# "Painlevé 34" equation: equivalence test 

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#### Abstract

We give the complete solution of the Equivalence Problem for "Painlevé 34" equation.


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## 1 Introduction. "Painlevé 34" equation

At the beginning of XX century P. Painlevé and others studied the following class of second order ODEs

$$
y^{\prime \prime}=F\left(x, y ; y^{\prime}\right),
$$

where the function $F$ is rational in $y^{\prime}$ and analytic in $x$. Their goal was to find all equations whose general solutions have no movable critical singularities, i.e. have the Painlevé property. They solved this problem completely and found 50 equations. Six of which were principally new - irreducible equations - (they did not allow reducing the order, and their solutions defined new special functions), they are currently called the Painlevé equations (PI-PVI equations), see [1, 22. In some books all forenamed 50 equations are named "Painlevé equation 1-50". The complete list of them is in books [3], 4].

A distinctive feature of the "Painlevé 34 " equation is that its general solution and the PII solution

$$
\begin{equation*}
\text { PII : } \quad \tilde{y}^{\prime \prime}=2 \tilde{y}^{3}+\tilde{x} \tilde{y}+a, \quad a=\text { const } \tag{1}
\end{equation*}
$$

[^0]are expressed one into the other explicitely using the Bäcklund transformation, see [4], [5], [6, [7], 8]. They can be written in the form of a Hamiltonian system of ordinary differential equations with one degree of freedom, see [9].

Equation "Painlevé 34" from the book [4] is

$$
\begin{equation*}
X X X I V . \quad y^{\prime \prime}=\frac{y^{\prime 2}}{2 y}+4 a y^{2}-x y-\frac{1}{2 y}, \quad a=\text { const } \neq 0 \tag{2}
\end{equation*}
$$

In paper [9] this equation has some different form

$$
\begin{equation*}
y^{\prime \prime}=\frac{y^{\prime 2}}{2 y}-2 y^{2}-x y-\frac{(\alpha \pm 1 / 2)^{2}}{2 y}, \quad \alpha=\text { const } \tag{3}
\end{equation*}
$$

Note, that equation "Painlevé 34 " plays important role in the description of multi-ion electro-diffusion models, see [10].

In paper [11] was first stated the problem of deriving syzygies (relationships between the invariants) for every equation from the list of Gambier [4] i.e. for every "Painlevé equation 1-50". This work was continued in the recent paper [21], where was found some syzygies for the equation from the list of Gambier including equation (2).

The aim of this paper is constructing the equivalence test - necessary and sufficient conditions written in terms of invariants checking the equivalence of some equation (5) to the "Painlevé 34 " equation (3) under the general point transformations

$$
\begin{equation*}
\tilde{x}=\tilde{x}(x, y), \quad \tilde{y}=\tilde{y}(x, y) \tag{4}
\end{equation*}
$$

## 2 Implementation of the classification and calculation of the invariants

Equation (3) is from the following class of the second order ODE's

$$
\begin{equation*}
y^{\prime \prime}=P(x, y)+3 Q(x, y) y^{\prime}+3 R(x, y) y^{\prime 2}+S(x, y) y^{\prime 3} \tag{5}
\end{equation*}
$$

that is the closed under the general point transformations (44).
In the set of papers [12, 13, 14], review see [15], Ruslan Sharipov succeeded to construct the system of (pseudo)invariants which he calculated explicitly in the terms of the coefficients of equations (5). On the basis of this system he classified equations (5). In the present paper we use this classification for solving the equivalence problem of equation (3).

Step 1. At first we write equation (3) in the form

$$
\begin{equation*}
y^{\prime \prime}=\frac{y^{\prime 2}}{2 y}-2 y^{2}-x y-\frac{\beta^{2}}{2 y}, \quad \beta=\alpha \pm 1 / 2 \tag{6}
\end{equation*}
$$

Equation (6) has the form (5) with the coefficients

$$
P(x, y)=-2 y^{2}-x y-\frac{\beta^{2}}{2 y}, \quad Q(x, y)=0, \quad R(x, y)=\frac{1}{6 y}, \quad S(x, y)=0 .
$$

Step 2. Then calculate the basic objects characterizing the equation (6). Details are in the papers [12, 13, 14], [15].

Pseudotensorial field of weight $m$ and valence $(r, s)$ is an indexed set transformed under change of variables (4) by the rule

$$
F_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{r}}=(\operatorname{det} T)^{m} \sum_{p_{1} \ldots p_{r} q_{1} \ldots q_{s}} \sum_{p_{1}} S_{p_{1}}^{i_{1}} \ldots S_{p_{r}}^{i_{r}} T_{j_{1}}^{q_{1}} \ldots T_{j_{s}}^{q_{s}} \tilde{F}_{q_{1} \ldots q_{s}}^{p_{1} \ldots p_{r}},
$$

where $S$ and $T$ are direct and inverse transformations matrices for (4).
The first pseudovectorial field $\boldsymbol{\alpha}$ associated with equation (5) has weight 2 and the components $\alpha^{1}=B, \alpha^{2}=-A$, where

$$
\begin{align*}
A & =P_{0.2}-2 Q_{1.1}+R_{2.0}+2 P S_{1.0}+ \\
& +S P_{1.0}-3 P R_{0.1}-3 R P_{0.1}-3 Q R_{1.0}+6 Q Q_{0.1}  \tag{7}\\
B & =S_{2.0}-2 R_{1.1}+Q_{0.2}-2 S P_{0.1}- \\
& -P S_{0.1}+3 S Q_{1.0}+3 Q S_{1.0}+3 R Q_{0.1}-6 R R_{1.0}
\end{align*}
$$

We can check this fact applying the direct symbolic calculations. Hereinafter symbol $K_{i . j}$ denotes the partial differentiation: $K_{i . j}=\partial^{i+j} K / \partial x^{i} \partial y^{j}$.

The second pseudovectorial field $\boldsymbol{\beta}$ has weight 4 and the components $\beta^{1}=G, \beta^{2}=H$, where

$$
\begin{aligned}
G & =-B B_{1.0}-3 A B_{0.1}+4 B A_{0.1}+3 S A^{2}-6 R B A+3 Q B^{2}, \\
H & =-A A_{0.1}-3 B A_{1.0}+4 A B_{1.0}-3 P B^{2}+6 Q A B-3 R A^{2} .
\end{aligned}
$$

Their scalar product (using the skew-symmetric Gramian matrix) denoting the pseudoinvariant $F$ by the formula

$$
\begin{equation*}
3 F^{5}=A G+B H . \tag{8}
\end{equation*}
$$

For the equation (6) $\boldsymbol{\alpha}$ from (7) and $F$ from (8) are equal to

$$
A=-3-\frac{3 \beta^{2}}{8 y^{3}}, \quad B=0, \quad F=0
$$

Step 3. As if for the equation (6) the conditions $F=0$, but $A \neq 0$ or $B \neq 0$ are true, it relates to the Case of intermediate degeneration, for the details see [14]. In this case we can calculate another important pseudoinvariants $\Omega$ of weight $1, N$ of weight 2 and $M$ of weight 4 using explicit formulas, that are different in the cases $A \neq 0$ or $B \neq 0$.

As $A \neq 0$, the explicit formula for pseudoinvariant $\Omega$ reads as

$$
\begin{align*}
\Omega & =\frac{2 B A_{1.0}\left(B P+A_{1.0}\right)}{A^{3}}-\frac{\left(2 B_{1.0}+3 B Q\right) A_{1.0}}{A^{2}}+\frac{\left(A_{0.1}-2 B_{1.0}\right) B P}{A^{2}}- \\
& -\frac{B A_{2.0}+B^{2} P_{1.0}}{A^{2}}+\frac{B_{2.0}}{A}+\frac{3 B_{1.0} Q+3 B Q_{1.0}-B_{0.1} P-B P_{0.1}}{A}+  \tag{9}\\
& +Q_{0.1}-2 R_{1.0} .
\end{align*}
$$

And in the case $B \neq 0$ the similar formula is

$$
\begin{align*}
\Omega & =\frac{2 A B_{0.1}\left(A S-B_{0.1}\right)}{B^{3}}-\frac{\left(2 A_{0.1}-3 A R\right) B_{0.1}}{B^{2}}+\frac{\left(B_{1.0}-2 A_{0.1}\right) A S}{B^{2}}+ \\
& +\frac{A B_{0.2}-A^{2} S_{0.1}}{B^{2}}-\frac{A_{0.2}}{B}+\frac{3 A_{0.1} R+3 A R_{0.1}-A_{1.0} S-A S_{1.0}}{B}+  \tag{10}\\
& +R_{1.0}-2 Q_{0.1} .
\end{align*}
$$

In the cases $A \neq 0$ and $B \neq 0$ the pseudoinvariant $N$ is given by the formulas

$$
\begin{equation*}
N=-\frac{H}{3 A}, \quad N=\frac{G}{3 B} \tag{11}
\end{equation*}
$$

The pseudoinvariant $M$ in the case $A \neq 0$ reads as

$$
\begin{align*}
M= & -\frac{12 B N\left(B P+A_{1.0}\right)}{5 A}+\frac{24}{5} B N Q+\frac{6}{5} N B_{1.0}+\frac{6}{5} N A_{0.1}-  \tag{12}\\
& -A N_{0.1}+B N_{1.0}-\frac{12}{5} A N R .
\end{align*}
$$

And in the case $B \neq 0$ reads as

$$
\begin{align*}
M= & -\frac{12 A N\left(A S-B_{0.1}\right)}{5 B}+\frac{24}{5} A N R-\frac{6}{5} N A_{0.1}-\frac{6}{5} N B_{1.0}+ \\
& +B N_{1.0}-A N_{0.1}-\frac{12}{5} B N Q \tag{13}
\end{align*}
$$

For the equation (6) pseudoinvariants $\Omega$ from (9), $N$ from (11) and $M$ from (12) are equal to

$$
\Omega=0, \quad N=\frac{5 \beta^{2}}{4 y^{4}}-\frac{1}{2 y}, \quad M=\frac{9}{10 y^{2}}-\frac{63 \beta^{2}}{4 y^{5}}
$$

Step 4. It is easy to see that the pseudoinvariant $M$ given by (12), (13) for the equation (6) is not vanishing.

As if $M \neq 0$ for the equation (6), then it relates to the First case of intermediated degeneration, see [14]. In this case the basic invariants are

$$
\begin{equation*}
I_{1}=\frac{M}{N^{2}}, \quad I_{2}=\frac{\Omega^{2}}{N}, \quad I_{3}=\frac{\hat{\Gamma}_{22}^{1}}{M}, \quad \text { where } \tag{14}
\end{equation*}
$$

$$
\begin{aligned}
\hat{\Gamma}_{22}^{1}= & \frac{\gamma^{1} \gamma^{2}\left(\gamma_{1.0}^{1}-\gamma_{0.1}^{2}\right)}{M}+\frac{\left(\gamma^{2}\right)^{2} \gamma_{0.1}^{1}-\left(\gamma^{1}\right)^{2} \gamma_{1.0}^{2}}{M}+ \\
& +\frac{P\left(\gamma^{1}\right)^{3}+3 Q\left(\gamma^{1}\right)^{2} \gamma^{2}+3 R \gamma^{1}\left(\gamma^{2}\right)^{2}+S\left(\gamma^{2}\right)^{3}}{M} .
\end{aligned}
$$

Here $\gamma$ is a new pseudovectorial field of weight 3 associated with equation (5) relating to the First case of intermediate degeneration

As $A \neq 0$, the components of the pseudovectorial field $\gamma$ reads as

$$
\begin{align*}
\gamma^{1}= & -\frac{6 B N\left(B P+A_{1.0}\right)}{5 A^{2}}+\frac{18 N B Q}{5 A}+\frac{6 N\left(B_{1.0}+A_{0.1}\right)}{5 A}- \\
& -N_{0.1}-\frac{12}{5} N R-2 \Omega B  \tag{15}\\
\gamma^{2}= & -\frac{6 N\left(B P+A_{1.0}\right)}{5 A}+N_{1.0}+\frac{6}{5} N Q+2 \Omega A .
\end{align*}
$$

As $B \neq 0$, reads as

$$
\begin{align*}
\gamma^{1}= & -\frac{6 N\left(A N-B_{0.1}\right)}{5 B}-N_{0.1}+\frac{6}{5} N R-2 \Omega B, \\
\gamma^{2}= & -\frac{6 A N\left(A S-B_{0.1}\right)}{5 B^{2}}+\frac{18 N A R}{5 B}-\frac{6 N\left(A_{0.1}+B_{1.0}\right)}{5 B}+  \tag{16}\\
& +N_{1.0}-\frac{12}{5} N Q+2 \Omega A .
\end{align*}
$$

The additional invariants are computed by differentiating the basic invariants (14) along pseudovectorial fields $\boldsymbol{\alpha}$ from (7) and $\boldsymbol{\gamma}$ from (15), (16)

$$
\begin{align*}
I_{4}=\frac{B\left(I_{1}\right)_{x}^{\prime}-A\left(I_{1}\right)_{y}^{\prime}}{N}, & I_{6}=\frac{B\left(I_{3}\right)_{x}^{\prime}-A\left(I_{3}\right)_{y}^{\prime}}{N} \\
I_{7}=\frac{\left(\gamma^{1}\left(I_{1}\right)_{x}^{\prime}+\gamma^{2}\left(I_{1}\right)_{y}^{\prime}\right)^{2}}{N^{3}}, & I_{9}=\frac{\left(\gamma^{1}\left(I_{3}\right)_{x}^{\prime}+\gamma^{2}\left(I_{3}\right)_{y}^{\prime}\right)^{2}}{N^{3}}  \tag{17}\\
I_{15}=\frac{\left(\gamma^{1}\left(I_{6}\right)_{x}^{\prime}+\gamma^{2}\left(I_{6}\right)_{y}^{\prime}\right)^{2}}{N^{3}}, & I_{21}=\frac{\left(\gamma^{1}\left(I_{9}\right)_{x}^{\prime}+\gamma^{2}\left(I_{9}\right)_{y}^{\prime}\right)^{2}}{N^{3}}
\end{align*}
$$

For the equation (6), invariants $I_{1}, I_{2}$ from (14) and $I_{7}$ from (17) are

$$
\begin{equation*}
I_{1}=-\frac{36}{5} \frac{y^{3}\left(35 \beta^{2}-2 y^{3}\right)}{\left(5 \beta^{2}-2 y^{3}\right)^{2}}, \quad I_{2}=0, \quad I_{7}=0 \tag{18}
\end{equation*}
$$

As we can see, the invariant $I_{7}$ is vanishing, so the equation (6) relating to the Case 1.4 of intermediate degeneration, for details see [15].

## 3 Equivalence test

It may be two different possibilities, $I_{1}$ given by (18) is a constant or not.

### 3.1 Case $I_{1}=$ const

Equations (5) relating to the First case of intermediate degeneration with the conditions $I_{1}=$ const, $I_{2}=0$ from (14) were described in paper [17].

Let us represent formula for $I_{1}$ from (18) in the form

$$
I_{1}=\frac{18}{5}-\frac{90 \beta^{2}\left(2 y^{3}+\beta^{2}\right)}{\left(5 \beta^{2}-2 y^{3}\right)^{2}}
$$

It is not difficult to see that the only way $I_{1}$ to be a constant is $\beta=0$.
Then we calculate the invariant $I_{3}$ from (14), the additional invariants $I_{6}, I_{9}$ from (17) and a new invariant $J$, where

$$
\begin{equation*}
J=\frac{4+10 I_{6}-60 I_{3}}{50 \sqrt{I_{9}}} . \tag{19}
\end{equation*}
$$

For the equation (6) with the zero parameter $\beta$ these invariants are

$$
I_{3}=\frac{1}{30} \frac{2 y+x}{y}, \quad I_{6}=\frac{1}{5} \frac{x}{y}, \quad I_{9}=-\frac{1}{1250} \frac{1}{y^{3}}, \quad I_{21}=0, \quad J=0 .
$$

In papers [15]- 17] the following Theorem was proved. (In paper [15] the condition $I_{21}=0$ is unfortunately missed.)

Theorem 1. Equation (5) is equivalent to Painleve II equation (1) with the parameter $a= \pm J$ if and only if the following conditions hold:

1. equation corresponds to the Case of intermediate degeneration: $A \neq 0$ or $B \neq 0$ in (7), but $F=0$ in (8);
2. equation corresponds to the First case of intermediate degeneration: $M \neq 0$ in (12), (13), $\Omega=0$ in (9), (10);
3. $I_{1}=18 / 5$ in (14), $I_{9} \neq 0, I_{21}=0$ in (17), invariant $J=$ const in (19). Among the invariants $I_{3}, I_{6}$ and $I_{9}$ from (14), (17) one can find two functionally independent.

The invariant point transformation is

$$
\tilde{y}=\frac{1}{\sqrt[6]{2500 I_{9}}}, \quad \tilde{x}=\frac{5 I_{6}}{\sqrt[6]{2500 I_{9}}}-\frac{3}{2} J \sqrt[6]{2500 I_{9}}
$$

For the equation (6) with zero parameter $\beta$ all conditions of Theorem 1 are hold. So it is equivalent to Painlevé II equation (1) with zero parameter $a$. The corresponding change of variables $x=-\sqrt[3]{2} \tilde{x}, y=-\sqrt[3]{2} \tilde{y}^{2}$ transforms equation (6) (that is written in variables $(x, y)$ ) into equation (1) (that is written into variables ( $\tilde{x}, \tilde{y})$ ) with the parameter $a=0$.

### 3.2 Case $I_{1} \neq$ const

It was proved above that in this case the parameter $\beta$ is not vanishing. Let us make the following change of variables,

$$
y=\tilde{y}^{1 / 3} \beta^{2 / 3}, \quad x=\tilde{x} \beta^{2 / 3},
$$

then the equation (6) takes the form

$$
\begin{equation*}
\tilde{y}^{\prime \prime}=\frac{5 \tilde{y}^{\prime 2}}{6 \tilde{y}}-\beta^{2} \tilde{y}^{1 / 3}\left(6 \tilde{y}+3 \tilde{x} \tilde{y}^{2 / 3}+\frac{3}{2}\right) . \tag{20}
\end{equation*}
$$

To simplify the notation below we do not write the tildes over the variables $x$ and $y$ in the equation (20). This equation also has form (5) with the coefficients

$$
P=-\beta^{2} y^{1 / 3}\left(6 y+3 x y^{2 / 3}+\frac{3}{2}\right), \quad Q=0, \quad R=\frac{5}{18 y}, \quad S=0 .
$$

Let us calculate invariants $I_{1}, I_{3}$ from (14), $I_{4}, I_{7}, I_{9}, I_{15}, I_{21}$ from (17) for the equation (20)

$$
\begin{align*}
& I_{1}=\frac{36}{5} \frac{y(2 y-35)}{(2 y-5)^{2}}, \quad I_{3}=\frac{y\left(4 y+2 x y^{2 / 3}+1\right)(2 y-35)}{15(2 y+1)^{3}}, \\
& I_{4}=-\frac{3240(2 y+7) y(2 y+1)}{(2 y-5)^{4}}, \quad I_{9}=-\frac{64}{625} \frac{y^{6}(2 y-35)^{4}}{\beta^{2}(2 y+1)^{8}(2 y-5)^{3}}, \\
& I_{6}=\frac{4 y^{5 / 3} x\left(4 y^{2}-296 y+175\right)-6 y\left(300 y^{2}-136 y-35\right)}{5(2 y-5)(2 y+1)^{3}}, \quad I_{7}=0, \\
& I_{15}=-\frac{2304 y^{6}\left(4 y^{2}-296 y+175\right)^{2}(2 y-35)^{2}}{625(2 y-5)^{2}(2 y+1)^{8} \beta^{2}}, \quad I_{21}=0 . \tag{21}
\end{align*}
$$

Then we regard the symbols $I_{1}$ and $I_{4}$ as the parameters in order to convert formulas for $I_{1}$ and $I_{4}$ from (21) into polynomials. We get two polynomials depending only on the variable $y$
$\mathbf{P}_{\mathbf{1}}=36 y(-35+2 y)-5 I_{1}(2 y-5)^{2}, \quad \mathbf{P}_{\mathbf{2}}=3240(2 y+7) y(2 y+1)+I_{4}(2 y-5)^{4}$.
By implementation of Buchberger's algorithm, see [18], we reduce polynomials $\mathbf{P}_{\mathbf{1}}$ and $\mathbf{P}_{\mathbf{2}}$ with respect to the variable $y$. We obtain a new invariant $K$, that is vanishing for the equation (20)

$$
\begin{align*}
K= & 500 I_{1}^{4}-7275 I_{1}^{3}+500 I_{4} I_{1}^{2}+32940 I_{1}^{2}- \\
& -5475 I_{4} I_{1}-47628 I_{1}+125 I_{4}^{2}+13230 I_{4}=0 . \tag{22}
\end{align*}
$$

From the next-to-last step of Buchberger's algorithm, we get a formula for the variable $y$ in terms of invariants

$$
y=\frac{125\left(2322 I_{1}+3 I_{4}+20 I_{4} I_{1}-915 I_{1}^{2}+75 I_{1}^{3}\right)}{2\left(-1469664+1250 I_{4} I_{1}-13875 I_{4}+691470 I_{1}-90825 I_{1}^{2}+3375 I_{1}^{3}\right)} .
$$

The variable $x$ we find using the formula of $I_{3}$ from (21)

$$
x=\frac{\left(120 I_{3}-8\right) y^{3}+\left(138+180 I_{3}\right) y^{2}+\left(90 I_{3}+35\right) y+15 I_{3}}{2 y^{5 / 3}(2 y-35)} .
$$

The parameter $\beta^{2}$ we find using the formula of $I_{9}$ from (21)

$$
\beta^{2}=-\frac{64}{625} \frac{y^{6}(2 y-35)^{4}}{I_{9}(2 y+1)^{8}(2 y-5)^{3}} .
$$

The invariants $I_{6}, I_{3}, I_{1}$ and $I_{15}, I_{9}$ are related by the formulas

$$
\begin{aligned}
I_{6} & =\frac{6\left(4 y^{2}-296 y+175\right) I_{3}}{(2 y-5)(2 y-35)}-\frac{(2 y-1)(2 y-5) I_{1}}{9(2 y+1)^{2}}, \\
I_{15} & =\frac{36\left(4 y^{2}-296 y+175\right)^{2} I_{9}}{(2 y-35)^{2}(2 y-5)^{2}} .
\end{aligned}
$$

Theorem 2. Equation (5) is equivalent to the "Painlevé 34 " equation (20) with the parameter $\beta \neq 0$ if and only if the following conditions hold:

1. the equation corresponds to the Case of intermediate degeneration: $A \neq 0$ or $B \neq 0$ from (7), but $F=0$ from (8);
2. the equation corresponds to the Case 1.4 of intermediate degeneration: $M \neq 0$ from (12), (13), $I_{2}=0$ from (14), $I_{7}=0, I_{21}=0$ from (17);
3. the invariant $K=0$ from (22);
4. there exists a non-degenerate invariant change of variables that connects equations (5) and (20)

$$
\begin{align*}
& \tilde{y}=\frac{125}{2} . \\
& \cdot \frac{\left(3+20 I_{1}\right) I_{4}+3 I_{1}\left(5 I_{1}-18\right)\left(5 I_{1}-43\right)}{125\left(10 I_{1}-111\right) I_{4}+3\left(5 I_{1}-18\right)\left(225 I_{1}^{2}-5245 I_{1}+27216\right)}  \tag{23}\\
& \tilde{x}=\frac{\left(120 I_{3}-8\right) \tilde{y}^{3}+\left(138+180 I_{3}\right) \tilde{y}^{2}+\left(90 I_{3}+35\right) \tilde{y}+15 I_{3}}{2 \tilde{y}^{5 / 3}(2 \tilde{y}-35)}, \tag{24}
\end{align*}
$$

5. the following invariant is a constant

$$
\begin{equation*}
\beta^{2}=-\frac{64}{625} \frac{\tilde{y}^{6}(2 \tilde{y}-35)^{4}}{I_{9}(2 \tilde{y}+1)^{8}(2 \tilde{y}-5)^{3}} . \tag{25}
\end{equation*}
$$

6. invariants $K_{1}=0$ and $K_{2}=0$, where

$$
\begin{align*}
& K_{1}=I_{6}-\frac{6\left(4 \tilde{y}^{2}-296 \tilde{y}+175\right) I_{3}}{(2 \tilde{y}-5)(2 \tilde{y}-35)}+\frac{(2 \tilde{y}-1)(2 \tilde{y}-5) I_{1}}{9(2 \tilde{y}+1)^{2}},  \tag{26}\\
& K_{2}=\frac{I_{15}}{I_{9}}-\frac{36\left(4 \tilde{y}^{2}-296 \tilde{y}+175\right)^{2}}{(2 \tilde{y}-35)^{2}(2 \tilde{y}-5)^{2}} .
\end{align*}
$$

Here in the formulas (24), (25), (26) we should substitute the expression of $\tilde{y}$ via the invariants $I_{1}$ and $I_{4}$ from (23).

Example 1. Let us return to equation (2). All conditions of Theorem 2 are true. The point transformation

$$
\tilde{y}=-2 a y^{3}, \quad \tilde{x}=\frac{x}{(2 a)^{2 / 3}}, \quad a \neq 0
$$

transforms equation (2) (that is written in variables $(x, y)$ ) into equation (20) ) (that is written into variables $(\tilde{x}, \tilde{y})$ ) with the parameter $\beta^{2}=4 a^{2}$.

Let's note, that in the case $a=0$ equation (21) is equivalent to $y^{\prime \prime}=y^{-3}$, see [15] for the details.

Example 2. Equation (3) is not equivalent to Painlevé IV equation

$$
y^{\prime \prime}=\frac{y^{\prime 2}}{2 y}+\frac{3 y^{3}}{2}+4 x y^{2}+2\left(x^{2}-\alpha\right) y-\frac{\beta^{3}}{2 y}, \quad \alpha, \beta=\text { const } .
$$

Indeed, for the equation PIV the invariants $I_{7} \neq 0$. See [19].
Example 3. Equation describing 3-ion case (3a) from 10
$w^{\prime \prime}-\frac{w^{2}}{2 w}+\nu_{1}^{2}\left(-2 k_{1} w^{2}-(C x+K) w+\frac{k_{2}}{w}\right)=0, \quad \nu_{1}, k_{1}, k_{2}, C, K=\mathrm{const}$
is equivalent to "Painlevé 34 " equation (20) with the parameter $\beta^{2}=2 k_{2} \nu_{1}^{2} k_{1}^{2} / C^{2}$ if $\nu_{1} \neq 0, k_{1} \neq 0, k_{2} \neq 0, C \neq 0$. All conditions of Theorem 2 are true. The following point transformation

$$
\tilde{y}=-\frac{k_{1} w^{3}}{2 k_{2}}, \quad \tilde{x}=-\frac{C x+K}{2^{1 / 3} k_{1}^{2 / 3} k_{2}^{1 / 3}}
$$

transforms this equation (that is written in variables $(x, w))$ into the equation (20)) (that is written into variables $(\tilde{x}, \tilde{y})$ ).

And it is equivalent to Painlevé II equation (1) with the parameter $a=0$ if $\nu_{1} \neq 0, k_{1} \neq 0, k_{2}=0, C \neq 0$. All conditions of Theorem 1 are true. The following point transformation

$$
\tilde{y}=\frac{\sqrt[3]{\nu_{1}} \sqrt{k_{1} w}}{\sqrt[6]{2} \sqrt[3]{C}}, \quad \tilde{x}=\frac{(C x+K) \sqrt[3]{\nu_{1}}}{\sqrt[6]{2} \sqrt[3]{C} \sqrt{k_{1} w}}
$$

transforms this equation (that is written in variables $(x, w)$ ) into the equation (11) (that is written into variables $(\tilde{x}, \tilde{y})$ ).

Example 4. Equation describing 3-ion case (3b) from [10]

$$
\begin{aligned}
\left(w+\frac{C x+K}{k_{1}}\right) w^{\prime \prime} & -\frac{w^{\prime 2}}{2}-\frac{C w^{\prime}}{k_{1}}-2 k_{1} \nu_{1}^{2} w^{3}- \\
& -4 \nu_{1}^{2}(C x+K) w^{2}-2 \nu_{1}^{2}(C x+K)^{2} \frac{w}{k_{1}}=0
\end{aligned}
$$

$\nu_{1}, k_{1}, C, K=$ const is equivalent to "Painlevé 34 " equation (20) with $\beta^{2}=1 / 4$ if $\nu_{1} \neq 0, k_{1} \neq 0, C \neq 0$. All conditions of Theorem 2 are true. The following point transformation

$$
\tilde{y}=-\frac{\nu_{1}^{2}\left(k_{1} w+C x+K\right)^{3}}{C^{2}}, \quad \tilde{x}=\frac{2 \nu_{1}^{2 / 3}(C x+K)}{C^{2 / 3}}
$$

transforms this equation (that is written in variables $(x, w)$ ) into the equation (20) (that is written into variables $(\tilde{x}, \tilde{y})$ ).

Example 5. Reduction of Nonlinear Schroëdinger equation, see [20]. Here function $V(x)$ is a potential.

$$
y^{\prime \prime}=V(x) y-y^{3}+\frac{k^{2}}{y^{3}}, \quad k=\text { const } .
$$

If $k \neq 0$, invariants are

$$
\begin{aligned}
& A=-6 y+\frac{12 k^{2}}{y^{5}}, \quad B=0, \quad F=0, \quad M=\frac{72}{5}+\frac{1008 k^{2}}{y^{6}} \\
& I_{2}=0, \quad I_{7}=0, \quad I_{21}=-\frac{288 V^{\prime 2}(x) y^{88}\left(y^{6}+70 k^{2}\right)^{10} V^{\prime \prime 2}(x)}{9765625\left(y^{6}-2 k^{2}\right)^{18}\left(y^{6}+10 k^{2}\right)^{9}}=0, \\
& K=0, \quad \beta^{2}=-\frac{k^{2}}{V^{2}(x)}=\text { const }, \quad K_{1}=0, \quad K_{2}=0 .
\end{aligned}
$$

So, all conditions of Theorem 2 are true if $k \neq 0$ and $V(x)$ is a certain linear function, then this equation is equivalent to "Painlevé 34" equation (20). The following point transformation

$$
\tilde{y}=-\frac{y^{6}}{4 k^{2}}, \quad \tilde{x}=\sqrt[3]{2} k V(x)
$$

transforms this equation (that is written in variables $(x, y)$ ) into the equation (20) (that is written into variables $(\tilde{x}, \tilde{y})$ ).

If $k=0$ then the invariants are

$$
\begin{aligned}
& I_{1}=\frac{18}{5}, \quad I_{3}=\frac{1}{15}-\frac{V(x)}{15 y^{2}}, \quad I_{6}=-\frac{2 V(x)}{5 y^{2}}, \quad I_{9}=-\frac{2 V^{\prime 2}(x)}{625 y^{6}} \\
& I_{21}=-\frac{288 V^{\prime 2}(x) V^{\prime \prime 2}(x)}{9765625 y^{14}}=0, \\
& J=0 .
\end{aligned}
$$

All conditions of Theorem 1 are true if $V(x)$ is a certain linear function, $V(x) \neq$ const. The following linear point transformation

$$
\tilde{y}=\frac{\sqrt[6]{-1} y}{\sqrt{2} \sqrt[3]{V^{\prime}(x)}}, \quad \tilde{x}=\frac{V(x)}{\sqrt[3]{V^{\prime 2}(x)}}
$$

transforms this equation (that is written in variables $(x, y)$ ) into the equation Painlevé II (11) with zero parameter $a$ (that is written into variables $(\tilde{x}, \tilde{y})$ ).

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