The inf-sup constant for *hp*-Crouzeix-Raviart triangular elements

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December 22, 2022

Abstract

In this paper, we consider the discretization of the two-dimensional stationary Stokes equation by Crouzeix-Raviart elements for the velocity of polynomial order $k \geq 1$ on conforming triangulations and discontinuous pressure approximations of order $k-1$. We will bound the inf-sup constant from below independent of the mesh size and show that it depends only logarithmically on k. Our assumptions on the mesh are very mild: for odd k we require that the triangulations contain at least one inner vertex while for even k we assume that the triangulations consist of more than a single triangle.

AMS Subject Classification: 65N30, 65N12, 76D07, 33C45,
Key Words:Non-conforming finite elements, Crouzeix-Raviart elements, high order finite elements, Stokes equation.

1 Introduction

In this paper we consider the numerical discretization of the two-dimensional stationary Stokes problem by Crouzeix-Raviart elements. They were introduced in the seminal paper [16] in 1973 by Crouzeix and Raviart with the goal to obtain a stable and economic discretization of the Stokes equation. They can be considered as an non-conforming enrichment of conforming finite elements of polynomial degree k for the velocity and discontinuous pressures of degree $k-1$. It is well known that the conforming $(k, k-1)$ pair of finite elements can be unstable; for two-dimensions the proof of the inf-sup stability of Crouzeix-Raviart discretizations of general order k has been evolved over the last 50 years, the inf-sup stability for $k = 1$ has been proved in [16] and only recently the last open case $k = 3$, has been proved in [11]. We mention the papers [21], [32], [15], [7], [27], [10] which contain essential milestones in this development. There is a vast of literature on various further aspects of Crouzeix-Raviart elements; we omit to present a comprehensive review here but refer to the overview article [9] instead.

Since higher order methods are becoming increasingly popular a natural question arises how the inf-sup constant depends on the polynomial degree k. It is the goal of this paper to investigate this dependency.

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The paper is organized as follows. In Section 1.1 we introduce the Stokes problem and the Crouzeix-Raviart discretization of polynomial order k. We state our main theorem that the discrete inf-sup constant can be estimated from below by $c\left(\log\left(k+1\right)\right)^{-\alpha}$, where the positive constant c depends only on the shape-regularity of the mesh and on the maximal outer angle of the domain Ω. The explicit value of α depends on the mesh topology. The simplest case is that each triangle in triangulation contains at least one inner vertex and then $\alpha = 1/2$ holds. We will give the value of α also for more general triangulations.

The proof is given in Section 2. The key ingredient is to show that for any discrete pressure q, there exists a velocity field $\mathbf{v}_q$ from the Crouzeix-Raviart space such that $\tilde{q} := q - \operatorname{div}_{\mathcal{T}} \mathbf{v}_q$ belongs to the *Scott-Vogelius pressure space* [37, R.1, R.2], [32, R.1, R.2] (see (2.25)) and $\mathbf{v}_q$ depends continuously on q. These properties allow us to construct a conforming velocity field $\tilde{\mathbf{v}}_{\tilde{q}}$ of order k with $\operatorname{div} \tilde{\mathbf{v}}_{\tilde{q}} = \tilde{q}$. For this step, the construction in [32, Thm. 5.1], [27, Thm. 1] is modified such that the norm of the right-inverse does not deteriorate if a triangle vertex is a "nearly-critical" point – a notion which will be introduced in Definition 2.11. This key result is proved for odd polynomial degree in Section 2.2 and for even polynomial degree in Section 2.3.

In the conclusions (Sec. 3) we summarize our main findings and compare our results with existing results in the literature on some other pairs of finite elements for the Stokes equation.

In the appendices, we prove a technical result for a Gram matrix related to the bilinear form $\left(\operatorname{div}_{\mathcal{T}} \cdot, \cdot\right)_{L^2(\Omega)}$ applied to the Crouzeix-Raviart element and discontinuous pressure space (see §A), an estimate of traces of non-conforming Crouzeix-Raviart basis functions (see §B), give some explicit formulae for integrals related to orthogonal polynomials (see §C), and prove a discrete Friedrichs inequality for Crouzeix-Raviart spaces (see §D).

1.1 The Stokes problem and its numerical discretization

Let $\Omega \subset \mathbb{R}^2$ denote a bounded polygonal Lipschitz domain with boundary $\partial\Omega$. For a vertex $\mathbf{z}$ in $\partial\Omega$, we denote by $\alpha_{\mathbf{z}}$ the exterior angle between the two segments in $\partial\Omega$ with joint $\mathbf{z}$. The minimal outer angle at the boundary vertices is given by

$$\alpha_\Omega := \min_{\mathbf{z} \text{ is a vertex in } \partial\Omega} \alpha_{\mathbf{z}} \tag{1.1}$$

and satisfies $0 < \alpha_\Omega < 2\pi$ since Ω is Lipschitz. We consider the Stokes equation

$$\begin{array}{llll} -\Delta \mathbf{u} & -\nabla p & = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} & & = 0 & \text{in } \Omega \end{array}$$

with Dirichlet boundary conditions for the velocity and a usual normalization condition for the pressure

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \quad \text{and} \quad \int_\Omega p = 0.$$

To formulate this equation in a variational form we first introduce the relevant function spaces. Throughout the paper we restrict to vector spaces over the field of real numbers.

For $s \geq 0$, $1 \leq p \leq \infty$, $W^{s,p}\left(\Omega\right)$ denote the classical Sobolev spaces of functions with norm $\left\|\cdot\right\|_{W^{s,p}(\Omega)}$. As usual we write $L^p\left(\Omega\right)$ instead of $W^{0,p}\left(\Omega\right)$ and $H^s\left(\Omega\right)$ for $W^{s,2}\left(\Omega\right)$. For $s \geq 0$, we denote by $H_0^s\left(\Omega\right)$ the closure of the space of infinitely smooth functions with compact support in Ω with respect to the $H^s\left(\Omega\right)$ norm. Its dual space is denoted by $H^{-s}\left(\Omega\right)$. For the pressure p, the space $L_0^2\left(\Omega\right) := \left\{u \in L^2\left(\Omega\right) : \int_\Omega u = 0\right\}$ will be relevant.

The scalar product and norm in $L^2(\Omega)$ are denoted respectively by

$$(u,v)_{L^2(\Omega)} := \int_\Omega uv \quad \text{and} \quad \|u\|_{L^2(\Omega)} := (u,u)_{L^2(\Omega)}^{1/2} \quad \text{in } L^2(\Omega).$$

Vector-valued and 2×2 tensor-valued analogues of the function spaces are denoted by bold and blackboard bold letters, e.g., $\mathbf{H}^s(\Omega) = (H^s(\Omega))^2$ and $\mathbb{H}^s = (H^s(\Omega))^{2\times 2}$.

The $\mathbf{L}^2(\Omega)$ scalar product and norm for vector valued functions are given by

$$(\mathbf{u},\mathbf{v})_{\mathbf{L}^2(\Omega)} := \int_\Omega \langle \mathbf{u},\mathbf{v}\rangle \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} := (\mathbf{u},\mathbf{u})_{\mathbf{L}^2(\Omega)}^{1/2},$$

where $\langle \mathbf{u},\mathbf{v}\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. In a similar fashion, we define for $\mathbf{G},\mathbf{H} \in \mathbb{L}^2(\Omega)$ the scalar product and norm by

$$(\mathbf{G},\mathbf{H})_{\mathbb{L}^2(\Omega)} := \int_\Omega \langle \mathbf{G},\mathbf{H}\rangle \quad \text{and} \quad \|\mathbf{G}\|_{\mathbb{L}^2(\Omega)} := (\mathbf{G},\mathbf{G})_{\mathbb{L}^2(\Omega)}^{1/2},$$

where $\langle \mathbf{G},\mathbf{H}\rangle = \sum_{i,j=1}^3 G_{i,j}H_{i,j}$. We also need fractional order Sobolev norms on boundary of triangles and introduce the relevant notation; for details see, e.g., [29]. For a bounded Lipschitz domain $\omega \subset \mathbb{R}^2$ with boundary $\partial\omega$, let $L^2(\partial\omega)$ and $H^1(\partial\omega)$ denote the usual Lebesgue and Sobolev space on $\partial\omega$ with norm $\|\cdot\|_{L^2(\partial\omega)}$ and $\|\cdot\|_{H^1(\partial\omega)}$. For $0<s<1$ the fractional Sobolev space on $\partial\omega$ of order s is denoted by $H^s(\partial\omega)$ and equipped with the norm

$$\|v\|_{H^s(\partial\omega)} := \left(\|v\|_{L^2(\partial\omega)}^2 + |v|_{H^s(\partial\omega)}^2\right)^{1/2}$$

and seminorm

$$|v|_{H^s(\partial\omega)} := \left(\int_{\partial\omega}\int_{\partial\omega} \frac{|v(\mathbf{x})-v(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^{1+2s}} d\mathbf{y}d\mathbf{x}\right)^{1/2}.$$

We introduce the bilinear form $a : \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega) \to \mathbb{R}$ by

$$a(\mathbf{u},\mathbf{v}) := (\nabla\mathbf{u},\nabla\mathbf{v})_{\mathbb{L}^2(\Omega)}, \tag{1.2}$$

where $\nabla\mathbf{u}$ and $\nabla\mathbf{v}$ denote the derivatives of $\mathbf{u}$ and $\mathbf{v}$. The variational form of the Stokes problem is given by: For given $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$,

$$\text{find } (\mathbf{u},p) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega) \text{ s.t. } \begin{cases} a(\mathbf{u},\mathbf{v}) + (p,\operatorname{div}\mathbf{v})_{L^2(\Omega)} & = \mathbf{F}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (\operatorname{div}\mathbf{u},q)_{L^2(\Omega)} & = 0 \qquad\quad \forall q \in L_0^2(\Omega). \end{cases} \tag{1.3}$$

It is well-known (see, e.g., [22]) that (1.3) is well posed. Since we consider non-conforming discretizations we restrict the space $\mathbf{H}^{-1}(\Omega)$ for the right-hand side to a smaller space and assume from now on for simplicity that $\mathbf{F}(\mathbf{v}) = (\mathbf{f},\mathbf{v})_{\mathbf{L}^2(\Omega)}$ for some $\mathbf{f} \in \mathbf{L}^2(\Omega)$; for a more general setting we refer to [35], [36].

In the following a discretization for problem (1.3) is introduced. Let $\mathcal{T} = \{K_i : 1 \leq i \leq n\}$ denote a triangulation of Ω consisting of closed triangles which are *conforming*: the intersection of two different triangles is either empty, a common edge, or a common point. We also assume $\Omega = \operatorname{dom}\mathcal{T}$, where

$$\operatorname{dom}\mathcal{T} := \operatorname{int}\left(\bigcup_{K\in\mathcal{T}} K\right) \tag{1.4}$$

and $\operatorname{int}(M) := \overset{\circ}{M}$ denotes the interior of a set $M \subset \mathbb{R}^2$.

Piecewise versions of differential operators such as ∇ and div are defined for functions u and vector fields $\mathbf{w}$ which are sufficiently smooth in the interior of the triangles $K \subset \mathcal{T}$ by

$$(\nabla_{\mathcal{T}} u)|_{\overset{\circ}{K}} := \nabla \left(u|_{\overset{\circ}{K}} \right) \quad \text{and} \quad (\operatorname{div}_{\mathcal{T}} \mathbf{w})|_{\overset{\circ}{K}} := \operatorname{div} \left(\mathbf{w}|_{\overset{\circ}{K}} \right).$$

The values on ∂K are arbitrary since ∂K has measure zero.

An important measure for the quality of a finite element triangulation is the shape-regularity constant given by

$$\gamma_{\mathcal{T}} := \max_{K \in \mathcal{T}} \frac{h_K}{\rho_K} \tag{1.5}$$

with the local mesh width $h_K := \operatorname{diam} K$ and ρ_K denoting the diameter of the largest inscribed ball in K. The global mesh width is $h_{\mathcal{T}} := \max \{ h_K : K \in \mathcal{T} \}$.

Remark 1.1 *It is well known that the shape-regularity implies that there exists some minimal angle $\phi_{\mathcal{T}} > 0$ depending only on $\gamma_{\mathcal{T}}$ such that every triangle angle in $\mathcal{T}$ is bounded from below by $\phi_{\mathcal{T}}$. In turn, every triangle angle in $\mathcal{T}$ is bounded from above by $\pi - 2\phi_{\mathcal{T}}$.*

The set of edges in $\mathcal{T}$ is denoted by $\mathcal{E}(\mathcal{T})$, while the subset of boundary edges is $\mathcal{E}_{\partial\Omega}(\mathcal{T}) := \{ E \in \mathcal{E}(\mathcal{T}) : E \subset \partial (\operatorname{dom} \mathcal{T}) \}$; the subset of inner edges is given by $\mathcal{E}_{\Omega}(\mathcal{T}) := \mathcal{E}(\mathcal{T}) \backslash \mathcal{E}_{\partial\Omega}(\mathcal{T})$. For each edge $E \in \mathcal{E}$ we fix a unit vector $\mathbf{n}_E$ orthogonal to E with the convention that $\mathbf{n}_E$ is the outer normal vector for boundary edges $E \in \mathcal{E}_{\partial\Omega}$.

The set of triangle vertices in $\mathcal{T}$ is denoted by $\mathcal{V}(\mathcal{T})$, while the subset of inner vertices is $\mathcal{V}_{\Omega}(\mathcal{T}) := \{ \mathbf{V} \in \mathcal{V}(\mathcal{T}) : \mathbf{V} \notin \partial (\operatorname{dom} \mathcal{T}) \}$ and $\mathcal{V}_{\partial\Omega}(\mathcal{T}) := \mathcal{V}(\mathcal{T}) \backslash \mathcal{V}_{\Omega}(\mathcal{T})$. For $K \in \mathcal{T}$, the set of its vertices is denoted by $\mathcal{V}(K)$. For $E \in \mathcal{E}(\mathcal{T})$, we define the edge patch by

$$\mathcal{T}_E := \{ K \in \mathcal{T} : E \subset K \} \quad \text{and} \quad \omega_E := \bigcup_{K \in \mathcal{T}_E} K.$$

For $\mathbf{z} \in \mathcal{V}(\mathcal{T})$, the nodal patch is defined by

$$\mathcal{T}_{\mathbf{z}} := \{ K \in \mathcal{T} : \mathbf{z} \in K \} \quad \text{and} \quad \omega_{\mathbf{z}} := \bigcup_{K \in \mathcal{T}_{\mathbf{z}}} K \tag{1.6}$$

with local mesh width $h_{\mathbf{z}} := \max \{ h_K : K \in \mathcal{T}_{\mathbf{z}} \}$. For $K \in \mathcal{T}$, we set

$$\mathcal{T}_K := \{ K' \in \mathcal{T} \mid K \cap K' \neq \emptyset \} \quad \text{and} \quad \omega_K := \bigcup_{K' \in \mathcal{T}_K} K'. \tag{1.7}$$

For a subset $M \subset \mathbb{R}^2$, we denote by $[M]$ its convex hull; in this way an edge E with endpoints $\mathbf{a}, \mathbf{b}$ can be written as $E = [\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}]$.

Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, we employ the usual multiindex notation for $\boldsymbol{\mu} = (\mu_i)_{i=1}^m \in \mathbb{N}_0^m$ and points $\mathbf{x} = (x_i)_{i=1}^m \in \mathbb{R}^m$

$$|\boldsymbol{\mu}| := \mu_1 + \ldots + \mu_m, \quad \mathbf{x}^{\boldsymbol{\mu}} := \prod_{j=1}^m x_j^{\mu_j}.$$

Let $\mathbb{P}_{m,k}$ denote the space of m-variate polynomials of maximal degree k, consisting of functions of the form

$$\sum_{\substack{\boldsymbol{\mu}\in\mathbb{N}_0^m \\ |\boldsymbol{\mu}|\leq k}} a_{\boldsymbol{\mu}} \mathbf{x}^{\boldsymbol{\mu}}$$

for real coefficients $a_{\boldsymbol{\mu}}$. Formally, we set $\mathbb{P}_{m,-1} := \{0\}$. To indicate the domain explicitly in notation we write sometimes $\mathbb{P}_k(D)$ for $D \subset \mathbb{R}^m$ and skip the index m since it is then clear from the argument D.

We introduce the following finite element spaces

$$\begin{aligned} \mathbb{P}_k(\mathcal{T}) &:= \left\{ q \in L^2(\Omega) \mid \forall K \in \mathcal{T} : q|_{\overset{\circ}{K}} \in \mathbb{P}_k\left(\overset{\circ}{K}\right) \right\}, \\ \text{and (cf. (1.4))} \quad \mathbb{P}_{k,0}(\mathcal{T}) &:= \left\{ q \in \mathbb{P}_k(\mathcal{T}) : \textstyle\int_{\operatorname{dom}\mathcal{T}} q = 0 \right\}. \end{aligned} \tag{1.8}$$

Furthermore, let

$$\begin{aligned} S_k(\mathcal{T}) &:= \mathbb{P}_k(\mathcal{T}) \cap H^1(\operatorname{dom}\mathcal{T}), \\ \text{and} \quad S_{k,0}(\mathcal{T}) &:= S_k(\mathcal{T}) \cap H_0^1(\operatorname{dom}\mathcal{T}). \end{aligned}$$

The vector-valued versions are denoted by $\mathbf{S}_k(\mathcal{T}) := S_k(\mathcal{T})^2$ and $\mathbf{S}_{k,0}(\mathcal{T}) := S_{k,0}(\mathcal{T})^2$. Finally, we define the Crouzeix-Raviart space by

$$\mathrm{CR}_k(\mathcal{T}) := \left\{ v \in \mathbb{P}_k(\mathcal{T}) \mid \forall q \in \mathbb{P}_{k-1}(E) \quad \forall E \in \mathcal{E}_\Omega(\mathcal{T}) \quad \int_E [v]_E\, q = 0 \right\}, \tag{1.9a}$$

$$\mathrm{CR}_{k,0}(\mathcal{T}) := \left\{ v \in \mathrm{CR}_k(\mathcal{T}) \mid \forall q \in \mathbb{P}_{k-1}(E) \quad \forall E \in \mathcal{E}_{\partial\Omega}(\mathcal{T}) \quad \int_E vq = 0 \right\}. \tag{1.9b}$$

Here, $[v]_E$ denotes the jump of $v \in \mathbb{P}_k(\mathcal{T})$ across an edge $E \in \mathcal{E}_\Omega(\mathcal{T})$

$$[u]_E(\mathbf{x}) := \lim_{\varepsilon \searrow 0} \left(u(\mathbf{x} + \varepsilon \mathbf{n}_E) - u(\mathbf{x} - \varepsilon \mathbf{n}_E) \right).$$

and $\mathbb{P}_{k-1}(E)$ is the space of polynomials of maximal degree $k-1$ with respect to the local variable in E.

We have collected all ingredients for defining the Crouzeix-Raviart discretization for the Stokes equation. For $k \in \mathbb{N}$, let the discrete velocity space and pressure space be defined by

$$\mathbf{CR}_{k,0}(\mathcal{T}) := \left(\mathrm{CR}_{k,0}(\mathcal{T})\right)^2 \quad \text{and} \quad M_{k-1}(\mathcal{T}) := \mathbb{P}_{k-1,0}(\mathcal{T}).$$

Then, the discretization is given by: find $(\mathbf{u}_{\mathrm{CR}}, p_{\mathrm{disc}}) \in \mathbf{CR}_{k,0}(\mathcal{T}) \times M_{k-1}(\mathcal{T})$ such that

$$\left\{ \begin{array}{lll} a_{\mathcal{T}}(\mathbf{u}_{\mathrm{CR}}, \mathbf{v}) - b_{\mathcal{T}}(\mathbf{v}, p_{\mathrm{disc}}) & = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega)} & \forall \mathbf{v} \in \mathbf{CR}_{k,0}(\mathcal{T}), \\ b_{\mathcal{T}}(\mathbf{u}_{\mathrm{CR}}, q) & = 0 & \forall q \in M_{k-1}(\mathcal{T}), \end{array} \right. \tag{1.10}$$

where the bilinear forms $a_{\mathcal{T}} : \mathbf{CR}_{k,0}(\mathcal{T}) \times \mathbf{CR}_{k,0}(\mathcal{T}) \to \mathbb{R}$ and $b_{\mathcal{T}} : \mathbf{CR}_{k,0}(\mathcal{T}) \times M_{k-1}(\mathcal{T}) \to \mathbb{R}$ are given by

$$a_{\mathcal{T}}(\mathbf{u}, \mathbf{v}) := (\nabla_{\mathcal{T}} \mathbf{u}, \nabla_{\mathcal{T}} \mathbf{v})_{\mathbb{L}^2(\Omega)} \quad \text{and} \quad b_{\mathcal{T}}(\mathbf{v}, q) := (\operatorname{div}_{\mathcal{T}} \mathbf{v}, q)_{L^2(\Omega)}.$$

It is well known that problem (1.10) is well-posed if (i): the bilinear form $a_{\mathcal{T}}(\cdot,\cdot)$ is coercive and (ii): $b_{\mathcal{T}}(\cdot,\cdot)$ satisfies the inf-sup condition.

To verify the condition (i) we introduce, for a conforming triangulation $\mathcal{T}$ of the domain Ω, the *broken Sobolev space*

$$H^1(\mathcal{T}) := \left\{u \in L^2(\Omega) \mid \forall K \in \mathcal{T} : u|_K \in H^1(K)\right\}$$

and define, for $u \in H^1(\mathcal{T})$, the *broken H^1–seminorm* by

$$\|u\|_{H^1(\mathcal{T})} := \|\nabla_{\mathcal{T}} u\|_{\mathbf{L}^2(\Omega)} = \left(\sum_{K\in\mathcal{T}} \|\nabla u\|^2_{\mathbf{L}^2(K)}\right)^{1/2}.$$

In [16, Lem. 2]) it is proved that $\|\cdot\|_{\mathbf{H}^1(\mathcal{T})}$ defines a norm in $\mathbf{CR}_{k,0}(\mathcal{T}) + \mathbf{H}^1_0(\Omega)$ which is equivalent to the norm $\left(\sum_{K\in\mathcal{T}} \|\mathbf{u}\|^2_{\mathbf{H}^1(K)}\right)^{1/2}$ with equivalence constants independent of the polynomial degree and the mesh width (see Theorem D.1). This directly implies the coercivity of $a_{\mathcal{T}}(\cdot,\cdot)$:

$$a_{\mathcal{T}}(\mathbf{u},\mathbf{u}) \geq \|\mathbf{u}\|^2_{\mathbf{H}^1(\mathcal{T})} \quad \forall \mathbf{u} \in \mathbf{CR}_{k,0}(\mathcal{T}).$$

Hence, well-posedness of (1.10) follows from the inf-sup condition for $b_{\mathcal{T}}(\cdot,\cdot)$.

Definition 1.2 *Let $\mathcal{T}$ denote a conforming triangulation for Ω. The pair $\mathbf{CR}_{k,0}(\mathcal{T}) \times M_{k-1}(\mathcal{T})$ is* inf-sup stable *if there exists a constant $c_{\mathcal{T},k}$ such that*

$$\inf_{p\in M_{k-1}(\mathcal{T})\backslash\{0\}} \sup_{\mathbf{v}\in\mathbf{CR}_{k,0}(\mathcal{T})\backslash\{\mathbf{0}\}} \frac{(p, \operatorname{div}_{\mathcal{T}} \mathbf{v})_{L^2(\Omega)}}{\|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{T})} \|p\|_{L^2(\Omega)}} \geq c_{\mathcal{T},k} > 0. \tag{1.11}$$

We are now in the position to formulate our main theorem.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal Lipschitz domain and let $\mathcal{T}$ denote a conforming triangulation of Ω consisting of more than a single triangle. Let $k \in \mathbb{N}$. If $k \geq 3$ is odd we assume that $\mathcal{T}$ contains at least one inner vertex. Then, the inf-sup condition (1.11) holds:*

$$c_{\mathcal{T},k} \geq c_{\mathcal{T}} (\log(k+1))^{-\alpha} \tag{1.12}$$

for a constant $c_{\mathcal{T}} > 0$ depending only on the shape-regularity of the mesh and on the maximal outer angle α_Ω. In particular $c_{\mathcal{T}}$ is independent of the mesh width $h_{\mathcal{T}}$ and the polynomial degree k. The value of $\alpha \geq 0$ is given by

$$\alpha = \begin{cases} 1/2 & \begin{cases} \text{if } k \text{ is even,} \\ \text{or } k \geq 3 \text{ is odd and all triangles in } \mathcal{T} \text{ have at least one inner vertex,} \end{cases} \\ (1+L)/2 & \text{otherwise,} \end{cases}$$

where L depends only on the mesh topology via the number of steps involved in the step-by-step construction introduced in (2.13).

Proof. The estimate $c_{\mathcal{T},k} > 0$ follows, for $k = 1$ from [16], for $k = 2$ from [12, Thm. 3.1], for even $k \geq 4$ from [7], for odd $k \geq 5$ from [10], and for $k = 3$ from [11]. We set

$$c_{\mathcal{T},\text{low}} := \min\{c_{\mathcal{T},k} : 1 \leq k \leq 3\}.$$

Estimate (1.12) for some $c_{\mathcal{T}} := c^{\text{odd}}_{\mathcal{T},\text{high}} > 0$ for odd $k \geq 5$ is proved in Section 2.2, Lem. 2.22, while the estimate for some $c_{\mathcal{T}} := c^{\text{even}}_{\mathcal{T},\text{high}} > 0$ for even $k \geq 4$ is proved in Section 2.3. Both constants $c^{\mathrm{odd}}_{\mathcal{T},\mathrm{high}}$, $c^{\mathrm{even}}_{\mathcal{T},\mathrm{high}}$ depend only on the shape-regularity of the mesh and α_Ω. Hence, $c_{\mathcal{T}} \geq \min\left\{c_{\mathcal{T},\mathrm{low}}, c^{\mathrm{odd}}_{\mathcal{T},\mathrm{high}}, c^{\mathrm{even}}_{\mathcal{T},\mathrm{high}}\right\}$. ■

We emphasize that the original definition in [16] allows for slightly more general finite element spaces, more precisely, the spaces $\mathrm{CR}_k(\mathcal{T})$ can be enriched by locally supported functions. From this point of view, the definition (1.9) describes a minimal Crouzeix-Raviart space.

The possibility for enrichment has been used frequently in the literature to prove inf-sup stability for the arising finite element spaces (see, e.g., [16], [26], [28]). In contrast, we will prove the k-explicit estimate of the inf-sup constant for the Crouzeix-Raviart space $\mathrm{CR}_k(\mathcal{T})$.

2 Proof of Theorem 1.3

In this section, we will analyse the k-dependence of the inf-sup constant in the form (1.12), first for odd polynomial degree $k \geq 5$ and then for even degree $k \geq 4$.

2.1 Barycentric coordinates and basis functions for the velocity

In this section, we introduce basis functions for the finite element spaces in Section 1.1. We begin with introducing some general notation.

Notation 2.1 *For vectors $\mathbf{a}_i \in \mathbb{R}^n$, $1 \leq i \leq m$, we write $[\mathbf{a}_1 \mid \mathbf{a}_2 \mid \ldots \mid \mathbf{a}_m]$ for the $n \times m$ matrix with column vectors $\mathbf{a}_i$. For $\mathbf{v} = (v_1, v_2)^T \in \mathbb{R}^2$ we set $\mathbf{v}^\perp := (v_2, -v_1)^T$. Let $\mathbf{e}_{k,i} \in \mathbb{R}^k$ be the i-th canonical unit vector in $\mathbb{R}^k$.*

For $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\|$ is the Euclidean vector norm while the induced matrix norm is given for $\mathbf{B} \in \mathbb{R}^{n\times n}$ by $\|\mathbf{B}\| := \sup\{\|\mathbf{Bx}\| / \|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}\}$.

Vertices in a triangle are numbered counterclockwise. In a triangle K with vertices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ the angle at $\mathbf{A}_i$ is called α_i. If a triangle is numbered by an index (e.g., K_ℓ), the angle at $A_{\ell,i}$ is called $\alpha_{\ell,i}$. For quantities in a triangle K as, e.g., angles α_j, $1 \leq j \leq 3$, we use the cyclic numbering convention $\alpha_{3+1} := \alpha_1$ and $\alpha_{1-1} := \alpha_3$.

For a d-dimensional measurable set D, we write $|D|$ for its measure; for a discrete set, say $\mathcal{J}$, we denote by $|\mathcal{J}|$ its cardinality.

In the proofs, we consider frequently nodal patches $\mathcal{T}_{\mathbf{z}}$ for inner vertices $\mathbf{z} \in \mathcal{V}_\Omega(\mathcal{T})$. The number m denotes the number of triangles in $\mathcal{T}_{\mathbf{z}}$. Various quantities in this patch such as, e.g., the triangles in $\mathcal{T}_{\mathbf{z}}$, have an index which runs from 1 to m. Here, we use the cyclic numbering convention $K_{m+1} := K_1$ and $K_{1-1} := K_m$ and apply this analogously for other quantities in the nodal patch.

Let the closed reference triangle $\widehat{K}$ be the triangle with vertices $\hat{\mathbf{A}}_1 := (0,0)^T$, $\hat{\mathbf{A}}_2 := (1,0)^T$, $\hat{\mathbf{A}}_3 := (0,1)^T$. The nodal points on the reference element of order $k \in \mathbb{N}_0$ are given by

$$\widehat{\mathcal{N}}_k := \begin{cases} \left\{\frac{1}{k}\boldsymbol{\mu} \mid \boldsymbol{\mu} \in \mathbb{N}_0^2 : \quad |\boldsymbol{\mu}| \leq k\right\} & k \geq 1, \\ \left\{\left(\frac{1}{3}, \frac{1}{3}\right)\right\} & k = 0. \end{cases}$$

For a triangle $K \subset \mathbb{R}^2$, we denote by $\chi_K : \widehat{K} \to K$ an affine bijection. The mapped nodal points of order $k \in \mathbb{N}_0$ on K are given by

$$\mathcal{N}_k(K) := \left\{ \chi_K(\mathbf{z}) : \mathbf{z} \in \widehat{\mathcal{N}}_k \right\}.$$

Nodal points of order k on $\mathcal{T}$ are defined by

$$\mathcal{N}_k(\mathcal{T}) := \bigcup_{K \in \mathcal{T}} \mathcal{N}_k(K), \quad \mathcal{N}_{\partial\Omega}^k(\mathcal{T}) := \mathcal{N}_k(\mathcal{T}) \cap \partial\Omega, \quad \text{and} \quad \mathcal{N}_{k,\Omega}(\mathcal{T}) := \mathcal{N}_k(\mathcal{T}) \cap \Omega.$$

We introduce the Lagrange basis for the space $S_k(\mathcal{T})$, which is indexed by the nodal points $\mathbf{z} \in \mathcal{N}_k(\mathcal{T})$ and characterized by

$$B_{k,\mathbf{z}} \in S_k(\mathcal{T}) \quad \text{and} \quad \forall \mathbf{z}' \in \mathcal{N}_k(\mathcal{T}) \qquad B_{k,\mathbf{z}}(\mathbf{z}') = \delta_{\mathbf{z},\mathbf{z}'}, \tag{2.1}$$

where $\delta_{\mathbf{z},\mathbf{z}'}$ is the Kronecker delta. A basis for the space $S_{k,0}(\mathcal{T})$ is given by $B_{k,\mathbf{z}}$, $\mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T})$.

Let K denote a triangle with vertices $\mathbf{A}_i$, $1 \leq i \leq 3$, and let $\lambda_{K,\mathbf{A}_i} \in \mathbb{P}_1(K)$ be the *barycentric coordinate* for the node $\mathbf{A}_i$ defined by

$$\lambda_{K,\mathbf{A}_i}(\mathbf{A}_j) = \delta_{i,j} \quad 1 \leq i, j \leq 3. \tag{2.2}$$

If the numbering of the vertices in K is fixed, we write $\lambda_{K,i}$ short for $\lambda_{K,\mathbf{A}_i}$. For the barycentric coordinate on the reference element $\widehat{K}$ for the vertex $\hat{\mathbf{A}}_j$ we write $\widehat{\lambda}_j$, $j = 1, 2, 3$. Elementary calculation yield (see, e.g., [10, Appendix A])

$$\partial_{\mathbf{n}_k} \lambda_{K,\mathbf{A}_i} = \frac{|E_i|}{2|K|} \times \begin{cases} -1 & i = k, \\ \cos\alpha_\ell & \ell \text{ s.t. } \{\ell, i, k\} = \{1, 2, 3\}, \end{cases} \tag{2.3}$$

where E_i is the edge of K opposite to $\mathbf{A}_i$, $\mathbf{n}_k$ the outward unit normal at E_k, and α_ℓ the angle in K at $\mathbf{A}_\ell$.

Definition 2.2 *Let L_k denote the usual univariate Legendre polynomial of degree k (see [17, Table 18.3.1]). Let $k \in \mathbb{N}$ be even and $K \in \mathcal{T}$. Then, the* non-conforming triangle bubble *is given by*

$$B_{k,K}^{\mathrm{CR}} := \begin{cases} \dfrac{1}{2}\left(-1 + \displaystyle\sum_{i=1}^{3} L_k(1 - 2\lambda_{K,i})\right) & \text{on } K, \\ 0 & \text{on } \Omega \backslash K. \end{cases}$$

For k odd and $E \in \mathcal{E}(\mathcal{T})$, the non-conforming edge bubble *is given by*

$$B_{k,E}^{\mathrm{CR}} := \begin{cases} L_k\left(1 - 2\lambda_{K,\mathbf{A}_{K,E}}\right) & \text{on } K \text{ for } K \in \mathcal{T}_E, \\ 0 & \text{on } \Omega \backslash \omega_E, \end{cases} \tag{2.4}$$

where $\mathbf{A}_{K,E}$ denotes the vertex in K opposite to E.

Different representations of the functions $B_{k,E}^{\mathrm{CR}}$, $B_{k,K}^{\mathrm{CR}}$ exist in the literature, see [34], [5], [12, for $p = 4, 6.$], [13] while the formula for $B_{k,K}^{\mathrm{CR}}$ has been introduced in [7] and the one for $B_{k,E}^{\mathrm{CR}}$ in [10].

Proposition 2.3 *A basis for the space* $\mathrm{CR}_{k,0}(\mathcal{T})$ *is given*

1. *for even* k *by*
$$\{B_{k,\mathbf{z}} \mid \mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T})\} \cup \left\{B^{\mathrm{CR}}_{k,K} \mid K \in \mathcal{T}\right\},$$

2. *for odd* k *by*
$$\{B_{k,\mathbf{z}} \mid \mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T}) \backslash \mathcal{V}_\Omega(\mathcal{T})\} \cup \left\{B^{\mathrm{CR}}_{k,E} \mid E \in \mathcal{E}_\Omega(\mathcal{T})\right\}.$$

The proof of this proposition and the following corollary can be found, e.g., in [34, Rem. 3], [13, Thm. 22], [10, Cor. 3.4].

Corollary 2.4 *A basis for the space* $\mathbf{CR}_{k,0}(\mathcal{T})$ *is given*

1. *for even* k *by*
$$\begin{aligned} &\{B_{k,\mathbf{z}}\mathbf{v}_\mathbf{z} \mid \mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T})\} \cup \{B_{k,\mathbf{z}}\mathbf{w}_\mathbf{z} \mid \mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T})\} \\ &\cup \left\{B^{\mathrm{CR}}_{k,K}\mathbf{v}_K \mid K \in \mathcal{T}\right\} \cup \left\{B^{\mathrm{CR}}_{k,K}\mathbf{w}_K \mid K \in \mathcal{T}\right\}, \end{aligned} \tag{2.5}$$

2. *for odd* k *by*
$$\begin{aligned} &\{B_{k,\mathbf{z}}\mathbf{v}_\mathbf{z} \mid \mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T}) \backslash \mathcal{V}_\Omega(\mathcal{T})\} \cup \{B_{k,\mathbf{z}}\mathbf{w}_\mathbf{z} \mid \mathbf{z} \in \mathcal{N}_{k,\Omega}(\mathcal{T}) \backslash \mathcal{V}_\Omega(\mathcal{T})\} \\ &\cup \left\{B^{\mathrm{CR}}_{k,E}\mathbf{v}_E \mid E \in \mathcal{E}_\Omega(\mathcal{T})\right\} \cup \left\{B^{\mathrm{CR}}_{k,E}\mathbf{w}_E \mid E \in \mathcal{E}_\Omega(\mathcal{T})\right\}. \end{aligned} \tag{2.6}$$

Here, for any nodal point $\mathbf{z}$*, the linearly independent vectors* $\mathbf{v}_\mathbf{z}, \mathbf{w}_\mathbf{z} \in \mathbb{R}^2$ *can be chosen arbitrarily. The same holds for any triangle* K *for the vectors* $\mathbf{v}_K, \mathbf{w}_K \in \mathbb{R}^2$ *in (2.5) and for any* $E \in \mathcal{E}_\Omega(\mathcal{T})$ *for the vectors* $\mathbf{v}_E, \mathbf{w}_E \in \mathbb{R}^2$ *in (2.6).*

Remark 2.5 *The original definition of Crouzeix-Raviart spaces by [16] is implicit and given for conforming simplicial finite element meshes in* $\mathbb{R}^d$*,* $d = 2, 3$*. For their practical implementation, a basis is needed and Corollary 2.4 provides a simple definition. A basis for Crouzeix-Raviart finite elements in* $\mathbb{R}^3$ *is introduced in [20] for* $k = 2$*, a general construction is given in [14], and a basis for a minimal Crouzeix-Raviart spaces in general dimension* d *is presented in [30].*

2.2 The case of odd $k \geq 5$

In this section, we assume for the following

$$\begin{array}{ll} \text{a)} & k \geq 5 \text{ is odd and} \\ \text{b)} & \mathcal{T} \text{ is a conforming triangulation and has at least one inner vertex.} \end{array} \tag{2.7}$$

This section is structured as follows. In §2.2.1 we generalize the concept of critical points (see [37], [32]) to η-critical points which turn out to be essential for estimates with constants depending on the mesh only via the shape-regularity constant and α_Ω. We split these η-critical points into a set of "obtuse" $\eta-$critical points and "acute" $\eta-$critical points. In §2.2.2, we provide the proof of Theorem 1.3 for a maximal partial triangulation that does not contain acute $\eta-$critical points and satisfies (2.7). Finally, in §2.2.3 we present the argument to allow for acute $\eta-$critical points.

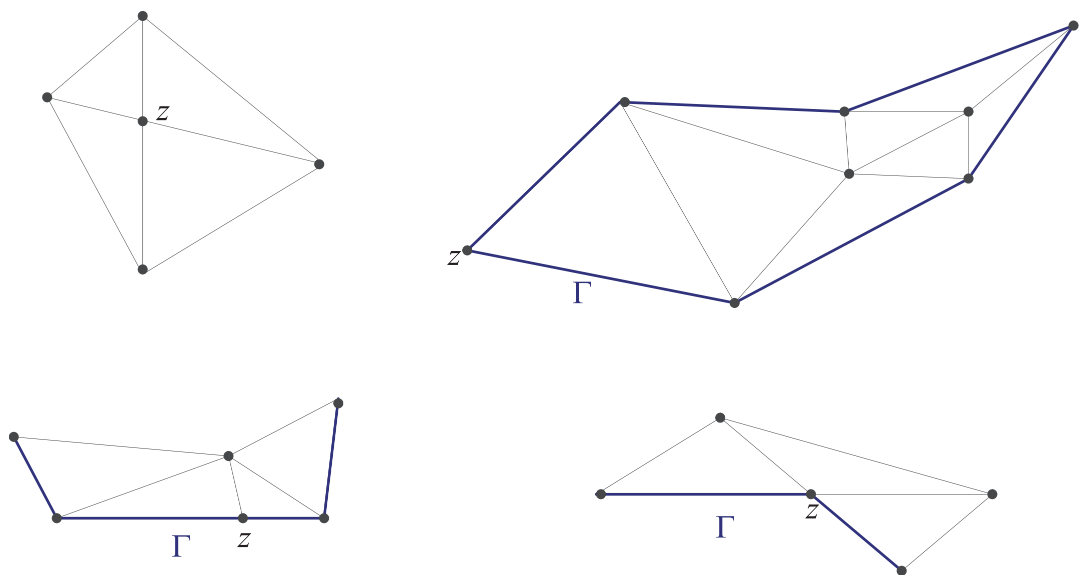

Figure 1: Illustration of the four critical cases as in Remark 2.7. Left top: inner critical point, right top: acute critical point, left bottom: flat critical point, right bottom: obtuse critical point.

2.2.1 Geometric preliminaries

For the analysis of the inf-sup constant we start with the definition of *critical points* (see [37], [32]).

Definition 2.6 *Let $\mathcal{T}$ denote a triangulation as in §1.1. For $\mathbf{z} \in \mathcal{V}(\mathcal{T})$, let*

$$\mathcal{E}_{\mathbf{z}} := \{E \in \mathcal{E}(\mathcal{T}) : \mathbf{z} \text{ is an endpoint of } E\}.$$

The point $\mathbf{z} \in \mathcal{V}(\mathcal{T})$ is a critical point *for $\mathcal{T}$ if there exist two straight infinite lines L_1, L_2 in $\mathbb{R}^2$ such that all edges $E \in \mathcal{E}_{\mathbf{z}}$ satisfy $E \subset L_1 \cup L_2$. The set of all critical points in $\mathcal{T}$ is $\mathcal{C}_{\mathcal{T}}$.*

Remark 2.7 *Geometric configurations where critical points occur are well studied in the literature (see, e.g., [32]). Any critical point $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}$ belongs to one of the following cases (see Fig. 1):*

1. *$\mathbf{z} \in \mathcal{V}_{\Omega}(\mathcal{T})$ and $\mathcal{T}_{\mathbf{z}}$ consists of four triangles and $\mathbf{z}$ is the intersections of the two diagonals in the quadrilateral $\omega_{\mathbf{z}}$.*
2. *$\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ and $\operatorname{card} \mathcal{E}_{\mathbf{z}} = 2$, i.e., both edges $E \in \mathcal{E}_{\mathbf{z}}$ are boundary edges with joint $\mathbf{z}$.*
3. *$\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ and $\operatorname{card} \mathcal{E}_{\mathbf{z}} = 3$ and two edges $E \in \mathcal{E}_{\mathbf{z}}$ are boundary edges which lie on a straight boundary piece.*
4. *$\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ and $\operatorname{card} \mathcal{E}_{\mathbf{z}} = 4$ and each of the two boundary edges is aligned with one edge of $\mathcal{E}_{\mathbf{z}} \cap \mathcal{E}_{\Omega}(\mathcal{T})$.*

Definition 2.8 *Let $\mathcal{T}$ denote a triangulation as in §1.1. Let $\mathbf{z} \in \mathcal{V}(\mathcal{T})$ and the nodal patch $\mathcal{T}_{\mathbf{z}}$ as in (1.6). Let the triangles K_ℓ, $1 \le \ell \le m$, in $\mathcal{T}_{\mathbf{z}}$ be numbered counterclockwise and denote the angle in K_ℓ at $\mathbf{z}$ by ω_ℓ. Then,*

$$\Theta(\mathbf{z}) := \begin{cases} \max\{|\sin(\omega_1+\omega_2)|, |\sin(\omega_2+\omega_3)|, \ldots, |\sin(\omega_m+\omega_1)|\} & \text{if } \mathbf{z} \in \mathcal{V}_\Omega(\mathcal{T}), \\ \max\{|\sin(\omega_1+\omega_2)|, |\sin(\omega_2+\omega_3)|, \ldots, |\sin(\omega_{m-1}+\omega_m)|\} & \text{if } \mathbf{z} \in \Gamma \wedge m > 1, \\ 0 & \text{if } \mathbf{z} \in \Gamma \wedge m = 1. \end{cases}$$

Remark 2.9 *It is easy to see that $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}$ if and only if $\Theta(\mathbf{z}) = 0$.*

Lemma 2.10 *Let $\phi_{\mathcal{T}}$ be as in Remark 1.1. Set*

$$\eta_0 := \min\left\{\frac{1}{2}, c_1, \frac{3\phi_{\mathcal{T}}}{\pi}, \sin\phi_{\mathcal{T}}\right\}$$

with

$$c_1 := \begin{cases} \min\{\sin 2\phi_{\mathcal{T}}, |\sin(2\pi - 4\phi_{\mathcal{T}})|\} & \phi_{\mathcal{T}} \le \pi/8, \\ \sin 2\phi_{\mathcal{T}} & \pi/8 < \phi_{\mathcal{T}} \le \pi/4, \\ 1 & \phi_{\mathcal{T}} > \pi/4. \end{cases}$$

Let $0 \le \eta < \eta_0$ be fixed. If, for $\mathbf{z} \in \mathcal{V}(\mathcal{T})$, it holds $\Theta(\mathbf{z}) \le \eta$, then, for any edge $E = [\mathbf{z}, \mathbf{z}'] \in \mathcal{E}_\Omega(\mathcal{T})$ it holds

$$\Theta(\mathbf{z}') \ge \eta_0.$$

Proof. Let $\mathbf{z} \in \mathcal{V}(\mathcal{T})$ and consider an edge $E = [\mathbf{z}, \mathbf{z}'] \in \mathcal{E}_\Omega(\mathcal{T})$. Then, there are two triangles $K, K' \in \mathcal{T}$ which are adjacent to E. The angle in K resp. K' at $\mathbf{z}$ is denoted by ω resp. ω'.

1st case. Let $\omega + \omega' \le \pi/2$ or $\omega + \omega' \ge \frac{3}{2}\pi$. Then, we conclude from Remark 1.1 that

$$2\phi_{\mathcal{T}} \le \omega + \omega' \le \frac{\pi}{2} \quad \text{or} \quad \frac{3}{2}\pi \le \omega + \omega' \le 2\pi - 4\phi_{\mathcal{T}}.$$

For the left inequality to hold, the minimal angle must satisfy $\phi_{\mathcal{T}} \le \pi/4$ while for the right inequality, it must hold $\phi_{\mathcal{T}} \le \pi/8$. For $\phi_{\mathcal{T}} > \pi/4$, the **1st case** is empty. For $\phi_{\mathcal{T}} \le \pi/4$ we get

$$\Theta(\mathbf{z}) \ge \left\{\begin{array}{ll} \min\{\sin 2\phi_{\mathcal{T}}, |\sin(2\pi - 4\phi_{\mathcal{T}})|\} & \phi_{\mathcal{T}} \le \pi/8 \\ \sin 2\phi_{\mathcal{T}} & \pi/8 < \phi_{\mathcal{T}} \le \pi/4 \end{array}\right\} \ge c_1 \ge \eta_0.$$

Since $\eta < \eta_0 \le c_1$ this case cannot appear.

2nd case. Let $\pi/2 < \omega + \omega' < 3\pi/2$. The condition $|\sin(\omega + \omega')| \le \eta$ implies that $\omega + \omega' = \pi + \delta$ with

$$|\delta| \le \arcsin\eta \overset{[17,\, 4.24.1]}{=} \eta \sum_{\ell=0}^{\infty} \eta^{2\ell} \frac{(2\ell)!}{(\ell!)^2 4^\ell (2\ell+1)} \overset{\eta_0 \le 1/2}{\le} \eta \sum_{\ell=0}^{\infty} 2^{-2\ell} \frac{(2\ell)!}{(\ell!)^2 4^\ell (2\ell+1)} = \frac{\pi\eta}{3}. \quad (2.8)$$

Consequently the two angles α in K and α' in K' at $\mathbf{z}'$ satisfy

$$\alpha + \alpha' = 2\pi - \omega - \omega' - \beta - \beta' = \pi - \delta - \beta - \beta' \le \pi + \frac{\pi\eta}{3} - 2\phi_{\mathcal{T}} \overset{\pi\eta/3 \le \phi_{\mathcal{T}}}{\le} \pi - \phi_{\mathcal{T}},$$

where β (resp. β') denotes the third angle in K (resp. K'). Hence, in this case

$$\Theta(\mathbf{z}') \ge |\sin(\pi - \phi_{\mathcal{T}})| = \sin\phi_{\mathcal{T}} \ge \eta_0.$$

■

Definition 2.11 *Let η_0 be as in Lemma 2.10. For $0 \leq \eta < \eta_0$, the set of* η-critical points $\mathcal{C}_{\mathcal{T}}(\eta)$ *is given by*

$$\mathcal{C}_{\mathcal{T}}(\eta) := \{\mathbf{z} \in \mathcal{V}(\mathcal{T}) \mid \Theta(\mathbf{z}) \leq \eta\}.$$

A point $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta) \backslash \mathcal{C}_{\mathcal{T}}(0)$ is called a nearly critical point. *An η-critical point $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta)$ is* isolated *if all edge $[\mathbf{z}, \mathbf{z}'] \in \mathcal{E}(\mathcal{T})$ satisfy: $\mathbf{z}'$ is not an η-critical point.*

By perturbing the geometric configurations in Remark 2.7 we obtain the following subcases (see Fig. 2).

Definition 2.12 *Let η_0 be as in Lemma 2.10 and $0 \leq \eta < \eta_0$. If $\mathbf{z} \in \mathcal{V}(\mathcal{T})$ satisfies*

1. $\mathbf{z} \in \mathcal{V}_{\Omega}(\mathcal{T})$ *and* $\operatorname{card} \mathcal{T}_{\mathbf{z}} = 4$ *and* $\Theta(\mathbf{z}) \leq \eta$. *Then* $\mathbf{z}$ *is an* inner η-critical point. *Let*

$$\mathcal{C}_{\mathcal{T}}^{\text{inner}}(\eta) := \{\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta) : \mathbf{z} \text{ is an inner } \eta\text{-critical point}\}.$$

2. $\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ *and* $\operatorname{card} \mathcal{E}_{\mathbf{z}} = 2$. *Then* $\mathbf{z}$ *is an* acute critical point. *Let*[1]

$$\mathcal{C}_{\mathcal{T}}^{\text{acute}} := \{\mathbf{z} \in \mathcal{C}_{\mathcal{T}} : \mathbf{z} \text{ is an acute critical point}\}.$$

3. $\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ *and* $\operatorname{card} \mathcal{E}_{\mathbf{z}} = 3$ *and* $\Theta(\mathbf{z}) \leq \eta$. *Then* $\mathbf{z}$ *is* flat η-critical point. *Let*

$$\mathcal{C}_{\mathcal{T}}^{\text{flat}}(\eta) := \{\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta) : \mathbf{z} \text{ is a flat } \eta\text{-critical point}\}.$$

4. $\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ *and* $\operatorname{card} \mathcal{E}_{\mathbf{z}} = 4$ *and* $\Theta(\mathbf{z}) \leq \eta$. *Then* $\mathbf{z}$ *is a (locally)* concave η-critical point. *Let*

$$\mathcal{C}_{\mathcal{T}}^{\text{concave}}(\eta) := \{\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta) : \mathbf{z} \text{ is a concave } \eta\text{-critical point}\}.$$

The acute critical points require some special treatment and we denote the union of the others by

$$\mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) := \mathcal{C}_{\mathcal{T}}^{\text{inner}}(\eta) \cup \mathcal{C}_{\mathcal{T}}^{\text{flat}}(\eta) \cup \mathcal{C}_{\mathcal{T}}^{\text{concave}}(\eta).$$

The following lemma states that for a possibly adjusted η_0, still depending only on the shape-regularity of the mesh and the maximal outer angle α_{Ω}, the η-critical points belong to one of the four categories described in Definition 2.12.

Lemma 2.13 *Let $\mathcal{T}$ be a conforming triangulation such that $D := \operatorname{dom} \mathcal{T}$ is a Lipschitz domain.*

Then, there exists some $\eta_0' \in \left]0, \eta_0\right]$ depending only on the shape-regularity of the mesh and the minimal outer angle α_D such that for $0 \leq \eta < \eta_0'$ any η-critical point belongs to one of the four categories described in Definition 2.12.

Proof. Let $\mathbf{z}$ be an η-critical point. We set $m := \operatorname{card} \mathcal{T}_{\mathbf{z}}$ and choose a counterclockwise numbering for the triangles in $\mathcal{T}_{\mathbf{z}}$, i.e., K_i, $1 \leq i \leq m$. The shape-regularity of the mesh implies that there is some $m_{\max}$ depending only on $\phi_{\mathcal{T}}$ such that $m \leq m_{\max}$. Denote by ω_i

[1] Note that the set of acute critical points is independent of η.

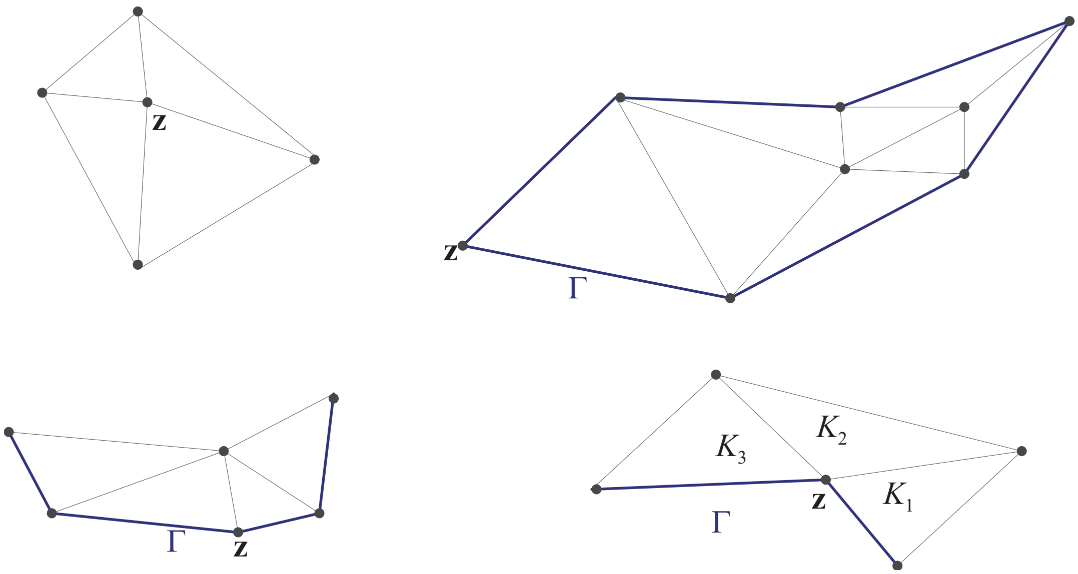

Figure 2: Illustration of the four η-critical cases as in Remark 2.12. Left top: inner $\eta-$critical point, right top: acute critical point, left bottom: flat η-critical point, right bottom: concave η-critical point.

the angle in K_i at $\mathbf{z}$. Let $\eta_0' \in]0, \eta_0]$ which will be fixed later and assume $0 \leq \eta < \eta_0'$. Since $\mathbf{z}$ is an η-critical it holds

$$|\sin(\omega_i + \omega_{i+1})| \leq \eta < \eta_0' \quad \forall 1 \leq i \leq m' \quad \text{for } m' := \begin{cases} m & \text{if } \mathbf{z} \in \mathcal{V}_\Omega(\mathcal{T}), \\ m-1 & \text{if } \mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T}). \end{cases}$$

The shape-regularity implies $\phi_\mathcal{T} \leq \omega_i \leq \pi - 2\phi_\mathcal{T}$ and, for $\delta = \arcsin \eta_0'$, we get

$$\omega_i + \omega_{i+1} \in [2\phi_\mathcal{T}, \delta] \cup [\pi - \delta, \pi + \delta] \cup [2\pi - \delta, 2\pi - 4\phi_\mathcal{T}] \quad \text{for all } 1 \leq i \leq m'. \tag{2.9}$$

Since $\arcsin : [0, 1[\to \mathbb{R}_{\geq 0}$ is monotonously increasing with $\arcsin 0 = 0$ and $\lim_{x\to 1} \arcsin x = +\infty$, we can select η_0' such $0 < \delta < 2\phi_\mathcal{T}$. In turn, the first and last interval in (2.9) are empty and

$$\omega_i + \omega_{i+1} =: \pi + \delta_i \quad \text{for some } \delta_i \text{ with } |\delta_i| \leq \delta \text{ for all } 1 \leq i \leq m'. \tag{2.10}$$

Case 1: $\mathbf{z} \in \mathcal{V}_\Omega(\mathcal{T})$.
In this case we obtain

$$4\pi = \sum_{i=1}^{m} (\omega_i + \omega_{i+1}) \overset{(2.10)}{=} m\pi + \sum_{i=1}^{m} \delta_i. \tag{2.11}$$

By adjusting η_0' such that $m_{\max}\delta < \pi$ we conclude that $m = 4$ and $\sum_{i=1}^m \delta_i = 0$. Hence, $\mathbf{z}$ is an inner η-critical point according to Definition 2.12(1).

Case 2a: $\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ and $m \leq 3$.

These cases correspond to acute critical/flat η-critical/concave η-critical points according to Definition 2.12(2-4).

Case 2b: $\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$ and $m \geq 4$.

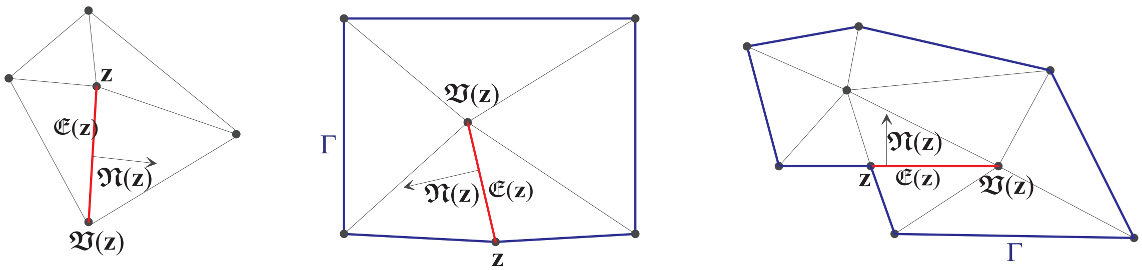

Figure 3: Three types of obtuse η-critical points $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta)$ with associated inner edge $\mathfrak{E}(\mathbf{z})$, normal vector $\mathfrak{N}(\mathbf{z})$ and opposite endpoint $\mathfrak{V}(\mathbf{z})$ of $\mathfrak{E}(\mathbf{z})$; left: inner η-critical point, middle: flat η-critical point, right: concave η-critical point.

We argue as in Case **1** but take into account that the patch $\mathcal{T}_{\mathbf{z}}$ is not "closed" since $\mathbf{z}$ is a boundary point. Let $\alpha := 2\pi - \sum_{i=1}^{m} \omega_i$ be the "outer angle" of the domain at $\mathbf{z}$. Then

$$\omega_1 + \omega_m + 2\sum_{i=2}^{m-1} \omega_i = \sum_{i=1}^{m-1} (\omega_i + \omega_{i+1}) = (m-1)\pi + \sum_{i=1}^{m-1} \delta_i.$$

By the definition of α, we obtain

$$\begin{aligned}(m-1)\pi + \sum_{i=1}^{m-1} \delta_i &= 2\pi - \alpha + \sum_{i=2}^{m-1} \omega_i = 2\pi - \alpha + \sum_{\ell=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} (\omega_{2\ell} + \omega_{2\ell+1}) + Q(m)\,\omega_{m-1} \\ &= 2\pi - \alpha + \left\lfloor \frac{m-2}{2} \right\rfloor \pi + \sum_{\ell=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \delta_{2\ell} + Q(m)\,\omega_{m-1},\end{aligned}$$

where $Q(m) = 0$ if m is even and $Q(m) = 1$ if m is odd. By rearranging the terms we get

$$Q(m)\,\omega_{m-1} + \Delta_m = \left(m - 3 - \left\lfloor \frac{m-2}{2} \right\rfloor\right)\pi + \alpha \quad \text{for} \quad \Delta_m := \sum_{\ell=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \delta_{2\ell} - \sum_{i=1}^{m-1} \delta_i. \tag{2.12}$$

We adjust η_0' such that $\delta = \arcsin \eta_0'$ satisfies $m_{\max}\delta < \alpha$ and, in turn, $|\Delta_m| \leq m_{\max}\delta < \alpha$. Then, it is easy to verify that

$$|Q(m)\,\omega_{m-1} + \Delta_m| < Q(m)\,\pi + \alpha \leq \left(m - 3 - \left\lfloor \frac{m-2}{2} \right\rfloor\right)\pi + \alpha$$

holds for all $m \geq 4$. Hence, (2.12) cannot hold and there exists no η-critical boundary point $\mathbf{z}$ for $m \geq 4$.

■

Next, we collect the $\eta-$critical points in pairwise disjoint, edge-connected sets which we will define in the following. We say two points $\mathbf{y}, \mathbf{y}' \in \mathcal{V}(\mathcal{T})$ are *edge-connected* if there is an edge $E \in \mathcal{E}(\mathcal{T})$ with endpoints $\mathbf{y}, \mathbf{y}'$. A subset $\mathcal{V}' \subset \mathcal{V}(\mathcal{T})$ is edge-connected if there is a numbering of the points in $\mathcal{V}' = \{\mathbf{y}_j : 1 \leq j \leq n\}$ such that $\mathbf{y}_{j-1}$, $\mathbf{y}_j$ are edge-connected for all $2 \leq j \leq n$. A point $\mathbf{z} \in \mathcal{V}(\mathcal{T})$ is edge-connected to $\mathcal{V}'$ if $\mathbf{z} \in \mathcal{V}'$ or there is $\mathbf{y} \in \mathcal{V}'$ such that $\mathbf{z}$, $\mathbf{y}$ are edge-connected.

From Lemma 2.10 we know that two edge-connected points $\mathbf{z}, \mathbf{z}' \in \mathcal{V}(\mathcal{T})$ can be both critical only if the connecting edge E belongs to $\mathcal{E}_{\partial\Omega}(\mathcal{T})$; in this case it holds $\mathbf{z}, \mathbf{z}' \in \mathcal{V}_{\partial\Omega}(\mathcal{T})$. Next, we will group the points in $\mathcal{C}_{\mathcal{T}}(\eta)$ into subsets called *fans*.

From Lemma 2.10 it follows that the points in $\mathcal{C}_{\mathcal{T}}^{\text{inner}}(\eta)$ are isolated (see Def. 2.11). All other η-critical points lie on the boundary. Next, we define mappings $\mathfrak{E} : \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) \rightarrow \mathcal{E}_{\Omega}(\mathcal{T})$, $\mathfrak{N} : \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) \rightarrow \mathbb{S}_2$, and $\mathfrak{V} : \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) \rightarrow \mathcal{V}(\mathcal{T}) \backslash \mathcal{C}_{\mathcal{T}}(\eta)$. The construction is illustrated in Figure 3.

For $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta)$, Definition 2.12 implies that $|\mathcal{E}_{\mathbf{z}}| \geq 3$ and hence $\mathcal{E}_{\mathbf{z}} \cap \mathcal{E}_{\Omega}(\mathcal{T}) \neq \emptyset$. We fix one edge $E \in \mathcal{E}_{\mathbf{z}} \cap \mathcal{E}_{\Omega}(\mathcal{T})$ and set $\mathfrak{E}(\mathbf{z}) := E$. Note that the choice of E is unique for $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}^{\text{flat}}(\eta)$. For $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}^{\text{inner}}(\eta)$ the choice is arbitrary. For $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}^{\text{concave}}(\eta)$, the set $\mathcal{E}_{\mathbf{z}} \cap \mathcal{E}_{\Omega}(\mathcal{T})$ consists of two edges, say E_1, E_2. We fix one of them and set $\mathfrak{E}(\mathbf{z}) := E_2$. Let $\mathbf{z}' \in \mathcal{V}(\mathcal{T})$ be such that $\mathfrak{E}(\mathbf{z}) = [\mathbf{z}, \mathbf{z}']$. Then $\mathfrak{V}(\mathbf{z}) := \mathbf{z}'$. Lemma 2.10 implies that $\mathbf{z}'$ is not an η-critical point. A unit vector $\mathfrak{N}(\mathbf{z})$ orthogonal to $\mathfrak{E}(\mathbf{z})$ is defined by the condition that $\mathbf{z}' - \mathbf{z}$ and $\mathfrak{N}(\mathbf{z})$ form a right-handed system.

Definition 2.14 *We decompose $C_{\mathcal{T}}^{\text{obtuse}}(\eta)$ into disjoint* fans $\mathcal{C}_{\mathcal{T},\ell}(\eta)$, $\ell \in \mathcal{J}$, *such that the following conditions are satisfies*

1. $\mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) = \bigcup\limits_{\ell \in \mathcal{J}} \mathcal{C}_{\mathcal{T},\ell}(\eta)$,

2. *for any $\ell \in \mathcal{J}$, the set $\mathcal{C}_{\mathcal{T},\ell}(\eta)$ is edge-connected,*

3. *for any $\ell \in \mathcal{J}$, there is $\mathbf{z}_\ell \in \mathcal{V}(\mathcal{T}) \backslash \mathcal{C}_{\mathcal{T}}(\eta)$ such that for all $\mathbf{z} \in \mathcal{C}_{\mathcal{T},\ell}(\eta)$ it holds $\mathfrak{V}(\mathbf{z}) = \mathbf{z}_\ell$ and, vice versa:*

4. *any $\mathbf{z}' \in \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta)$ which is edge-connected to some $\mathcal{C}_{\mathcal{T},\ell}(\eta)$ and satisfies $\mathfrak{V}(\mathbf{z}') = \mathbf{z}_\ell$ belongs to $\mathcal{C}_{\mathcal{T},\ell}(\eta)$.*

The following lemma will allow us to construct a right-inverse for the divergence operator separately for each fan.

Lemma 2.15 *Let η_0 be as in in Lemma 2.10 and let $0 \leq \eta < \eta_0$ be fixed.*

a. *Then, the mapping $\mathfrak{E} : \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) \rightarrow \mathcal{E}_{\Omega}(\mathcal{T})$ is injective.*

b. *For $\ell \in \mathcal{J}$, let $\omega_\ell := \bigcup\limits_{\mathbf{z} \in \mathcal{C}_{\mathcal{T},\ell}(\eta)} \omega_{\mathfrak{E}(\mathbf{z})}$. The domains ω_ℓ have pairwise disjoint interior.*

Proof. Part a. The injectivity of the mapping $\mathfrak{E} : \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta) \rightarrow \mathcal{E}_{\Omega}(\mathcal{T})$ follows from Lemma 2.10: if $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta)$ and $\mathbf{z}' \in \mathcal{V}(\mathcal{T})$ is such that $E := [\mathbf{z}, \mathbf{z}'] \in \mathcal{E}_{\Omega}(\mathcal{T})$ then $\mathbf{z}' \notin \mathcal{C}_{\mathcal{T}}(\eta)$.

Part b. The following construction is illustrated in Figure 4. For fixed $\ell \in \mathcal{J}$, we number the points in $\mathcal{C}_{\mathcal{T},\ell}$ by $\mathbf{z}_{\ell,j}$, $1 \leq j \leq n_\ell$, such that $\mathbf{z}_{\ell,j-1}$ and $\mathbf{z}_{\ell,j}$ are edge-connected for all $2 \leq j \leq n_\ell$ and $\mathbf{z}_{\ell,1}$ and $\mathbf{z}_{\ell,n_\ell}$ are the endpoints in the polygonal line through these points. Let $K_{\ell,j}$ be the triangle with vertices $\mathbf{z}_{\ell,j-1}$, $\mathbf{z}_\ell$, $\mathbf{z}_{\ell,j}$, $2 \leq j \leq n_\ell$ and let $\mathcal{T}_\ell := \{K_{\ell,j} : 2 \leq j \leq n_\ell\}$. Note that this set is empty if $|\mathcal{C}_{\mathcal{T},\ell}| = 1$. Let $K_{\ell,1}, K_{\ell,n_\ell+1} \in \mathcal{T} \backslash \mathcal{T}_\ell$ be two different triangles such that $E_{\ell,1} := [\mathbf{z}_{\ell,1}, \mathbf{z}_\ell] \subset \partial K_{\ell,1}$ and $E_{\ell,n_\ell} := [\mathbf{z}_{\ell,n_\ell}, \mathbf{z}_\ell] \subset \partial K_{\ell,n_\ell+1}$. Let $\mathbf{z}_{\ell,0}$ be the third

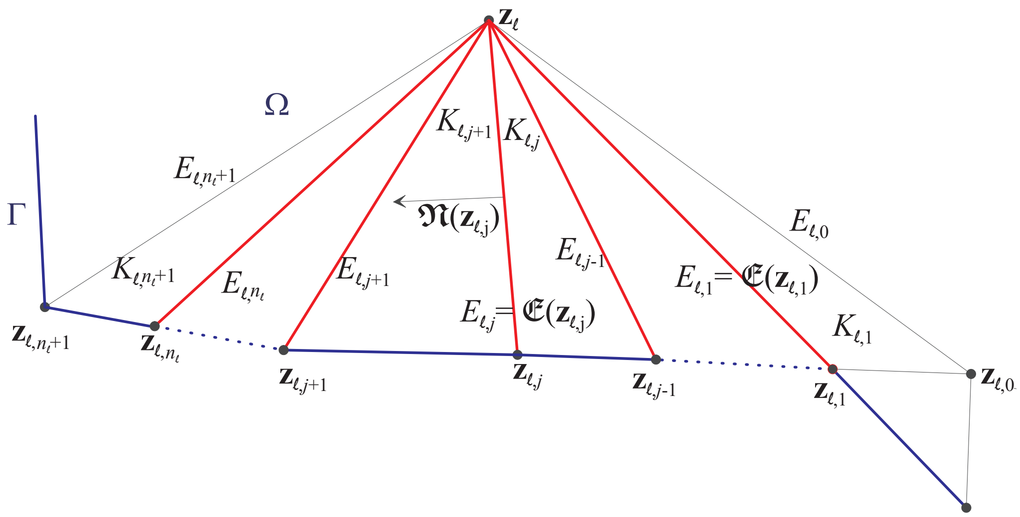

Figure 4: Nodal patch, illustrating edge-connected obtuse η-critical points of a fan $\mathcal{C}_{\mathcal{T},\ell}(\eta)$. In this example, the left-most η-critical point is $\mathbf{z}_{\ell,n_\ell}$ and of type "flat", the right-most is $\mathbf{z}_{\ell,1}$ of type "concave" with $\mathfrak{E}(\mathbf{z}_{\ell,1}) = [\mathbf{z}_{\ell,1}, \mathbf{z}]$. The extremal points $\mathbf{z}_{\ell,0}$ and $\mathbf{z}_{\ell,n_\ell+1}$ do not belong to $\mathcal{C}_{\mathcal{T},\ell}(\eta)$. The edge connecting $\mathbf{z}_\ell$ with $\mathbf{z}_{\ell,j}$ is denoted by $E_{\ell,j}$.

vertex in $K_{\ell,1}$ and observe that it does not belong to $\mathcal{C}_{\mathcal{T},\ell}$. Since $E_{\ell,1}$ is an inner edge and $\mathbf{z}_{\ell,1}$ is an η-critical point, Lemma 2.10 implies that $\mathbf{z}_\ell$ is not an η-critical point. Next, we show that $E_{\ell,0} := [\mathbf{z}_{\ell,0}, \mathbf{z}_\ell]$ does not belong to $\mathcal{E}_{\mathcal{T}}^{\text{obtuse}} := \left\{\mathfrak{E}(\mathbf{z}) : \mathbf{z} \in \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta)\right\}$; from this the assertion follows. We assume $E_{\ell,0} \in \mathcal{E}_{\mathcal{T}}^{\text{obtuse}}$ and derive a contradiction. Since $\mathbf{z}_\ell \notin \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta)$, this assumption implies that $\mathbf{z}_{\ell,0} \in \mathcal{C}_{\mathcal{T}}^{\text{obtuse}}(\eta)$. If $\mathfrak{V}(\mathbf{z}_{\ell,0}) = \mathbf{z}_\ell$ then Definition 2.14(4) implies that $\mathbf{z}_{\ell,0} \in \mathcal{C}_{\mathcal{T},\ell}$ and this is a contradiction. If $\mathfrak{V}(\mathbf{z}_{\ell,0}) \neq \mathbf{z}_\ell$, then $E_{\ell,0} \notin \mathcal{E}_{\mathcal{T}}^{\text{obtuse}}$. ■

Since the acute critical points need some special treatment we define a sequence of triangulations $\mathcal{T}_i$, $1 \le i \le L$, with the properties

1.
$$\mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots \subset \mathcal{T}_L = \mathcal{T}, \tag{2.13}$$

2. $\mathcal{T}_1$ is a maximal subset of $\mathcal{T}$ such that $\mathcal{C}_{\mathcal{T}_1}^{\text{acute}} = \emptyset$,

3. for $j = 1, 2, \ldots, L$,
$$\mathcal{T}_j = \{K \in \mathcal{T} \mid \text{at least one edge of } K \text{ belongs to } \mathcal{E}(\mathcal{T}_{j-1})\}.$$

By this step-by-step procedure, triangles are attached to a previous triangulation $\mathcal{T}_{j-1}$ which have an edge in common with the set of edges in $\mathcal{T}_{j-1}$. The proof of Theorem 1.3 under assumption (2.7) then consists of first proving the inf-sup stability for $\mathcal{T}_1$ and then to investigate the effect of attaching a triangle to an inf-sup stable triangulation. A sufficient condition for $L = 0$ is that every triangle in $\mathcal{T}$ has an interior point.

2.2.2 The case $\mathcal{C}_{\mathcal{T}}^{\text{acute}} = \emptyset$

In this section, we prove the inf-sup stability for the triangulation $\mathcal{T}_1$ in (2.13) where $\mathcal{C}_{\mathcal{T}_1}^{\text{acute}} = \emptyset$. For simplicity we skip the index 1 and write $\mathcal{T}$, $\mathcal{C}_{\mathcal{T}}(\eta)$, etc.

Next, we define some fundamental non-conforming Crouzeix-Raviart vector fields which will be used to eliminate the critical pressures in the Stokes element $(\mathbf{S}_{k,0}(\mathcal{T}), \mathbb{P}_{k-1,0}(\mathcal{T}))$.

Essential properties of the Crouzeix-Raviart function $B_{k,E}^{\text{CR}}$ are: it is a polynomial of degree k on each $K \subset \mathcal{T}_E$ and a Legendre polynomial on each edge $E' \subset \partial\omega_E$ so that the jump relations in (1.9b) are satisfied. Furthermore, $\left[B_{k,E}^{\text{CR}}\right]_E = 0$ and $\operatorname{supp} B_{k,E}^{\text{CR}} \subset \omega_E$.

In the first step, we modify the function $B_{k,E}^{\text{CR}}$ by adding a conforming edge bubble in $S_{k,0}(\mathcal{T})$ such that the $H^1(\Omega)$ norm of the modified function has an improved behaviour with respect to k.

Let $E \in \mathcal{E}_\Omega(\mathcal{T})$ with endpoints $\mathbf{V}_1$, $\mathbf{V}_2$. Set $\mathbf{t}_E = (\mathbf{V}_2 - \mathbf{V}_1) / \|\mathbf{V}_2 - \mathbf{V}_1\|$ and consider a function $w_E \in \mathbb{P}_k(\mathcal{T})$ with $\operatorname{supp} w_E = \omega_E$ and

$$w_E|_K|_{E'} = \left. B_{k,E}^{\text{CR}}\big|_K \right|_{E'} \quad \forall K \in \mathcal{T}_E \text{ and } E' \subset \partial K \cap \partial\omega_E, \tag{2.14a}$$

$$[w_E]_E = 0 \quad \text{and} \quad \partial_{\mathbf{t}_E} w_E(\mathbf{V}_1) = \partial_{\mathbf{t}_E} w_E(\mathbf{V}_2) = 0. \tag{2.14b}$$

Then, w_E also belongs to the space $\text{CR}_{k,0}(\mathcal{T})$ and

$$\nabla\left(w_E|_K\right)(\mathbf{z}) = \nabla\left(B_{k,E}^{\text{CR}}\big|_K\right)(\mathbf{z}) \quad \forall K \in \mathcal{T} \quad \forall \mathbf{z} \in \mathcal{V}(K). \tag{2.15}$$

The last relation can be derived from the following reasoning. For $K \in \mathcal{T}$ and $\mathbf{z} \in \mathcal{V}(K)$, set $\mathbf{t}_{\mathbf{y}} := (\mathbf{y} - \mathbf{z}) / \|\mathbf{y} - \mathbf{z}\|$ for all $\mathbf{y} \in \mathcal{V}(K) \setminus \{\mathbf{z}\}$. Let $\mathbf{c} \in \mathbb{R}^2$ be arbitrary. Clearly, $\mathbf{c} = \sum_{\mathbf{y} \in \mathcal{V}(K)\setminus\{\mathbf{z}\}} \alpha_{\mathbf{y}} \mathbf{t}_{\mathbf{y}}$ for some $\alpha_{\mathbf{y}} \in \mathbb{R}$. The conditions in (2.14) imply that for $\mathbf{y} \in \mathcal{V}(K) \setminus \{\mathbf{z}\}$ it holds

$$\frac{\partial\, w_E|_K}{\partial \mathbf{t}_{\mathbf{y}}} = \frac{\partial\, B_{k,E}^{\text{CR}}\big|_K}{\partial \mathbf{t}_{\mathbf{y}}}.$$

Hence,

$$\begin{aligned} \left\langle \nabla\left(w_E|_K\right), \mathbf{c} \right\rangle(\mathbf{z}) &= \sum_{\mathbf{y} \in \mathcal{V}(K)\setminus\{\mathbf{z}\}} \alpha_{\mathbf{y}} \frac{\partial\, w_E|_K}{\partial \mathbf{t}_{\mathbf{y}}}(\mathbf{z}) \\ &= \sum_{\mathbf{y} \in \mathcal{V}(K)\setminus\{\mathbf{z}\}} \alpha_{\mathbf{y}} \frac{\partial\, B_{k,E}^{\text{CR}}\big|_K}{\partial \mathbf{t}_{\mathbf{y}}}(\mathbf{z}) = \left\langle \nabla\left(\left(B_{k,E}^{\text{CR}}\right)\big|_K\right), \mathbf{c} \right\rangle(\mathbf{z}). \end{aligned} \tag{2.16}$$

Since $\mathbf{c}$ was arbitrary, (2.15) follows.

Lemma 2.16 *Let $k \geq 5$ be odd and for $E \in \mathcal{E}_\Omega(\mathcal{T})$, let $B_{k,E}^{\text{CR}}$ be as in (2.4). Then, there exists a function $\tilde{B}_{k,E}^{\text{CR}} \in \text{CR}_{k,0}(\mathcal{T})$ with*

i. $\operatorname{supp} \tilde{B}_{k,E}^{\text{CR}} = \omega_E$,

ii. for all $K \in \mathcal{T}$, for all $\mathbf{z} \in \mathcal{V}(K)$:

$$\nabla\left(\tilde{B}_{k,E}^{\text{CR}}\big|_K\right)(\mathbf{z}) = \nabla\left(B_{k,E}^{\text{CR}}\big|_K\right)(\mathbf{z}), \tag{2.17}$$

iii. for all $K \in \mathcal{T}_E$, for all $E' \subset \partial K \cap \partial\omega_E$:

$$\tilde{B}^{\mathrm{CR}}_{k,E}\Big|_K\Big|_{E'} = B^{\mathrm{CR}}_{k,E}\big|_K\Big|_{E'} \quad \text{and} \quad \left[\tilde{B}^{\mathrm{CR}}_{k,E}\right]_E = 0,$$

iv. for all $K \in \mathcal{T}$

$$\int_K \operatorname{div}_{\mathcal{T}}\left(\tilde{B}^{\mathrm{CR}}_{k,E}\mathbf{n}_E\right) = 0. \tag{2.18}$$

v. The piecewise gradient is bounded:

$$\left\|\nabla_{\mathcal{T}}\tilde{B}^{\mathrm{CR}}_{k,E}\right\|_{\mathbf{L}^2(\Omega)} \leq C\sqrt{\log(k+1)}. \tag{2.19}$$

Proof. We employ the reference triangle as in [6] in order to apply the polynomial extension theorem therein. Let $\tilde{K}$ be the equilateral triangle with vertices $\tilde{\mathbf{A}}_1 := (-1,0)^T$, $\tilde{\mathbf{A}}_2 := (1,0)^T$, $\tilde{\mathbf{A}}_3 := \left(0,\sqrt{3}\right)^T$ and let $\tilde{E}_j$ denote the edge in $\tilde{K}$ opposite to $\tilde{\mathbf{A}}_j$, $1 \leq j \leq 3$.

Let $E \in \mathcal{E}_\Omega(\mathcal{T})$ with endpoints $\mathbf{V}_1$, $\mathbf{V}_2$, and let $K \in \mathcal{T}_E$. Choose an affine pullback $\phi_K : \tilde{K} \to K$ such that $\phi_K\left(\tilde{E}_3\right) = E$. We employ the function $\tilde{\psi}_k^{\pm} \in \mathbb{P}_k([-1,1])$ given by

$$\tilde{\psi}_k^-(x) := c_k^{-1}\frac{(1+x)(1-x)^2}{4}P^{(3,3)}_{k-3}(x) \quad \text{and} \quad \tilde{\psi}_k^+(x) := -\tilde{\psi}_k^-(-x)$$

with the Jacobi polynomials $P^{(\alpha,\beta)}_n$ (see, e.g., [17, §18.3]) and the normalisation factor $c_k := (-1)^{k-1}\binom{k}{3}$. These functions have been analysed in the proof of Lemma A.1 in [3, denoted by F_k] and we recall relevant properties. It holds $\tilde{\psi}_k^{\pm}(\pm 1) = \left(\tilde{\psi}_k^-\right)'(+1) = \left(\tilde{\psi}_k^+\right)'(-1) = 0$ and $\left(\tilde{\psi}_k^-\right)'(-1) = \left(\tilde{\psi}_k^+\right)'(+1) = 1$ (cf. [17, §18.3]). Their norms can be estimated by

$$\left\|\tilde{\psi}_k^{\pm}\right\|_{L^2([-1,1])} \leq Ck^{-3} \quad \text{and} \quad \left\|\left(\tilde{\psi}_k^{\pm}\right)'\right\|_{L^2([-1,1])} \leq Ck^{-1}.$$

We set

$$\tilde{\varphi}_k(x) = L_{k-1}(x) - L'_{k-1}(-1)\,\tilde{\psi}_k^-(x) - L'_{k-1}(1)\,\tilde{\psi}_k^+(x). \tag{2.20}$$

Clearly, it holds

$$\tilde{\varphi}_k(\pm 1) = 1, \quad \tilde{\varphi}_k'(\pm 1) = 0.$$

By using Lemma C.1, we get

$$\|\tilde{\varphi}_k\|_{L^2([-1,1])} \leq Ck^{-1/2}, \quad \|\tilde{\varphi}_k\|_{H^1([-1,1])} \leq Ck.$$

From [6, Thm. 7.4] we conclude that there is $\tilde{w}_E \in \mathbb{P}_k\left(\tilde{K}\right)$ with

$$\tilde{w}_E|_{\tilde{E}_j} = B^{\mathrm{CR}}_{k,E}\big|_K \circ \phi_K\Big|_{\tilde{E}_j}, \quad j = 1,2 \quad \text{and} \quad \tilde{w}_E|_{\tilde{E}_3} = \tilde{\varphi}_k$$

which satisfies

$$\|\tilde{w}_E\|_{H^1(\tilde{K})} \leq C\,\|\tilde{w}_E\|_{H^{1/2}(\partial\tilde{K})}$$

for a constant C independent of k. Lemma B.2 implies the following estimate of the $H^{1/2}$ norm of $\tilde{w}_E$:

$$\|\tilde{w}_E\|_{H^{1/2}(\partial\tilde{K})} \leq C\sqrt{\log(k+1)}. \tag{2.21}$$

In turn, we get

$$\|\tilde{w}_E\|_{H^1(\tilde{K})} \leq C\sqrt{\log(k+1)}. \tag{2.22}$$

By using the affine lifting ϕ_K to the triangle K we define the function w_E by

$$w_E|_K := \begin{cases} \tilde{w}_E \circ \phi_K^{-1} & \text{if } K \in \mathcal{T}_E, \\ 0 & \text{otherwise.} \end{cases}$$

This function is continuous across E (with value $\tilde{\varphi}_k \circ \phi_K^{-1}|_E$) and, on $E' \subset \partial\omega_E$, it is a lifted Legendre polynomial. This implies property (iii) for w_E. The function w_E vanishes outside ω_E so that (i) holds. Since the construction implies that the derivative of w_E in the direction of E, evaluated at the endpoints $\mathbf{V}_1$, $\mathbf{V}_2$ of E, is zero, we may apply the reasoning in (2.16) to obtain property (ii) for w_E.

From (2.22) we obtain by the transformation rule for integrals and the chain rule for differentiation

$$\|\nabla w_E\|_{\mathbf{L}^2(K)} \leq C\|\tilde{w}_E\|_{H^1(\tilde{K})} \leq C\sqrt{\log(k+1)} \quad \forall K \in \mathcal{T}_E.$$

Next, we modify w_E such that property (iv) holds without affecting the other properties. Let $\psi_E \in S_{4,0}(\mathcal{T})$ with $\operatorname{supp}\psi_E = \omega_E$ and

$$\psi_E|_K := \alpha_K \lambda_{K,\mathbf{V}_1}^2 \lambda_{K,\mathbf{V}_2}^2 \quad \text{with} \quad \alpha_K := \left(\int_K \partial_{\mathbf{n}_E} w_E\right) / \left(\int_K \partial_{\mathbf{n}_E}\left(\lambda_{K,\mathbf{V}_1}^2 \lambda_{K,\mathbf{V}_2}^2\right)\right) \quad \forall K \in \mathcal{T}_E. \tag{2.23}$$

The modified function $\tilde{B}_{k,E}^{\mathrm{CR}}$ finally is defined by

$$\tilde{B}_{k,E}^{\mathrm{CR}} = w_E - \psi_E. \tag{2.24}$$

Since $\operatorname{div}_{\mathcal{T}}\left(\tilde{B}_{k,E}^{\mathrm{CR}} \mathbf{n}_E\right)\Big|_K = \partial_{\mathbf{n}_E}\left(\tilde{B}\,{}_{k,E}^{\mathrm{CR}}\big|_K\right)$ property (iv) follows by construction. The gradient $\nabla_{\mathcal{T}}\psi_E$ vanishes in the vertices of K so that $(\partial_{\mathbf{n}_E}\psi_E)(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathcal{V}(K)$ and (ii) is inherited from w_E. Properties (i), (iii) are obvious. Next, we verify (v). Let $\mathbf{V}_3$ denote the vertex in K opposite to E. We first compute

$$\begin{aligned}\int_K \partial_{\mathbf{n}_E}\left(\lambda_{K,\mathbf{V}_1}^2 \lambda_{K,\mathbf{V}_2}^2\right) &= \sum_{j=1}^{2} 2\partial_{\mathbf{n}_E}\lambda_{K,\mathbf{V}_j}\int_K \lambda_{K,\mathbf{V}_1}\lambda_{K,\mathbf{V}_2}\lambda_{K,\mathbf{V}_{3-j}} = \int_K \lambda_{K,\mathbf{V}_1}^2\lambda_{K,\mathbf{V}_2}\sum_{j=1}^{2} 2\partial_{\mathbf{n}_E}\lambda_{K,\mathbf{V}_j} \\ &= -\frac{1}{15}\partial_{\mathbf{n}_E}\lambda_{K,\mathbf{V}_3}|K| \overset{(2.3)}{=} \frac{|E|}{30}, \\ \left|\int_K \partial_{\mathbf{n}_E} w_E\right| &\leq |K|^{1/2}\|\nabla w_E\|_{\mathbf{L}^2(K)} \leq C|K|^{1/2}\sqrt{\log(k+1)}.\end{aligned}$$

In this way, $|\alpha_K| \leq C\sqrt{\log(k+1)}$ and an inverse inequality for quartic polynomials gives us

$$\|\nabla\psi_E\|_{\mathbf{L}^2(K)} \leq Ch_K^{-1}\|\psi_E\|_{L^2(K)} \leq Ch_K^{-1}\sqrt{\log(k+1)}\,\|1\|_{L^2(K)} \leq C\sqrt{\log(k+1)}.$$

Hence, property (iv) follows. ■

Next we recall a result which goes back to Vogelius [37] and Scott-Vogelius [32], see also [27, Proof of Thm. 1].

Definition 2.17 *Let η_0 be as in Lemma 2.10. For $0 \leq \eta < \eta_0$, the subspace $M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})$ of the pressure space $M_{k-1}(\mathcal{T})$ is given by*

$$M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T}) := \{q \in M_{k-1}(\mathcal{T}) \mid \forall \mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta) : A_{\mathcal{T},\mathbf{z}}(q) = 0\}, \tag{2.25}$$

where, for $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta)$, the functional $A_{\mathcal{T},\mathbf{z}}(q)$ is as follows: fix the counterclockwise numbering K_ℓ, $1 \leq \ell \leq m$, of the triangles in the patch $\mathcal{T}_{\mathbf{z}}$ by the condition $K_1 \cap K_2 = \mathfrak{E}(\mathbf{z})$ and set

$$A_{\mathcal{T},\mathbf{z}}(q) = \sum_{\ell=1}^{m} (-1)^\ell \left(q|_{K_\ell} \right)(\mathbf{z}). \tag{2.26}$$

Note that $M_{0,k-1}^{\mathrm{SV}}(\mathcal{T})$ is the pressure space introduced by Vogelius [37] and Scott-Vogelius [32] and the following inclusions hold: for $0 \leq \eta \leq \eta' \leq \eta_0$

$$M_{\eta',k-1}^{\mathrm{SV}}(\mathcal{T}) \subset M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T}) \subset M_{0,k-1}^{\mathrm{SV}}(\mathcal{T}) = Q_h^{k-1}$$

with the pressure space Q_h^{k-1} in [27, p. 517].

For the Scott-Vogelius pressure space $M_{0,k-1}^{\mathrm{SV}}(\mathcal{T})$, the existence of a continuous right-inverse of the divergence operator into $\mathbf{S}_{k,0}(\mathcal{T})$ was proved in [37] and [32].

Proposition 2.18 (Scott-Vogelius) *For any $p \in M_{0,k-1}^{\mathrm{SV}}(\mathcal{T})$ there exists some $\mathbf{v} \in \mathbf{S}_{k,0}(\mathcal{T})$ such that*

$$\operatorname{div} \mathbf{v} = q \quad \textit{and} \quad \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C \|q\|_{L^2(\Omega)},$$

for a constant which only depends on the shape-regularity of the mesh, the polynomial degree k, and on $\Theta_{\min}^{-1}$, where

$$\Theta_{\min} := \min_{\mathbf{z} \in \mathcal{V}(\mathcal{T}) \backslash \mathcal{C}_{\mathcal{T}}} \Theta(\mathbf{z}). \tag{2.27}$$

In particular, the constant C is independent of h.

In Lemma 2.20, we will show that, by subtracting the divergence of a suitable Crouzeix-Raviart velocity from a given pressure in $M_{k-1}(\mathcal{T})$, the resulting modified pressure belongs to the reduced pressure space $M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})$. As a preliminary, we need a bound of the functional $A_{\mathcal{T},\mathbf{z}}$ in (2.26) which is explicit with respect to the local mesh size and polynomial degree.

Lemma 2.19 *There exists a constant C which only depends on the shape-regularity of the mesh such that*

$$|A_{\mathcal{T},\mathbf{z}}(q)| \leq C \frac{k^2}{h_{\mathbf{z}}} \|q\|_{L^2(\omega_{\mathbf{z}})} \quad \forall q \in \mathbb{P}_{k-1}(\mathcal{T})$$

for any $k \in \mathbb{N}$.

Proof. Let $\mathbf{z} \in \mathcal{V}(\mathcal{T})$ and $K \in \mathcal{T}_{\mathbf{z}}$. The affine pullback to the reference triangle is denoted by $\chi_K : \widehat{K} \to K$. For $q \in \mathbb{P}_k(K)$, let $\widehat{q} := q \circ \chi_K$ and $\hat{\mathbf{z}} := \chi_K^{-1}(\mathbf{z})$. Then

$$|q(\mathbf{z})| = |\widehat{q}(\hat{\mathbf{z}})| \overset{[38],\,[2,\text{ Lem. } 6.1]}{\leq} \frac{(k+1)(k+2)}{\sqrt{2}} \|\widehat{q}\|_{L^2(\widehat{K})} = \binom{k+2}{2} |K|^{-1/2} \|q\|_{L^2(K)}. \tag{2.28}$$

A summation over all $K \in \mathcal{T}_{\mathbf{z}}$ leads to

$$|A_{\mathcal{T},\mathbf{z}}(q)| \leq \binom{k+2}{2} \sum_{\ell=1}^{m} |K_\ell|^{-1/2} \|q\|_{L^2(K_\ell)} \leq C \binom{k+2}{2} h_{\mathbf{z}}^{-1} \|q\|_{L^2(\omega_{\mathbf{z}})},$$

where C only depends on the shape-regularity of the mesh. ■

The following lemma shows that the non-conforming Crouzeix-Raviart elements allow us to modify a general pressure $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$ in such a way that the result belongs to $M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})$ *provided* $\mathcal{C}_{\mathcal{T}}^{\mathrm{acute}} = \emptyset$.

Lemma 2.20 *Let assumption (2.7) be satisfied and $\mathcal{C}_{\mathcal{T}}^{\mathrm{acute}} = \emptyset$. There exists a constant $\eta_2 > 0$ which only depends on the shape-regularity of the mesh and α_Ω (see (1.1)) such that for any fixed $0 \leq \eta < \eta_2$ and any $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$, there exists some $\mathbf{v}_q \in \mathbf{CR}_{k,0}(\mathcal{T})$ such that*

$$\int_K \operatorname{div} \mathbf{v}_q = 0 \quad \forall K \in \mathcal{T}, \tag{2.29}$$

$$q - \operatorname{div}_{\mathcal{T}} \mathbf{v}_q \in M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T}) \tag{2.30}$$

and

$$\|\nabla_{\mathcal{T}} \mathbf{v}_q\|_{\mathbb{L}^2(\Omega)} \leq C_{\mathrm{CR}} \sqrt{\log(k+1)}\, \|q\|_{L^2(\Omega)}. \tag{2.31}$$

The constant C_{CR} depends only on the shape-regularity of the mesh and α_Ω.

Proof. Let $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$. Let the fans $\mathcal{C}_{\mathcal{T},\ell}(\eta)$, $\ell \in \mathcal{J}$, be as in Definition 2.14. For each fan $\mathcal{C}_{\mathcal{T},\ell}(\eta)$ we employ an ansatz

$$\mathbf{v}_\ell := \sum_{\mathbf{z} \in \mathcal{C}_{\mathcal{T},\ell}(\eta)} \alpha_{\ell,\mathbf{z}} \tilde{B}_{k,\mathfrak{E}(\mathbf{z})}^{\mathrm{CR}} \mathfrak{N}(\mathbf{z}) \tag{2.32}$$

for $\tilde{B}_{k,\mathfrak{E}(\mathbf{z})}^{\mathrm{CR}}$ as in (2.24), where the coefficients $\alpha_{\ell,\mathbf{z}} \in \mathbb{R}$ are defined next. The global function $\mathbf{v}_q$ is then given by

$$\mathbf{v}_q = \sum_{\ell=1}^{N} \mathbf{v}_\ell.$$

Property (2.29) follows from this ansatz by using Lemma 2.16. Next, we define the coefficients $\alpha_{\ell,\mathbf{z}}$ in (2.32) such that (2.30) holds and prove the norm estimates for $\mathbf{v}_q$. Our construction of the fans implies that open interiors of the supports of $\mathbf{v}_\ell$ are pairwise disjoint (see Lem. 2.15); as a consequence the definition of $(\alpha_{\ell,\mathbf{z}})_{\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta)}$ and the estimate of $\nabla_{\mathcal{T}} \mathbf{v}_q$ can be performed for each fan separately.

For $\mathbf{z} \in \mathcal{C}_{\mathcal{T},\ell}(\eta)$, let $E := \mathfrak{E}(\mathbf{z})$, $\mathbf{n}_E := \mathfrak{N}(\mathbf{z})$. Let $K_{\mathbf{z}}^-$, $K_{\mathbf{z}}^+$, denote the triangles in $\mathcal{T}_E$ with the convention that $\mathbf{n}_E$ points into $K_{\mathbf{z}}^+$. The vertex in $K_{\mathbf{z}}^\pm$ opposite to E is denoted by $\mathbf{A}^\pm$. We use

$$\operatorname{div}\left(\tilde{B}_{k,E}^{\mathrm{CR}} \mathbf{n}_E\Big|_K\right)(\mathbf{y}) \overset{(2.17)}{=} \operatorname{div}\left(B_{k,E}^{\mathrm{CR}} \mathbf{n}_E\big|_K\right)(\mathbf{y}) \quad \forall \mathbf{y} \in \mathcal{V}(K)$$

and compute the divergence of $B_{k,E}^{\mathrm{CR}} \mathbf{n}_E$

$$\operatorname{div}\left(B_{k,E}^{\mathrm{CR}} \mathbf{n}_E\big|_K\right) = \begin{cases} \mp \frac{|E|}{|K|} L_k'(1 - 2\lambda_{K,\mathbf{A}^\pm}) & \text{on } K = K_{\mathbf{z}}^\pm, \quad i = 1,2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

Well-known properties of Legendre polynomials applied to (2.33) imply that for any vertex $\mathbf{y}$ of K and odd polynomial degree k

$$\operatorname{div}\left(\tilde{B}_{k,E}^{\mathrm{CR}} \mathbf{n}_E\Big|_K\right)(\mathbf{y}) = \mp \binom{k+1}{2} \times \begin{cases} \frac{|E|}{|K|} & \forall \mathbf{y} \in \mathcal{V}(K), \quad \text{if } K = K_{\mathbf{z}}^\pm \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

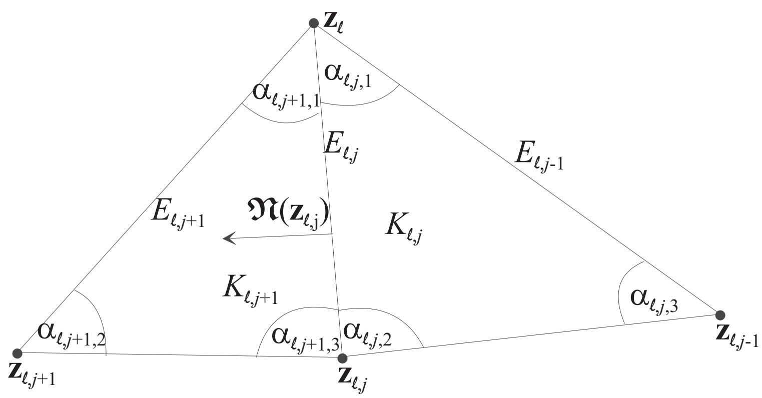

Figure 5: Local numbering convention of the angles in $K_{\ell,j}$ and $K_{\ell,j+1}$. The angle in $K_{\ell,j}$ at $\mathbf{z}_\ell$ is denoted by $\alpha_{\ell,j,1}$, at $\mathbf{z}_{\ell,j}$ by $\alpha_{\ell,j,2}$, at $\mathbf{z}_{\ell,j-1}$ by $\alpha_{\ell,j,3}$ and in $K_{\ell,j+1}$ accordingly.

Hence the condition $A_{\mathcal{T},\mathbf{y}}\left(q-\operatorname{div}_{\mathcal{T}}\mathbf{v}_\ell\right)=0$ for all $\mathbf{y}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)$ is equivalent to the system of linear equation

$$\mathbf{M}_\ell\boldsymbol{\alpha}_\ell=\mathbf{r}_\ell \tag{2.35}$$

with

$$\mathbf{M}_\ell:=\left(A_{\mathcal{T},\mathbf{y}}\left(\operatorname{div}_{\mathcal{T}}\left(B^{\mathrm{CR}}_{k,\mathfrak{E}(\mathbf{z})}\mathfrak{N}(\mathbf{z})\right)\right)\right)_{\substack{\mathbf{y}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)\\ \mathbf{z}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)}},\quad \boldsymbol{\alpha}_\ell:=\left(\alpha_{\ell,\mathbf{z}}\right)_{\mathbf{z}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)},\quad \mathbf{r}_\ell:=\left(A_{\mathcal{T},\mathbf{y}}(q)\right)_{\mathbf{y}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)}. \tag{2.36}$$

The matrix $\mathbf{M}_\ell$ is explicitly given by

$$\mathbf{M}_\ell:=-\binom{k+1}{2}\begin{bmatrix}\frac{|E_{\ell,1}|}{|K_{\ell,1}|}+\frac{|E_{\ell,1}|}{|K_{\ell,2}|} & \frac{|E_{\ell,2}|}{|K_{\ell,2}|} & 0 & \cdots & 0\\ \frac{|E_{\ell,1}|}{|K_{\ell,2}|} & \frac{|E_{\ell,2}|}{|K_{\ell,2}|}+\frac{|E_{\ell,2}|}{|K_{\ell,3}|} & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & & 0\\ \vdots & \ddots & & & \frac{|E_{\ell,n_\ell}|}{|K_{\ell,n_\ell}|}\\ 0 & \cdots & 0 & \frac{|E_{\ell,n_\ell-1}|}{|K_{\ell,n_\ell}|} & \frac{|E_{\ell,n_\ell}|}{|K_{\ell,n_\ell}|}+\frac{|E_{\ell,n_\ell}|}{|K_{\ell,n_{\ell+1}}|}\end{bmatrix}.$$

We use (cf. Fig. 5)

$$\frac{|E_{\ell,j}|}{|K_{\ell,j}|}=\frac{2\sin\left(\alpha_{\ell,j,1}+\alpha_{\ell,j,2}\right)}{|E_{\ell,j}|\sin\alpha_{\ell,j,1}\sin\alpha_{\ell,j,2}}\quad\text{and}\quad\frac{|E_{\ell,j}|}{|K_{\ell,j+1}|}=\frac{2\sin\left(\alpha_{\ell,j+1,1}+\alpha_{\ell,j+1,3}\right)}{|E_{\ell,j}|\sin\alpha_{\ell,j+1,1}\sin\alpha_{\ell,j+1,3}}$$

(cf. [10, formula before (3.29)]) and obtain

$$\mathbf{M}_\ell=\mathbf{D}_\ell\left(\mathbf{T}_\ell+\boldsymbol{\Delta}_\ell\right) \tag{2.37}$$

with $\mathbf{D}_\ell = -k\,(k+1)\,\operatorname{diag}\left[|E_{\ell,j}|^{-1} : 1 \le j \le n_\ell\right]$ and

$$\mathbf{T}_\ell := \begin{bmatrix} \frac{\sin\left(\alpha_{\ell,1,1}+\alpha_{\ell,2,1}\right)}{\sin\alpha_{\ell,1,1}\sin\alpha_{\ell,2,1}} & \frac{1}{\sin\alpha_{\ell,2,1}} & 0 & \dots & 0 \\ \frac{1}{\sin\alpha_{\ell,2,1}} & \frac{\sin\left(\alpha_{\ell,2,1}+\alpha_{\ell,3,1}\right)}{\sin\alpha_{\ell,2,1}\sin\alpha_{\ell,3,1}} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & & & \frac{1}{\sin\alpha_{\ell,n_\ell,1}} \\ 0 & \dots & 0 & \frac{1}{\sin\alpha_{\ell,n_\ell,1}} & \frac{\sin\left(\alpha_{\ell,n_\ell,1}+\alpha_{\ell,n_\ell+1,1}\right)}{\sin\alpha_{\ell,n_\ell,1}\sin\alpha_{\ell,n_\ell+1,1}} \end{bmatrix}, \tag{2.38}$$

$$\boldsymbol{\Delta}_\ell := \operatorname{diag}\left[\frac{\sin\left(\alpha_{\ell,j,2}+\alpha_{\ell,j+1,3}\right)}{\sin\alpha_{\ell,j,2}\sin\alpha_{\ell,j+1,3}} : 1 \le j \le n_\ell\right].$$

In Lemma A.1, we will prove that the matrix $\mathbf{T}_\ell + \boldsymbol{\Delta}_\ell$ is invertible and the inverse is bounded by a constant independent of $h_{\mathcal{T}}$ and k. Hence,

$$\|\boldsymbol{\alpha}_\ell\| \le \tilde{C}\frac{h_{\mathbf{z}_\ell}}{k\,(k+1)}\|\mathbf{r}_\ell\|. \tag{2.39}$$

Let $\mathcal{T}_\ell := \{K_{\ell,j} : 1 \le j \le n_\ell + 1\}$ and $D_\ell := \operatorname{dom}(\mathcal{T}_\ell)$. We estimate the function $\mathbf{v}_\ell$ in (2.32) by

$$\begin{aligned} \|\nabla_{\mathcal{T}}\mathbf{v}_\ell\|_{\mathbb{L}^2(D_\ell)} &\le \left(\sum_{\mathbf{z}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)} |\alpha_{\ell,\mathbf{z}}|^2 \left\|\nabla_{\mathcal{T}}\tilde{B}^{\mathrm{CR}}_{k,\mathfrak{E}(\mathbf{z})}\mathfrak{N}(\mathbf{z})\right\|^2_{\mathbb{L}^2(D_\ell)}\right)^{1/2} \\ &\le \max_{\mathbf{z}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)} \left\|\nabla_{\mathcal{T}}\tilde{B}^{\mathrm{CR}}_{k,\mathfrak{E}(\mathbf{z})}\mathfrak{N}(\mathbf{z})\right\|_{\mathbb{L}^2(D_\ell)} \|\boldsymbol{\alpha}_\ell\| \\ &\overset{(2.19)}{\le} C h_{\mathbf{z}_\ell}\frac{\sqrt{\log(k+1)}}{(k+1)^2}\|\mathbf{r}_\ell\|. \end{aligned}$$

The constant C only depends on the shape-regularity of the mesh. We use Lemma 2.19 and conclude that

$$\|\nabla_{\mathcal{T}}\mathbf{v}_\ell\|_{\mathbb{L}^2(D_\ell)} \le C\sqrt{\log(k+1)}\,\|q\|_{L^2(D_\ell)}.$$

Since the interiors of the supports D_ℓ have pairwise empty intersection the estimate

$$\|\nabla_{\mathcal{T}}\mathbf{v}_q\|_{\mathbb{L}^2(\Omega)} \le C\sqrt{\log(k+1)}\,\|q\|_{L^2(\Omega)}$$

follows. ■

Definition 2.21 *Let assumption (2.7) be satisfied and $\eta_2 > 0$ as in Lemma 2.20. Fix $\eta \in [0, \eta_2[$. For $q \in \mathbb{P}_{k,0}(\mathcal{T})$, the linear map $\Pi^{\mathrm{CR}}_k : \mathbb{P}_{k-1,0}(\mathcal{T}) \to \mathbf{CR}_{k,0}(\mathcal{T})$ is given by*

$$\Pi^{\mathrm{CR}}_k q := \sum_{\ell\in\mathcal{J}}\sum_{\mathbf{z}\in\mathcal{C}_{\mathcal{T},\ell}(\eta)} \alpha_{\ell,\mathbf{z}}\tilde{B}^{\mathrm{CR}}_{k,\mathfrak{E}(\mathbf{z})}\mathfrak{N}(\mathbf{z})$$

with $\boldsymbol{\alpha}_\ell$ as in (2.35).

For the proof of the following lemma we recall the definitions of some linear maps from the literature which are related to the right inverse of the divergence operator acting on some polynomial spaces.

Bernardi and Raugel introduced in [8, Lem. II.4] a linear mapping $\Pi^{\mathrm{BR}} : M_{k-1}(\mathcal{T}) \to \mathbf{S}_{2,0}(\mathcal{T})$ with the property: for any $q \in M_{k-1}(\mathcal{T})$, the function $\Pi^{\mathrm{BR}} q \in \mathbf{S}_{2,0}(\mathcal{T})$ satisfies

$$\int_K q = \int_K \operatorname{div}\left(\Pi^{\mathrm{BR}} q\right) \quad \forall K \in \mathcal{T} \tag{2.40}$$

and

$$\left\|\Pi^{\mathrm{BR}} q\right\|_{\mathbf{H}^1(\Omega)} \leq C_{\mathrm{BR}} \|q\|_{L^2(\Omega)} \tag{2.41}$$

for a constant C_{BR} which is independent of the mesh width and the polynomial degree.

Next we consider some right inverse of the divergence operator on the space

$$M_{k-1}^{\mathrm{V}}(\mathcal{T}) := \left\{ q \in M_{k-1}(\mathcal{T}) \mid \begin{pmatrix} \int_K q = 0 & \forall K \in \mathcal{T} \\ q|_K(\mathbf{y}) = 0 & \forall \mathbf{y} \in \mathcal{V}(K) \end{pmatrix} \right\}.$$

Let

$$\mathbf{S}_k^{\mathrm{V}}(\mathcal{T}) := \left\{ \mathbf{u} \in \mathbf{S}_k(\mathcal{T}) \mid \forall K \in \mathcal{T} : \quad \mathbf{u}|_{\partial K} = \mathbf{0} \right\}.$$

There exists a linear operator $\Pi^{\mathrm{V}} : M_{k-1}^{\mathrm{V}}(\mathcal{T}) \to \mathbf{S}_k^{\mathrm{V}}(\mathcal{T})$ such that for all $q \in M_{k-1}^{\mathrm{V}}(\mathcal{T})$

$$\begin{aligned} q &= \operatorname{div} \Pi^{\mathrm{V}} q, \\ \left\|\Pi^{\mathrm{V}} q\right\|_{\mathbf{H}^1(\Omega)} &\leq C_{\mathrm{V}} \|q\|_{L^2(\Omega)}, \end{aligned} \tag{2.42}$$

where the constant C_{V} is independent of the mesh width and the polynomial degree. Note that in the original paper [37, Lem. 2.5] by M. Vogelius, the right-hand side in the estimate (2.42) contains an additional factor $k^{\beta_{\mathrm{V}}}$ for some positive β_{V} (independent of the mesh width). In [3, Thm. 3.4], the operator in [37, Lem. 2.5] is modified and the estimate in the form (2.42) is proved for the modified operator.

Finally, we reconsider the linear operator $\Pi^{\mathrm{GS}} : M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T}) \to \mathbf{S}_{4,0}(\mathcal{T})$ introduced by Guzmán and Scott in [27, Proof of Lem. 6 and Lem. 7] with the property that, for any $q \in M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})$, it holds

$$\left(I - \operatorname{div} \Pi^{\mathrm{GS}}\right) q \in M_{k-1}^{\mathrm{V}}(\mathcal{T}), \tag{2.43a}$$

$$\left\|\nabla \Pi^{\mathrm{GS}} q\right\|_{\mathbb{L}^2(\Omega)} \leq C_{\mathrm{GS}} k^{\kappa} \left(\theta_{\min} + \eta\right)^{-1} \|q\|_{L^2(\Omega)} \quad \text{for } \kappa = 2. \tag{2.43b}$$

We emphasize that in [27, Lemma 7] the constant $\Theta_{\min}^{-1}$ (cf. (2.27)) instead of $\left(\Theta_{\min} + \eta\right)^{-1}$ appears in (2.43b) so that the estimate of $\left\|\nabla \Pi^{\mathrm{GS}} q\right\|_{\mathbb{L}^2(\Omega)}$ for $q \in M_{0,k-1}^{\mathrm{SV}}(\mathcal{T})$ deteriorates in cases where the $\mathbf{z}$ is a *nearly critical* point, i.e., very close to the geometric situations described in Remark 2.7. The proof of [27, Lemma 7] is split into an estimate related to points $\mathbf{z}$ with $A_{\mathcal{T},\mathbf{z}}(q) = 0$ (cf. (2.26)) and an estimate for the remaining points $\mathbf{z}$ with $A_{\mathcal{T},\mathbf{z}}(q) \neq 0$. Only in this second part, the constant $\Theta_{\min}^{-1}$ is involved. The result has been improved in [24, Lem. 4.5] and it was shown that there is an operator $\Pi_{\eta,k-1} : M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T}) \to \mathbf{S}_{k,0}(\mathcal{T})$ such that the properties in (2.43) hold for $\kappa = 0$: for any $q \in M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})$ it holds

$$\begin{aligned} &\left(I - \operatorname{div} \Pi_{\eta,k-1}\right) q \in M_{k-1}^{\mathrm{V}}(\mathcal{T}), \\ &\left\|\nabla \Pi_{\eta,k-1} q\right\|_{\mathbb{L}^2(\Omega)} \leq C_{\pi} \left(\Theta_{\min} + \eta\right)^{-1} \|q\|_{L^2(\Omega)}. \end{aligned}$$

From Lemma 2.20, we conclude that $q - \operatorname{div}_{\mathcal{T}} \left(\Pi_k^{\mathrm{CR}} q\right) \in M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})$ and the second part of the proof in [27, Lemma 7] is applied only to points with

$$\min_{\mathbf{z} \in \mathcal{V}(\mathcal{T}) \backslash \mathcal{C}_{\mathcal{T}}(\eta)} \Theta(\mathbf{z}) \geq \max\{\eta, \Theta_{\min}\}.$$

Hence, (2.43b) follows for η depending only on the shape-regularity of the mesh.

Lemma 2.22 *Let assumption (2.7) be satisfied and let $\mathcal{C}_{\mathcal{T}}^{\text{acute}} = \emptyset$. There exists a constant $\eta_2 > 0$ which only depends on the shape-regularity of the mesh and α_Ω as in (1.1) such that for any fixed $0 \leq \eta < \eta_2$ and any $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$, there exists some $\mathbf{w}_q \in \mathbf{CR}_{k,0}(\mathcal{T})$ such that*

$$q = \operatorname{div}_{\mathcal{T}} \mathbf{w}_q$$

and

$$\|\mathbf{w}_q\|_{\mathbf{H}^1(\mathcal{T})} \leq C\sqrt{\log(k+1)}\,(\Theta_{\min} + \eta)^{-1} \|q\|_{L^2(\Omega)}.$$

The constant C only depends on the shape-regularity of the mesh and α_Ω but is independent of the mesh width and the polynomial degree k.

Proof. For the construction of $\mathbf{w}_q$ we follow and modify the lines of proof in [27, Thm. 1] by a) involving the operator Π_k^{CR} and b) employing the concept of η-critical points.

For given $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$, we employ the operators Π_k^{CR}, Π^{BR}, $\Pi_{\eta,k-1}$, Π^{V} in the definition of the function $\mathbf{w}_q$

$$\begin{aligned}
&\mathbf{w}_q = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4, \\
&\mathbf{T}_1 := \Pi^{\mathrm{BR}} q, \\
&\mathbf{T}_2 := \Pi_k^{\mathrm{CR}} \left(I - \operatorname{div} \Pi^{\mathrm{BR}}\right) q, \\
&\mathbf{T}_3 := \Pi_{\eta,k-1} \left(I - \operatorname{div}_{\mathcal{T}} \Pi_k^{\mathrm{CR}}\right) \left(I - \operatorname{div} \Pi^{\mathrm{BR}}\right) q, \\
&\mathbf{T}_4 := \Pi^{\mathrm{V}} \left(I - \operatorname{div} \Pi_{\eta,k-1}\right) \left(I - \operatorname{div}_{\mathcal{T}} \Pi_k^{\mathrm{CR}}\right) \left(I - \operatorname{div} \Pi^{\mathrm{BR}}\right) q.
\end{aligned} \tag{2.44}$$

By construction we have

$$\operatorname{div}_{\mathcal{T}} \mathbf{w}_q = q.$$

The first two summands in (2.44) satisfy

$$\|\mathbf{T}_1\|_{\mathbf{H}^1(\Omega)} \overset{(2.41)}{\leq} C_{\mathrm{BR}} \|q\|_{L^2(\Omega)}, \tag{2.45}$$

$$\begin{aligned}
\|\mathbf{T}_2\|_{\mathbf{H}^1(\mathcal{T})} &\overset{(2.31)}{\leq} C_{\mathrm{CR}}\sqrt{\log(k+1)} \left(\|q\|_{L^2(\Omega)} + \left\|\Pi^{\mathrm{BR}} q\right\|_{\mathbf{H}^1(\Omega)}\right) \\
&\overset{(2.45)}{\leq} C_{\mathrm{CR}}\sqrt{\log(k+1)}\,(1 + C_{\mathrm{BR}}) \|q\|_{L^2(\Omega)}.
\end{aligned} \tag{2.46}$$

For the third term in (2.44) we get

$$\begin{aligned}
\|\nabla \mathbf{T}_3\|_{\mathbb{L}^2(\Omega)} &\leq C_\pi (\Theta_{\min} + \eta)^{-1} \left\|\left(I - \operatorname{div}_{\mathcal{T}} \Pi_k^{\mathrm{CR}}\right) \left(I - \operatorname{div} \Pi^{\mathrm{BR}}\right) q\right\|_{L^2(\Omega)} \\
&\leq C_\pi (\Theta_{\min} + \eta)^{-1} \left(\left\|\left(I - \operatorname{div} \Pi^{\mathrm{BR}}\right) q\right\|_{L^2(\Omega)} + \left\|\operatorname{div}_{\mathcal{T}} \Pi_k^{\mathrm{CR}} \left(I - \operatorname{div} \Pi^{\mathrm{BR}}\right) q\right\|_{L^2(\Omega)}\right).
\end{aligned} \tag{2.47}$$

The combination with (2.41), (2.31) leads to

$$\left\|\nabla \mathbf{T}_3\right\|_{\mathbb{L}^2(\Omega)} \leq C C_\pi \left(1+C_{\mathrm{BR}}\right)\left(1+C_{\mathrm{CR}}\right) \frac{\sqrt{\log (k+1)}}{\Theta_{\min}+\eta}\|q\|_{L^2(\Omega)} . \tag{2.48}$$

For the fourth term we get in a similar way

$$\begin{aligned}
\left\|\nabla \mathbf{T}_4\right\|_{\mathbb{L}^2(\Omega)} & \overset{(2.42)}{\leq} C_{\mathrm{V}}\left(\left\|\left(I-\operatorname{div}_{\mathcal{T}} \Pi_k^{\mathrm{CR}}\right)\left(I-\operatorname{div} \Pi^{\mathrm{BR}}\right) q\right\|_{L^2(\Omega)}+\left\|\operatorname{div} \mathbf{T}_3\right\|_{L^2(\Omega)}\right) \\
& \leq C_{\mathrm{V}}\left(\|q\|_{L^2(\Omega)}+\left\|\operatorname{div} \mathbf{T}_1\right\|_{L^2(\Omega)}+\left\|\operatorname{div}_{\mathcal{T}} \mathbf{T}_2\right\|_{L^2(\Omega)}+\left\|\operatorname{div} \mathbf{T}_3\right\|_{L^2(\Omega)}\right) \\
& \overset{(2.45),\,(2.46),\,(2.48)}{\leq} C_{\mathrm{V}}\left(1+C_{\mathrm{BR}}\right)\left(2+C_{\mathrm{CR}}\right) \sqrt{\log (k+1)}\left(1+C C_\pi\left(\Theta_{\min}+\eta\right)^{-1}\right)\|q\|_{L^2(\Omega)} .
\end{aligned} \tag{2.49}$$

The combination of (2.45), (2.46), (2.48), (2.49) with (2.44) leads to the assertion. ■

Lemma 2.22 implies that for conforming triangulations $\mathcal{T}$ which satisfy (2.7) and $\mathcal{C}_{\mathcal{T}}^{\text{acute}}=\emptyset$, there exists a bounded linear operator

$$\Pi_{\mathcal{T},k}^{\mathrm{inv}}: \mathbb{P}_{k-1,0}(\mathcal{T}) \rightarrow \mathbf{CR}_{k,0}(\mathcal{T})$$

such that $\operatorname{div}_{\mathcal{T}} \circ \Pi_{\mathcal{T},k}^{\mathrm{inv}}$ is the identity on $\mathbb{P}_{k-1,0}(\mathcal{T})$ and

$$\left\|\Pi_{\mathcal{T},k}^{\mathrm{inv}} q\right\|_{\mathbf{H}^1(\mathcal{T})} \leq C_{\mathrm{inv}} \sqrt{\log (k+1)}\|q\|_{L^2(\Omega)}$$

for a constant C_{inv} which only depends on the shape-regularity of the mesh and α_Ω (cf. (1.1)).

2.2.3 The case $\mathcal{C}_{\mathcal{T}}^{\text{acute}}(\eta) \neq \emptyset$

In this section, we remove the condition $\mathcal{C}_{\mathcal{T}}^{\text{acute}}=\emptyset$ and construct a bounded right-inverse of the piecewise divergence operator for odd $k \geq 5$ and conforming triangulations which contain at least one inner point. The construction is based on the step-by-step procedure (cf. (2.13)) from the triangulation $\mathcal{T}_1$ to $\mathcal{T}$. Inductively, we assume that there is a triangulation $\mathcal{T}_j$ along a bounded right-inverse $\Pi_{j,k}^{\mathrm{inv}}: \mathbb{P}_{k-1,0}(\mathcal{T}_j) \rightarrow \mathbf{CR}_{k,0}(\mathcal{T}_j)$ of the piecewise divergence operator. A single extension step is analysed by the following lemma.

Lemma 2.23 *Let $\mathcal{T}$ denote a conforming triangulation for the domain $\Omega:=\operatorname{dom} \mathcal{T}$ and let $\mathcal{T}^{\prime} \subset \mathcal{T}$ be a subset such that every triangle $K \in \mathcal{T} \backslash \mathcal{T}^{\prime}$ has one edge, say E, which belongs to $\mathcal{E}(\mathcal{T}^{\prime})$. We assume that $\mathcal{T}^{\prime}$ has at least one inner vertex and set $\Omega^{\prime}:=\operatorname{dom} \mathcal{T}^{\prime}$. Assume that there exists a bounded linear operator $\Pi_{\mathcal{T}^{\prime},k}^{\mathrm{inv}}: \mathbb{P}_{k-1,0}(\mathcal{T}^{\prime}) \rightarrow \mathbf{CR}_{k,0}(\mathcal{T}^{\prime})$ with $\operatorname{div}_{\mathcal{T}^{\prime}} \circ \Pi_{\mathcal{T}^{\prime},k}^{\mathrm{inv}}=\mathrm{Id}$ on $\mathbb{P}_{k-1,0}(\mathcal{T}^{\prime})$ and*

$$\left\|\Pi_{\mathcal{T}^{\prime},k}^{\mathrm{inv}} q\right\|_{\mathbf{H}^1(\Omega^{\prime})} \leq C_{\mathcal{T}^{\prime}}\|q\|_{L^2(\Omega^{\prime})} .$$

Then, there exists a linear operator $\Pi_{\mathcal{T},k}^{\mathrm{inv}}: \mathbb{P}_{k-1,0}(\mathcal{T}) \rightarrow \mathbf{CR}_{k,0}(\mathcal{T})$ with $\operatorname{div}_{\mathcal{T}} \circ \Pi_{\mathcal{T},k}^{\mathrm{inv}}=\mathrm{Id}$ on $\mathbb{P}_{k-1,0}(\mathcal{T})$ and

$$\left\|\Pi_{\mathcal{T},k}^{\mathrm{inv}} q\right\|_{\mathbf{H}^1(\Omega)} \leq C_{\mathcal{T}}\|q\|_{L^2(\Omega)} \quad \text{with} \quad C_{\mathcal{T}}:=C_3 \sqrt{\log (k+1)} C_{\mathcal{T}^{\prime}}$$

for a constant C_3 which depends only on the shape-regularity of the mesh and on α_Ω.

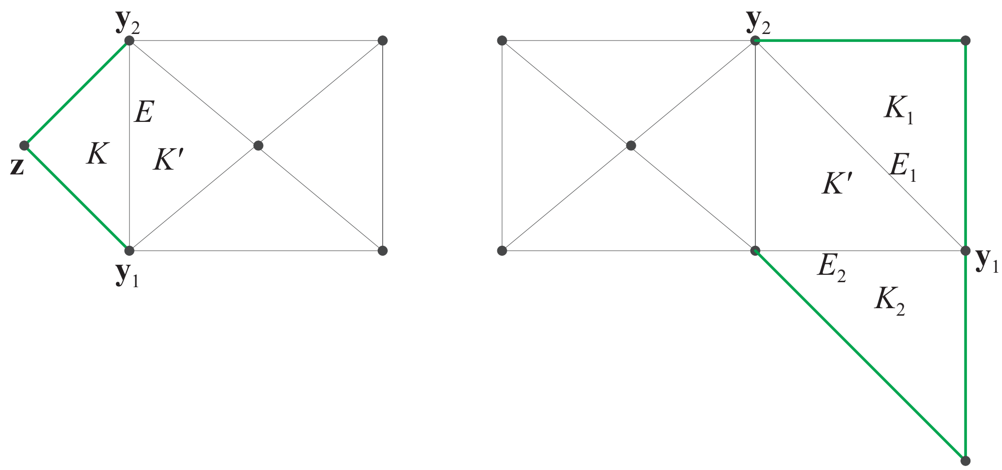

Figure 6: The black triangles form the triangulation $\mathcal{T}'$. Left: One triangle K is attached to $\mathcal{T}'$ having a common side E with $K' \in \mathcal{T}'$ and $\mathcal{T}_{\text{out}}(K') = \{K\}$. Right: Two triangles K_1, $K_2 \notin \mathcal{T}'$ are attached to a triangle $K' \in \mathcal{T}'$ and $\mathcal{T}_{\text{out}}(K') = \{K_1, K_2\}$.

Proof. Let $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$. We set $\mathbf{v}_0 := \Pi^{\text{BR}} q$ where the operator Π^{BR} is as in (2.40) and satisfies

$$\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \leq C_{\text{BR}} \|q\|_{L^2(\Omega)}.$$

Hence, $q_1 := q - \operatorname{div} \mathbf{v}_0$ belongs to $\mathbb{P}_{k-1,0}(\mathcal{T})$ and has trianglewise integral mean zero.

The following construction is illustrated in Figure 6. Let $K' \in \mathcal{T}'$ be such that there exists a non-empty subset $\mathcal{T}_{\text{out}}(K') \subset \mathcal{T} \backslash \mathcal{T}'$ having the property that any $K \in \mathcal{T}_{\text{out}}(K')$ shares an edge with K'. We have $|\mathcal{T}_{\text{out}}(K')| \leq 2$; indeed, if $|\mathcal{T}_{\text{out}}(K')| = 3$, then all three edges of K' are boundary edges which implies $\mathcal{T}' = \{K'\}$ and violates the condition that $\mathcal{T}'$ must contain an inner vertex.

For $K \in \mathcal{T}_{\text{out}}(K')$, let $\mathbf{z}$ denote the vertex in K opposite to E and set $\omega_E = K \cup K'$. The endpoints of E are denoted by $\mathbf{y}_1$, $\mathbf{y}_2$. We employ the ansatz (cf. (2.24))

$$\mathbf{v}_1 := \alpha \tilde{B}_{k,E}^{\text{CR}} \mathbf{n}_E \tag{2.50}$$

with the convention that $\mathbf{n}_E$ is the unit vector orthogonal to E and directed into K'. By construction it holds $\mathbf{v}_1 \in \mathbf{CR}_{k,0}(\mathcal{T})$ and $\operatorname{supp} \mathbf{v}_1 \subset \omega_E$. We determine α in (2.50) such that $\operatorname{div}(\mathbf{v}_1|_K)(\mathbf{z}) = q_1(\mathbf{z})$ and employ (2.34) to get

$$\left|\operatorname{div}\left(\tilde{B}_{k,E}^{\text{CR}} \mathbf{n}_E\right)(\mathbf{z})\right| = \binom{k+1}{2} \frac{|E|}{|K|}.$$

Hence $|\alpha| = |q_1(\mathbf{z})| \frac{|K|}{|E|} / \binom{k+1}{2}$ and we conclude as in the proof of Lemma 2.20 that

$$\|\nabla_{\mathcal{T}} \mathbf{v}_1\|_{\mathbb{L}^2(\omega_E)} \leq C_{\text{CR}} \sqrt{\log(k+1)} \|q_1\|_{L^2(K)}.$$

We set

$$q_2 := q_1 - \operatorname{div}_{\mathcal{T}} \mathbf{v}_1 \quad \text{so that } q_1 = \operatorname{div}_{\mathcal{T}} \mathbf{v}_1 + q_2$$

and note that

$$\|q_2\|_{L^2(K)} \leq \left(1 + C_{\mathrm{CR}}\sqrt{\log(k+1)}\right)\|q_1\|_{L^2(K)}. \tag{2.51}$$

The construction implies $q_2 \in \mathbb{P}_{k-1,0}(\mathcal{T})$, q_2 has trianglewise integral mean zero, and $q_2(\mathbf{z}) = 0$. Next, we employ the vector field defined in [24, Lem. 4.9] which is a modification of the cubic vector field defined in [27, (3.5)] but allows for better k-explicit estimates. We recall the relevant lemma from [24] for the existence of such vector fields and collect important properties.

Lemma 2.24 ([24, Lem. 4.9]) *Let $\mathcal{T}$ be a conforming triangulation of Ω and let $k \geq 3$. Let $E \in \mathcal{E}(\mathcal{T})$ with endpoints $\mathbf{y}_1$, $\mathbf{y}_2$. Then there exist vector fields $\mathbf{v}_{E,j}$, $j \in \{1,2\}$, with the following properties*

$$\begin{array}{ll}
\mathbf{v}_{E,j} \in \mathbf{S}_k(\mathcal{T}), & \operatorname{supp}\mathbf{v}_{E,j} \subset \omega_E, \\
\int_K \operatorname{div}\mathbf{v}_{E,j} = 0 & \forall K \in \mathcal{T}, \quad j \in \{1,2\} \\
\left(\operatorname{div}\mathbf{v}_{E,j}|_K\right)(\mathbf{v}) = \left\{\begin{array}{ll} 1 & \text{if } K \in \mathcal{T}(E) \wedge \mathbf{v} = \mathbf{v}_j \\ 0 & \text{otherwise,}\end{array}\right. & \forall K \in \mathcal{T}, \quad \forall \mathbf{v} \in \mathcal{V}(K), \quad \forall j \in \{1,2\} \\
\|\nabla \mathbf{v}_{E,j}\|_{\mathbb{L}^2(\omega_E)} \leq C h_E k^{-2}. &
\end{array} \tag{2.52}$$

We employ this vector field to the edge $E = K \cap K'$, set

$$\mathbf{v}_2 := \sum_{j=1}^{2} q_2|_K(\mathbf{y}_j)\,\mathbf{v}_{E,j}$$

and define

$$q_3 := q_2 - \operatorname{div}\mathbf{v}_2 \quad \text{so that } q_2 = \operatorname{div}\mathbf{v}_2 + q_3. \tag{2.53}$$

The function $q_3 \in \mathbb{P}_{k-1,0}(\mathcal{T})$ has trianglewise integral mean zero and $q_3|_K$ vanishes in all vertices of K. The norm $\|\nabla \mathbf{v}_2\|_{\mathbb{L}^2(K' \cup K)}$ can be estimated in the same way as the function $\mathbf{T}_3$ in (2.48); however the factor $(\Theta_{\min} + \eta)^{-1}$ does not appear as in (2.48) since the last estimate in (2.52) does not depend on these quantities. In this way, we get

$$\begin{aligned}
\|\nabla_{\mathcal{T}} \mathbf{v}_2\|_{\mathbb{L}^2(\omega_E)} &\leq C_1 \|q_2\|_{L^2(K)} && (2.54)\\
&\overset{(2.51)}{\leq} C_1\left(1 + C_{\mathrm{CR}}\sqrt{\log(k+1)}\right)\|q_1\|_{L^2(K)}. && (2.55)
\end{aligned}$$

Hence, from [3, Thm. 3.4] we deduce that there exists $\mathbf{v}_3 \in \mathbf{S}_{k,0}(\mathcal{T})$ with $\operatorname{supp}\mathbf{v}_3 = K$ such that $\operatorname{div}\mathbf{v}_3 = q_3$ on K and

$$\begin{aligned}
h_K^{-1}\|\mathbf{v}_3\|_{\mathbf{L}^2(K)} + \|\nabla \mathbf{v}_3\|_{\mathbb{L}^2(K)} \leq & C_{\mathrm{V}}\|q_3\|_{L^2(K)} \overset{(2.53),\,(2.54)}{\leq} C_{\mathrm{V}}(1 + C_1)\|q_2\|_{L^2(K)} \\
&\overset{(2.51)}{\leq} C_{\mathrm{V}}(1 + C_1)\left(1 + C_{\mathrm{CR}}\sqrt{\log(k+1)}\right)\|q_1\|_{L^2(K)}.
\end{aligned}$$

In this way we have constructed the function $\mathbf{v}_K \in \mathbf{CR}_{k,0}(\mathcal{T})$ by

$$\mathbf{v}_K = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$$

such that $\operatorname{div}\mathbf{v}_K = q_1$ on K, $\operatorname{supp}\mathbf{v}_K \subset \omega_E$ and

$$\begin{aligned}
\|\nabla_{\mathcal{T}}\mathbf{v}_K\|_{\mathbb{L}^2(\Omega)} = \|\nabla_{\mathcal{T}}\mathbf{v}_K\|_{\mathbb{L}^2(\omega_E)} &\leq \sum_{\ell=1}^{3}\|\nabla_{\mathcal{T}}\mathbf{v}_\ell\|_{\mathbb{L}^2(\Omega)} \\
&\leq C_2\sqrt{\log(k+1)}\,\|q_1\|_{L^2(K)} \leq C_2\sqrt{\log(k+1)}\left(\|q\|_{L^2(K)} + \|\nabla\mathbf{v}_0\|_{\mathbb{L}^2(K)}\right),
\end{aligned}$$

where C_2 only depends on the shape-regularity of the mesh and α_Ω through the constants C_{V}, C_{CR}, C_1. Let $\mathbf{v}_q := \mathbf{v}_0 + \sum_{K\in\mathcal{T}\backslash\mathcal{T}'}\mathbf{v}_K$ and note that by construction

$$\operatorname{div}_{\mathcal{T}}\mathbf{v}_q = q \quad \text{on } \Omega\backslash\overline{\Omega'}$$

and

$$\begin{aligned}
\|\nabla_{\mathcal{T}}\mathbf{v}_q\|_{\mathbb{L}^2(\Omega)} &\leq \|\nabla\mathbf{v}_0\|_{\mathbb{L}^2(\Omega)} + \sum_{K\in\mathcal{T}\backslash\mathcal{T}'}\|\nabla_{\mathcal{T}}\mathbf{v}_K\|_{\mathbb{L}^2(\Omega)} \\
&\leq \|\nabla\mathbf{v}_0\|_{\mathbb{L}^2(\Omega)} + 2C_2\sum_{K\in\mathcal{T}\backslash\mathcal{T}'}\sqrt{\log(k+1)}\left(\|q\|_{L^2(K)} + \|\nabla\mathbf{v}_0\|_{\mathbb{L}^2(K)}\right) \\
&\leq C\sqrt{\log(k+1)}\,\|q\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, the linear map $\Pi^{\mathrm{inv}}_{\mathcal{T},k} : \mathbb{P}_{k-1,0}(\mathcal{T}) \to \mathbf{CR}_{k,0}(\mathcal{T})$ is defined by

$$\Pi^{\mathrm{inv}}_{\mathcal{T},k}q = \mathbf{v}_q + \Pi^{\mathrm{inv}}_{\mathcal{T}',k}\left(\left.(q - \operatorname{div}\mathbf{v}_q)\right|_{\Omega'}\right)$$

and satisfies $\operatorname{div}\circ\Pi^{\mathrm{inv}}_{\mathcal{T},k} = \mathrm{Id}$ on $\mathbb{P}_{k-1,0}(\mathcal{T})$ and

$$\left\|\nabla_{\mathcal{T}}\Pi^{\mathrm{inv}}_{\mathcal{T},k}q\right\|_{\mathbb{L}^2(\Omega)} \leq C_3\sqrt{\log(k+1)}C_{\mathcal{T}'}\,\|q\|_{L^2(\Omega)}$$

for some C_3 which only depends on the shape-regularity of the mesh and α_Ω. ∎

By iterating this argument we can prove Theorem 1.3 for the case (2.7).

Theorem 2.25 *Let $\mathcal{T}$ be a conforming triangulation which contains at least one interior vertex. Let $k \geq 5$ be odd and let $L \in \mathbb{N}_0$ the number of steps in the construction (2.13). Then, the inf-sup constant for the corresponding Crouzeix-Raviart discretization satisfies*

$$c_{\mathcal{T},k} \geq c_{\mathcal{T}}\left(\log(k+1)\right)^{-(L+1)/2}$$

for some constant $c_{\mathcal{T}}$ which depends only on the shape-regularity of the mesh and α_Ω. If every triangle in $\mathcal{T}$ has an interior vertex, then $L = 0$.

2.3 The case of even k

This case is slightly simpler that the case of odd k since the non-conforming Crouzeix-Raviart functions for even k have smaller support (i.e., one triangle) compared to two triangles (which share an edge) for odd k.

In this section, we assume

$$\begin{array}{ll}\text{a)} & k \geq 4 \text{ is even and} \\ \text{b)} & \mathcal{T} \text{ is a conforming triangulation and contains more than a single triangle.}\end{array} \tag{2.56}$$

Remark 2.26 *It is easy to verify that* $|\mathcal{T}| > 1$ *implies that there exists a mapping* $\mathfrak{K} : \mathcal{C}_{\mathcal{T}}(\eta) \rightarrow \mathcal{T}$ *with* $\mathfrak{K}(\mathbf{z}) \in \mathcal{T}_{\mathbf{z}}$ *and not all vertices of* $\mathfrak{K}(\mathbf{z})$ *are* η*-critical.*

For an η-critical point $\mathbf{z} \in \mathcal{C}_{\mathcal{T}}(\eta)$, let $n_{\mathbf{z}} := |\mathcal{T}_{\mathbf{z}}|$ and fix a counterclockwise numbering of the triangles in

$$\mathcal{T}_{\mathbf{z}} = \left\{K_j^{\mathbf{z}} : 1 \leq j \leq n_{\mathbf{z}}\right\} \tag{2.57}$$

such that $K_j^{\mathbf{z}}$ and $K_{j+1}^{\mathbf{z}}$ share an edge for all $1 \leq j \leq n_{\mathbf{z}} - 1$. With this notation at hand, the functional $A_{\mathcal{T},\mathbf{z}}$ is given by

$$A_{\mathcal{T},\mathbf{z}} q := \sum_{j=1}^{n_{\mathbf{z}}} (-1)^j \left. q \right|_{K_j^{\mathbf{z}}} (\mathbf{z}) \quad \forall q \in \mathbb{P}_{k-1,0}(\mathcal{T}).$$

Lemma 2.27 *Let assumption (2.56) be satisfied. There exists a constant* $\eta_2 > 0$ *which only depends on the shape-regularity of the mesh and* α_Ω *such that for any fixed* $0 \leq \eta < \eta_2$ *and any* $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$, *there exists some* $\mathbf{v}_q \in \mathbf{CR}_{k,0}(\mathcal{T})$ *such that*

$$\int_K \operatorname{div} \mathbf{v}_q = 0 \quad \forall K \in \mathcal{T}, \tag{2.58}$$

$$q - \operatorname{div}_{\mathcal{T}} \mathbf{v}_q \in M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T}) \tag{2.59}$$

and

$$\|\mathbf{v}_q\|_{\mathbf{H}^1(\mathcal{T})} \leq C_{\mathrm{CR}} \sqrt{\log(k+1)}\, \|q\|_{L^2(\Omega)}. \tag{2.60}$$

The constant C_{CR} *depends only on the shape-regularity of the mesh and* α_Ω.

Proof. For a triangle $K \in \mathcal{T}$, we set $\mathcal{C}_K(\eta) := \{\mathbf{z} \in \mathcal{V}(K) \cap \mathcal{C}_{\mathcal{T}}(\eta)\}$ and $\mathcal{C}_K^{\mathrm{active}}(\eta) := \{\mathbf{z} \in \mathcal{C}_K(\eta) : K = \mathfrak{K}(\mathbf{z})\}$. Their cardinalities are denoted by $n_K := |\mathcal{C}_K(\eta)|$ and $n_K^{\mathrm{active}} := \left|\mathcal{C}_K^{\mathrm{active}}(\eta)\right|$. Note that $0 \leq n_K^{\mathrm{active}} \leq n_K \leq 2$ (cf. Rem. 2.26). We number the vertices $\mathbf{V}_j$ in K with the convention $\{\mathbf{V}_j : 1 \leq j \leq n_K\} = \mathcal{C}_K(\eta)$ and $\left\{\mathbf{V}_j : 1 \leq j \leq n_K^{\mathrm{active}}\right\} = \mathcal{C}_K^{\mathrm{active}}(\eta)$. The angle in K at $\mathbf{V}_j$ is denoted by α_j. Let E_j be the edge in K opposite to $\mathbf{V}_j$ and let $\mathbf{n}_j$ denote the outward unit normal vector at E_j.

In a similar way as for the construction of $\tilde{B}_{k,E}^{\mathrm{CR}}$ in Lemma 2.16 (and employing Lemma B.1 instead of Lemma B.2 for (2.21)) there exists a function $\tilde{B}_{k,K}^{\mathrm{CR}} \in \mathrm{CR}_{k,0}(\mathcal{T})$ with

1. $\operatorname{supp} \tilde{B}_{k,K}^{\mathrm{CR}} = K$,

2. for all $K \in \mathcal{T}$, for all $\mathbf{z} \in \mathcal{V}(K)$

$$\nabla \left(\left. \tilde{B}_{k,K}^{\mathrm{CR}} \right|_K \right)(\mathbf{z}) = \nabla \left(\left. B_{k,K}^{\mathrm{CR}} \right|_K \right)(\mathbf{z}), \tag{2.61}$$

3. for all $K \in \mathcal{T}$

$$\left. \left. \tilde{B}_{k,K}^{\mathrm{CR}} \right|_K \right|_{\partial K} = \left. \left. B_{k,K}^{\mathrm{CR}} \right|_K \right|_{\partial K}, \tag{2.62}$$

4. for all $K \in \mathcal{T}$ and any $\mathbf{c} \in \mathbb{R}^2$

$$\int_K \operatorname{div}_{\mathcal{T}} \left(\tilde{B}_{k,K}^{\mathrm{CR}} \mathbf{c} \right) = 0. \tag{2.63}$$

5. The piecewise gradient is bounded by

$$\left\| \tilde{B}_{k,K}^{\mathrm{CR}} \right\|_{H^1(\mathcal{T})} \leq C\sqrt{\log (k+1)}. \tag{2.64}$$

For $j = 1, 2$, we define

$$\boldsymbol{\psi}_{k,K}^{\mathrm{CR},j} := \begin{cases} \tilde{B}_{k,K}^{\mathrm{CR}} \mathbf{n}_j & \text{in } K, \\ 0 & \text{in } \Omega \backslash K, \end{cases} \tag{2.65}$$

i.e., we fix $\mathbf{v}_K := \mathbf{n}_1$ and $\mathbf{w}_K = \mathbf{n}_2$ in (2.5). The divergence of $\boldsymbol{\psi}_{k,K}^{\mathrm{CR},j}$ evaluated at a vertex $\mathbf{V}_s$, $s = 1, 2$, is given by

$$\begin{aligned} \operatorname{div}\left(\boldsymbol{\psi}_{k,K}^{\mathrm{CR},j} \Big|_K \right) (\mathbf{V}_s) &\overset{(2.61)}{=} \operatorname{div}\left(B_{k,K}^{\mathrm{CR}} \mathbf{n}_j \big|_K \right) (\mathbf{V}_s) = -\sum_{i=1}^{3} L_k' \left(1 - 2\lambda_{K,i}\left(\mathbf{V}_s\right)\right) \partial_{\mathbf{n}_j} \lambda_{K,i} \\ &= k\,(k+1)\, \partial_{\mathbf{n}_j} \lambda_{K,s} \overset{(2.3)}{=} \binom{k+1}{2} \frac{|E_s|}{|K|} \times \begin{cases} -1 & j = s, \\ \cos \alpha_3 & j \neq s. \end{cases} \end{aligned}$$

Let $q \in \mathbb{P}_{k-1,0}(\mathcal{T})$. We choose $\boldsymbol{\delta}_K := \left(\delta_{K,j}\right)_{j=1}^2$ by the conditions for $s = 1, 2$

$$A_{\mathcal{T},\mathbf{V}_s} \left(\operatorname{div}_{\mathcal{T}} \left(\sum_{j=1}^{2} \delta_{K,j} \boldsymbol{\psi}_{k,K}^{\mathrm{CR},j} \right) \right) \overset{!}{=} \begin{cases} A_{\mathcal{T},\mathbf{V}_s}(q) & \text{if } \mathbf{V}_s \in \mathcal{C}_K^{\text{active}}(\eta), \\ 0 & \text{otherwise.} \end{cases} \tag{2.66}$$

For $s = 1, 2$, let ℓ_s be defined by $K_{\ell_s}^{\mathbf{V}_s} = K$ (cf. (2.57)). Then,

$$\begin{aligned} A_{\mathcal{T},\mathbf{V}_s} \left(\operatorname{div}_{\mathcal{T}} \left(\sum_{j=1}^{2} \delta_{K,j} \boldsymbol{\psi}_{k,K}^{\mathrm{CR},j} \right) \right) &= (-1)^{\ell_s} \sum_{j=1}^{2} \delta_{K,j} \left(\operatorname{div} \boldsymbol{\psi}_{k,K}^{\mathrm{CR},j} \Big|_K \right) (\mathbf{V}_s) \\ &= \mathbf{M}_K \boldsymbol{\delta}_K \end{aligned}$$

for

$$\mathbf{M}_K = (-1)^{\ell_s+1} \binom{k+1}{2} |K|^{-1} \begin{bmatrix} |E_1| & -|E_1| \cos \alpha_3 \\ -|E_2| \cos \alpha_3 & |E_2| \end{bmatrix}.$$

We define $\mathbf{r}_K = \left(r_{K,s}\right)_{s=1}^2$ by

$$r_{K,s} := \begin{cases} A_{\mathcal{T},\mathbf{V}_s}(q) & \text{if } \mathbf{V}_s \in \mathcal{C}_K^{\text{active}}(\eta), \\ 0 & \text{otherwise} \end{cases}$$

so that $\boldsymbol{\delta}_K$ is the solution of

$$\mathbf{M}_K \boldsymbol{\delta}_K = \mathbf{r}_K.$$

Observe that

$$\det \begin{bmatrix} |E_1| & -|E_1| \cos \alpha_3 \\ -|E_2| \cos \alpha_3 & |E_2| \end{bmatrix} = |E_1|\,|E_2| \sin^2 \alpha_3 = 2\,|K| \sin \alpha_3$$

and $\sin \alpha_3 \geq \sin \phi_{\mathcal{T}} > 0$ due to the shape-regularity of the mesh. For the coefficient $\boldsymbol{\delta}_K$ we get explicitly

$$\boldsymbol{\delta}_K = \frac{(-1)^{\ell_s+1}}{(k+1)\, k \sin \alpha_3} \begin{bmatrix} |E_2| & |E_1| \cos \alpha_3 \\ |E_2| \cos \alpha_3 & |E_1| \end{bmatrix} \mathbf{r}_K \tag{2.67}$$

with an estimate

$$\|\boldsymbol{\delta}_K\| \leq C \frac{h_K}{k\,(k+1)} \|\mathbf{r}_K\| \overset{\text{Lem. 2.19}}{\leq} C \|q\|_{L^2(\omega_K)} \quad \text{if } \mathcal{C}_K^{\text{active}}(\eta) \neq \emptyset,$$

where C only depends on the shape-regularity of the mesh. Note that this is the analogue for even k to (2.39). If $\mathcal{C}_K^{\text{active}}(\eta) = \emptyset$, it holds $\boldsymbol{\delta}_K = \mathbf{0}$.

We define the global function

$$\mathbf{v}_q := \sum_{\substack{K\in\mathcal{T} \\ \mathcal{C}_K^{\text{active}}(\eta)\neq\emptyset}} \sum_{j=1}^{2} \delta_{K,j} \boldsymbol{\psi}_{k,K}^{\text{CR},j}. \tag{2.68}$$

From (2.63) we conclude that $\mathbf{v}_q$ satisfies (2.58).

Next, we verify (2.59). Let $\mathbf{y} \in \mathcal{C}_{\mathcal{T}}(\eta)$ and recall the notation and convention as in (2.57). Let $K_\ell^{\mathbf{y}} = \mathfrak{K}(\mathbf{y})$. Then (2.59) follows from

$$\begin{aligned} A_{\mathcal{T},\mathbf{y}}(\operatorname{div}_{\mathcal{T}} \mathbf{v}_q) &= (-1)^\ell \left(\operatorname{div} \mathbf{v}_q|_{K_\ell^{\mathbf{y}}}\right)(\mathbf{y}) = (-1)^\ell \left(\operatorname{div} \sum_{j=1}^{2} \delta_{K_\ell^{\mathbf{y}},j} \boldsymbol{\psi}_{k,K_\ell^{\mathbf{y}}}^{\text{CR},j}\Big|_{K_\ell^{\mathbf{y}}}\right)(\mathbf{y}) \\ &= A_{\mathcal{T},\mathbf{y}}(q). \end{aligned}$$

The estimate

$$\left\|\nabla \boldsymbol{\psi}_{k,K}^{\text{CR},j}\right\|_{\mathbb{L}^2(K)} \leq C\sqrt{\log(k+1)}$$

for a constant C which only depends on the shape-regularity of the mesh and α_Ω follows directly from (2.64) and the final estimate (2.60) is derived by repeating the arguments as in the proof of Lemma 2.20. ■

This lemma allows us to extend Definition 2.21 to the case of even k by defining the coefficients $\boldsymbol{\delta}_K$ by (2.67) and the functions $\boldsymbol{\psi}_{k,K}^{\text{CR}}$ by (2.65) and set (cf. (2.68)) $\Pi_k^{\text{CR}} q := \mathbf{v}_q$. Since $\left(I - \Pi_k^{\text{CR}}\right) q \in M_{\eta,k-1}^{\text{SV}}(\mathcal{T})$ we may apply the further steps in the proof of Lemma 2.22 to obtain the inf-sup stability for even k.

Theorem 2.28 *Let $\mathcal{T}$ be a conforming triangulation satisfies (2.56). Then, the inf-sup constant for the corresponding Crouzeix-Raviart discretization satisfies*

$$c_{\mathcal{T},k} \geq c_{\mathcal{T}} (\log(k+1))^{-1/2}$$

for some constant $c_{\mathcal{T}}$ which depends only on the shape-regularity of the mesh and α_Ω.

3 Conclusion

In this paper, we have derived lower bounds for the inf-sup constant for Crouzeix-Raviart elements for the Stokes equation which are explicit with respect to the polynomial degree k and are independent of the mesh size.

1. The inf-sup constant can be bounded from below by $c_{\mathcal{T},k} \geq c_{\mathcal{T}} (\log(k+1))^{-1/2}$ if

   (a) for odd $k \geq 3$,

i. $\mathcal{T}$ has at least one interior point and

ii. for $k \geq 5$, $\mathcal{T}$ has no acute critical point,

(b) for even $k \geq 4$, $\mathcal{T}$ contains more than one triangle,

2. If for odd k, condition 1.a.ii. is not satisfied but a step-by-step construction (2.13) for some $L \geq 1$ is possible, then, $c_{\mathcal{T},k} \geq c_{\mathcal{T}} (\log (k+1))^{-(L+1)/2}$.

Finally, we compare these findings with some other stable pairs of Stokes elements on triangulations in the literature. The element $(\mathbf{S}_{k,0}(\mathcal{T}), \mathbb{P}_{k-2,0}(\mathcal{T}))$ has a discrete inf-sup constant which can be estimated from below by Ck^{-3} (see [31], [33]). The discrete inf-sup constant for the Scott-Vogelius element $\left(\mathbf{S}_{k,0}(\mathcal{T}), M_{0,k-1}^{\mathrm{SV}}(\mathcal{T})\right)$ for $k \geq 4$ can be estimated from below by $c\Theta_{\min} k^{-m}$ for some integer m sufficiently large (see [32], [37]). The pressure-wired Stokes element $\left(\mathbf{S}_{k,0}(\mathcal{T}), M_{\eta,k-1}^{\mathrm{SV}}(\mathcal{T})\right)$ in **??** (again for $k \geq 4$) is a mesh-robust generalization of the Scott-Vogelius element with a lower bound of the inf-sup constant of the form $c(\Theta_{\min} + \eta)$. In [3], a conforming stable pair $(\mathbf{X}_k(\mathcal{T}), M_{k-1}(\mathcal{T}))$ of Stokes elements on triangulations is introduced and it is proved that the discrete inf-sup constant can be estimated from below by $c/\tilde{\Theta}_{\min}$ for a constant c independent of h and k and $\tilde{\Theta}_{\min} := \min_{\mathbf{z} \in \mathcal{V}_{\partial\Omega}(\mathcal{T}) \setminus \mathcal{C}_{\mathcal{T}}} \Theta(\mathbf{z})$. However, the implementation requires finite elements for the velocity with C^1 continuity at the triangle vertices and pressures which are continuous in the triangle vertices.

Acknowledgement 3.1 *Thanks are due to Benedikt Gräßle, HU Berlin, for fruitful discussions on Lemma 2.13.*

I am grateful to my colleagues from TU Vienna, Profs. Joachim Schöberl and Markus Melenk. Joachim showed by numerical experiments that the lower bound of the inf-sup constant in the first arxiv version of the paper, namely $k^{-1/4}$, might be too pessimistic and "it should be at least $k^{-1/6}$" and Markus raised the suspicion that the interpolation argument might be too pessimistic for Legendre polynomials.

A The inverse of the matrix $\mathbf{T}_\ell + \mathbf{\Delta}_\ell$ in (2.37)

Lemma A.1 *There exists $\eta_2 > 0$ which only depends on the shape-regularity of the mesh and α_Ω such that for any $0 \leq \eta < \eta_2$ the matrix $\mathbf{T}_\ell + \mathbf{\Delta}_\ell$ in (2.37) is invertible and there exists a constant C depending only on the shape-regularity of the mesh and α_Ω such that (cf. Notation 2.1)*

$$\left\| (\mathbf{T}_\ell + \mathbf{\Delta}_\ell)^{-1} \right\| \leq C.$$

Note that the matrix $\mathbf{T}_\ell$ in (2.38) is the same as the matrix $\mathbf{T}_{n,\boldsymbol{\alpha}}$ which has been analysed in [10, (3.36)]. In particular, the formula

$$\det \mathbf{T}_\ell = \frac{\sin\left(\sum_{j=1}^{n_\ell+1} \alpha_{\ell,j,1}\right)}{\prod_{j=1}^{n_\ell+1} \sin \alpha_{\ell,j,1}}$$

was proved.

Next we show that the sum $\sum_{j=1}^{n_\ell+1} \alpha_{\ell,j,1}$ is bounded away from 0 and π. The bound $\sum_{j=1}^{n_\ell+1} \alpha_{\ell,j,1} \geq \varphi_{\mathcal{T}}$ follows from Remark 1.1. Since $\mathbf{z}_{\ell,j}$, $1 \leq j \leq n_\ell$ are $\eta-$critical points the sum of both angles adjacent to $E_{\ell,j}$ at $\mathbf{z}_{\ell,j}$ satisfy

$$\sin\left(\alpha_{\ell,j,2} + \alpha_{\ell,j+1,3}\right) \leq \eta.$$

We write $\alpha_{\ell,j,2} + \alpha_{\ell,j+1,3} =: \pi + \delta_j$. From the proof of Lemma 2.10, in particular from the estimate (2.8) we conclude that $|\delta_j| \leq c_2\eta$.

Since all points $\mathbf{z}_{\ell,j}$ are edge-connected to the same point $\mathbf{z}_\ell$, the number n_ℓ is bounded from above by a constant $n_{\max}$ which only depends on the shape-regularity of the mesh. Hence,

$$\begin{aligned}
\sum_{j=1}^{n_\ell+1} \alpha_{\ell,j,1} &= \sum_{j=1}^{n_\ell+1} \left(\pi - \alpha_{\ell,j,2} - \alpha_{\ell,j,3}\right) = \left(n_\ell + 1\right)\pi - \alpha_{\ell,1,3} - \alpha_{\ell,n_\ell+1,2} - \sum_{j=1}^{n_\ell} \left(\alpha_{\ell,j,2} + \alpha_{\ell,j+1,3}\right) \\
&= \left(n_\ell + 1\right)\pi - \alpha_{\ell,1,3} - \alpha_{\ell,n_\ell+1,2} - \sum_{j=1}^{n_\ell} \left(\pi + \delta_j\right) = \pi - \alpha_{\ell,1,3} - \alpha_{\ell,n_\ell+1,2} - \sum_{j=1}^{n_\ell} \delta_j \\
&\leq \pi - 2\varphi_{\mathcal{T}} + n_{\max} c_2 \eta.
\end{aligned}$$

By adjusting the constant η_0 in Lemma 2.10 to $\eta_1 := \min\left\{\eta_0, \varphi_{\mathcal{T}} / \left(n_{\max} c_2\right)\right\}$ it follows that

$$\sum_{j=1}^{n_\ell+1} \alpha_{\ell,j,1} \leq \pi - \varphi_{\mathcal{T}}.$$

By using the trivial estimate $0 < \sin\alpha_{\ell,j,1} \leq 1$, we may conclude that

$$\det \mathbf{T}_\ell = \frac{\sin\left(\sum_{j=1}^{n_\ell+1} \alpha_{\ell,j,1}\right)}{\prod_{j=1}^{n_\ell+1} \sin\alpha_{\ell,j,1}} \geq \sin\varphi_{\mathcal{T}} > 0. \tag{A.1}$$

Note that the entries in the matrix $\mathbf{T}_\ell$ (cf. 2.38) satisfy

$$\left|\left(\mathbf{T}_\ell\right)_{\mathbf{y},\mathbf{z}}\right| \leq \frac{1}{\sin^2\varphi_{\mathcal{T}}} \tag{A.2}$$

and hence the Frobenius norm $\|\mathbf{T}_\ell\|_{\mathrm{F}}$ can be estimated by

$$\|\mathbf{T}_\ell\|_{\mathrm{F}} \leq \frac{3n_\ell}{\sin^2\varphi_{\mathcal{T}}} \leq \frac{3n_{\max}}{\sin^2\varphi_{\mathcal{T}}}.$$

It is well known that $\|\mathbf{T}_\ell\| \leq \|\mathbf{T}_\ell\|_{\mathrm{F}}$ and hence the bound on $\|\mathbf{T}_\ell\|$ follows.

We combine (A.1), (A.2), and $n_\ell \leq n_{\max}$ to obtain by Cramer's rule that there exists a constant C which only depends on the shape-regularity such that

$$\left|\left(\mathbf{T}_\ell^{-1}\right)_{\mathbf{y},\mathbf{z}}\right| \leq C.$$

By the same arguments as before we conclude that $\left\|\mathbf{T}_\ell^{-1}\right\| \leq \tilde{C}$ for a constant $\tilde{C}$ which only depends on the shape-regularity of the mesh.

Next we estimate $\|\mathbf{\Delta}_\ell\|$. Since $\mathbf{\Delta}_\ell$ is diagonal it suffices to estimate the diagonal entries

$$\left|\frac{\sin\left(\alpha_{\ell,j,2}+\alpha_{\ell,j+1,3}\right)}{\sin\alpha_{\ell,j,2}\sin\alpha_{\ell,j+1,3}}\right| = \left|\frac{\sin\left(\pi+\delta_\ell\right)}{\sin\alpha_{\ell,j,2}\sin\alpha_{\ell,j+1,3}}\right| = \left|\frac{\sin\delta_\ell}{\sin\alpha_{\ell,j,2}\sin\alpha_{\ell,j+1,3}}\right| \leq \frac{c_2\eta}{\sin^2\varphi_{\mathcal{T}}}.$$

We write $\mathbf{T}_\ell+\mathbf{\Delta}_\ell = \mathbf{T}_\ell\left(\mathbf{I}+\mathbf{T}_\ell^{-1}\mathbf{\Delta}_\ell\right)$ and obtain

$$\left\|\mathbf{T}_\ell^{-1}\mathbf{\Delta}_\ell\right\| \leq \left\|\mathbf{T}_\ell^{-1}\right\|\left\|\mathbf{\Delta}_\ell\right\| \leq \tilde{C}\frac{c_2\eta}{\sin^2\varphi_{\mathcal{T}}}.$$

Next, we adjust the upper bound η_1 by setting $\eta_2 := \min\left\{\eta_1, \frac{\sin^2\varphi_{\mathcal{T}}}{2\tilde{C}c_2}\right\}$ to obtain $\left\|\mathbf{T}_\ell^{-1}\mathbf{\Delta}_\ell\right\| \leq 1/2$ with implies the invertibility of $\mathbf{T}_\ell+\mathbf{\Delta}_\ell$ with bound

$$\left\|\left(\mathbf{T}_\ell+\mathbf{\Delta}_\ell\right)^{-1}\right\| \leq 2\tilde{C}.$$

■

B Estimate of the $H^{1/2}$ norm of traces of non-conforming Crouzeix-Raviart functions

In this appendix, we prove the norm estimate (2.64) for $\tilde{B}_{k,K}^{\mathrm{CR}}$ and (2.19) for $\tilde{B}_{k,E}^{\mathrm{CR}}$. We first introduce some norms and semi-norms on the unit interval $I := [-1,1]$ in a formal way:

$$\begin{aligned}
|u|_{H^{1/2}(I)} &:= \left(\int_{-1}^{1}\int_{-1}^{1}\left|\frac{u(s)-u(t)}{s-t}\right|^2 dsdt\right)^{1/2}, & \|u\|_{H^{1/2}(I)} &:= \left(\|u\|_{L^2(I)}^2+|u|_{H^{1/2}(I)}^2\right)^{1/2},\\
|u|_{H^{1/2}_{(0,}(I)} &:= \left(\int_{-1}^{1}\frac{|u(s)|^2}{1+s}ds\right)^{1/2}, & \|u\|_{H^{1/2}_{(0,}(I)} &:= \left(\|u\|_{H^{1/2}(I)}^2+|u|_{H^{1/2}_{(0,}(I)}^2\right)^{1/2},\\
|u|_{H^{1/2}_{,0)}(I)} &:= \left(\int_{-1}^{1}\frac{|u(s)|^2}{1-s}ds\right)^{1/2}, & \|u\|_{H^{1/2}_{,0)}(I)} &:= \left(\|u\|_{H^{1/2}(I)}^2+|u|_{H^{1/2}_{,0)}(I)}^2\right)^{1/2},\\
|u|_{H^{1/2}_{00}(I)} &:= \left(|u|_{H^{1/2}_{(0,}(I)}^2+|u|_{H^{1/2}_{,0)}(I)}^2\right)^{1/2}, & \|u\|_{H^{1/2}_{00}(I)} &:= \left(\|u\|_{H^{1/2}(I)}^2+|u|_{H^{1/2}_{00}(I)}^2\right)^{1/2}.
\end{aligned}\tag{B.1}$$

Lemma B.1 *Let $k \geq 4$ be even and $K \in \mathcal{T}$. Let $\left.\tilde{B}_{k,K}^{\mathrm{CR}}\right|_{\partial K}$ be defined by (2.62). Then there exists an absolute constant C depending only on the shape regularity of $\mathcal{T}$ such that*

$$\left\|\tilde{B}_{k,K}^{\mathrm{CR}}\right\|_{H^{1/2}(\partial K)} \leq C\sqrt{\log(k+1)}.$$

Proof. We first prove the estimate for the reference element $\widehat{K}$. By construction (see (2.62)) the function $\tilde{B}_{k,\widehat{K}}^{\mathrm{CR}}$ coincides with $B_{k,\widehat{K}}^{\mathrm{CR}}$ on $\partial\widehat{K}$. Let the vertices of $\widehat{K}$ be numbered counterclockwise and denoted by $\widehat{\mathbf{z}}_i$, $1 \leq i \leq 3$. The edge opposite to $\widehat{\mathbf{z}}_i$ is $\widehat{E}_i = [\widehat{\mathbf{z}}_{i+1}, \widehat{\mathbf{z}}_{i-1}]$ (with cyclic numbering convention $\mathbf{z}_{3+1} := \mathbf{z}_1$ and $\mathbf{z}_{1-1} := \mathbf{z}_3$). We choose the pullbacks to $[-1,1]$ by

$$\phi_i(s) := \widehat{\mathbf{z}}_{i+1} + s\left(\widehat{\mathbf{z}}_{i-1}-\widehat{\mathbf{z}}_{i+1}\right) \quad i = 1,2,3 \tag{B.2}$$

and observe $\tilde{B}^{\mathrm{CR}}_{k,\widehat{K}}\Big|_{\widehat{E}_i} \circ \phi_i = L_k$. From [25] (see also [4, p 1870]) we deduce that the $H^{1/2}\left(\partial\widehat{K}\right)$ norm is equivalent to

$$|||v|||_{H^{1/2}(\partial\widehat{K})} := \left(\sum_{i=1}^{3}\left(\|v_i\|^2_{H^{1/2}(I)} + |d_i|^2_{H^{1/2}_{(0,}(I)}\right)\right)^{1/2}, \tag{B.3}$$

where for $v \in H^{1/2}\left(\partial\widehat{K}\right)$:

$$v_i := v|_{E_i} \circ \phi_i, \quad d_i(s) := v_{i-1}(s) - v_{i+1}(-s). \tag{B.4}$$

In our application (and even k) we have $v_{i-1}(s) = L_k(s) = L_k(-s) = v_{i+1}(-s)$ so that $d_i = 0$. We use $\|L_k\|_{L^2(I)} = \sqrt{2/(2k+1)}$ to get

$$|||L_k|||_{H^{1/2}(\partial\widehat{K})} \leq \left(\frac{6}{2k+1} + 3\,|L_k|^2_{H^{1/2}(I)}\right)^{1/2}. \tag{B.5}$$

This integral can be evaluated analytically for $v = L_k$ (see Lem. C.2) and we obtain

$$|L_k|_{H^{1/2}(I)} = 2\left(\sum_{\ell=1}^{k}\frac{1}{\ell}\right)^{1/2} \leq \sqrt{C\log(k+1)} \qquad \forall k = 1, 2, \ldots \tag{B.6}$$

for a generic constant $C > 0$. This leads to the final estimate on the reference element:

$$\begin{aligned}\left\|\tilde{B}^{\mathrm{CR}}_{k,\widehat{K}}\right\|_{H^{1/2}(\partial\widehat{K})} &\leq C\left|\left|\left|\tilde{B}^{\mathrm{CR}}_{k,\widehat{K}}\right|\right|\right|_{H^{1/2}(\partial\widehat{K})} \overset{\text{(B.5), (B.6)}}{\leq} C\left(\frac{1}{2k+1} + \log(k+1)\right)^{1/2} \\ &\leq C\sqrt{\log(k+1)}.\end{aligned}$$

For a triangle $K \in \mathcal{T}$, let $\phi_K : \widehat{K} \to K$ be an affine pullback and set $\tilde{B}^{\mathrm{CR}}_{k,\widehat{K}} = \tilde{B}^{\mathrm{CR}}_{k,K} \circ \phi_K$. Then, the transformation rule for integrals yields

$$\begin{aligned}\left\|\tilde{B}^{\mathrm{CR}}_{k,K}\right\|_{H^{1/2}(\partial K)} &= \left(\left\|\tilde{B}^{\mathrm{CR}}_{k,K}\right\|^2_{L^2(\partial K)} + \left|\tilde{B}^{\mathrm{CR}}_{k,K}\right|^2_{H^{1/2}(\partial K)}\right)^{1/2} \\ &= C\left(h_K\left\|\tilde{B}^{\mathrm{CR}}_{k,\widehat{K}}\right\|^2_{L^2(\partial\widehat{K})} + \left|\tilde{B}^{\mathrm{CR}}_{k,\widehat{K}}\right|^2_{H^{1/2}(\partial\widehat{K})}\right)^{1/2},\end{aligned} \tag{B.7}$$

where C only depends on the shape regularity of the mesh. The leads to the claim. ■

The case of odd $k \geq 5$ is considered in the following lemma.

Lemma B.2 *Let $k \geq 5$ be odd. For $E \in \mathcal{E}_\Omega(\mathcal{T})$, let the function $\tilde{B}^{\mathrm{CR}}_{k,E}$ be as in the proof of Lemma 2.16. Then there exists a generic constant C depending only on the shape regularity of $\mathcal{T}$ such that for any $K \in \mathcal{T}_E$, it holds*

$$\left\|\tilde{B}^{\mathrm{CR}}_{k,E}\right\|_{H^{1/2}(\partial K)} \leq C\sqrt{\log(k+1)}.$$

Proof. Let $E \in \mathcal{E}_\Omega(\mathcal{T})$ and $K \in \mathcal{T}_E$. Similarly as in (B.7) we have

$$\left\|\tilde{B}_{k,E}^{\mathrm{CR}}\right\|_{H^{1/2}(\partial K)} \leq C \left\|\tilde{B}_{k,\widehat{E}}^{\mathrm{CR}}\right\|_{H^{1/2}\left(\partial\widehat{K}\right)}$$

with $\tilde{B}_{k,\widehat{E}}^{\mathrm{CR}} = \tilde{B}_{k,E}^{\mathrm{CR}} \circ \phi_K$ and affine pullback $\phi_K : \widehat{K} \to K$. Number the vertices in $\widehat{K}$ counterclockwise $\widehat{\mathbf{z}}_i$, $1 \leq i \leq 3$, such that $\widehat{\mathbf{z}}_3$ is opposite to $\widehat{E} := \phi_K^{-1}(E)$. The edge in $\partial\widehat{K}$ opposite to $\widehat{\mathbf{z}}_i$ is denoted by $\widehat{E}_i$ and this implies $\widehat{E} = \widehat{E}_3$. The edgewise pullbacks ϕ_i are defined as in (B.2). We employ the equivalence of the $H^{1/2}\left(\partial\widehat{K}\right)$ norm with $\left\|\!\left|\cdot\right|\!\right\|_{H^{1/2}\left(\partial\widehat{K}\right)}$ (see (B.3)) and obtain

$$\left\|\tilde{B}_{k,\widehat{E}}^{\mathrm{CR}}\right\|_{H^{1/2}\left(\partial\widehat{K}\right)} \leq C \left(\sum_{i=1}^{3}\left(\|v_i\|_{H^{1/2}(I)}^2 + |d_i|_{H_{(0,}^{1/2}(I)}^2\right)\right)^{1/2}$$

with

$$v_i := \left.\tilde{B}_{k,\widehat{E}}^{\mathrm{CR}}\right|_{\widehat{E}_i} \circ \phi_i \quad \text{and} \quad d_i(s) := v_{i-1}(s) - v_{i+1}(-s).$$

Note that

$$v_1(s) = L_k(-s), \quad v_2(s) = L_k(s), \quad v_3 = L_{k-1} - \tilde{w}_3 \tag{B.8}$$

with $\tilde{w}_3 := L_{k-1}'(-1)\,\tilde{\psi}_k^- - L_{k-1}'(1)\,\tilde{\psi}_k^+$ and $\tilde{\psi}_k^\pm$ as in (2.20). The antisymmetry of the Legendre polynomial for odd k implies $d_3 = 0$ and

$$\left\|\tilde{B}_{k,\widehat{E}}^{\mathrm{CR}}\right\|_{H^{1/2}\left(\partial\widehat{K}\right)} \leq C \left(\|L_k\|_{H^{1/2}(I)}^2 + \|L_{k-1}\|_{H^{1/2}(I)}^2 + \|\tilde{w}_3\|_{H^{1/2}(I)}^2 + \sum_{i=1}^{2}|d_i|_{H_{(0,}^{1/2}(I)}^2\right)^{1/2}. \tag{B.9}$$

The estimates $\|L_j\|_{H^{1/2}(I)}^2 \leq C\log(k+1)$ for $j \in \{k-1, k\}$ follow from (B.6). For the last term, we employ

$$\begin{aligned}
|d_i|_{H_{(0,}^{1/2}(I)} &\overset{\text{(B.4), (B.8)}}{\leq} |L_k + L_{k-1}|_{H_{(0,}^{1/2}(I)} + |\tilde{w}_3|_{H_{(0,}^{1/2}(I)} \\
&\overset{\text{(B.1)}}{\leq} |L_k + L_{k-1}|_{H_{(0,}^{1/2}(I)} + |\tilde{w}_3|_{H_{00}^{1/2}(I)}.
\end{aligned}$$

The last term can be estimated by taking into account $|L_k'(\pm1)| = \binom{k+1}{2}$ (obtained, e.g., by evaluating and differentiating [17, 18.5.8 for the choice $\alpha = \beta = 0$ and 18.5.10 for the choice $\lambda = 1/2$.] at ±1):

$$\|\tilde{w}_3\|_{H_{00}^{1/2}(I)} \leq C(k+1)^2\left(\left\|\tilde{\psi}_k^-\right\|_{H_{00}^{1/2}(I)} + \left\|\tilde{\psi}_k^+\right\|_{H_{00}^{1/2}(I)}\right) \overset{\text{[3, (A.5)]}}{\leq} C.$$

For the third term in (B.9), we simply employ $\|\tilde{w}_3\|_{H^{1/2}(I)} \leq \|\tilde{w}_3\|_{H_{00}^{1/2}(I)}$ so that

$$\left\|\tilde{B}_{k,\widehat{E}}^{\mathrm{CR}}\right\|_{H^{1/2}\left(\partial\widehat{K}\right)} \leq C\left(\sqrt{\log(k+1)} + |L_k + L_{k-1}|_{H_{(0,}^{1/2}(I)}\right).$$

The last integral can be evaluated analytically: By using the recurrence relation (cf. [17, 18.9.1])

$$L_k(x) = \frac{2k-1}{k}xL_{k-1}(x) - \frac{k-1}{k}L_{k-2}(x)$$

we obtain

$$L_k(x) + L_{k-1}(x) = \frac{2k-1}{k}(x+1)L_{k-1}(x) - \frac{k-1}{k}(L_{k-1}(x) + L_{k-2}(x)). \tag{B.10}$$

This and the orthogonality properties of the Legendre polynomials lead to

$$\begin{aligned} I_k &:= |L_k + L_{k-1}|^2_{H^{1/2}_{(0,}(I)} = \int_{-1}^{1} \frac{\left(\frac{2k-1}{k}(x+1)L_{k-1}(x) - \frac{k-1}{k}(L_{k-1}(x)+L_{k-2}(x))\right)^2}{x+1} dx \\ &= \left(\frac{2k-1}{k}\right)^2 \int_{-1}^{1} (x+1) L_{k-1}^2(x)\,dx - \frac{4(k-1)}{k^2} + \left(\frac{k-1}{k}\right)^2 I_{k-1}. \end{aligned} \tag{B.11}$$

The integral in (B.11) will be evaluated in (C.1). We obtain

$$\left(\frac{2k-1}{k}\right)^2 \int_{-1}^{1} (x+1) L_{k-1}^2(x)\,dx = \frac{2(2k-1)}{k^2}$$

and the explicit recursion formula

$$I_k = \frac{2}{k^2} + \left(\frac{k-1}{k}\right)^2 I_{k-1}$$

with starting value

$$I_1 = \int_{-1}^{1} \frac{(L_1(x) + L_0(x))^2}{x+1} dx = \int_{-1}^{1} \frac{(1+x)^2}{x+1} dx = 2.$$

It is easy to verify that $I_k = 2/k$ solves the recursion and hence,

$$|L_k + L_{k-1}|_{H^{1/2}_{(0,}(I)} = \sqrt{2/k}.$$

From this, the assertion follows. ■

C Analytic evaluation of some integrals involving Legendre polynomials

In the proof of Lemma 2.20 some integrals over Legendre polynomials appear and we present here their explicit evaluation.

Lemma C.1 *For $k \geq 0$, it holds*

$$\int_{-1}^{1} L_k^2(t) = \int_{-1}^{1} (t+1) L_k^2(t)\,dt = \frac{2}{2k+1} \tag{C.1}$$

and

$$\int_{-1}^{1} (L_k'(t))^2\,dt = \int_{-1}^{1} (t+1)(L_k'(t))^2\,dt = k(k+1). \tag{C.2}$$

Proof. The relation $\int_{-1}^{1} L_k^2(t)\,dt = 2/(2k+1)$ follows from [23, 7.221(1)].

For $k=0$, these relations follows from $L_0(t)=1$. Let $k\geq 1$. The recurrence relation in [17, 18.9.1, Table 18.9.1] imply

$$(t+1)\,L_k(t) = \frac{k+1}{2k+1}L_{k+1}(t) + L_k(t) + \frac{k}{2k+1}L_{k-1}(t)\,. \tag{C.3}$$

Substituting $(t+1)\,L_k(t)$ under the integral in (C.1) by this and taking into account the orthogonality relations of the Legendre polynomials leads to

$$\int_{-1}^{1}(t+1)\,L_k^2(t)\,dt = \int_{-1}^{1}L_k^2(t)\,dt = \frac{2}{2k+1}.$$

For the second integral (C.2) we employ integration by parts:

$$\int_{-1}^{1}(t+1)\left(L_k'(t)\right)^2 dt = \left.(t+1)\,L_k'(t)\,L_k(t)\right|_{-1}^{1} - \int_{-1}^{1} g_k(t)\,L_k(t)\,dt \quad \text{for } g(t) := \left((t+1)\,L_k'(t)\right)'. \tag{C.4}$$

Since $g\in\mathbb{P}_{k-1}$ the orthogonality properties of Legendre polynomials imply that the integral in the right-hand side of (C.4) is zero. By using $L_k(\pm1) = (\pm1)^k$ (cf. [17, Table 18.6.1]) and $L_k'(\pm1) = (\pm1)^{k+1}\binom{k+1}{2}$ (cf. [17, combine 18.9.15 with Table 18.6.1]) we get

$$\int_{-1}^{1}(t+1)\left(L_k'(t)\right)^2 dt = 2L_k'(1)\,L_k(1) = k\,(k+1)\,.$$

Finally

$$\int_{-1}^{1}\left(L_k'(t)\right)^2 dt = \left.L_k'(t)\,L_k(t)\right|_{t=-1}^{1} - \int_{-1}^{1}L_k''(t)\,L_k(t)\,dt.$$

The last integral is zero due the orthogonality of the Legendre polynomials. The endpoint values of L_k and L_k' lead to the assertion. ■

In the final part of this section, we will compute the value of $|L_k|_{H^{1/2}(I)}$ explicitly. We set

$$I_k(s) := |L_k|^2_{H^s(I)} = \int_{-1}^{1}\int_{-1}^{1}\frac{\left(L_k(x)-L_k(y)\right)^2}{|x-y|^{1+2s}}\,dydx.$$

Lemma C.2 *It holds*

$$I_k(1/2) = 4\sum_{\ell=1}^{k}\frac{1}{\ell}.$$

Proof. We write

$$\left(L_k(y)-L_k(x)\right)^2 = \sum_{\ell=2}^{2k}\frac{\kappa_\ell(x)}{\ell!}\,(y-x)^\ell$$

with

$$\kappa_\ell(x) := \frac{d^\ell}{dy^\ell}\left((L_k(y)-L_k(x))^2\right)\Big|_{y\leftarrow x}$$
$$= \sum_{r=0}^{\ell}\binom{\ell}{r}\left(\frac{d^r}{dy^r}(L_k(y)-L_k(x))\right)\frac{d^{\ell-r}}{dy^{\ell-r}}(L_k(y)-L_k(x))\Big|_{y\leftarrow x}$$
$$= \sum_{r=1}^{\ell-1}\binom{\ell}{r}\left(\frac{d^r}{dy^r}(L_k(y)-L_k(x))\right)\frac{d^{\ell-r}}{dy^{\ell-r}}(L_k(y)-L_k(x))\Big|_{y\leftarrow x}$$
$$= \sum_{r=1}^{\ell-1}\binom{\ell}{r}L_k^{(r)}(x)\,L_k^{(\ell-r)}(x) = \left(L_k^2\right)^{(\ell)}(x) - 2L_k(x)\,L_k^{(\ell)}(x).$$

Hence,

$$I_k(s) := I_k^{\mathrm{I}}(s) - I_k^{\mathrm{II}}(s)$$

with

$$I_k^{\mathrm{I}}(s) = \sum_{\ell=2}^{2k}\frac{1}{\ell!}\int_{-1}^{1}\int_{-1}^{1}\frac{(L_k^2)^{(\ell)}(x)\,(y-x)^\ell}{|x-y|^{1+2s}}dydx,$$
$$I_k^{\mathrm{II}}(s) = 2\sum_{\ell=2}^{2k}\frac{1}{\ell!}\int_{-1}^{1}\int_{-1}^{1}\frac{L_k(x)\,L_k^{(\ell)}(x)\,(y-x)^\ell}{|x-y|^{1+2s}}dydx.$$

We perform the integration with respect to y explicitly and get

$$I_k^{\mathrm{I}}(s) = \sum_{\ell=2}^{2k}\frac{1}{\ell!}\int_{-1}^{1}\left(L_k^2\right)^{(\ell)}(x)\,w_{\ell,s}(x)\,dx,$$
$$I_k^{\mathrm{II}}(s) = 2\sum_{\ell=2}^{2k}\frac{1}{\ell!}\int_{-1}^{1}L_k(x)\,L_k^{(\ell)}(x)\,w_{\ell,s}(x)\,dx$$

with

$$w_{\ell,s}(x) := \frac{(-1)^\ell(1+x)^{\ell-2s}+(1-x)^{\ell-2s}}{\ell-2s}.$$

From now on we restrict to the case $s = 1/2$. Since $w_{\ell,1/2}L_k^{(\ell)}$ is a polynomial of maximal degree $k-1$ the second integral vanishes: $I_k^{\mathrm{II}}(1/2) = 0$. We apply recursively integration by parts and use the orthogonality of the Legendre polynomials to obtain

$$I_k(1/2) = I_k^{\mathrm{I}}(1/2) = \sum_{\ell=2}^{2k}\sum_{m=0}^{2k-\ell}\frac{(-1)^m\,2^{m+2}}{(m+\ell)(m+\ell-1)}\frac{1}{(m+1)!}\left(L_k^2\right)^{(m+1)}(1).$$

We interchange the ordering of the summation, introduce the new variable $t = \ell+m-2$, and obtain

$$I_k(1/2) = \sum_{m=0}^{2k-2}\sum_{\ell=2}^{2k-m}\frac{(-1)^m\,2^{m+2}}{(m+\ell)(m+\ell-1)}\frac{1}{(m+1)!}\left(L_k^2\right)^{(m+1)}(1)$$
$$= \sum_{m=0}^{2k-2}\frac{(-1)^m\,2^{m+2}}{(m+1)!}\left(L_k^2\right)^{(m+1)}(1)\sum_{t=m}^{2k-2}\frac{1}{(t+2)(t+1)}.$$

By using a telescoping sum argument, it is easy to verify that the inner sum equals

$$\sum_{t=m}^{2k-2} \frac{1}{(t+2)(t+1)} = \sum_{t=m}^{2k-2} \left(\frac{1}{t+1} - \frac{1}{t+2} \right) = \frac{1}{m+1} - \frac{1}{2k}.$$

Hence,

$$I_k(1/2) = \frac{1}{k} I_k^{\mathrm{III}}(1/2) - 2 I_k^{\mathrm{IV}}(1/2), \tag{C.5}$$

for

$$I_k^{\mathrm{III}}(1/2) := -\sum_{m=0}^{2k-2} \frac{(-1)^m 2^{m+1}}{(m+1)!} \left(L_k^2\right)^{(m+1)}(1),$$
$$I_k^{\mathrm{IV}}(1/2) := \sum_{m=0}^{2k-2} \frac{(-1)^{m+1} 2^{m+1}}{(m+1)!\,(m+1)} \left(L_k^2\right)^{(m+1)}(1).$$

For I_k^{III} and $k \geq 1$ we get

$$\begin{aligned} I_k^{\mathrm{III}}(1/2) &= \sum_{m=1}^{2k-1} \frac{(-1)^m 2^m}{m!} \left(L_k^2\right)^{(m)}(1) \\ &= -L_k^2(1) - \frac{2^{2k}}{(2k)!} \left(L_k^2\right)^{(2k)}(1) + \sum_{m=0}^{2k} \frac{(-1)^m 2^m}{m!} \left(L_k^2\right)^{(m)}(1). \end{aligned} \tag{C.6}$$

We use the endpoint formula $L_k^{(m)}(\pm 1) = 0$ for $m > k$ and for $m \in \{0, 1, \ldots, k\}$:

$$L_k^{(m)}(\pm 1) \overset{[1,\ 22.5.37,\ 22.4.2]}{=} (\pm 1)^{k+m} (2m-1)!! \binom{k+m}{k-m} = (\pm 1)^{k+m} \frac{1}{(2m)!!} \frac{(k+m)!}{(k-m)!}.$$

This leads to $L_k^2(1) = 1$ and the Leibniz rule for differentiation yields

$$\begin{aligned} \frac{2^{2k}}{(2k)!} \left(L_k^2\right)^{(2k)}(1) &= \frac{2^{2k}}{(2k)!} \sum_{\ell=0}^{2k} \binom{2k}{\ell} L_k^{(\ell)}(1)\, L_k^{(2k-\ell)}(1) = \frac{2^{2k}}{(2k)!} \binom{2k}{k} \left(L_k^{(k)}(1) \right)^2 \\ &= \frac{2^{2k}}{(2k)!} \binom{2k}{k} \left(\frac{(2k)!}{(2k)!!} \right)^2 = \frac{2^{2k}}{(2k)!} \frac{(2k)!}{(k!)^2} \frac{(2k)!^2}{2^{2k}\,(k)!^2} = \frac{(2k)!^2}{(k)!^4}. \end{aligned}$$

The last sum in (C.6) is the Taylor expansion of L_k^2 about $x = 1$, evaluated at $x = -1$, i.e.,

$$\sum_{m=0}^{2k} \frac{(-1)^m 2^m}{m!} \left(L_k^2\right)^{(m)}(1) = \left(L_k^2\right)(-1) = 1.$$

Hence,

$$I_k^{\mathrm{III}}(1/2) = -\frac{(2k)!^2}{(k)!^4}. \tag{C.7}$$

For the quantity I_k^{IV} we obtain by similar arguments

$$\begin{aligned}
I_k^{\mathrm{IV}}(1/2) &= \sum_{m=1}^{2k-1} \frac{(-1)^m 2^m}{m!m} \left(L_k^2\right)^{(m)}(1) \\
&= -\frac{2^{2k-1}}{(2k)!k} \left(L_k^2\right)^{(2k)}(1) - \int_{-1}^{1} \frac{1}{s-1} \sum_{m=1}^{2k} \frac{1}{m!} \left(L_k^2\right)^{(m)}(1)(s-1)^m ds \\
&= -\frac{1}{2k}\frac{(2k)!^2}{(k)!^4} - \int_{-1}^{1} \frac{L_k^2(s) - L_k^2(1)}{s-1} ds.
\end{aligned}$$

The combination of this with (C.5), (C.7) leads to

$$I_k(1/2) = 2\int_{-1}^{1} \frac{L_k^2(s)-1}{s-1} ds = 2\int_{-1}^{1} \frac{L_k(s)-1}{s-1}\left(L_k(s)+1\right) ds.$$

Since $\frac{L_k(s)-1}{s-1}$ is a polynomial of maximal degree $k-1$ the orthogonality of L_k leads to

$$I_k(1/2) = 2\int_{-1}^{1} \frac{L_k(s)-1}{s-1} ds. \tag{C.8}$$

We employ the recursion formula in [17, Table 18.9.1] for $k \geq 2$

$$L_k(s) = \frac{2k-1}{k} s L_{k-1}(s) - \frac{k-1}{k} L_{k-2}(s)$$

so that

$$\begin{aligned}
I_k(1/2) &= 2\int_{-1}^{1} \frac{\frac{2k-1}{k}\left(sL_{k-1}(s)-1\right) - \frac{k-1}{k}\left(L_{k-2}(s)-1\right)}{s-1} ds \\
&= \frac{2k-1}{k} 2\int_{-1}^{1} \left(L_{k-1}(s) + \frac{L_{k-1}(s)-1}{s-1}\right) ds - \frac{k-1}{k} I_{k-2}(1/2) \\
&= \frac{2k-1}{k} I_{k-1}(1/2) - \frac{k-1}{k} I_{k-2}(1/2).
\end{aligned} \tag{C.9}$$

From (C.8) and $L_0(s) = 1$, $L_1(s) = s$, $L_2(s) = (3s^2-1)/2$ we get

$$\begin{aligned}
&I_0(1/2) = 0, \\
&I_1(1/2) = 2\textstyle\int_{-1}^{1} \frac{L_1(s)-1}{s-1} ds = 4, \\
&I_2(1/2) = 2\textstyle\int_{-1}^{1} \frac{L_2(s)-1}{s-1} ds = 6.
\end{aligned} \tag{C.10}$$

Now, it is easy to verify that $I_k = 4\sum_{\ell=1}^{k} \frac{1}{\ell}$ satisfies the recursion (C.9) and the initial value conditions (C.10). ■

D Norm equivalence for Crouzeix-Raviart spaces

It is well known that for $V := H_0^1(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T})$, the norms $\left(\|\nabla_{\mathcal{T}} u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)^{1/2}$ and $\|\nabla_{\mathcal{T}} u\|_{L^2(\Omega)}$ are equivalent. In this section, we state estimates for the constants in these equivalencies – the proof is a repetition of the well-known arguments for the case $k=1$ (see, e.g., [19, Lem. 36.6]).

Theorem D.1 *There exists a constant $C > 0$ depending only on the shape-regularity of the mesh and the domain Ω such that*

$$\|u\|_{H^1(\mathcal{T})} \leq \left(\|\nabla_{\mathcal{T}} u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{1/2} \leq C \|u\|_{H^1(\mathcal{T})} \quad \forall u \in V.$$

In particular C is independent of the polynomial degree $k \geq 1$ and the mesh size $h_{\mathcal{T}}$.

Proof. We prove this result only under the regularity assumption that the Poisson problem:

$$\text{find } \phi \in H^1_0(\Omega) \quad \text{s.t.} \quad (\nabla \phi, \nabla v)_{\mathbf{L}^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega)$$

is H^2 regular. For less regularity we refer to [19, Lem. 36.6]. For $u \in V$, we have

$$\|u\|_{L^2(\Omega)} = \sup_{v \in L^2(\Omega) \backslash \{0\}} \frac{(u, v)_{L^2(\Omega)}}{\|v\|_{L^2(\Omega)}}. \tag{D.1}$$

For $v \in L^2(\Omega)$, there exists some $\mathbf{w} \in \mathbf{H}^1(\Omega)$ such that $\operatorname{div} \mathbf{w} = v$ and $\|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \leq C_\Omega \|v\|_{L^2(\Omega)}$ for a constant C_Ω which only depends on Ω. Hence,

$$(u, v)_{L^2(\Omega)} = (u, \operatorname{div} \mathbf{w})_{L^2(\Omega)} = -(\nabla_{\mathcal{T}} u, \mathbf{w})_{L^2(\Omega)} + \sum_{K \in \mathcal{T}} \int_{\partial K} \langle \mathbf{w}, \mathbf{n}_K \rangle u,$$

where $\mathbf{n}_K$ is the unit normal vector pointing to the exterior of K. Next, we rewrite the sum over the triangle boundaries as a sum over the edges. For $E \in \mathcal{E}_\Omega(\mathcal{T})$ we fix the direction of a unit vector $\mathbf{n}_E$ which is orthogonal to E and for $E \in \mathcal{E}_{\partial\Omega}(\mathcal{T})$ let $\mathbf{n}_E$ denote the unit vector, orthogonal to E, pointing to the exterior of Ω. Then

$$(u, v)_{L^2(\Omega)} = (u, \operatorname{div} \mathbf{w})_{L^2(\Omega)} = -(\nabla_{\mathcal{T}} u, \mathbf{w})_{L^2(\Omega)} - \sum_{E \in \mathcal{E}_\Omega(\mathcal{T})} \int_E \langle \mathbf{w}, \mathbf{n}_E \rangle [u]_E + \sum_{E \in \mathcal{E}_{\partial\Omega}(\mathcal{T})} \int_E \langle \mathbf{w}, \mathbf{n}_E \rangle u.$$

Let $K_E \in \mathcal{T}_E$ be fixed and let $\mathbf{q}_E \in (\mathbb{P}_0(K_E))^2$ be the function with constant value $\frac{1}{|E|} \int_E \mathbf{w}$. The orthogonality conditions of the Crouzeix-Raviart elements across edges (see (1.9b)) imply

$$\begin{aligned}(u, v)_{L^2(\Omega)} &= -(\nabla_{\mathcal{T}} u, \mathbf{w})_{L^2(\Omega)} - \sum_{E \in \mathcal{E}_\Omega(\mathcal{T})} \int_E \langle \mathbf{w} - \mathbf{q}_E, \mathbf{n}_E \rangle [u]_E + \sum_{E \in \mathcal{E}_{\partial\Omega}(\mathcal{T})} \int_E \langle \mathbf{w} - \mathbf{q}_E, \mathbf{n}_E \rangle u \\ &\leq \|\nabla_{\mathcal{T}} u\|_{\mathbf{L}^2(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \\ &\quad + \sum_{E \in \mathcal{E}_\Omega(\mathcal{T})} \|[u]_E\|_{L^2(E)} \|\mathbf{w} - \mathbf{q}_E\|_{\mathbf{L}^2(E)} + \sum_{E \in \mathcal{E}_{\partial\Omega}(\mathcal{T})} \|u\|_{L^2(E)} \|\mathbf{w} - \mathbf{q}_E\|_{\mathbf{L}^2(E)}.\end{aligned} \tag{D.2}$$

We employ first a weighted trace inequality (see, e.g., [18, Lem. 12.15]) and then a Poincaré-Steklov estimate (see, e.g., [19, (12.17) for $p = 2$ and $s = 1$.]) to get for $h_E := |E|$

$$\|\mathbf{w} - \mathbf{q}_E\|_{\mathbf{L}^2(E)} \leq C \left(h_E^{-1/2} \|\mathbf{w} - \mathbf{q}_E\|_{\mathbf{L}^2(K_E)} + h_E^{1/2} \|\nabla (\mathbf{w} - \mathbf{q}_E)\|_{\mathbb{L}^2(K_E)} \right) \leq C h_{K_E}^{1/2} \|\nabla \mathbf{w}\|_{\mathbb{L}^2(K_E)}, \tag{D.3}$$

where C only depends on the shape-regularity of the mesh.

Next we estimate the jump of u across E. For $E \in \mathcal{E}_\Omega(\mathcal{T})$, we define $u_E \in \mathbb{P}_0(\mathcal{T}_E)$ as the function with constant value $\frac{1}{|E|}\int_E u|_K$ on $K \in \mathcal{T}_E$ and observe $[u_E]_E = 0$. Hence,

$$\begin{aligned}\|[u]_E\|_{L^2(E)} &= \|[u-u_E]_E\|_{L^2(E)} \leq \sum_{K\in\mathcal{T}_E} \|(u-u_E)|_K\|_{L^2(E)} \\ &\leq \sum_{K\in\mathcal{T}_E}\left(h_E^{-1/2}\|u-u_E\|_{L^2(K)} + h_E^{1/2}\|\nabla u\|_{\mathbf{L}^2(K)}\right) \leq C\sum_{K\in\mathcal{T}_E} h_K^{1/2}\|\nabla u\|_{\mathbf{L}^2(K)},\end{aligned}$$

for a constant C which only depends on the shape-regularity of the mesh. For $E \in \mathcal{E}_{\partial\Omega}(\mathcal{T})$ the estimate $\|u\|_{L^2(E)} \leq h_K^{1/2}\|\nabla u\|_{\mathbf{L}^2(K)}$ for $K \in \mathcal{T}_E$ follows in a similar fashion. The combination of (D.2) with (D.3) and the two trace estimates for u leads to

$$\begin{aligned}(u,v)_{L^2(\Omega)} &\leq \|\nabla_{\mathcal{T}} u\|_{\mathbf{L}^2(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + C\sum_{E\in\mathcal{E}(\mathcal{T})}\|\nabla_{\mathcal{T}} u\|_{\mathbf{L}^2(\omega_E)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega_E)} \\ &\leq C\|\nabla_{\mathcal{T}} u\|_{\mathbf{L}^2(\Omega)}\|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \leq CC_\Omega\|\nabla_{\mathcal{T}} u\|_{\mathbf{L}^2(\Omega)}\|q\|_{L^2(\Omega)}.\end{aligned}$$

Using this estimate in (D.1) finishes the proof. ■

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