A row generation method for the general case of inverse continuous facility location problem

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Abstract

In a single facility location problem, a set of points is given and the goal is finding the optimal location of new facility respect to given criteria such as minimizing time, cost and distances between the clients and facilities. On the other side, the inverse models try to modify the parameters of the problem with the minimum cost such that a given point becomes optimal. In this paper, we introduce a novel algorithm for the general case of the inverse single facility location problem with variable weights in the plane. The convergence and optimality conditions of the algorithm are presented. Then in the special cases the inverse minisum and minimax single facility location problems are considered and the algorithm tested on some instances. The results indicate the efficiency of the algorithm on these instances.

Keywords: continuous facility location; row generation; inverse facility location; variable weights

1 Introduction

The single facility location problem is well known in operations research literature and plays an important role in practice. The continuous version of this problem asks for finding the location of a facility in the plane, such that the cost of servicing clients by the facility is minimized. In the classical version of facility location problems, a set of points, represent the location of clients, are given and a weight is assigned to each point. Two basic objective functions in the facility location problems are minimizing the sum of weighted distances and the maximum weighted distances between clients and the facility, which called minisum and minimax single facility location

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problems, respectively. These problems also called median and center problems in the literature of location theory. For more details in continuous facility location models, the reader is referred to [11].

In some real applications, the facilities may already exist and the goal is modifying the parameters of the problem with minimum cost, such that the given facility location become optimal. These kind of location models are called inverse facility location problems.

There are many researches in the graph version of the inverse facility location problems. Among them, Cai et al. [6] showed that the inverse center problem is NP-hard. Burkard et al. [5] proposed an O(nlogn) algorithm for the inverse 1-median problem on a tree. Then Galavii [9] improved the time complexity of this problem to linear time. Alizadeh and Burkard [1] showed that the inverse 1-center problem can be solved in $O(n^2)$ time. Guan and Zhang [10] and Nguyen and Sepasian [16] considered the inverse 1-median and 1-center problems on trees with Chebyshev and Hamming norms, respectively. Sepasian and Rahbarnia [18] solved the inverse 1-median problem with varying vertex and edge length on trees. When the underlying network is a block graph, Nguyan [15] proposed a solution algorithm for the inverse 1-median problem with variable vertex weight. Nazari et al. [12] investigated the inverse of backup 2-median problem with variable edge length and vertex weight. Recently, Omidi et al. [17] proposed an algorithm for solving the inverse balanced facility location models on a tree.

The continuous version of inverse facility location problem is rarely considered by authors. The minisum single facility location problem has been considered by Burkard et al. [4]. They showed the Euclidean case of this problem with variable vertex weights can be solved in O(nlogn) time. Baroughi-Bonab et al. [2] investigated the inverse minisum single facility location problem with variable coordinates. They showed the problem with rectilinear and Chebyshev norms are NP-hard. Nazari and Fathali [13] applied a meta-heuristic algorithm for solving the inverse backup 2-median problem with variable coordinates.

In this paper, we investigate the general case of inverse continuous single facility location problem with variable weights. This problem asks to modify the weights of existing points with minimum cost such that a given point becomes optimal. A novel row generation algorithm is introduced. Then, we show the algorithm converges to the solution of inverse facility location problem. The algorithm explained for the inverse minimax and minsum location models.

In what follows, the general form of continuous single facility location problem and its inverse are presented in Section 2. In Section 3, a general

algorithm is introduced and its convergence is investigated. Then solving two special cases of continuous inverse location models by the presented algorithm are explained in Subsections 3.1 and 3.2. Computational results are presented in Section 4. Section 5 contains the summary and the conclusion on this paper.

$\mathbf{2}$ The general form of continuous facility location problem and its inverse

Let **A** be the set of n points $\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n$ in the plane. Where $\mathbf{A}_i = (a_i, b_i)$ for i = 1, ..., n are the location of clients. For i = 1, ..., n, let the point \mathbf{A}_i has the weight w_i . The goal in a facility location problem is finding a point $\mathbf{x} = (x, y)$ in the plane, which is the location of a new facility such that the function $f(\mathbf{x}, \mathbf{w}, \mathbf{A})$ is minimized, i.e.

$$(P) \min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}, \mathbf{w}, \mathbf{A}),$$

where $\mathbf{w} = (w_1, ..., w_n)$ is the vector of weights of existing points.

In the inverse model the aim is modifying some parameters of the problem with minimum cost such that a given point $\bar{\mathbf{x}}$ become optimal. The changing parameters can be the weights, coordinates of points or both.

In this paper, we consider the weight modifying case. Thus, we want to modify the weights of existing points w_i , for i = 1, ..., n, to $\hat{w}_i = w_i + p_i - q_i$ with minimum cost. Where $p_i \geq 0$ and $q_i \geq 0$ are the values of augmenting and reduction of the weight of point \mathbf{A}_i and bounded from above by u_i^+ and u_i^- , respectively. Suppose that c_i^+ and c_i^- denote the costs of augmenting and reduction of per unit of w_i , respectively. Then the inverse problem with variable weights can be modeled as follow.

$$(P_w) \min_{i=1}^{n} (c_i^+ p_i + c_i^- q_i)$$
 (1)

$$f(\bar{\mathbf{x}}, \hat{\mathbf{w}}, \mathbf{A}) \le f(\mathbf{x}, \hat{\mathbf{w}}, \mathbf{A}) \quad \forall \mathbf{x} \in \mathbb{R}^2$$
 (2)

$$\hat{w}_i = w_i + p_i - q_i \quad i = 1, \dots, n,$$
 (3)

$$0 \le p_i \le u_i^+ \qquad i = 1, \dots, n,$$

$$0 \le q_i \le u_i^- \qquad i = 1, \dots, n,$$

$$\hat{u}_i \ge 0 \qquad i = 1, \dots, n.$$
(5)
(6)

$$0 \le q_i \le u_i^- \qquad i = 1, \cdots, n, \tag{5}$$

$$\hat{w}_i > 0$$
 $i = 1, ..., n.$ (6)

This model is a problem with infinite constraints and difficult to solve.

In the next section we present a novel row generation algorithm for solving the inverse single facility location problem.

3 The general algorithm for the inverse problem

As mentioned in Section 2, the general form of inverse model contains infinite constraints. In this section we present a method with finite constraints for solving the inverse problem.

Let $\mathbf{x}^{(0)}$ be the optimal solution of problem (P). The weights of existing points should be modified such that the given point $\bar{\mathbf{x}}$ become better than any other point. The point $\mathbf{x}^{(0)}$ is optimal with respect to the current wights. Therefore, in the first step we modify the weights such that $\bar{\mathbf{x}}$ become better than $\mathbf{x}^{(0)}$. Thus we consider a new model by replacing the infinite constraints (2) with the following single constraint,

$$f(\bar{\mathbf{x}}, \hat{\mathbf{w}}, \mathbf{A}) \le f(\mathbf{x}^{(0)}, \hat{\mathbf{w}}, \mathbf{A}).$$

Then the new weights of points are obtained by solving this new model. In the next step, $\mathbf{x}^{(1)}$ the optimal solution of problem (P) with respect to the new weights, is calculated. Then the following new constraint is added to the model

$$f(\bar{\mathbf{x}}, \hat{\mathbf{w}}, \mathbf{A}) \le f(\mathbf{x}^{(1)}, \hat{\mathbf{w}}, \mathbf{A}),$$

and the new weights are obtained by solving this model. This procedure continues until the weights do not changed.

For i = 1, ..., n, let $p_i^{(k)}$ and $q_i^{(k)}$ be the values of modifying the weights of existing points in iteration k then

$$C^{(k)} = \sum_{i=1}^{n} (c_i^+ p_i^{(k)} + c_i^- q_i^{(k)})$$

is the value of objective function for the current solution.

These ideas lead us to the following algorithm.

Algorithm [A1].

- 1. **Find** $\mathbf{x}^{(0)}$, the optimal solution of problem (P).
- 2. **Set** k = 1 and $\mathbf{w}^{(0)} = \mathbf{w}$.
- 3. If $f(\bar{\mathbf{x}}, \mathbf{w}^{(0)}, \mathbf{A}) = f(\mathbf{x}^{(0)}, \mathbf{w}^{(0)}, \mathbf{A})$ stop, the current solution is optimal

4. Find $(\hat{w}_i, p_i^{(1)}, q_i^{(1)}), i = 1, ..., n$, the optimal solution of the following problem,

$$(P_0) \min \sum_{i=1}^{n} (c_i^+ p_i + c_i^- q_i)$$
 (7)

$$f(\bar{\mathbf{x}}, \hat{\mathbf{w}}, \mathbf{A}) \le f(\mathbf{x}^{(0)}, \hat{\mathbf{w}}, \mathbf{A}) \tag{8}$$

$$\hat{w}_{i} - p_{i} + q_{i} = w_{i}^{(0)} \quad i = 1, \dots, n,$$

$$0 \leq p_{i} \leq u_{i}^{+} \qquad i = 1, \dots, n,$$

$$0 \leq q_{i} \leq u_{i}^{-} \qquad i = 1, \dots, n,$$

$$\hat{w}_{i} \geq 0 \qquad i = 1, \dots, n.$$
(10)
(11)

$$0 \le p_i \le u_i^+ \qquad i = 1, \cdots, n, \tag{10}$$

$$0 \le q_i \le u_i^- \qquad i = 1, \cdots, n, \tag{11}$$

$$\hat{w}_i \ge 0 \qquad \qquad i = 1, ..., n. \tag{12}$$

If this problem is infeasible stop, the main problem is infeasible.

- 5. Set $\mathbf{w}^{(1)} = \mathbf{w}^{(0)} + \mathbf{p}^{(1)} \mathbf{q}^{(1)}$.
- 6. While $||\mathbf{w}^{(k)} \mathbf{w}^{(k-1)}|| > \epsilon \text{ do}$
 - (a) Find $\mathbf{x}^{(\mathbf{k})}$ the optimal solution of the following sub-problem

$$\min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}, \mathbf{w}^{(k)}, \mathbf{A}).$$

(b) **Add** the following constraint to problem (P_{k-1})

$$f(\bar{\mathbf{x}}, \hat{\mathbf{w}}, \mathbf{A}) \le f(\mathbf{x}^{(k)}, \hat{\mathbf{w}}, \mathbf{A}),$$

and call the new problem (P_k) .

- (c) If problem (P_k) is infeasible stop, the main problem is infeasible.
- (d) Find $(w_i^{(k+1)}, p_i^{(k)}, q_i^{(k)}), i = 1, ..., n$, the optimal solution of the problem (P_k) .
- (e) **Set** k = k + 1.

End while

7. For i = 1, ..., n set

$$(w_i^*, p_i^*, q_i^*) = (w_i^{(k+1)}, p_i^{(k)}, q_i^{(k)}).$$

8. **Set**

$$C^* = \sum_{i=1}^{n} (c_i^+ p_i^* + c_i^- q_i^*).$$

End of algorithm

Note that in each iteration, one constraint has been added to the previous model and the best weights are chosen, thus the algorithm obey the row generation rule. Moreover, instead of solving a problem with infinite constraints, the algorithm solve some problems with finite constraints. Indeed, the problem is reduced to the discrete version of the inverse location problem.

The following theorem shows that if in two iterations the weights of points do not changes then an optimal solution is found.

Theorem 3.1. Let t be an iteration of Algorithm A1 such that $\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)}$, then $\hat{\mathbf{w}} = \mathbf{w}^{(\mathbf{t}+1)}$ is the optimal solution of (P_w) .

Proof In the iteration t of the Algorithm A1, the following problem is solved

$$(P_t) \min \sum_{i=1}^{n} (c_i^+ p_i + c_i^- q_i)$$

$$f(\bar{\mathbf{x}}, \hat{\mathbf{w}}, \mathbf{A}) \le f(\mathbf{x}^{(\mathbf{k})}, \hat{\mathbf{w}}, \mathbf{A}) \qquad k = 0, ..., t$$

$$\hat{w_i} = w_i^{(0)} + p_i - q_i \quad i = 1, \cdots, n,$$

$$0 \le p_i \le u_i^+ \qquad i = 1, \cdots, n,$$

$$0 \le q_i \le u_i^- \qquad i = 1, \cdots, n,$$

$$\hat{w_i} \ge 0 \qquad i = 1, ..., n.$$

Where $\mathbf{x}^{(\mathbf{k})}$ is the solution of k-th sub-problem. Specially, for k=t,

$$f(\mathbf{x}^{(t)}, \mathbf{w}^{(t)}, \mathbf{A}) = \min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}, \mathbf{w}^{(t)}, \mathbf{A}).$$

By the algorithm $(\mathbf{w^{(t+1)}}, \mathbf{p^{(t)}}, \mathbf{q^{(t)}})$ is the optimal solution of (P_t) , then

$$f(\bar{\mathbf{x}}, \mathbf{w}^{(t+1)}, \mathbf{A}) \le f(\mathbf{x}^{(t)}, \mathbf{w}^{(t+1)}, \mathbf{A}),$$
$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(0)} + \mathbf{p}^{(t)} - \mathbf{q}^{(t)}.$$

Since $\mathbf{w}^{(t)} = \mathbf{w}^{(t+1)}$ then

$$f(\mathbf{x}^{(t)}, \mathbf{w}^{(t+1)}, \mathbf{A}) = \min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}, \mathbf{w}^{(t+1)}, \mathbf{A}).$$

Therefore,

$$f(\bar{\mathbf{x}}, \mathbf{w}^{(t+1)}, \mathbf{A}) \le f(\mathbf{x}, \mathbf{w}^{(t+1)}, \mathbf{A}) \ \forall \mathbf{x} \in \mathbb{R}^2,$$

It means $(\mathbf{w^{(t+1)}}, \mathbf{p^{(t)}}, \mathbf{q^{(t)}})$ is a feasible solution of (P_w) . Now let $(\mathbf{w'}, \mathbf{p'}, \mathbf{q'})$ be a feasible solution of (P_w) then

$$f(\bar{\mathbf{x}}, \mathbf{w}', \mathbf{A}) \le f(\mathbf{x}, \mathbf{w}', \mathbf{A}) \quad \forall \mathbf{x} \in \mathbb{R}^2,$$

$$\mathbf{w}' = \mathbf{w}^{(0)} + \mathbf{p}' - \mathbf{q}'.$$

This implies that $(\mathbf{w}', \mathbf{p}', \mathbf{q}')$ is also feasible for (P_t) . Since $(\mathbf{w}^{(t+1)}, \mathbf{p}^{(t)}, \mathbf{q}^{(t)})$ is the optimal solution of (P_t) then

$$\sum_{i=1}^{n} (c_i^+ p_i^{(t)} + c_i^- q_i^{(t)}) \le \sum_{i=1}^{n} (c_i^+ p_i' + c_i^- q_i').$$

It means $(\mathbf{w^{(t+1)}}, \mathbf{p^{(t)}}, \mathbf{q^{(t)}})$ is the optimal solution of (P_w) .

The stopping condition of the algorithm is based on Theorem 3.1. This stopping condition can be replaced by another one which is the case that the value of objective function in sub-problem is equal to $f(\bar{\mathbf{x}}, \mathbf{w}^{(t)}, \mathbf{A})$.

Theorem 3.2. If in the iteration t of Algorithm A1, $f(\bar{\mathbf{x}}, \mathbf{w}^{(t)}, \mathbf{A}) = f(\mathbf{x}^{(t)}, \mathbf{w}^{(t)}, \mathbf{A})$, then $\mathbf{w}^{(t)}$ is an optimal solution of (P_w) .

Proof Note that $\mathbf{x}^{(t)}$ is the optimal solution of sub-problem, thus

$$f(\mathbf{x}^{(t)}, \mathbf{w}^{(t)}, \mathbf{A}) = \min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}, \mathbf{w}^{(t)}, \mathbf{A}).$$

Since $f(\bar{\mathbf{x}}, \mathbf{w}^{(t)}, \mathbf{A}) = f(\mathbf{x}^{(t)}, \mathbf{w}^{(t)}, \mathbf{A})$, therefore

$$f(\bar{\mathbf{x}}, \mathbf{w}^{(t)}, \mathbf{A}) \le f(\mathbf{x}, \mathbf{w}^{(t)}, \mathbf{A}) \ \forall \mathbf{x} \in \mathbb{R}^2,$$

so $\mathbf{w}^{(t)}$ is a feasible solution for problem (P_w) . The proof of optimality is the same as proof of Theorem 3.1.

Moreover, if $\bar{\mathbf{x}} = \mathbf{x}^{(t)}$ then $f(\bar{\mathbf{x}}, \mathbf{w}^{(t)}, \mathbf{A}) = f(\mathbf{x}^{(t)}, \mathbf{w}^{(t)}, \mathbf{A})$. Thus the following corollary is immediately concluded.

Corollary 3.3. If in the iteration t of Algorithm A1, $\bar{\mathbf{x}} = \mathbf{x}^{(t)}$, then $\mathbf{w}^{(t)}$ is optimal for (P_w) .

The following theorem indicates the convergence of the algorithm.

Theorem 3.4. Algorithm A1 converges to an optimal solution of (P_w) .

Since in each iteration of Algorithm A1, a constraint is added to make $\bar{\mathbf{x}}$ better than some new points in the plane. Therefore, the feasibility improved. Thus if $k \to \infty$ then the algorithm will terminate with finding the optimal solution.

In some special cases such as inverse minisum problem with Euclidean norm, we know that the optimal solution of problem (P) lies in the convex hull of the existing points (see e.g. [11]). Therefore in this case, if the given point \bar{x} does not lie in the convex hull, then the inverse model is infeasible. The following lemma shows the relation between infeasibility of problem (P_w) and problem (P_k) in Algorithm A1.

Lemma 3.5. In any iteration k of Algorithm A1, if problem (P_k) is infeasible then problem (P_w) is infeasible, too.

Note that the time complexity of the algorithm depends on sub-problem (P) and problem (P_k) , in iteration k. One of inverse location models that can be solved in polynomial time by Algorithm A1, is inverse minisum location problem. In the next sections we explain the presented algorithm for solving the inverse minisum and minimax location problems.

3.1The inverse minisum location problem

In this section we consider the special case that the objective function of location problem is minimum the weighted sum of distances between existing points and the new facility, i.e.

$$(P_{ms}) \min_{\mathbf{x} \in \mathbb{R}^2} \sum_{i=1}^n w_i d(\mathbf{x}, A_i),$$

where d(x, A) is the distance between the points x and A. This problem called Fermat-weber problem.

In this case, the inverse model with variable weights can be written as follow.

$$(P_{ws}) \min \sum_{i=1}^{n} (c_i^+ p_i + c_i^- q_i)$$
(13)

$$\sum_{i=1}^{n} \hat{w}_i d(\bar{x}, A_i) \le \sum_{i=1}^{n} \hat{w}_i d(x, A_i) \qquad \forall \mathbf{x} \in \mathbb{R}^2$$
 (14)

$$\hat{w}_i = w_i + p_i - q_i \quad i = 1, \dots, n,$$
 (15)

$$0 \le p_i \le u_i^+$$
 $i = 1, \dots, n,$ (16)
 $0 \le q_i \le u_i^ i = 1, \dots, n.$ (17)

$$0 \le q_i \le u_i^- \qquad i = 1, \cdots, n. \tag{17}$$

Table 1: The parameters of considered instance in Example 3.6

i	$A_i = (a_i, b_i)$	w_i	u_i^-	u_i^+	c_i^-	c_i^+
1	(1,0)	0	0	5	$\sqrt{2}$	$\sqrt{2}$
2	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	0	0	5	7	7
3	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	0	0	5	1	1
4	$(0, -1)^2$	$\frac{10}{\sqrt{2}}$	0	0	0	0

This is a linear programming model with infinite constraints. Although, in the case that distances are measured by Euclidean norm, Burkard et al. [4] presented a linear model with finite constraints, however, in the general L_p norm there isn't any efficient method. Therefore, our proposed algorithm is the first for solving the inverse minisum location problem with L_p norm.

Using Algorithm A1, in the iteration k, the following sub-problem should be solved.

$$\min_{\mathbf{x} \in \mathbb{R}^2} \sum_{i=1}^n w_i^{(k)} d(\mathbf{x}, A_i), \tag{18}$$

When the distances are measured by L_p norms, this sub-problem can be solved by the iterative Weiszfeld algorithm (see e.g. [11]).

Also, the following constraint should be added to problem (P_{k-1}) in iteration k,

$$\sum_{i=1}^{n} \hat{w}_i(d(\bar{\mathbf{x}}, A_i) - d(\mathbf{x}^{(k)}, A_i)) \le 0, \tag{19}$$

which is a linear constraint. Since the linear programs and sub-problems can be solved in a polynomial time, therefore the time complexity of Algorithm A1 for the inverse minisum location problem with variable weights, is polynomial.

Example 3.6. Consider the instance of the inverse minisum location problem given in Table 1, which is the only instance that solved in [4]. The given point is $\bar{\mathbf{x}} = (0,0)$. Burkard et al. [4] found that the optimal solution is $\mathbf{w}^* = (0,5,5,\frac{10}{\sqrt{2}})$. We examined our algorithm for finding the solution of this instance.

The iteration results of Algorithm A1 are shown in Table 2. In this table the column with heading $\mathbf{x}^{(k)}$ indicates the solution of sub-problem (18) in iteration k with respect to $\mathbf{w}^{(k)}$. The algorithm find the optimal solution with tolerance $\epsilon = 0.01$, after 11 iterations. The results show $\mathbf{x}^{(k)}$ converges to the given point $\bar{\mathbf{x}} = (0,0)$. The cost of modifying $\mathbf{w}^{(0)}$ to $\mathbf{w}^{(11)}$ is $C^* = 39.9836$.

Table 2:	The iteration	results of	of Algorithm A	1 on instance	of Example 3.6.

						1
iteration	$\mathbf{x}^{(k)} = (x^{(k)}, y^{(k)})$	$w_1^{(k)}$	$w_2^{(k)}$	$w_3^{(k)}$	$w_4^{(k)}$	$ \mathbf{w}^{(k)} - \mathbf{w}^{(k-1)} $
0	(0.0000, -1.0000)	5.0000	0.8979	0.0000	7.0711	7.1278
1	(0.1697, -0.3725)	0.8790	2.9114	5.0000	7.0711	4.5866
2	(0.0206, -0.3787)	2.1373	3.4609	5.0000	7.0711	1.3730
3	(0.0877, -0.1749)	0.5318	3.9241	5.0000	7.0711	1.6710
4	(-0.0125, -0.1498)	1.0139	4.3648	5.0000	7.0711	0.6532
5	(-0.0469, 0.0698)	0.3567	4.4951	5.0000	7.0711	0.6700
6	(0.0008, -0.0541)	0.3567	4.8101	5.0000	7.0711	0.3150
7	(0.0168, -0.0132)	0.1549	4.8136	5.0000	7.0711	0.2019
8	(0.0014, -0.0077)	0.0000	4.9647	5.0000	7.0711	0.2164
9	(-0.0015, -0.0015)	0.0264	4.9724	5.0000	7.0711	0.0275
10	(0.0004, -0.0011)	0.0000	4.9918	5.0000	7.0711	0.0328
11	(-0.0003, -0.0003)	0.0041	4.9950	5.0000	7.0711	0.0052

3.2 The inverse minimax location problem

In this section, the main steps of Algorithm A1 for the case that the objective function of location problem is minimum the weighted maximum distance from existing points to the new facility, is explained. The minimax location problem is as follow.

$$(P_{mm}) \min_{\mathbf{x} \in \mathbb{R}^2} \max_{i=1,\dots,n} w_i d(\mathbf{x}, A_i).$$

Then the inverse model with variable weights can be written as follow.

$$(P_{wm})\min\sum_{i=1}^{n}(c_{i}^{+}p_{i}+c_{i}^{-}q_{i})$$
(20)

$$\max_{i=1,\dots,n} \{ \hat{w}_i d(\bar{x}, A_i) \} \le \max_{i=1,\dots,n} \{ \hat{w}_i d(x, A_i) \} \quad \forall \mathbf{x} \in \mathbb{R}^2$$
 (21)

$$\hat{w}_i = w_i + p_i - q_i \quad i = 1, \dots, n,$$
 (22)

$$0 \le p_i \le u_i^+$$
 $i = 1, \dots, n,$ (23)
 $0 \le q_i \le u_i^ i = 1, \dots, n.$ (24)

$$0 \le q_i \le u_i^- \qquad i = 1, \cdots, n. \tag{24}$$

This problem is a nonlinear model with infinite constraints. As far as we know, this is an open problem and there isn't any efficient method for solving it. Using our proposed algorithm, some models with finite constraints should be solved. The following sub-problem should be solved in the iteration k of Algorithm A1.

$$\min_{\mathbf{x} \in \mathbb{R}^2} \max_{i=1,\dots,n} w_i^{(k)} d(\mathbf{x}, A_i). \tag{25}$$

This minimax sub-problem can be solved in a linear time (see, e.g. [7] and references therein).

Also, in the iteration k, the following constraint should be added to problem (P_{k-1}) ,

$$\max_{i=1,\dots,n} \{ \hat{w}_i(d(\bar{\mathbf{x}}, A_i) - d(\mathbf{x}^{(k)}, A_i)) \} \le 0.$$
 (26)

Although the added constraints in the inverse minisum model are linear, these constraints in the inverse minimax case are nonlinear. However, the problem (P_k) is a discrete version of the inverse minimax location problem which called inverse 1-center problem. Nguyen et al. [14] showed that the discrete inverse 1-center problem can be solved in $O(n^2 log n)$ time. Therefore, the time complexity of our proposed algorithm for inverse minimax location problem is polynomial and it is more efficient than solving a problem with infinite constrains.

4 Computational results

In this section we examine some test problems on our presented algorithm. The algorithm was written in MATLAB 2014. All the experiments were run on a PC with Intel Core i7 processor, 8 GB of RAM and CPU 2.4 GHz. The proposed algorithm was tested on 4 test problems with varying given points and L_p norms. In the Euclidean case the results are compared with those obtained by the linear programming method of Burkard et al. [4]. In the other cases, there isn't any efficient method in the literature, therefore just the results of our algorithm are presented. In tables 4 and 6, the notation C_B indicates the optimal costs which obtained by the method of Burkard et al. [4]. After finding the new weights by the method of Burkard et al. [4], to verify its feasibility, we applied the Wiszfeld method with the new weights. The obtained solution is shown by \mathbf{x}_B .

The results of all test problems are reported for $\epsilon = 0.01$. The number of the last iteration is shown by t.

As the first test problem, consider the points and their required data that are given in Table 3. The coordinate of these points are given from [8].

The results for varying given points with Euclidean norm are shown in Table 4. Each column of this table, contains a given point $\bar{\mathbf{x}}$ and the obtained results for this point. The results indicates that in all cases, the algorithm find the optimal solution of problem (P_w) with tolerance ϵ .

We also examined the proposed algorithm for 3 instances Ruspini 75, Bongartz 287 and TSPLIB 654, which can be found in Beasley [3]. All

$(a_i, b_i, w_i, c_i^-, c_i^+, u_i^-, u_i^+)$	$(a_i, b_i, w_i, c_i^-, c_i^+, u_i^-, u_i^+)$	$(a_i, b_i, w_i, c_i^-, c_i^+, u_i^-, u_i^+)$
(1, 2, 3, 2, 1, 3, 5)	(4,4,1,2,1,1,3)	(7, 1, 2, 1, 1, 2, 6)
(1, 3, 2, 2, 3, 2, 7)	(4,9,2,1,1,2,2)	(7,2,3,3,5,3,7)
(2, 5, 1, 3, 4, 1, 9)	(5,3,2,2,5,2,6)	(8,5,5,2,3,4,8)
(3,6,3,2,1,3,8)	(5,5,1,4,3,1,5)	(8,8,3,4,2,3,3)
(4, 8, 2, 2, 2, 2, 4)	(6,6,3,5,2,3,2)	(9,7,4,2,5,4,9)
(4, 1, 3, 1, 3, 2, 8)	(6,3,3,1,4,2,4)	(9,6,5,5,5,5,1)

Table 3: Data for 18 points problem.

the weights, costs and upper bounds are randomly generated in the interval [1,10]. The range of coordinates of the points in these instances are given in the third column of Table 5. Since the median is in the convex hull of the existing points, therefore, for each instances, the given point $\bar{\mathbf{x}}$ should be chosen in the range of existing points, otherwise the inverse problem is infeasible. Table 5 also shows the median points and their objective functions which obtained by the Weiszfeld algorithm with tolerance $\epsilon = 0.001$.

Tables 6 to 9 contains the results obtained by Algorithm A1 for varying L_p norms and given points $\bar{\mathbf{x}}$. In these tables the columns with heading $\Delta w^{(t)}$ indicate the norm of difference between the weights in the last two iterations of algorithm, i.e. $\Delta w^{(t)} = ||\mathbf{w}^{(t)} - \mathbf{w}^{(t-1)}||$. Table 6 shows the results those obtained by our algorithm and method of Burkard et al. [4] for L_2 norm. Comparing the results indicates that the method of Burkard et al. [4] is faster than our algorithm. However, their method can just be applied for L_2 norm.

As we mentioned in Theorem 3.3, the stopping condition can be replaced by $||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}|| \le \epsilon$. The results show $\mathbf{x}^{(t)}$ is very close to $\bar{\mathbf{x}}$ and in the most cases this alternative stopping condition also holds. To make more visible the comparing of the two stopping conditions, the variations of difference of weights and obtaining points in two consecutive iterations for instance TSPLIB with L_3 norm and $\bar{\mathbf{x}} = (1500, 1500)$ are shown in Figure 1.

5 Summary and conclusion

In this paper we presented a row generation algorithm for general case of inverse continuous location problems. The convergence and optimality condition have been proved for the given algorithm. In special cases we considered the inverse minisum and minimax location models and some test problems have been solved by the presented algorithm. The results show

Table 4: The results of Algorithm A1 on the instance with 18 points.

table 4. The resu	105 Of Migorithm	TII OII UIC IIISUAII	ice with to points
$\bar{\mathbf{x}}$	(2,2)	(3,5)	(7,7)
$\mathbf{x}^{(t)}$	(2.0978, 2.0105)	(2.9972, 4.9992)	(6.9831, 6.9826)
\mathbf{x}_B	(2.0681, 1.9962)	(2.9954, 4.9983)	(6.9950, 6.9987)
C^*	100.1798	72.2736	57.6731
C_B	101.2458	72.7461	58.48071
Iteration(t)	17	14	17
$w_1^{(t)}$	8.0000	8.0000	0.0000
$w_2^{(t)}$	1.3723	2.0000	0.0000
$ w_2^{(\iota)} $	0.0000	8.7530	1.0000
-1 $w^{(c)}$	0.0000	3.0000	1.2235
$m_{\zeta^{(l)}}$	0.0000	2.0000	4.8117
$w_c^{(c)}$	3.9030	1.5466	1.0000
$ w_7^{(i)} $	0.0000	0.0000	0.0000
$ w_{2}^{(i)} $	0.0000	2.0000	4.0000
$w_9^{(t)}$	8.0000	0.0000	0.0000
$w_{10}^{(c)}$	0.0000	1.0000	0.4531
$w_{11}^{(t)}$	0.0000	3.0000	3.0000
$w_{12}^{(t)}$	0.0000	1.0000	1.0000
$w_{13}^{(t)}$	0.0581	0.0000	0.0000
$w_{14}^{(t)}$	3.0000	0.0000	0.0000
$w_{15}^{(t)}$	1.0000	1.0000	1.0000
$w_{16}^{(t)}$	0.0000	3.0000	6.0000
$w_{17}^{(t)}$	0.0000	0.0000	4.0000
$w_{18}^{(t)}$	0.6396	5.0000	5.0000
$ \mathbf{w}^{(t)} - \mathbf{w}^{(t-1)} $	0.0000	0.0066	0.0011

Table 5: The coordinate ranges of existing points and median points of instances from [3] with varying L_p norms and random weights.

Instance	\overline{n}	Coordinate ranges	p	$\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)})$	$f(\mathbf{x}^{(0)}, \mathbf{w}^{(0)}, \mathbf{A})$
Ruspini	75	(5,5) to (110,110)	2	(56.7635, 102.9328)	23630.4385
			3	(62.3770, 103.0802)	22543.1684
			5	(67.2228, 103.7886)	21855.8203
			8	(68.9591, 104.1744)	21566.5435
Bongartz	287	(5,5) to $(48,48)$	2	(22.3657, 32.2146)	13946.0271
			3	(22.2903, 32.2109)	13226.8711
			5	(22.2175, 32.1861)	12843.64547
			8	(22.1757, 32.1678)	12709.5519
TSPLIB	654	(1000, 1000) to (5000, 5000)	2	(3410.0620, 3689.1907)	9085118.3175
			3	(3455.5845, 3704.2810)	8463911.9145
			5	(3389.6068, 3116.5265)	8327910.9097
			8	(3379.4386, 3085.4853)	8203351.1022

Table 6: The results for the instances from [3] for L_2 norm.

					[-]					
			Method of Bu	rkard et al. [4	1]		Algo	orithm A1		
Instance	n	$\bar{\mathbf{x}} = (\bar{x}, \bar{y})$	x _B	C_B	CPU	$\mathbf{x}^{(t)}$	t	C^*	$\Delta w^{(t)}$	CPU
					(in sec)					(in sec)
Ruspini	75	(50, 50)	(49.964, 50.038)	918.572	0.0263	(50.003, 50.006)	21	918.125	0.0058	1.0773
		(80, 20)	(80.052, 20.034)	2124.434	0.0300	(80.054, 20.068)	20	2121.925	0.0048	0.8295
		(20, 80)	(20.011, 80.022)	1226.341	0.0411	(20.025, 80.014)	23	1225.508	0.0087	1.0798
Bongartz	287	(15, 35)	(14.999, 34.996)	5111.552	0.0554	(15.000,34.995)	19	5110.484	0.0000	1.7481
		(10, 40)	(9.996, 39.993)	7836.151	0.0605	(10.007,40.004)	27	7835.434	0.0000	4.1123
		(30, 20)	(29.965, 20.003)	7813.039	0.0820	(29.956,20.016)	20	7811.124	0.0030	2.3278
TSPLIB	654	(2000, 4000)	(2000.036, 3999.988)	1668.822	0.1054	(2000.041,3999.944)	25	1668.666	0.0098	10.2197
		(1500, 1500)	(1500.192, 1500.108)	19772.922	0.1288	(1500.284,1500.631)	27	19772.592	0.0064	10.3044
		(3500, 3500)	(3499.858, 3499.994)	680.945	0.1460	(3500.050,3500.037)	27	680.716	0.0042	10.0377

Table 7: The results for the instances from [3] for L_3 norm.

			3			
n	$\mathbf{\bar{x}} = (\bar{x}, \bar{y})$	$\mathbf{x}^{(t)}$	t	C^*	$\Delta w^{(t)}$	CPU
						(in sec)
75	(50, 50)	(49.9796,50.0447)	21	797.8499	0.0036	0.7683
	(80, 20)	(79.9370, 19.9753)	20	2095.7514	0.0084	0.8642
	(20, 80)	(20.0039, 80.0050)	22	989.6491	0.0007	0.7568
287	(15, 35)	(15.0005, 35.0006)	30	5021.4630	0.0000	4.5220
	(10, 40)	(10.0045, 40.0008)	25	7812.9340	0.0081	3.8931
	(25, 25)	(24.9987, 25.0107)	19	4545.8444	0.005	2.9389
654	(2000, 4000)	(2000.0102, 3999.9955)	27	2573.2310	0.0054	10.7947
	(1500, 1500)	(1500.1864, 1500.1759)	34	19698.5450	0.0084	15.5954
	(3500, 3500)	(3500.0449, 3500.0118)	35	1236.0234	0.0054	15.7235
	75	$\begin{array}{ccc} 75 & (50,50) \\ & (80,20) \\ & (20,80) \\ \hline 287 & (15,35) \\ & (10,40) \\ & (25,25) \\ \hline 654 & (2000,4000) \\ & (1500,1500) \\ \end{array}$	$ \begin{array}{c cccc} n & \bar{\mathbf{x}} = (\bar{x}, \bar{y}) & \mathbf{x}^{(t)} \\ \hline 75 & (50, 50) & (49.9796, 50.0447) \\ & (80, 20) & (79.9370, 19.9753) \\ & (20, 80) & (20.0039, 80.0050) \\ \hline 287 & (15, 35) & (15.0005, 35.0006) \\ & (10, 40) & (10.0045, 40.0008) \\ & (25, 25) & (24.9987, 25.0107) \\ \hline 654 & (2000, 4000) & (2000.0102, 3999.9955) \\ & (1500, 1500) & (1500.1864, 1500.1759) \\ \hline \end{array} $	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$

Table 8: The results for the instances from [3] for L_5 norm.

100	,ic 0.	THE TESTING I	of the instances from [e	, 101	L ₀ norm.		
Instance	n	$\mathbf{\bar{x}} = (\bar{x}, \bar{y})$	$\mathbf{x}^{(t)}$	t	C^*	$\Delta w^{(t)}$	CPU
							(in sec)
Ruspini	75	(50, 50)	(49.9887, 50.0299)	15	803.0411	0.0064	0.6311
		(80, 20)	(79.8436, 19.9586)	18	2047.4155	0.0094	1.3337
		(20, 80)	(20.0017, 80.0112)	17	776.3165	0.0078	0.6362
Bongartz	287	(15, 35)	(15.0004, 35.0002)	30	4858.1739	0.0023	4.3586
		(10, 40)	(9.9993, 39.9963)	24	7750.9010	0.0065	3.6254
		(25, 25)	(24.9986, 25.0089)	18	4305.7498	0.0039	1.6300
TSPLIB	654	(2000, 4000)	(2000.0134, 3999.9885)	21	3791.6483	0.0091	8.2931
		(1500, 3500)	(1500.0089, 3500.0034)	36	8707.6269	0.0049	20.4489
		(1500, 4000)	(1500.0035, 4000.0007)	35	7868.2359	0.0078	17.2575

Table 9: The results for the instances from [3] for L_8 norm.

Instance	n	$\mathbf{\bar{x}} = (\bar{x}, \bar{y})$	$\mathbf{x}^{(t)}$	t	C^*	$\Delta w^{(t)}$	CPU
							(in sec)
Ruspini	75	(50, 50)	(49.9987, 50.0048)	15	860.9878	0.0022	0.5870
		(80, 20)	(79.9531, 20.0733)	18	2025.5144	0.0054	1.5873
		(20, 80)	(19.7536, 80.2957)	10	624.4426	0.0098	0.5051
Bongartz	287	(15, 35)	(15.0013,35.0000)	26	4729.2314	0.0086	3.2491
		(10, 40)	(9.9986, 39.9981)	26	7687.0110	0.0079	3.7591
		(25, 25)	(25.0002, 25.0075)	22	4161.0671	0.0011	2.8497
TSPLIB	654	(1500, 3500)	(3500.0117, 3499.9914)	40	9191.1963	0.0099	25.3061
			I .				

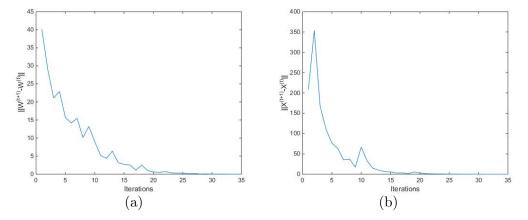


Figure 1: The changes of difference weights and obtaining points in two consecutive iterations for instance TSPLIB with L_3 norm and $\bar{\mathbf{x}} = (3500, 3500)$. (a) The change of difference weights. (b) The change of difference obtaining points.

the algorithm could efficiently find the optimal solutions.

Note that, the presented algorithm can be applied for solving inverse of other cases of the location problems such as inverse covering problem, inverse line location problem and inverse circle location problem. The algorithm can also be used for solving other kind of inverse location models such as hamming and rectilinear norms. However, the efficiency of the algorithm depends on the solvability of two problems: 1- finding the optimal solution of problem (P), 2- problem (P_k) , in iteration k.

The inverse continuous location problems with variable coordinates can also be solved by the same structure algorithm. However, it will be more harder, since problem (P_k) is not linear even for the inverse minisum location problem (Baroughi-Bonab et al. [2] showed this problem with rectilinear and Chebyshev norms is NP-hard). This case of inverse location models can be considered in the future works.

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