On the numerical computation of the Mittag-Leffler function

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A B S T R A C T

Recently simple limiting functions establishing upper and lower bounds on the Mittag-Leffler function were found. This paper follows those expressions to design an efficient algorithm for the approximate calculation of expressions usual in fractional-order control systems. The numerical experiments demonstrate the superior efficiency of the proposed method.

Keywords:
Mittag-Leffler function
Fractional Calculus
Fractional control

1. Introduction

During the last decades Fractional Calculus (FC) became a major area of research and development and we can mention its application in many scientific areas ranging from mathematics and physics, up to biology, engineering, and earth sciences [18,22,14,19,6,9,21,11,2,1,13,7]. The Mittag-Leffler function (MLf) plays an important role in FC, being often called by scholars the "queen" of the FC functions. Nine decades after its first formulation by the Swedish Mathematician [3] Magnus Gösta Mittag-Leffler (1846–1927), the MLf became a relevant topic, not only from the pure mathematical point of view, but also from the perspective of its applications.

Bearing these ideas in mind, this short communication addresses the application of the MLf and real-time calculation in control systems of the expression $e_\alpha(t) = E_\alpha(-t^\alpha)$ where α denotes the fractional order, t stands for time and E_α represents the one parameter MLf to be recalled in the sequel.

The paper is organized as follows. Section 2 introduces the fundamental aspects of $E_\alpha(t)$ and $e_\alpha(t)$. Section 3 develops the approximation for the numerical calculation of $e_\alpha(t)$ and analyses its computational load. Finally, Section 4 outlines the main conclusions.

2. Fundamental aspects

2.1. The Mittag-Leffler function

The MLf, defined as

$$E_\alpha(t) = \sum_{n=0}^{+\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)} \tag{1}$$

is a special function, first studied and discussed in [16,15,17], which generalises the standard exponential $e^t = \sum_{n=0}^{+\infty} \frac{t^n}{\Gamma(n+1)}$. It can in its turn be generalised [25,26] as

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{+\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n + \beta)} \tag{2}$$

which is the two-parameter MLf. Its main properties and applications can be found in [5] and in chapter 18 of [4]; it is of great importance in Fractional Calculus (and thus in the study of dynamic systems of fractional order) [24]. It also appears when studying related fields such as Lévy flights, random walks, viscoelasticity, or superdiffusive transport. Further generalisations of the MLf to three and more parameters (up to ten) have also been used [8], but are not needed in what follows.

The computation of the MLf is not trivial since it poses numerical problems that may compromise the result; several strategies are known to deal with such numerical problems whenever they appear [10]. One of them is the use of asymptotic approximations.

2.2. The Mittag-Leffler function in control systems

Consider function

$$e_\alpha(t) = E_\alpha(-t^\alpha) = E_{\alpha,1}(-t^\alpha) = \sum_{n=0}^{+\infty} (-1)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad t > 0, \quad 0 < \alpha < 1 \tag{3}$$

which is a particular case of the MLf often appearing in control applications.

In fact, for the elementary fractional-order control system represented in Fig. 1, where s represents the Laplace variable, when the reference input is a unit step $r = 1,\ t \geqslant 0$ the output results $c(t) = e_\alpha(t) = E_\alpha(-t^\alpha)$.

2.3. Asymptotic approximations of the Mittag-Leffler function

Function $e_\alpha(t)$ is often computed using the two following approximations:

$$e_\alpha(t) \approx e_\alpha^0(t) = e^{-\frac{t^\alpha}{\Gamma(1+\alpha)}}, \quad t \approx 0 \tag{4}$$

$$e_\alpha(t) \approx e_\alpha^\infty(t) = \frac{t^\alpha}{\Gamma(1-\alpha)}, \quad t \gg 0 \tag{5}$$

Recently the two following alternative rational approximations were proposed [12] and proved [23]:

$$e_\alpha(t) \approx f_\alpha(t) = \frac{1}{1 + \frac{t^\alpha}{\Gamma(1+\alpha)}}, \quad t \approx 0 \tag{6}$$

$$e_\alpha(t) \approx g_\alpha(t) = \frac{1}{1 + t^\alpha \Gamma(1-\alpha)}, \quad t \gg 0 \tag{7}$$

The three functions $e_\alpha(t),\ f_\alpha(t)$ and $g_\alpha(t)$ are shown in Fig. 2.

In this study we propose to approximate $e_\alpha(t)$ interpolating $f_\alpha(t)$ and $g_\alpha(t)$:

$$e_\alpha(t) \approx h_\alpha(t) = \phi_\alpha(t) f_\alpha(t) + (1 - \phi_\alpha(t)) g_\alpha(t) \tag{8}$$

Here $\phi_\alpha(t)$ is a weight function. We will find an explicit expression for $\phi_\alpha(t)$ and show why $h_\alpha(t)$ is a good approximation of $e_\alpha(t)$.

3. Approximation and numerical calculation

3.1. Determining the weight function $\phi_\alpha(t)$

Function $e_\alpha(t)$ was calculated using Matlab and the routine in [20] for reference purposes. Weights $\phi_\alpha(t)$ were determined for $\alpha \in]0, 1[$ with a step of $\Delta\alpha = 0.01$, and for $t \in [10^{-5}, 10^5]$ with 20 logarithmically-spaced points per decade. They are shown in Fig. 3: those for $\alpha < 0.5 \wedge t > 1$ were not further considered because numerical instability arises spoiling the results.

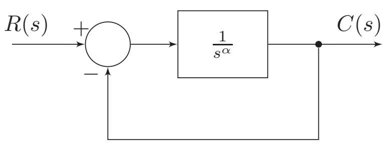


Fig. 1. Elementary fractional-order control system.

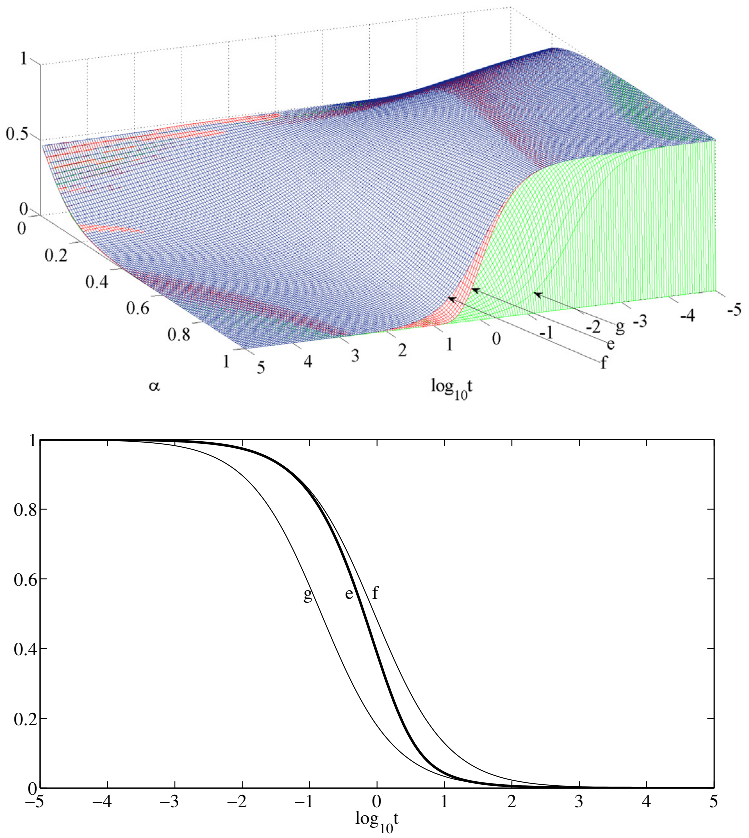


Fig. 2. Function $e_\alpha(t)$ and its approximations $f_\alpha(t)$ and $g_\alpha(t)$ for $0 < \alpha < 1$ (left) and for the particular case $\alpha = 0.8$ (right).

Cuts of this surface for constant values of α can be approximated by functions of the type

$$\phi_\alpha(t) = \frac{1}{1 + e^{-x_1(\alpha)\log_{10}t + x_2(\alpha)}} \tag{9}$$

The curve fitting was performed using the Nelder–Mead simplex method. Parameters x_1 and x_2 are seen to depend on the value of α as third-order polynomials. The final approximate expressions for $x_1(\alpha)$ and $x_2(\alpha)$ yield:

$$x_1(\alpha) = -3.0438\alpha^3 + 2.2634\alpha^2 - 1.749\alpha + 0.033976 \tag{10}$$

$$x_2(\alpha) = -0.35668\alpha^3 + 0.43597\alpha^2 - 0.61079\alpha + 0.012472 \tag{11}$$

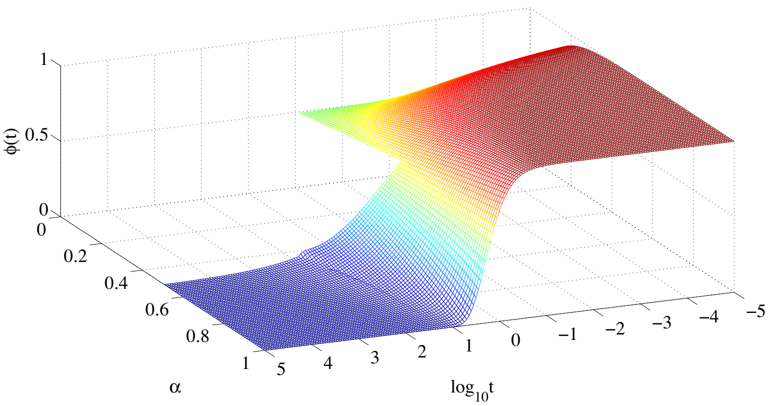


Fig. 3. Values of $\phi_\alpha(t)$ calculated numerically.

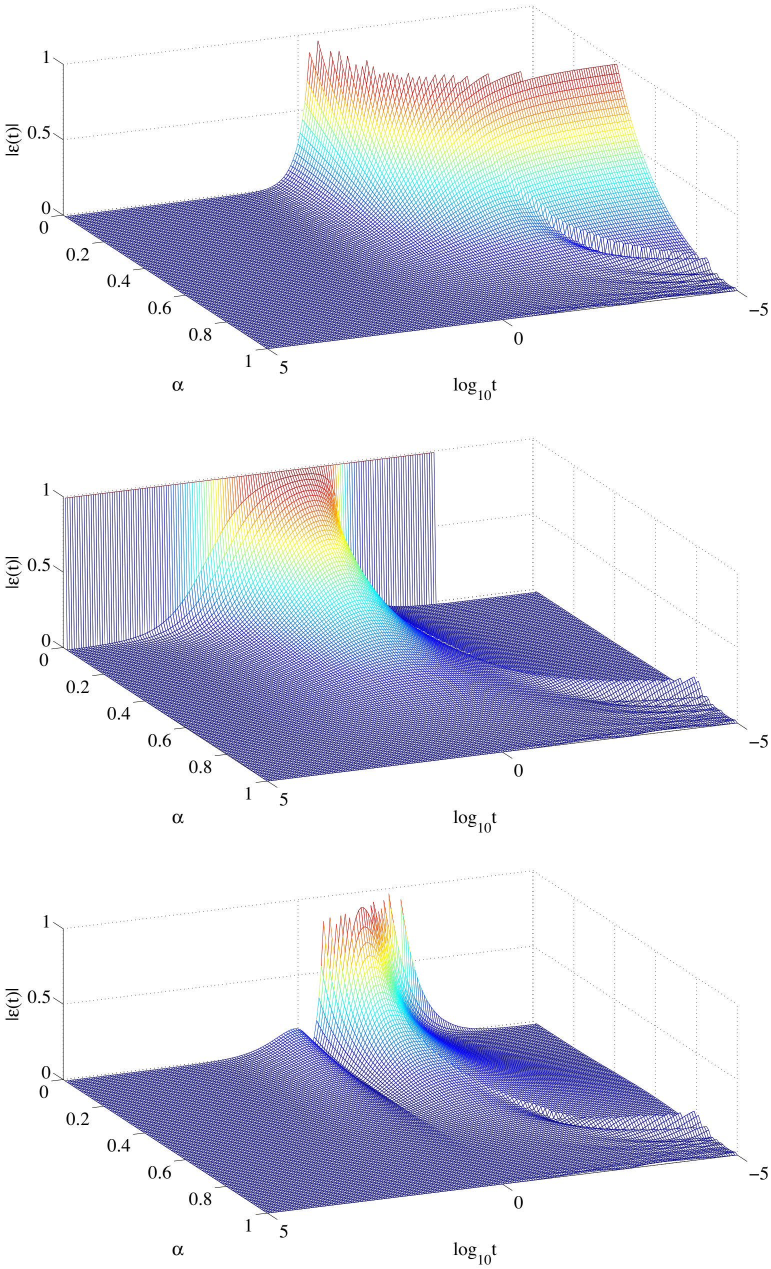
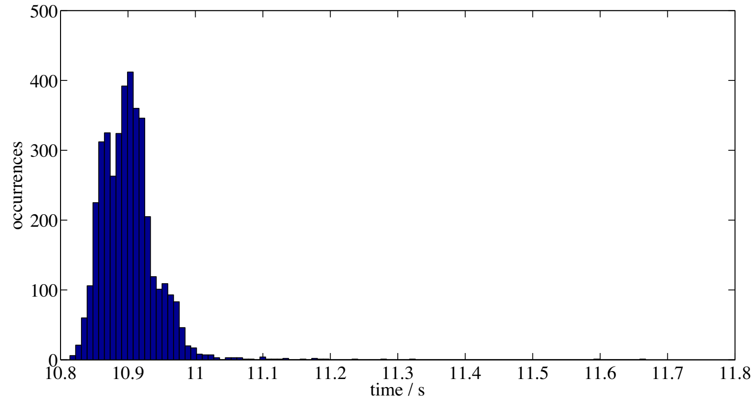


Fig. 4. Absolute values of the relative error, $|\epsilon(t)| = \left|\frac{\epsilon_\alpha(t)}{e_\alpha(t)}\right|$, calculated numerically for $f_\alpha(t)$ (top left), $g_\alpha(t)$ (top right) and $h_\alpha(t)$ (bottom). Points where the absolute relative error is larger than 1 are not shown.

Table 1
Time to calculate functions $e_\alpha(t)$ and $h_\alpha(t)$ 20,100 times.

	$e_\alpha(t)$	$h_\alpha(t)$
Average/s	10.9016	0.0048
Std. dev./s	0.0438	0.0002
Minimum/s	10.8139	0.0047
Maximum/s	11.6674	0.0105



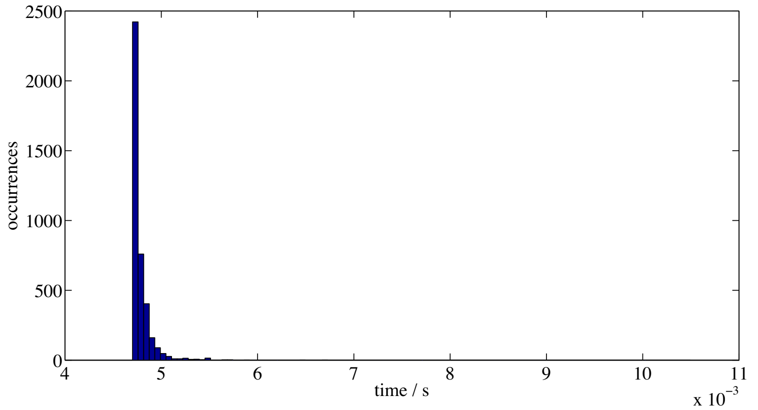


Fig. 5. Histograms of the time to calculate functions $e_\alpha(t)$ (left) and $h_\alpha(t)$ (right) 20,100 times, using 100 equally spaced containers.

3.2. Computational performance

Expressions (8)–(11) define $h_\alpha(t)$, which is an approximation of $e_\alpha(t)$. This approximation $e_\alpha(t) \approx h_\alpha(t)$ presents good results on two accounts: first, errors committed by $h_\alpha(t)$ are much inferior to those of the approximations $f_\alpha(t)$ and $g_\alpha(t)$ on which it is based; second, it can be computed much faster than $e_\alpha(t)$.

Concerning errors, given by

$$\epsilon_\alpha(t) = e_\alpha(t) - h_\alpha(t) \tag{12}$$

for $h_\alpha(t)$, we have $\max_{\alpha,t}|\epsilon_\alpha(t)| = 0.0861$, against 0.2036 for $f_\alpha(t)$, and 1.0 for $g_\alpha(t)$. Absolute relative errors

$$|\epsilon(t)| = \left|\frac{\epsilon_\alpha(t)}{e_\alpha(t)}\right| \tag{13}$$

are shown in Fig. 4; it can be seen that results are also far superior for $h_\alpha(t)$.

As to computational speed, function $h_\alpha(t)$ runs rather faster than function $e_\alpha(t)$: hence its usefulness in applications where many values or $e_\alpha(t)$ have to be computed in real time. To exemplify this, the two functions were calculated for 100 values of α (from $\alpha = 0.01$ to $\alpha = 1$, with a spacing of $\Delta\alpha = 0.01$) and 201 values of t (logarithmically spaced from $t = 10^{-5}$ to $t = 10^5$); these are actually the values shown in the first plot of Fig. 2. These 20,100 calculations were repeated 4000 times in a Pentium@2.10 GHz computer running Windows 7 and Matlab R2010b; the statistics for the time each of these 4000 iterations took to run are presented in Table 1 and Fig. 5. Notice that calculating $h_\alpha(t)$ is in average 2277 times faster than calculating $e_\alpha(t)$.

4. Conclusions

The MLf is an important function in mathematics, numerical calculus, engineering and applied sciences that are studied with the formalism of FC. Currently new phenomena are discovered and analysed adopting the FC perspective and requiring efficient computational schemes for expressions involving the MLf. This paper joined two recently proposed asymptotic expressions for deriving a fast numerical procedure useful in expressions common in fractional order control algorithms.

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