Tg_2), \operatorname{\mathbf{curl}}(Tg_1))$$
$$= (\mathbf{u}_1, \mathbf{u}_2)_V - (\mathbf{u}_2, \operatorname{\mathbf{curl}}(Tg_1))$$
$$= (\varepsilon^{-1} \operatorname{\mathbf{curl}}(Tg_2), \operatorname{\mathbf{curl}} g_1) = (g_1, Tg_2)_Q.$$

In analogy to what has been done in the case of the Laplace eigenvalue problem in [5], it is then natural to consider the following variational formulation in order to describe the solutions of the eigenvalue problem (7). Find $\lambda \in \mathbb{R}$ and $p \in Q$ with $p \neq 0$ such that for some $\mathbf{u} \in \mathbf{V}$ it holds

(12)
$$\begin{cases} (\varepsilon \mathbf{u}, \mathbf{v}) + (\mu^{-1} \operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} p) = \lambda(p, \operatorname{rot} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ - (\mathbf{u}, \operatorname{\mathbf{curl}} q) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} p, \operatorname{\mathbf{curl}} q) = 0 & \forall q \in Q. \end{cases}$$

In Problem (12) we look for real eigenvalues and use real functional spaces. This is justified by the fact that the underlying operator is self-adjoint (see Proposition 5). Moreover, we have the following orthogonality properties.

Proposition 6. Let $\lambda_i \neq \lambda_j$ be two eigenvalues of (12), and (p_i, \mathbf{u}_i) and (p_j, \mathbf{u}_j) the corresponding eigenfunctions. Then the following orthogonalities are satisfied

$$(p_i, p_j)_Q = (\varepsilon^{-1} \operatorname{curl}(p_i), \operatorname{curl}(p_j)) = 0$$

$$(\mathbf{u}_i, \mathbf{u}_j)_V = (\varepsilon \mathbf{u}_i, \mathbf{u}_j) + (\mu^{-1} \operatorname{rot}(\mathbf{u}_i), \operatorname{rot}(\mathbf{u}_j)) = 0$$

In case of multiple eigenvalues, the corresponding eigenfunctions can be chosen so that the same orthogonalities are satisfied.

Proof. The result follows in a standard way by testing (12) with $\mathbf{v} = \mathbf{u}_j$ and $q = p_j$ when $\mathbf{u} = \mathbf{u}_i$, $p = p_i$, and $\lambda = \lambda_i$. The same equation (12) with the roles of *i* and *j* swapped, gives

$$(\lambda_i + 1)(p_i, \operatorname{rot} \mathbf{u}_j) = (\lambda_j + 1)(p_j, \operatorname{rot} \mathbf{u}_i)$$

which gives the results, observing that

$$(p_i, \operatorname{rot} \mathbf{u}_i) = (\varepsilon^{-1} \operatorname{curl}(p_i), \operatorname{curl}(p_i)) = (p_i, \operatorname{rot} \mathbf{u}_i).$$

When λ_i is different from λ_j this implies the first orthogonality. The second one follows by inserting the first one into Equation (12).

Remark 2. From the above orthogonalities, it follows by standard arguments that $Q = \operatorname{span}\{p_i, i = 1, \ldots\}$. Moreover, from the second equation of (12) we have that $\mathbf{u}_i = \varepsilon^{-1} \operatorname{curl} p_i + \operatorname{grad} \phi_i$. Substituting \mathbf{u}_i in the first equation and taking $\mathbf{v} = \operatorname{grad} \phi_i$ gives ($\varepsilon \operatorname{grad} \phi_i, \operatorname{grad} \phi_i$) = 0 which implies $\operatorname{grad} \phi_i = \mathbf{0}$. It follows that the set of the \mathbf{u}_i 's generates the subspace of \mathbf{V} containing the vectorfields \mathbf{v} with $\operatorname{div}(\varepsilon \mathbf{v}) = 0$.

3.1. On the structure of the spectrum. Eigenvalue problems in the form of (12) are not standard and, to the best of our knowledge, have been mainly used when discussing the spectrum of operators arising from the Least-Squares finite element method.

Problem (12) can be written as follows in terms of operators:

(13)
$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^{\top} \\ \mathsf{B} & \mathsf{C} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = \lambda \begin{pmatrix} \mathsf{0} & \mathsf{D} \\ \mathsf{0} & \mathsf{0} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}$$

The aim of this subsection is to collect some results about the structure of the eigensolutions, discussing in particular the consequences of the possible degeneracy of the right hand side of (13).

First of all, even if the problem does not seem symmetric, it originates from (7) which is associated with a self-adjoint solution operator. Indeed, after observing

that $D = -B^{\top}$, we can argue as in [5] to show that (13) is equivalent to the symmetric problem

$$\begin{pmatrix} \mathsf{A} & \mathsf{0} \\ \mathsf{0} & \mathsf{0} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = (\lambda + 1) \begin{pmatrix} \mathsf{0} & -\mathsf{B}^\top \\ -\mathsf{B} & -\mathsf{C} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}$$

and to the symmetric Schur complement formulations

$$A\mathbf{x} = (\lambda + 1)B^{\top}C^{-1}B\mathbf{x}$$
$$C\mathbf{y} = (\lambda + 1)BA^{-1}B^{\top}\mathbf{y}.$$

We now proceed with some comments that are particularly important in view of the numerical approximation, and of the three-dimensional extension.

Let us assume that $(\mathbf{x}, \mathbf{y})^{\top}$ is such that the right hand side of (13) is vanishing. Then, $\mathbf{B}^{\top}\mathbf{y} = \mathbf{0}$ which implies $\mathbf{curl} p = 0$, that is p = 0 due to zero mean value condition on p. This is not admissible, since Problem (12) seeks a non vanishing p. On the other hand, the numerical approximation of (13) will have the analogous form of a generalized algebraic eigenvalue problem involving matrices and vectors that we denote with the same symbols. It happens that, if we follow the same argument as before (see also Remark 2), we may have solutions that correspond to a generic \mathbf{x} and to $\mathbf{y} = \mathbf{0}$. These solutions correspond to eigenvalues $\lambda = \infty$ that should be discarded in view of the condition $p \neq 0$. This should be taken into account when the numerical results are performed.

It is also interesting to observe what happens if we relax the zero mean value condition on the space Q. In such case there exists a non vanishing y such that $B^{\top}y = -Dy = 0$. Such y corresponds to a **curl**-free p, that is p constant. If we combine $y \in \ker(B^{\top}) = \ker(D)$ with a vanishing x, we can see a singular behavior of (13) that can be summarized by the equation

 $0 = \lambda 0$,

that is, λ cannot be determined. Notice, that this singular behavior doesn't occur if, for instance, the zero mean value condition is dropped in the case of the standard Galerkin approximation of the Neumann Laplace eigenproblem: in such case the original spectrum is modified by adding a vanishing eigenvalue that corresponds to the constant eigenfunction.

4. Two dimensional finite element analysis

Let us consider finite dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$. The approximation of (11) consists in finding $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that

(14)
$$\begin{cases} (\varepsilon \mathbf{u}_h, \mathbf{v}) + (\mu^{-1} \operatorname{rot} \mathbf{u}_h, \operatorname{rot} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} p_h) = (g, \operatorname{rot} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_h \\ - (\mathbf{u}_h, \operatorname{\mathbf{curl}} q) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} p_h, \operatorname{\mathbf{curl}} q) = 0 & \forall q \in Q_h \end{cases}$$

and, correspondingly, the discrete eigenvalue problem we are interested in, reads: find $\lambda_h \in \mathbb{R}$ and $p_h \in Q_h$ with $p_h \neq 0$ such that for some $\mathbf{u}_h \in \mathbf{V}_h$ it holds

(15)
$$\begin{cases} (\varepsilon \mathbf{u}_h, \mathbf{v}) + (\mu^{-1} \operatorname{rot} \mathbf{u}_h, \operatorname{rot} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} p_h) = \lambda_h(p_h, \operatorname{rot} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_h \\ - (\mathbf{u}_h, \operatorname{\mathbf{curl}} q) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} p_h, \operatorname{\mathbf{curl}} q) = 0 & \forall q \in Q_h. \end{cases}$$

We start our analysis of the discrete problem by discussing the convergence of the solution of (14) towards the solution of (11). This is a standard result in the

framework of finite element Least-Squares approximations that follows from the coercivity of the system recalled in Proposition 4.

In the following theorem we recall the a priori error analysis that is a standard consequence of Céa's lemma.

Theorem 1. Given $g \in Q$, let (\mathbf{u}, p) be the solution of (11) and (\mathbf{u}_h, p_h) the corresponding discrete solution of (14). Then the following estimate holds true

$$\|\mathbf{u} - \mathbf{u}_h\|_{\boldsymbol{V}} + \|p - p_h\|_Q \le C \inf_{(\mathbf{v},q)\in\boldsymbol{V}_h\times Q_h} (\|\mathbf{u} - \mathbf{v}\|_{\boldsymbol{V}} + \|p - q\|_Q)$$

In order to study the convergence of the eigensolutions of (15) towards those of (12) a standard tool is the convergence in norm of the sequence of discrete solution operators towards the continuous one. Indeed, in [11] it was shown that for symmetric problems the convergence in norm is not only sufficient but also necessary for the convergence of the eigensolutions.

Analogously to what we have done in the continuous case, let $T_h: Q \to Q$ be the discrete solution operator which associated to $g \in Q$ the component $p_h \in Q_h \subset Q$ of the solution of (14). A necessary and sufficient condition for the convergence of our eigensolutions is the existence of $\rho(h)$, tending to zero as h goes to zero, such that

(16)
$$||(T - T_h)g||_Q \le \rho(h)||g||_Q.$$

The most natural choice of finite element spaces is to consider Nédélec *edge* elements for V_h and standard Lagrange *nodal* elements for Q_h , that is, for $k \ge 0$,

(17)
$$\boldsymbol{V}_{h} = \{ \boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v} |_{K} \in \mathcal{N}_{k}(K) \; \forall K \in \mathcal{T}_{h} \}$$
$$\boldsymbol{Q}_{h} = \{ \boldsymbol{q} \in \boldsymbol{Q} : \boldsymbol{q} |_{K} \in \mathcal{P}_{k+1}(K) \; \forall K \in \mathcal{T}_{h} \},$$

where $\mathcal{P}_k(K)$ is the space of polynomials on K of degree not exceeding k and

$$\mathcal{N}_k(K) = [\mathcal{P}_k(K)]^2 + \mathcal{P}_k(K)[(x,y)]^\top.$$

Other possible choices would involve different order of approximation for the two spaces.

From the standard approximation properties of these spaces, and assumption (5), Theorem 1 gives

$$\|(T - T_h)g\|_Q = \|p - p_h\|_Q \le C \inf_{(\mathbf{v},q)\in \mathbf{V}_h\times Q_h} (\|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}} + \|p - q\|_Q).$$

From Remark 1 we can take s > 1/2 and proceed as follows.

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|_{\boldsymbol{V}}^2 &= \|\varepsilon^{1/2}(\mathbf{u} - \mathbf{v})\|_0^2 + \|\mu^{-1/2}\operatorname{rot}(\mathbf{u} - \mathbf{v})\|_0^2 \\ &\leq C\overline{\varepsilon}h^{2s}\|\mathbf{u}\|_s^2 + \|\mu^{-1/2}\operatorname{rot}(\mathbf{u} - \mathbf{v})\|_0^2. \end{aligned}$$

We assume that μ is piecewise regular and that the mesh is compatible in the sense that μ is smooth in each element, so that rot **u** belongs to $H^{s}(K)$ for each $K \in \mathcal{T}_{h}$. Then the estimate reads

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|_{\boldsymbol{V}}^2 &\leq C\overline{\varepsilon}h^{2s} \|\mathbf{u}\|_s^2 + \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\mu}^{-1/2}\operatorname{rot}(\mathbf{u} - \mathbf{v})\|_{0,K}^2 \\ &\leq C\overline{\varepsilon}h^{2s} \|\mathbf{u}\|_s^2 + C\underline{\mu}^{-1}\sum_{K \in \mathcal{T}_h} h_K^{2s} \|\operatorname{rot} \mathbf{u}\|_{s,K}^2 \\ &\leq C\overline{\varepsilon}h^{2s} \|\mathbf{u}\|_s^2 + C(\varepsilon, \mu)h^{2s} \|g\|_Q. \end{aligned}$$

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For the variable p standard approximation properties give

$$||p-q||_Q \le C \underline{\varepsilon}^{-1/2} h^s ||p||_{1+s},$$

so that the final estimate reads

(18)
$$\|(T - T_h)g\|_Q \le Ch^s \|g\|_Q.$$

Hence, we can conclude the convergence of the eigensolutions (and the absence of spurious modes). In the next theorem we state this result, together with the optimal rate of convergence of the eigenmodes.

Remark 3. The definition of the finite element spaces (17) considers a balanced choice of the polynomial degrees (see also (18)). For the well posedness of the discrete problem, however, there are no compatibility conditions between the spaces and more general choices could be made.

Theorem 2. For $k \ge 0$, let V_h and Q_h be as in the above definitions, and assume that λ is an eigenvalue of (12) of multiplicity m with associated eigenspace E. Then there exist exactly m eigenvalues $\lambda_{1,h} \le \cdots \le \lambda_{m,h}$ converging to λ . Moreover, let us denote by E_h the space spanned by eigenfunctions associated with the m discrete eigenvalues. Then

$$\begin{aligned} |\lambda - \lambda_{i,h}| &\leq C\epsilon(h)^2 \quad (i = 1, \dots, m) \\ \hat{\delta}(E, E_h) &\leq C\epsilon(h), \end{aligned}$$

where

$$\epsilon(h) = \sup_{\substack{p \in E \\ \|p\|_1 = 1}} \|(T - T_h)p\|_Q$$

and $\hat{\delta}(A, B)$ denotes the gap between the subspaces A and B of Q.

Remark 4. The eigenspace E and the space E_h refer to the components p and p_h of the solutions of (12) and (15), respectively. However, the variables we are interested in when discussing the eigenproblem associated with Maxwell's equations, are \mathbf{u} and \mathbf{u}_h . On the other hand, problems (12) and (15) provide us also with the other (unique) component of the solution which satisfies the following approximation property. Let us denote by F the space spanned by \mathbf{u} associated with λ in (12) and by F_h the discrete space spanned by the corresponding $\{\mathbf{u}_{1,h}, \ldots, \mathbf{u}_{m,h}\}$ obtained from (15). Then we have

$$\hat{\delta}_{\rm rot}(F, F_h) \le C\epsilon(h)$$

where $\hat{\delta}_{rot}(A, B)$ denotes the gap between subspaces A and B of $H_0(rot; \Omega)$.

We also observe that if $E \subset H^{r_1+1}$ and $F \subset H^{r_2}(\operatorname{rot})$ then $\epsilon(h) = O(h^t)$ with $t = \min(k+1, r_1, r_2)$.

One of the main messages that are conveyed when discussing Least-Squares finite element methods, is that, thanks to the coercivity of the formulation, any choice of finite element spaces is admissible, without the need of satisfying a compatibility condition.

Several authors have tried to approximate the eigensolutions of the resonant cavity by using standard *nodal* elements. With this in mind, we might think of choosing as V_h a space of Lagrange nodal elements in each component and as Q_h a space of Lagrange nodal elements as well. For instance, an equal order approximation would involve the following spaces:

$$\boldsymbol{V}_h = \{ \mathbf{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{V} : \mathbf{v}|_K \in (\mathcal{P}_k(K))^2 \; \forall K \in \mathcal{T}_h \}$$

$$Q_h = \{ q \in Q : q|_K \in \mathcal{P}_k(K) \; \forall K \in \mathcal{T}_h \}.$$

In this case, the analogous of (18) can be derived from Theorem 1 and the approximation properties of the finite element spaces as follows

(19)
$$\| (T - T_h)g \|_Q = \| p - p_h \|_Q \le C \inf_{(\mathbf{v},q) \in \mathbf{V}_h \times Q_h} (\| \mathbf{u} - \mathbf{v} \|_{\mathbf{V}} + \| p - q \|_Q)$$
$$\le C \left(\inf_{\mathbf{v} \in \mathbf{V}_h} \| \mathbf{u} - \mathbf{v} \|_{\mathbf{V}} + h^s \| p \|_{1+s} \right).$$

The estimate of the first term in the right hand side of (19) is not as immediate as in the case of edge elements. Indeed, in the case of nodal element, the best approximation in $\boldsymbol{H}(\operatorname{rot};\Omega)$ is not in general better than the best approximation in $\boldsymbol{H}^1(\Omega)$. It follows that (19) gives

$$||(T - T_h)g||_Q \le Ch^s(||\mathbf{u}||_{1+s} + ||p||_{1+s}).$$

On the other hand, since $\varepsilon \mathbf{u} = \operatorname{\mathbf{curl}} p$, we get

$$||(T - T_h)g||_Q \le Ch^s ||p||_{2+s}$$

and we can obtain the convergence in norm (16) if Problem (9) satisfies the a priori bound

$$||p||_{2+s} \le C ||g||_Q.$$

From Remark 1 we know that this regularity holds true only in very particular circumstances. In particular, we can prove the optimal convergence of the eigensolutions (and the absence of spurious modes) in the case of ε and μ constant, and when Ω is a square. More general configurations are controversial from the theoretical point of view. On the other hand, our numerical simulations presented in the next section show that the method is pretty robust also in presence of strongly singular solutions.

5. Numerical examples in two dimensions

Analogously to what was observed in Section 3.1, the algebraic system associated with the discrete eigenvalue problem has the form

$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & \mathsf{C} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = \lambda \begin{pmatrix} \mathsf{0} & \mathsf{D} \\ \mathsf{0} & \mathsf{0} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix},$$

where the blocks of the matrices correspond to the pieces in (15) according to the following mapping

$$\begin{aligned} \mathsf{A}: & (\varepsilon \mathbf{u}_h, \mathbf{v}_h) + (\mu^{-1} \operatorname{rot} \mathbf{u}_h, \operatorname{rot} \mathbf{v}_h) \\ \mathsf{B}: & -(\mathbf{u}_h, \operatorname{\mathbf{curl}} q_h) \\ \mathsf{C}: & (\varepsilon^{-1} \operatorname{\mathbf{curl}} p_h, \operatorname{\mathbf{curl}} q_h) \\ \mathsf{D}: & (p_h, \operatorname{rot} \mathbf{v}_h). \end{aligned}$$

It turns out that this is a degenerate generalized eigenvalue problem. It is out of the scope of this paper to discuss the best strategy for its resolution. In our FEniCS code [19, 18] we call the SLEPc solver [22]. Our options include the use

Exact		Computed (r	ate)
1.00000	1.01090	1.00273(2.00)	1.00068(2.00)
1.00000	1.01268	1.00316(2.00)	1.00079(2.00)
2.00000	2.04063	2.01017(2.00)	2.00254(2.00)
4.00000	4.11271	4.02792(2.01)	4.00696(2.00)
4.00000	4.11276	4.02793(2.01)	4.00697(2.00)
5.00000	5.14696	5.03683(2.00)	5.00922(2.00)
5.00000	5.23986	5.05920(2.02)	5.01476(2.00)
8.00000	8.49059	8.12382(1.99)	8.03103(2.00)
9.00000	9.49795	9.12178(2.03)	9.03028(2.01)
9.00000	9.51605	9.12582(2.04)	9.03127(2.01)
Mesh	1/16	1/32	1/64

TABLE 1. Edge elements on a uniform mesh of the square

of mumps [2] for dealing with the singular matrix on the right hand side and shift and invert with shift equal to zero to compute the smallest eigenvalues.

A crucial comment, which will become essential for the three dimensional extension of our tests, refers to the gauge condition of the variables p and q. According to the variational formulation (15), we should impose on p and q the zero mean value condition after we multiply them by μ . This can be easily performed by adding a Lagrange multiplier at the expense of increasing by one the dimension of the system. The full variational formulation would read: find $\lambda_h \in \mathbb{R}$ and $p_h \in \tilde{Q}_h$, with $p_h \neq 0$, such that for some $\mathbf{u}_h \in \mathbf{V}_h$ and $\phi_h \in \mathbb{R}$

(20)
$$\begin{cases} (\varepsilon \mathbf{u}_h, \mathbf{v}) + (\mu^{-1} \operatorname{rot} \mathbf{u}_h, \operatorname{rot} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} p_h) = \lambda_h(p_h, \operatorname{rot} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_h \\ - (\mathbf{u}_h, \operatorname{\mathbf{curl}} q) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} p_h, \operatorname{\mathbf{curl}} q) + (\phi_h, \mu q) = 0 & \forall q \in \widetilde{Q}_h \\ (\mu p_h, \psi) = 0 & \forall \psi \in \mathbb{R}, \end{cases}$$

where \hat{Q}_h is the same finite element space as in (15) without imposing the zero mean value condition.

Clearly, problems (15) and (20) are equivalent.

Remark 5. Similarly to what we observed at the end of Section 3.1, if we solve the unconstrained problem (15) with Q_h replaced by \widetilde{Q}_h instead of (20), the only difference is the introduction of an additional singular pencil corresponding to the equation $\mathbf{0} = \lambda \mathbf{0}$ with $\mathbf{u}_h = \mathbf{0}$ and p constant.

The first test that we present is a sanity check on the square $(0, \pi)^2$ and $\mu = \varepsilon = 1$. It is well known that the eigenvalues are $m^2 + n^2$ for $m, n \ge 0$ and m + n > 0. Table 1 confirms the optimal convergence of the edge element approximation. Since the theoretical results clearly indicate that edge elements are optimally convergent, we focus now on a deeper investigation when nodal elements are used.

In Tables 2 and 3 we perform the same test with nodal (continuous piecewise linear) elements on a structured and nonstructured mesh, respectively. It turns out that also in this case the convergence is optimal and there are no spurious modes.

From (19) it was clear that the regularity of the solution is essential in order to prove theoretically the convergence of the method. It is then essential to verify the approximation behavior in presence of singular solutions. We start with the

Exact		Computed (r	ate)
1.00000	1.00961	1.00240(2.00)	1.00060(2.00)
1.00000	1.00963	1.00241(2.00)	1.00060(2.00)
2.00000	2.03841	2.00962(2.00)	2.00241(2.00)
4.00000	4.11637	4.02886(2.01)	4.00721(2.00)
4.00000	4.11642	4.02886(2.01)	4.00721(2.00)
5.00000	5.15263	5.03849(1.99)	5.00966(1.99)
5.00000	5.23639	5.05860(2.01)	5.01464(2.00)
8.00000	8.49203	8.12453(1.98)	8.03125(1.99)
9.00000	9.56410	9.13748(2.04)	9.03424(2.01)
9.00000	9.56410	9.13752(2.04)	9.03425(2.01)
Mesh	1/16	1/32	1/64

TABLE 2. Nodal elements on a uniform mesh of the square

TABLE 3. Nodal elements on a nonuniform mesh of the square

Exact		Computed (ra	ate)
1.00000	1.00478	1.00118(2.01)	1.00029(2.01)
1.00000	1.00490	1.00120(2.03)	1.00030(2.02)
2.00000	2.01564	2.00385(2.02)	2.00095(2.01)
4.00000	4.05564	4.01362(2.03)	4.00336(2.02)
4.00000	4.05739	4.01389(2.05)	4.00341(2.03)
5.00000	5.08616	5.02078(2.05)	5.00512(2.02)
5.00000	5.08816	5.02096(2.07)	5.00515(2.03)
8.00000	8.21196	8.05101(2.05)	8.01255(2.02)
9.00000	9.26651	9.06385(2.06)	9.01583(2.01)
9.00000	9.27064	9.06476(2.06)	9.01591(2.02)
Mesh	1/16	1/32	1/64

TABLE 4. Nodal elements on a uniform mesh of the L-shaped domain

Exact		Computed (ra	.te)
1.47562	1.60421	1.52532(1.37)	1.49509(1.35)
3.53403	3.56787	3.54233(2.03)	3.53609(2.01)
9.86960	10.07466	9.92010(2.02)	9.88218(2.01)
9.86960	10.07466	9.92010(2.02)	9.88218(2.01)
11.38948	11.70401	11.46698(2.02)	11.40879(2.00)
Mesh	1/16	1/32	1/64

L-shaped domain obtained by the removing the upper right square $(0, 1)^2$ from the square $(-1, 1)^2$, where we consider the reference solution presented in [15]. Tables 4 and 5 show that also in this case the nodal element approximation of our problem performs optimally and that no spurious modes are present. It can be appreciated that singular solutions present the expected lower rate of convergence.

We continue our investigations by considering more and more challenging problems. The next one is the so called slit domain, that is the square $(-1,1)^2$ from

Exact			Computed (ra	ite)	
1.47562	1.61215	1.53721(1.15)	1.51774(0.55)	1.49583(1.06)	1.48561(1.02)
3.53403	3.55865	3.54038(1.96)	3.53568(1.95)	3.53440(2.18)	3.53414(1.76)
9.86960	9.99824	9.90032(2.07)	9.87722(2.01)	9.87151(2.00)	9.87008(2.01)
9.86960	10.00032	9.90054(2.08)	9.87739(1.99)	9.87152(2.03)	9.87008(2.01)
11.38948	11.56604	11.43105(2.09)	11.39996(1.99)	11.39204(2.03)	11.39012(1.99)
Mesh	1/16	1/32	1/64	1/128	1/256

TABLE 5. Nodal elements on a nonuniform mesh of the L-shaped domain

TABLE 6. Nodal elements on a uniform mesh of the slit domain

Exact		Computed (ra	ite)
1.03407	1.36608	1.19389(1.05)	1.11254(1.03)
2.46740	2.48205	2.47105(2.01)	2.46831(2.00)
4.04693	4.09184	4.05806(2.01)	4.04970(2.00)
9.86960	10.07466	9.92010(2.02)	9.88218(2.01)
9.86960	10.07466	9.92010(2.02)	9.88218(2.01)
10.84485	11.11915	10.91246(2.02)	10.86170(2.00)
12.26490	12.72449	12.43232(1.46)	12.36074(0.80)
12.33701	12.96253	12.51124(1.84)	12.36438(2.67)
19.73921	20.82226	20.00356(2.03)	19.80490(2.01)
21.24411	22.81415	21.74694(1.64)	21.43012(1.43)
Mesh	1/16	1/32	1/64

which we remove the segment $(0,1) \times \{0\}$. This is the so called *cracked domain* in [15] that we use as a reference solution. Also in this case, nodal elements behave optimally, in agreement with the regularity of the solution, see Table 6.

In order to make the solution even more singular, we consider the same slit domain with mixed boundary conditions, that is we modify the boundary conditions in the definition of the spaces V and Q as follows: functions in V have zero tangential component along the exterior boundary, while functions in Q are vanishing on the slit.

Since we do not have a reference solution in this case, we compare our results with the eigenvalues computed with a standard Galerkin approximation of the Laplace eigenvalue problem with mixed boundary conditions (Dirichlet on the slit and Neumann on the rest of the boundary). Table 7 shows that also in this case our solution is convergent and that no spurious modes are present.

Finally, we challenge our code with the computation of the problem with different materials, that is jumping coefficients ε and μ . In this case, we compare with the standard *curl curl* formulation discretized by edge elements. We take a uniform mesh compatible with the jump of the material. The test cases consider the square $(0, \pi)^2$ containing a different material in the bottom-left square $(0, \pi/2)^2$. Table 8 shows the comparison of the results when ε is equal to 100 on the different material and 1 otherwise, with $\mu = 1$ everywhere, while Table 9 deals with the case when $\varepsilon = 0$ everywhere and $\mu = 1/100$ on the different material and 1 otherwise. Also in this case our method performs quite well.

Rank	Computed with standard Galerkin			
1	1.27238	1.14957	1.09097	
2	2.48205	2.47105	2.46831	
3	4.09155	4.05803	4.04970	
4	5.00737	4.95283	4.93930	
5	11.11902	10.91244	10.86170	
6	12.72449	12.43232	12.35737	
7	12.93065	12.49669	12.36074	
8	22.78151	21.73189	21.42284	
9	23.23963	22.45620	22.26849	
10	25.15087	24.18459	23.95261	
Rank	Con	puted with Leas	st-Squares	
1	1.09561	1.06413	1.04894	
2	2.47402	2.46905	2.46781	
3	4.07072	4.05292	4.04843	
4	4.97718	4.94538	4.93744	
5	10.99818	10.88309	10.85441	
6	12.56704	12.39424	12.33447	
7	12.71164	12.43105	12.35130	
8	22.25050	21.59440	21.38219	
9	22.75004	22.34091	22.24009	
10	24.57890	24.05065	23.91967	
Rank		Difference (ra	te)	
1	0.17677	0.08544(1.05)	0.04203(1.02)	
2	0.00803	0.00199(2.01)	0.00050(2.00)	
3	0.02084	0.00511(2.03)	0.00127(2.01)	
4	0.03019	0.00746(2.02)	0.00186(2.00)	
5	0.12084	0.02935(2.04)	0.00728(2.01)	
6	0.15745	0.03808(2.05)	0.02289(0.73)	
7	0.21901	0.06564(1.74)	0.00944(2.80)	
8	0.53101	0.13749(1.95)	0.04065(1.76)	
9	0.48960	0.11529(2.09)	0.02840(2.02)	
10	0.57197	0.13394(2.09)	0.03294(2.02)	
Mesh	1/16	1/32	1/64	

TABLE 7. Nodal elements on a uniform mesh of the slit domainwith mixed boundary conditions

We conclude this section by showing some more numerical results which are not completely covered by the theory. Namely we consider the situation when the jump in the coefficient is not aligned with the mesh. In order to do so, we consider the same geometry as in the previous examples reported in Tables 8 and 9 with a nonuniform mesh of the domain Ω . It turns out that in this case the background mesh is not fitted with the material discontinuities. The results are pretty much convincing also in this case, as it can be seen from Table 10 in the case of jumping ε and in Table 11 when μ jumps.

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TABLE 8. Comparison of curl curl formulation (edge elements) and our formulation (nodal elements) on a uniform and fitted mesh of the square with jumping ε

Rank	Computed with standard Galerkin			
1	0.01294	0.01295	0.01295	
2	0.01425	0.01425	0.01425	
3	0.02579	0.02579	0.02579	
4	0.04612	0.04613	0.04613	
5	0.05126	0.05128	0.05129	
6	0.09252	0.09261	0.09264	
7	0.09407	0.09412	0.09413	
8	0.09971	0.09973	0.09973	
9	0.10740	0.10744	0.10746	
10	0.11554	0.11556	0.11557	
Rank	Cor	nputed with Lea	st-Squares	
1	0.01483	0.01444	0.01391	
2	0.01602	0.01464	0.01433	
3	0.02677	0.02616	0.02596	
4	0.04732	0.04645	0.04622	
5	0.05505	0.05286	0.05205	
6	0.09655	0.09458	0.09359	
7	0.09726	0.09480	0.09433	
8	0.10286	0.10054	0.09995	
9	0.11148	0.10893	0.10808	
10	0.12236	0.11778	0.11642	
Rank		Difference (ra	ate)	
1	0.00189	0.00149(0.34)	0.00096(0.64)	
2	0.00177	0.00039(2.19)	0.00007(2.46)	
3	0.00098	0.00037(1.40)	0.00016(1.18)	
4	0.00120	0.00031(1.93)	0.00009(1.87)	
5	0.00379	0.00158(1.26)	0.00075(1.07)	
6	0.00403	0.00197(1.04)	0.00094(1.06)	
7	0.00319	0.00068(2.24)	0.00020(1.77)	
8	0.00315	0.00082(1.94)	0.00022(1.90)	
9	0.00408	0.00149(1.45)	0.00062(1.27)	
10	0.00682	0.00222(1.62)	0.00085(1.39)	
Mesh	1/64	1/128	1/256	

6. The three dimensional case

We start by writing a possible version of the three dimensional variational problem associated with (6) in the case when the uniqueness of **p** is enforced by (4). Let $\mathbf{V} = \mathbf{H}_0(\mathbf{curl}; \Omega)$ and $\mathbf{Q}^0 = \mathbf{H}(\mathbf{curl}) \cap \mathbf{H}_0(\operatorname{div}^0; \Omega; \mu)$. The continuous problem reads: given $\mathbf{g} \in \mathbf{H}_0(\operatorname{div}^0; \Omega; \mu)$, find $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}^0$ such that

(21)
$$\begin{cases} (\varepsilon \mathbf{u}, \mathbf{v}) + (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} \mathbf{p}) = (\mathbf{g}, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ - (\mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{q}) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} \mathbf{p}, \operatorname{\mathbf{curl}} \mathbf{q}) = 0 & \forall \mathbf{q} \in \mathbf{Q}^0. \end{cases}$$

Rank	Computed with standard Galerkin			
1	4.44469	4.44455	4.44451	
2	5.32703	5.32940	5.33000	
3	11.85091	11.85115	11.85121	
4	16.81898	16.83221	16.83553	
5	17.54526	17.56033	17.56410	
6	24.83957	24.82828	24.82545	
7	25.87595	25.90293	25.90970	
8	36.72957	36.78732	36.80175	
9	37.55359	37.62456	37.64233	
10	39.98800	39.97241	39.96839	
Rank	Con	puted with Leas	st-Squares	
1	4.49865	4.47280	4.45740	
2	5.43938	5.37976	5.35001	
3	12.19870	12.03015	11.93174	
4	17.01187	16.90259	16.86498	
5	17.73776	17.62300	17.58675	
6	25.38670	25.04539	24.91860	
7	26.46349	26.16889	26.03923	
8	37.38132	36.96354	36.85391	
9	38.25990	37.81014	37.69398	
10	41.12399	40.40199	40.15998	
Rank		Difference (ra	te)	
1	0.05397	0.02826(0.93)	0.01288(1.13)	
2	0.11235	0.05035(1.16)	$0.02001 \ (1.33)$	
3	0.34779	0.17900(0.96)	0.08053(1.15)	
4	0.19290	0.07038(1.45)	0.02946(1.26)	
5	0.19249	0.06267(1.62)	0.02265(1.47)	
6	0.54712	0.21712(1.33)	0.09315(1.22)	
7	0.58754	0.26595(1.14)	0.12953(1.04)	
8	0.65176	0.17622(1.89)	0.05216(1.76)	
9	0.70631	0.18558(1.93)	0.05164(1.85)	
10	1.13599	0.42958(1.40)	0.19159(1.16)	
Mesh	1/64	1/128	1/256	

TABLE 9. Comparison of curl curl formulation (edge elements) and our formulation (nodal elements) on a uniform and fitted mesh of the square with jumping μ

The next proposition states the ellipticity of the bilinear form associated with the above problem. We observe in particular that the norm of Q^0 is the same as the H(curl) norm since the space contains divergence free vectorfields.

Proposition 7. Let

$$a: (\boldsymbol{V} imes \boldsymbol{Q}^0) imes (\boldsymbol{V} imes \boldsymbol{Q}^0)
ightarrow \mathbb{R}$$

be the bilinear form associated with the formulation (21), that is $a(\mathbf{u}, \mathbf{p}; \mathbf{v}, \mathbf{q}) = (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{v}) + (\varepsilon^{1/2} \mathbf{u} - \varepsilon^{-1/2} \operatorname{\mathbf{curl}} \mathbf{p}, \varepsilon^{1/2} \mathbf{v} - \varepsilon^{-1/2} \operatorname{\mathbf{curl}} \mathbf{q}).$

TABLE 10. Comparison of curl curl formulation (edge elements) and our formulation (nodal elements) on a nonuniform and unfitted mesh of the square with jumping ε

Rank	Computed with standard Galerkin			
1	0.01306	0.01299	0.01298	
2	0.01450	0.01437	0.01431	
3	0.02607	0.02592	0.02585	
4	0.04625	0.04619	0.04616	
5	0.05163	0.05141	0.05138	
6	0.09324	0.09286	0.09282	
7	0.09584	0.09496	0.09455	
8	0.10070	0.10018	0.09993	
9	0.10921	0.10818	0.10786	
10	0.11742	0.11644	0.11603	
Rank	Cor	nputed with Lea	st-Squares	
1	0.01566	0.01457	0.01407	
2	0.01876	0.01632	0.01528	
3	0.02909	0.02729	0.02653	
4	0.04754	0.04666	0.04635	
5	0.05450	0.05285	0.05226	
6	0.09631	0.09445	0.09377	
7	0.09978	0.09701	0.09569	
8	0.10723	0.10225	0.10066	
9	0.11593	0.11089	0.10925	
10	0.12434	0.11951	0.11753	
Rank		Difference (ra	ate)	
1	0.00259	0.00158(0.71)	0.00108(0.54)	
2	0.00426	0.00195(1.13)	0.00097(1.00)	
3	0.00302	0.00137(1.14)	0.00068(1.02)	
4	0.00130	0.00047(1.47)	0.00019(1.27)	
5	0.00288	0.00144(1.00)	0.00088(0.71)	
6	0.00307	0.00159(0.95)	0.00095(0.74)	
7	0.00394	0.00205(0.94)	0.00114(0.84)	
8	0.00652	0.00207(1.65)	0.00073(1.50)	
9	0.00672	0.00271(1.31)	0.00139(0.96)	
10	0.00691	0.00307(1.17)	0.00150(1.03)	
Mesh	1/64	1/128	1/256	

Then there exists $\alpha > 0$ such that

$$a(\mathbf{v}, \mathbf{q}; \mathbf{v}, \mathbf{q}) \ge \alpha \left(\|\mathbf{v}\|_{\mathbf{curl}}^2 + \|\mathbf{q}\|_{\mathbf{curl}}^2 \right).$$

Proof. The proof is obtaining by extending the two dimensional proof of Proposition 4. The only critical point is the estimate of $\|\mathbf{q}\|_0$ in terms of $\|\mathbf{curl q}\|_0$ which can be performed thanks to the following Friedrichs inequality (see, for instance [3,

Rank	Computed with standard Galerkin			
1	4.33537	4.39753	4.41930	
2	5.19513	5.26094	5.29734	
3	11.57250	11.73322	11.78452	
4	16.44114	16.64371	16.73990	
5	17.10908	17.34339	17.46013	
6	24.25033	24.57767	24.68472	
7	25.25419	25.63192	25.76661	
8	35.92482	36.36569	36.59130	
9	36.70768	37.20499	37.42931	
10	38.93117	39.50523	39.73828	
Rank	Con	puted with Leas	st-Squares	
1	4.34099	4.39904	4.41972	
2	5.20261	5.26322	5.29785	
3	11.60469	11.74193	11.78668	
4	16.49871	16.65967	16.74373	
5	17.17074	17.35981	17.46425	
6	24.37575	24.60994	24.69306	
7	25.38645	25.66805	25.77518	
8	36.18030	36.43259	36.60809	
9	36.97634	37.27509	37.44661	
10	39.23970	39.58511	39.75854	
Rank		Difference (ra	te)	
1	0.00562	0.00151(1.90)	0.00041(1.87)	
2	0.00749	0.00228(1.71)	0.00051(2.15)	
3	0.03219	0.00872(1.88)	0.00217(2.01)	
4	0.05757	0.01596(1.85)	0.00383(2.06)	
5	0.06166	0.01643(1.91)	0.00412(1.99)	
6	0.12542	0.03227(1.96)	0.00834(1.95)	
7	0.13227	0.03613(1.87)	0.00856(2.08)	
8	0.25548	0.06690(1.93)	0.01679(1.99)	
9	0.26866	0.07009(1.94)	0.01730(2.02)	
10	0.30854	0.07987(1.95)	0.02026(1.98)	
Mesh	1/64	1/128	1/256	

TABLE 11. Comparison of curl curl formulation (edge elements) and our formulation (nodal elements) on a nonuniform and unfitted mesh of the square with jumping μ

Cor. 3.16]): there exists a constant C_F such that

 $\|\mathbf{q}\|_0 \le C_F \|\operatorname{\mathbf{curl}} \mathbf{q}\|_0 \qquad \forall \mathbf{q} \in \boldsymbol{Q}^0.$

From the ellipticity we deduce from Lax–Milgram lemma that problem (21) admits a unique solution with the stability estimate

$$\|\mathbf{u}\|_{\mathbf{curl}} + \|\mathbf{p}\|_{\mathbf{curl}} \le C \|\mathbf{g}\|_0.$$

As already observed, dealing with the approximation of Q^0 is not easy, so that we propose the following formulation which imposes the divergence free condition with an additional Lagrange multiplier $\phi \in H^1(\Omega) \cap L^2_0(\Omega)$.

Let $V = H_0(\operatorname{curl}; \Omega)$, $Q = H(\operatorname{curl})$, and $W = H^1(\Omega) \cap L^2_0(\Omega)$. We consider the following problem: given $\mathbf{g} \in H_0(\operatorname{div}^0; \Omega; \mu)$, find $(\mathbf{u}, \mathbf{p}, \phi) \in \mathbf{V} \times \mathbf{Q} \times W$ such that

(22)
$$\begin{cases} (\varepsilon \mathbf{u}, \mathbf{v}) + (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} \mathbf{p}) = (\mathbf{g}, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ - (\mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{q}) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} \mathbf{p}, \operatorname{\mathbf{curl}} \mathbf{q}) + (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \phi) = 0 & \forall \mathbf{q} \in \mathbf{Q} \\ (\mu \mathbf{p}, \operatorname{\mathbf{grad}} \psi) = 0 & \forall \psi \in W. \end{cases}$$

The following proposition states the equivalence of Problems (22) and (21).

Proposition 8. Let (\mathbf{u}, \mathbf{p}) be a solution of (21), then $(\mathbf{u}, \mathbf{p}, \phi)$ solves (22) with $\phi = 0$. Conversely, if $(\mathbf{u}, \mathbf{p}, \phi)$ solves (22), then \mathbf{p} is in \mathbf{Q}^0 and (\mathbf{u}, \mathbf{p}) is a solution of (21).

Proof. If (\mathbf{u}, \mathbf{p}) is a solution of (21) and $\phi = 0$, then clearly the first two equations of (22) are satisfied. Moreover, if \mathbf{p} belongs to \mathbf{Q}^0 then it also satisfies the third equation.

Vice versa, the third equation of (22) implies that \mathbf{p} belongs to Q^0 . Indeed, by integration by parts we formally have

$$\mu \mathbf{p}, \mathbf{grad}\,\psi) = -(\operatorname{div}(\mu \mathbf{p}), \psi) + \langle (\mu \mathbf{p}) \cdot \mathbf{n}, \psi \rangle_{\partial \Omega}.$$

Taking ψ in $\mathscr{D}(\Omega)$ we obtain that $\mu \mathbf{p}$ is in $\mathbf{H}(\operatorname{div}; \Omega)$ and $\operatorname{div}(\mu \mathbf{p}) = 0$ in Ω . Then, a generic $\psi \in W$ gives the result on the boundary that the trace of $(\mu \mathbf{p}) \cdot \mathbf{n}$ is vanishing.

It remains to show that ϕ is equal to zero so that the second equation of (22) corresponds to the second equation of (21). Taking $\mathbf{q} = \mathbf{grad} \phi$ we have $\|\mu^{1/2} \mathbf{grad} \phi\|_0 = 0$, that is $\mathbf{grad} \phi = 0$, so that $\phi = 0$ because it is in $L_0^2(\Omega)$.

For completeness, we give a direct proof of the existence and uniqueness of the solution of (22).

Proposition 9. Let \mathbf{g} be in $\mathbf{H}_0(\operatorname{div}^0; \Omega; \mu)$, then there exists a unique solution $(\mathbf{u}, \mathbf{p}, \phi) \in \mathbf{V} \times \mathbf{Q} \times W$ of Problem (22) that satisfies $\phi = 0$ and

$$\|\mathbf{u}\|_{\mathbf{curl}} + \|\mathbf{p}\|_{\mathbf{curl}} \le C \|\mathbf{g}\|_0.$$

Proof. By extending to $\mathbf{V} \times \mathbf{Q}$ the bilinear form a defined in the statement of Proposition 7, we can rewrite Problem (22) as: find $(\mathbf{u}, \mathbf{p}, \phi) \in \mathbf{V} \times \mathbf{Q} \times W$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{p}; \mathbf{v}, \mathbf{q}) + (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \phi) = (\mathbf{g}, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall (\mathbf{v}, \mathbf{q}) \in \mathbf{V} \times \mathbf{Q} \\ (\mu \mathbf{p}, \operatorname{\mathbf{grad}} \psi) = 0 & \forall \psi \in W. \end{cases}$$

This is a typical saddle point problem for which we show the *ellipticity in the kernel* and the *inf-sup condition*.

The kernel is defined as

$$K = \{ (\mathbf{v}, \mathbf{q}) \in \mathbf{V} \times \mathbf{Q} : (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \psi) = 0 \ \forall \psi \in W \},\$$

that is, $K = \mathbf{V} \times \mathbf{Q}^0$ (see the proof of Proposition 8). Hence, the ellipticity in the kernel follows from the ellipticity result proved in Proposition 7.

The inf-sup condition

$$\inf_{\psi \in W} \sup_{\mathbf{q} \in \boldsymbol{Q}} \frac{(\mu \mathbf{q}, \operatorname{\mathbf{grad}} \psi)}{\|\mathbf{q}\|_{\operatorname{\mathbf{curl}}} \|\psi\|_1} \ge \beta_0$$

follows in a standard way by observing that, given $\psi \in W,\, \mathbf{q} = \mathbf{grad}\,\psi$ belongs to ${\pmb Q}$ and

$$\frac{(\mu \mathbf{q}, \operatorname{\mathbf{grad}} \psi)}{\|\mathbf{q}\|_{\operatorname{\mathbf{curl}}}} \ge \beta_0 \|\psi\|_1,$$

where β_0 depends on the Poincaré constant C_P .

ι

Hence, the existence and stability follows from the standard results of mixed problems (see [10], for instance) and ϕ is zero as observed in the proof of Proposition 8.

Let us introduce a general finite element discretization of Problem (22) by considering finite dimensional subspaces $V_h \subset V$, $Q_h \subset Q$ and $W_h \subset W$. Then the discrete counterpart of (22) reads: given $\mathbf{g} \in H_0(\operatorname{div}^0; \Omega; \mu)$, find $(\mathbf{u}_h, \mathbf{p}_h, \phi_h) \in$ $V_h \times Q_h \times W_h$ such that

(23)
$$\begin{cases} (\varepsilon \mathbf{u}_h, \mathbf{v}) + (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}_h, \operatorname{\mathbf{curl}} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} \mathbf{p}_h) = (\mathbf{g}, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{V}_h \\ - (\mathbf{u}_h, \operatorname{\mathbf{curl}} \mathbf{q}) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} \mathbf{p}_h, \operatorname{\mathbf{curl}} \mathbf{q}) + (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \phi_h) = 0 & \forall \mathbf{q} \in \boldsymbol{Q}_h \\ (\mu \mathbf{p}_h, \operatorname{\mathbf{grad}} \psi) = 0 & \forall \psi \in W_h. \end{cases}$$

The analysis of existence and uniqueness of the solution of this problem can be performed following the same lines of the proof of Proposition 9. The crucial point is to show the ellipticity of the bilinear form a on the discrete kernel. This is not true for any choice of discrete spaces, therefore we consider Nédélec edge elements for both V_h and Q_h and standard Lagrange nodal elements for W_h . For $K \in \mathcal{T}_h$ and $k \geq 0$, let $\mathcal{N}_k(K)$ be the following space:

$$\mathcal{N}_k(K) = [\mathcal{P}_k(K)]^3 + \mathcal{P}_k(K)[(x, y, z)]^\top,$$

then we introduce the following discrete spaces

(24) $V_{h} = \{ \mathbf{v} \in \mathbf{V} : \mathbf{v}|_{K} \in \mathcal{N}_{k}(K) \; \forall K \in \mathcal{T}_{h} \}$ $Q_{h} = \{ \mathbf{q} \in \mathbf{Q} : \mathbf{q}|_{K} \in \mathcal{N}_{k}(K) \; \forall K \in \mathcal{T}_{h} \}$ $W_{h} = \{ \psi \in W : \psi|_{K} \in \mathcal{P}_{k}(K) \; \forall K \in \mathcal{T}_{h} \}.$

With this choice for the finite element spaces we can show the following proposition:

Proposition 10. Let \mathbf{g} be in $\mathbf{H}_0(\operatorname{div}^0; \Omega; \mu)$, then there exists a unique solution $(\mathbf{u}_h, \mathbf{p}_h, \phi_h) \in \mathbf{V}_h \times \mathbf{Q}_h \times W_h$ of Problem (23) that satisfies $\phi_h = 0$ and

$$\|\mathbf{u}_h\|_{\mathbf{curl}} + \|\mathbf{p}_h\|_{\mathbf{curl}} \le C \|\mathbf{g}\|_0.$$

Moreover the following error estimates holds true:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{curl}} + \|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{curl}} \le C \inf_{(\mathbf{v}, \mathbf{q}) \in \mathbf{V}_h \times \mathbf{Q}_h} \left(\|\mathbf{u} - \mathbf{v}\|_{\mathbf{curl}} + \|\mathbf{p} - \mathbf{q}\|_{\mathbf{curl}} \right).$$

Proof. Using the bilinear form *a* defined in Proposition 7 we can write Problem (23) as a saddle point problem as follows: find $(\mathbf{u}_h, \mathbf{p}_h, \phi_h) \in \mathbf{V}_h \times \mathbf{Q}_h \times W_h$ such that

(25)
$$\begin{cases} a(\mathbf{u}_h, \mathbf{p}_h; \mathbf{v}, \mathbf{q}) + (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \phi_h) = (\mathbf{g}, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall (\mathbf{v}, \mathbf{q}) \in \mathbf{V}_h \times \mathbf{Q}_h \\ (\mu \mathbf{p}_h, \operatorname{\mathbf{grad}} \psi) = 0 & \forall \psi \in W_h. \end{cases}$$

The discrete kernel is given by

$$\mathbb{K}_{h} = \{ (\mathbf{v}_{h}, \mathbf{q}_{h}) \in \boldsymbol{V}_{h} \times \boldsymbol{Q}_{h} : (\mu \mathbf{q}_{h}, \operatorname{\mathbf{grad}} \psi) = 0 \ \forall \psi \in W_{h} \}.$$

Then the discrete ellipticity in the kernel holds true thanks to the fact that for functions $\mathbf{q}_h \in \mathbf{Q}_h$ such that $(\mu \mathbf{q}_h, \operatorname{\mathbf{grad}} \psi) = 0 \ \forall \psi \in W_h$ the following Friedrichs inequality holds true (see [4, Prop. 4.6]) for a suitable constant \tilde{C}_F not depending on h

$$\|\mathbf{q}_h\|_0 \le \tilde{C}_F \|\operatorname{\mathbf{curl}} \mathbf{q}_h\|_0.$$

Recalling that grad $W_h \subset Q_h$, we have also that the discrete infsup condition

$$\inf_{\psi \in W_h} \sup_{\mathbf{q} \in \boldsymbol{Q}_h} \frac{(\mu \mathbf{q}, \operatorname{\mathbf{grad}} \psi)}{\|\mathbf{q}\|_{\operatorname{\mathbf{curl}}} \|\psi\|_1} \geq \beta_1$$

with $\beta_1 > 0$ depending on the Poincaré constant but not on h. Indeed, given $\psi \in W_h$ it is enough to take $\mathbf{q} = \mathbf{grad} \ \psi \in \mathbf{V}_h$. Moreover, we observe that taking $(\mathbf{v}, \mathbf{q}) = (\mathbf{0}, \mathbf{grad} \ \phi_h)$ in the first equation in (25) we get $(\mu \mathbf{grad} \ \phi_h, \mathbf{grad} \ \phi_h = 0$ which implies that $\phi_h = 0$ since it has zero mean value.

Hence by the theory on the approximation of saddle point problems, (see, e.g., [10]), we obtain existence, uniqueness and stability of the solution of (25) together with the error estimate.

We can then write the eigenvalue problems associated with (21) and (23) and the corresponding continuous and discrete solution operators.

We recall that $\mathbf{V} = \mathbf{H}_0(\mathbf{curl}; \Omega)$, $\mathbf{Q} = \mathbf{H}(\mathbf{curl})$, and $W = H^1(\Omega) \cap L^2_0(\Omega)$. The continuous eigenvalue problem reads: find $\lambda \in \mathbb{R}$ such that for a non-vanishing $\mathbf{p} \in \mathbf{Q}$ there exists $(\mathbf{u}, \phi) \in \mathbf{V} \times W$ such that

(26)
$$\begin{cases} (\varepsilon \mathbf{u}, \mathbf{v}) + (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} \mathbf{p}) = \lambda(\mathbf{p}, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ - (\mathbf{u}, \operatorname{\mathbf{curl}} \mathbf{q}) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} \mathbf{p}, \operatorname{\mathbf{curl}} \mathbf{q}) + (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \phi) = 0 & \forall \mathbf{q} \in \mathbf{Q} \\ (\mu \mathbf{p}, \operatorname{\mathbf{grad}} \psi) = 0 & \forall \psi \in W. \end{cases}$$

The corresponding solution operator $T : \mathbf{Q} \to \mathbf{Q}$ is defined as $T\mathbf{g} = \mathbf{p}$ where \mathbf{p} is the second component of the solution to (22).

With the finite element spaces defined in (24), the discrete version of (26) reads: find $\lambda_h \in \mathbb{R}$ such that for a non-vanishing $\mathbf{p}_h \in \mathbf{Q}_h$ there exists $(\mathbf{u}_h, \phi_h) \in \mathbf{V}_h \times W_h$ such that (27)

$$\begin{cases} (\varepsilon \mathbf{u}_h, \mathbf{v}) + (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}_h, \operatorname{\mathbf{curl}} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} \mathbf{p}_h) = \lambda_h(\mathbf{p}_h, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{V}_h \\ - (\mathbf{u}_h, \operatorname{\mathbf{curl}} \mathbf{q}) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} \mathbf{p}_h, \operatorname{\mathbf{curl}} \mathbf{q}) + (\mu \mathbf{q}, \operatorname{\mathbf{grad}} \phi_h) = 0 & \forall \mathbf{q} \in \boldsymbol{Q}_h \\ (\mu \mathbf{p}_h, \operatorname{\mathbf{grad}} \psi) = 0 & \forall \psi \in W_h \end{cases}$$

and the corresponding solution operator $T_h : \mathbf{Q} \to \mathbf{Q}_h \subset \mathbf{Q}$ is defined as $T_h \mathbf{g} = \mathbf{p}_h$, where \mathbf{p}_h is the second component of the solution to (23).

Exact		Computed (ra	ate)
2.00000	2.12293	2.03055(2.01)	2.00762(2.00)
2.00000	2.12749	2.03140(2.02)	2.00782(2.01)
2.00000	2.12749	2.03140(2.02)	2.00782(2.01)
3.00000	3.25478	3.06315(2.01)	3.01576(2.00)
3.00000	3.25478	3.06315(2.01)	3.01576(2.00)
5.00000	5.50030	5.12147(2.04)	5.03016(2.01)
5.00000	5.50930	5.12395(2.04)	5.03079(2.01)
5.00000	5.50930	5.12395(2.04)	5.03079(2.01)
5.00000	5.71701	5.17116(2.07)	5.04231(2.02)
5.00000	5.71701	5.17116(2.07)	5.04231(2.02)
Mesh	1/8	1/16	1/32

TABLE 12. Edge elements on a uniform mesh of the cube

From the a priori estimates proved in Proposition 10, we then obtain the convergence of the discrete eigenmodes to the continuous one along the lines of the classical Babuška–Osborn theory.

Theorem 3. For $k \ge 0$, let V_h , Q_h , and W_h as defined in (24); assume that λ is an eigenvalue of (26) of multiplicity m with associated eigenspace E. Then there exist exactly m eigenvalues $\lambda_{1,h} \le \cdots \le \lambda_{m,h}$ of (27) converging to λ . Moreover, let us denote by E_h the space spanned by eigenfunctions associated with the m discrete eigenvalues. Then

$$\begin{aligned} |\lambda - \lambda_{i,h}| &\leq C\epsilon(h)^2 \quad (i = 1, \dots, m) \\ \hat{\delta}(E, E_h) &\leq C\epsilon(h), \end{aligned}$$

where

$$\epsilon(h) = \sup_{\substack{\mathbf{p}\in E\\ \|\mathbf{p}\|_1=1}} \|(T-T_h)\mathbf{p}\|_Q$$

and $\hat{\delta}(A, B)$ denotes the gap between the subspaces A and B of Q.

7. NUMERICAL EXAMPLES IN THREE DIMENSIONS

We conclude this paper with some three dimensional numerical results. We limit ourselves to the results on the cube $(0, \pi)^3$ where the exact results are known analytically.

Table 12 shows the results of the computation performed with formulation (27) and confirms the good behavior of the scheme.

Finally, we show some promising results related to the nodal approximation of our problem. Indeed, we are considering the following two field formulation: find $\lambda_h \in \mathbb{R}$ such that for a non-vanishing $\mathbf{p}_h \in \mathbf{Q}_h$ there exists $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\begin{cases} (\varepsilon \mathbf{u}_h, \mathbf{v}) + (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}_h, \operatorname{\mathbf{curl}} \mathbf{v}) - (\mathbf{v}, \operatorname{\mathbf{curl}} \mathbf{p}_h) = \lambda_h(\mathbf{p}_h, \operatorname{\mathbf{curl}} \mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{V}_h \\ - (\mathbf{u}_h, \operatorname{\mathbf{curl}} \mathbf{q}) + (\varepsilon^{-1} \operatorname{\mathbf{curl}} \mathbf{p}_h, \operatorname{\mathbf{curl}} \mathbf{q}) = 0 & \forall \mathbf{q} \in \boldsymbol{Q}_h, \end{cases}$$

where the spaces V_h and Q_h are both consisting of continuous piecewise linear elements in each component. No theory is available in this case and in general the obtained results can be wrong; in particular no gauge condition is imposed on **p** so that there is no uniqueness in the case of the source problem. Nevertheless, in the

TABLE 13. Nodal elements on a uniform mesh of the cube

Exact	Computed (rate)		
2.00000	2.12841	2.03172(2.02)	2.00792(2.00)
2.00000	2.12841	2.03172(2.02)	2.00792(2.00)
2.00000	2.13005	2.03210(2.02)	2.00801(2.00)
3.00000	3.27704	3.06924(2.00)	3.01734(2.00)
3.00000	3.27704	3.06924(2.00)	3.01734(2.00)
5.00000	5.56075	5.13631(2.04)	5.03393(2.01)
5.00000	5.56516	5.13703(2.04)	5.03411(2.01)
5.00000	5.56516	5.13703(2.04)	5.03411(2.01)
5.00000	5.85266	5.20337(2.07)	5.05035(2.01)
5.00000	5.86162	5.20337(2.08)	5.05035(2.01)
Mesh	1/8	1/16	1/32

TABLE 14. Nodal elements on a nonuniform mesh of the cube

Exact	Computed (rate)		
2.00000	2.39482	2.10329(1.93)	2.02523(2.03)
2.00000	2.41541	2.10486(1.99)	2.02538(2.05)
2.00000	2.42126	2.10563(2.00)	2.02546(2.05)
3.00000	3.99333	3.22212(2.16)	3.05435(2.03)
3.00000	4.00258	3.22486(2.16)	3.05464(2.04)
5.00000	7.58994	5.61356(2.08)	5.14744(2.06)
5.00000	7.64984	5.62135(2.09)	5.14764(2.07)
5.00000	7.75720	5.62405(2.14)	5.14818(2.07)
5.00000	7.80063	5.63059(2.15)	5.14861(2.09)
5.00000	7.86660	5.63186(2.18)	5.14964(2.08)
Mesh	1/8	1/16	1/32

case of the cube we obtain the nice and optimal results reported in Table 13 in the case of a uniform mesh and in Table 14 when a nonuniform mesh is considered.

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FAKULTÄT FÜR MATHEMATIK, TU CHEMNITZ, GERMANY Email address: fleurianne.bertrand@mathematik.tu-chemnitz.de URL: https://www.tu-chemnitz.de/mathematik/numapde/

King Abdullah University of Science and Technology (KAUST), Saudi Arabia and Università degli Studi di Pavia, Italy

Email address: daniele.boffi@kaust.edu.sa URL: https://cemse.kaust.edu.sa/people/person/daniele-boffi

DICATAM, UNIVERSITÀ DEGLI STUDI DI BRESCIA, ITALY AND IMATI-CNR, PAVIA, ITALY Email address: lucia.gastaldi@unibs.it URL: http://lucia-gastaldi.unibs.it