# Computational solutions of unified fractional reaction-diffusion equations with composite fractional time derivative

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#### Abstract

This paper deals with the investigation of the computational solutions of an unified fractional reaction-diffusion equation, which is obtained from the standard diffusion equation by replacing the time derivative of first order by the generalized fractional time-derivative defined by Hilfer (2000), the space derivative of second order by the Riesz-Feller fractional derivative and adding the function  $\phi(x,t)$  which is a nonlinear function governing reaction. The solution is derived by the application of the Laplace and Fourier transforms in a compact and closed form in terms of the H-function. The main result obtained in this paper provides an elegant extension of the fundamental solution for the space-time fractional diffusion equation obtained earlier by Mainardi et al. (2001, 2005) and a result very recently given by Tomovski et al. (2011). Computational representation of the fundamental solution is also obtained explicitly. Fractional order moments of the distribution are deduced. At the end, mild extensions of the derived results associated with a finite number of Riesz-Feller space fractional derivatives are also discussed.

**Keywords**: Mittag-Leffler function, Riesz-Feller fractional derivative, H-function, Riemann-Liouville fractional derivative, Caputo derivative, Laplace transform, Fourier transform, Riesz derivative.

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### 1. Introduction

Standard (not fractional) reaction-diffusion equations are an important class of partial differential equations to investigate nonlinear behavior of complex systems. Standard nonlinear reaction-diffusion equations can be simulated by numerical techniques, such as finite difference methods. The reaction-diffusion equation takes into account particle diffusion (diffusion constant and spatial Laplacian operator) and particle reaction (reaction constants and nonlinear reactive terms). Well-known special cases of such standard reaction-diffusion equations are the (i) Schloegl model, (ii) Fisher-Kolmogorov equation, (iii) real and complex Ginzburg-Landau equations, (iv) FitzHugh-Nagumo model, and (v) Gray-Scott model. These equations are known under their respective names both in the mathematical and physical literature. The nontrivial behavior of these equations arises from the competition between the reaction/relaxation and diffusion/transport. In recent years, interest is developed by several authors in the applications of reaction-diffusion models in pattern formation in physical and biological sciences and for describing non-Gaussian, non-Markovian, and non-Fickian phenomena in complex systems. In this connection, one can refer to Murray (2003), Kuramoto (2003), Wilhelmsson and Lazzaro (2001), and Hundsdorfer and Verwer (2003). These systems show that

diffusion can produce the spontaneous formation of spatio-temporal patterns. For details, see the work of Nicolis and Prigogine (1977) and Haken (2004). A general model for reaction-diffusion systems is investigated by Henry and Wearne (2000, 2002) and Henry et al. (2005), and Haubold et al. (2007, 2011, 2012).

In this paper, we investigate the solution of an unified model of reaction-diffusion system (3.1) in which the two-parameter fractional derivative  ${}_{0}D^{\mu,\nu}_{0+}$  acts as a time-derivative and the Riesz-Feller derivative  ${}_{x}D^{\alpha}_{\theta}$  as the space-derivative. This new model provides an extension of the models discussed earlier by Jesperson et al. (1999), Del-Castillo-Negrete et al. (2003), Mainardi et al. (2001, 2005), Kilbas et al. (2004), Haubold et al. (2007), Saxena (2012), Saxena et al. (2010), Saxena et al. (2006 a,b,c,d), and Tomovski et al. (2001). The results are obtained in a compact form, which are suitable for numerical computation. Computational representation of the fundamental solution is explicitly derived and fractional order moments are investigated. For recent and related works on fractional kinetic equations and reaction-diffusion problems, one can refer to papers by Haubold and Mathai (1995, 2000) and Saxena et al. (2002, 2004a,b,c, 2006a,d, 2008).

# 2. Unified fractional reaction-diffusion equations

In this section, we will investigate the solution of the unified reaction-diffusion model (2.1). The main result is given in the form of the following

Theorem 2.1. Consider an unified fractional reaction-diffusion model

$$_{0}D_{t}^{\mu,\nu}N(x,t) = \eta_{x}D_{\theta}^{\alpha}N(x,t) + \phi(x,t)$$
 (2.1)

where  $\eta, t > 0, x \in R; \alpha, \theta, \beta$  are real parameters with the constraints  $0 < \alpha \leq \min(\alpha, 2 - \alpha), {}_{0}D_{t}^{\mu,\nu}$  is the generalized Riemann-Liouville fractional derivative operator defined by (A7) with the initial conditions

$$[I_{0_{+}}^{(1-\nu)(1-\mu),0}]N(x,0_{+}) = N_{0}(x); \ 0 < \mu < 1, \ 0 \le \nu \le 1$$
(2.2)

involving the Riemann-Liouville fractional integral of order  $(1 - \nu)(1 - \mu)$  evaluated for  $t \to 0_+$  and  $\lim_{|x| \to \infty} N(x,t) = 0$ . Here  $_xD^{\alpha}_{\theta}$  is the Riesz-Feller space-fractional derivative of order  $\alpha$  and asymmetry  $\theta$  defined by (A11),  $\eta$  is a diffusion constant and  $\phi(x,t)$  is a nonlinear function belonging to the area of reaction-diffusion. Then for the solution of (2.1) subject to the above constraints, there holds the formula

$$N(x,t) = \int_0^x G_1(x-\tau,t)N_0(\tau)d\tau + \int_0^t (t-\xi)^{\beta-1} \int_0^\infty G_2(x-\tau,t-\xi)\phi(\tau,\xi)d\tau d\xi$$
 (2.3)

where

$$G_1(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(\mu+\nu(1-\mu),\frac{\mu}{\alpha}),(1,\rho)} \right]$$
(2.4)

 $\alpha > 0$  and

$$G_2(x,t) = \frac{1}{\alpha |x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(\mu,\frac{\mu}{\alpha}),(1,1),(1,\rho)} \right]$$
(2.5)

for  $\alpha>0$  where  $H_{3,3}^{2,1}$  is the familiar H-function [Mathai and Saxena (1978), Kilbas and Saigo (2004) and Mathai et al. (2010)],  $\Re(\mu)>0$ ,  $\Re(\mu+\nu(1-\mu))>0$ .

**Proof:** If we apply the Laplace transform (Erdélyi, et al. 1954) with respect to the time variable t and Fourier transform with respect to space variable x and use the initial conditions and the formulas (A8) and (A11), then the given equation transforms into the form

$$s^{\mu}\tilde{N}^{*}(k,s) - s^{\nu(\mu-1)}N_{0}^{*}(k) = -\eta \ \psi_{\alpha}^{\theta}(k)\tilde{N}^{*}(k,s) + \phi^{*}(k,s), \tag{2.6}$$

where according to the conventions followed, the symbol  $\sim$  will stand for the Laplace transform with respect to time variable t and \* represents the Fourier transform with respect to space variable x. Solving for  $\tilde{N}^*(k,s)$ , it yields

$$\tilde{N}^*(k,s) = \frac{N_0^*(k)s^{\nu(\mu-1)}}{s^{\mu} + \eta \,\psi_{\alpha}^{\theta}(k)} + \frac{\tilde{\phi}^*(k)}{s^{\mu} + \eta \,\psi_{\alpha}^{\theta}(k)}.$$
(2.7)

On taking the inverse Laplace transform of (2.7) and applying the formula

$$L^{-1}\left\{\frac{s^{\beta-1}}{a+s^{\alpha}}\right\} = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}),\tag{2.8}$$

where  $\Re(s) > 0, \Re(\alpha) > 0, \Re(\alpha - \beta) > -1$ , it is seen that

$$N^{*}(k,t) = N_{0}^{*}(k)t^{\mu+\nu(1-\mu)-1}E_{\mu,\mu+\nu(1-\mu)}(-\eta \ t^{\mu}\psi_{\alpha}^{\theta}(k))$$
$$+ \int_{0}^{t} \phi^{*}(k,t-\xi)\xi^{\mu-1}E_{\mu,\mu}(-\eta \ \psi_{\alpha}^{\theta}(k)\xi^{\mu})d\xi.$$
(2.9)

Taking the inverse Fourier transform of (2.9), we find

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} N_0^*(k) E_{\mu,\mu+\nu(1-\mu)}(-\eta \ t^{\mu}\psi_{\alpha}^{\theta}(k)) \exp(-ikx) dk + \frac{1}{2\pi} \int_0^t \xi^{\mu-1} \int_{-\infty}^{\infty} \phi^*(k,t-\xi) E_{\mu,\mu}(-\eta t^{\mu}\psi_{\alpha}^{\theta}(k)) \exp(-ikx) dk \ d\xi.$$
 (2.10)

If we now apply the convolution theorem of the Fourier transform to (2.10) and make use of the following inverse Fourier transform formula (Haubold et al. 2007):

$$F^{-1}[E_{\beta,\gamma}(-\eta t^{\beta}\psi_{\alpha}^{\theta}(k);x] = \frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}\right],$$
(2.11)

where  $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$  and  $\rho = (\frac{\alpha - \theta}{2\alpha})$ , it gives the desired solution in the form

$$N(x,t) = \int_0^x G_1(x-\tau,t)N_0(\tau)d\tau + \int_0^t (t-\xi)^{\beta-1} \int_0^x G_2(x-\tau,t-\xi)\phi(\tau,\xi)d\tau d\xi$$
 (2.12)

where

$$G_{1}(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\mu,\mu+\nu(1-\mu)}(-\eta t^{\mu} \psi_{\alpha}^{\theta}(k)) dk$$

$$= \frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,1),(1,\rho)} \right]$$
(2.13)

for  $\alpha > 0$  and

$$G_{2}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\mu,\mu}(-\eta t^{\mu} \psi_{\alpha}^{\theta}(k)) dk$$

$$= \frac{1}{\alpha |x|} H_{3.3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,1),(1,\rho)} \right], \qquad (2.14)$$

where  $\Re(\alpha) > 0, \Re(\mu) > 0, \Re(\mu + \nu(1-\mu)) > 0$ . This completes the proof of the theorem.

## 3. Special cases

When  $\theta = 0$ , the Riesz-Feller space derivative reduces to the Riesz space fractional derivative, and consequently we arrive at the following result recently given by Tomovski et al. (2011).

Corollary 3.1. The solution of extended fractional reaction-diffusion equation

$$_{0}D_{0\perp}^{\mu,\nu}N(x,t) - \eta _{x}D_{0}^{\alpha}N(x,t) = \phi(x,t), \ x \in R, \ t > 0, \ \eta > 0$$
 (3.1)

with initial conditions

$$0 \le \nu \le 1, \ 0 < \mu < 1 \text{ and } 0 < \alpha \le 2, \lim_{x \to \pm \infty} N(x, t) = 0$$
 (3.2)

$$[I_{0_{\perp}}^{(1-\nu)(\mu-1),0}]N(x,0_{+}) = N_{0}(x)$$
(3.3)

where  $\eta$  is a diffusion constant,  ${}_{0}D_{0+}^{\mu,\nu}$  is the generalized Riemann-Liouville fractional derivative operator defined by (A7),  ${}_{x}D_{0}{}^{\alpha}$  is the Riesz space fractional derivative operator defined by (A15) and  $\phi(x,t)$  is a nonlinear function belonging to the area of reaction-diffusion, is given by

$$N(x,t) = \int_0^x G_3(x-\tau,t)N_0(\tau)d\tau + \int_0^t (t-\xi)^{\beta-1} \int_0^x G_4(x-\tau,t-\xi)\phi(\tau,\xi)d\tau d\xi,$$
 (3.4)

where

$$G_3(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(\mu+\nu(1-\mu),\frac{\mu}{\alpha}),(1,\frac{1}{2})} \right]$$
(3.5)

for  $\alpha > 0$  and

$$G_4(x,t) = \frac{1}{\alpha|x|} H_{3.3}^{2.1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(1,1),(1,\frac{1}{2})} \right], \quad \alpha > 0.$$
 (3.6)

**Note 3.1.** Expressions for  $G_3$  and  $G_4$  can be obtained from  $G_1$  and  $G_2$  respectively by taking  $\rho = \frac{1}{2}$ .

Similarly, if we set  $N_0(x) = \delta(x)$ ,  $\phi = 0$ , where  $\delta(x)$  is the Dirac delta function, then the theorem reduces to the following interesting result:

Corollary 3.2. Consider the following reaction-diffusion model

$$_{0}D_{0\perp}^{\mu,\nu}N(x,t) = \eta_{x}D_{\theta}^{\alpha}N(x,t), \ \eta > 0, \ x \in R, \ 0 < \alpha \le 2,$$
 (3.7)

with the initial conditions

$$(I_{0_{+}}^{(1-\nu)(\mu-1),0})N(x,0_{+}) = \delta(x), \ 0 < \mu < 1, \ 0 \le \nu \le 1, \ \lim_{x \to +\infty} N(x,t) = 0$$
(3.8)

where  $\eta$  is a diffusion constant and  $\delta(x)$  is the Dirac delta function. Then for the fundamental solution of (3.7) with the initial conditions (3.8), there holds the formula

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,1),(1,\rho)} \right]$$
(3.9)

for  $\alpha > 0$  where  $\rho = \frac{\alpha - \theta}{2\alpha}$ .

Some interesting special cases of (3.9) are enumerated below.

- (i): When  $\nu = 1$ , then the operator  $D_{0+}^{\mu,\nu}$  reduces to Caputo fractional derivative  ${}_0^C D_t^{\mu}$  defined in (A5) and the result (3.9) yields the fundamental solution of the space-time fractional diffusion equation obtained by Mainardi et al. (2001) and Mainardi et al. (2005).
- (ii): **Neutral fractional diffusion**. We note that for  $\mu = \alpha$ , the solution (3.9) reduces to

$$N_{\alpha,\nu}^{\theta}(x,t) = \frac{t^{\alpha+\nu(1-\alpha)-1}}{\alpha|x|} H_{3,3}^{2,1} \left[ \frac{|x|}{(t \eta^{\frac{1}{\alpha}})} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(\alpha+\nu(1-\alpha),1),(1,\rho)} \right]. \tag{3.10}$$

If we further set  $\nu = 1$ , then the time-fractional operator becomes the Caputo operator and neutral fractional diffusion occurs. It will be denoted by the conventional symbol  $N_{\alpha}^{\theta}$ , (3.10) now simplifies into

$$N_{\alpha}^{\theta}(x,t) = \frac{1}{\alpha|x|} H_{2,2}^{1,1} \left[ \frac{|x|}{t \eta^{\frac{1}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,\rho)} \right], \tag{3.11}$$

which can be expressed in terms of a Mellin-Barnes type integral as

$$N_{\alpha}^{\theta}(x,t) = \frac{1}{\pi \alpha x} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha}) \sin[\frac{\pi}{2} \frac{s}{\alpha} (\alpha - \theta)] (x/t)^{s} ds, \tag{3.12}$$

 $(\eta = 1$ , taken for simplicity). If the poles of the gamma functions occurring in the integrand of (3.12) are all simple, then evaluating the integral as a sum of the residues at the simple poles of  $\Gamma(s/\alpha)$  at the points  $s = -\alpha n$ ,  $n \in N_0$  and  $\Gamma(1 - s/\alpha)$  at the points  $s = \alpha + \alpha n$ ,  $n \in N_0$  we obtain the series representations

$$N_{\alpha}^{\theta}(x,t) = \frac{1}{\pi x} \sum_{n=0}^{\infty} \sin[\frac{n\pi}{2}(\theta - \alpha)](-x/t)^{\alpha n}, \quad 0 < x < 1$$
 (3.13)

and

$$N_{\alpha}^{\theta}(x,t) = \frac{1}{\pi x} \sum_{n=0}^{\infty} \sin\left[\frac{n\pi}{2}(\theta - \alpha)^{-\alpha n}\right], \quad 1 < x < \infty.$$
(3.14)

Following the procedure adopted in Gorenflo and Mainardi (1997) and making use of the formula

$$\sum_{n=1}^{\infty} r^n \sin(n\alpha) = \Im\left[\sum_{n=1}^{\infty} r^n \exp(in\alpha)\right] = \Im\left[\frac{r \exp(i\alpha)}{1 - r \exp(i\alpha)}\right]$$
$$= \frac{r \sin \alpha}{1 - 2r \cos \alpha + r^2}, \text{ for } |r| < 1$$
(3.15)

where  $\alpha \in R$  and it yields the interesting result given by Mainardi et al. (2001, 2005)

$$N_{\alpha}^{\theta}(x,t) = \frac{1}{t\pi} \frac{y^{\alpha-1} \sin[\frac{\pi}{2}(\alpha - \theta)]}{1 + 2y^{\alpha} \cos[\frac{\pi}{2}(\alpha - \theta)] + y^{2\alpha}}, \quad y = \frac{x}{t}, 0 < x < \infty, 0 < \alpha \le 2.$$
 (3.16)

Next, we derive some stable densities in terms of the H-functions as special cases of the solution (3.12).

(iii): If we set  $\mu = \nu = 1$ ,  $0 < \alpha < 2$ ,  $\theta \le \min\{\alpha, 2 - \alpha\}$  then (3.7) reduces to space-fractional diffusion equation, which we denote by  $L_{\alpha}^{\theta}(x)$ , is the fundamental solution of the following space-time fractional diffusion model:

$$\frac{\partial N(x,t)}{\partial t} = \eta \ _x D_{\theta}^{\alpha} N(x,t), \ \eta > 0, x \in R, \tag{3.17}$$

with the initial conditions  $N(x, t = 0) = \delta(x)$ ,  $\lim_{x \to \pm \infty} N(x, t) = 0$ , where  $\eta$  is the diffusion constant and  $\delta(x)$  is the Dirac delta function. Hence for the fundamental solution of (3.17) there holds the formula

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1} \left[ \frac{(\eta t)^{\frac{1}{\alpha}}}{|x|} \Big|_{(1,\frac{1}{\alpha}),(1,\rho)}^{(1,1),(1,\rho)} \right], \ 0 < \alpha < 1, \ |\theta| \le \alpha, \tag{3.18}$$

where  $\rho = \frac{\alpha - \theta}{2\alpha}$ . The density represented by the above expression is known as  $\alpha$ -stable Lévy density. By virtue of the H-function formula (Mathai et al., 2010) another form of this density is given by

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1} \left[ \frac{|x|}{(\eta t)^{\frac{1}{\alpha}}} \Big|_{(0,1),(0,\rho)}^{(0,\frac{1}{\alpha}),(0,\frac{1}{\alpha})} \right], \ 1 < \alpha \le 2, \ |\theta| \le 2 - \alpha.$$
 (3.19)

Remark 3.1. A general representation of all stable distributions in terms of special functions has been given by Schneider (1986). It is shown that the stable probability functions can be expressed by means of the H-functions, also see Uchaikin and Zolotarev (1999). We further note that Feller (1952) had derived the representations of the stable probability functions in terms of convergent and asymptotic power series in 1952. His results are revisited by Schneider, who had shown in case of  $L_{\alpha}^{\theta}(x)$  to restrict our attention to x > 0, since the evaluation for x < 0 can be done by using the symmetry property  $L_{\alpha}^{\theta}(-x) = L_{\alpha}^{-\theta}(x)$ .

(iv): Next, if we take  $\alpha = 2$ ,  $0 < \mu < 1$ ,  $\nu = 1$ ,  $\theta = 0$  then we obtain the time-fractional diffusion, which is governed by the following time fractional diffusion model:

$$\frac{\partial^{\mu}}{\partial t^{\mu}}N(x,t) = \eta \frac{\partial^{2}}{\partial x^{2}}N(x,t), \ \eta > 0, \ x \in R, \tag{3.20}$$

with the initial conditions  $N(x, t = 0) = \delta(x)$ ,  $\lim_{x \to \pm \infty} N(x, t) = 0$  where  $\eta$  is a diffusion constant and  $\delta(x)$  is the Dirac delta function, whose fundamental solution is given by the equation

$$N(x,t) = \frac{1}{2|x|} H_{1,1}^{1,0} \left[ \frac{|x|}{(\eta t)^{\frac{1}{2}}} \Big|_{(1,1)}^{(1,\frac{\mu}{2})} \right]. \tag{3.21}$$

(v): Further, if we set  $\alpha=2, \mu=\nu=1$  and  $\theta\to 0$  then for the fundamental solution of the standard diffusion equation

$$\frac{\partial}{\partial t}N(x,t) = \eta \frac{\partial^2}{\partial x^2}N(x,t), \tag{3.22}$$

with initial conditions

$$N(x, t = 0) = \delta(x), \lim_{x \to \pm \infty} N(x, t) = 0,$$
 (3.23)

there holds the formula

$$N(x,t) = \frac{1}{2|x|} H_{1,1}^{1,0} \left[ \frac{|x|}{(\eta t)^{\frac{1}{2}}} \Big|_{(1,1)}^{(1,\frac{1}{2})} \right] = (4\pi\eta t)^{-\frac{1}{2}} \exp\left[-\frac{|x|^2}{4\eta t}\right], \tag{3.24}$$

which is the classical Gaussian density. For further details and importance of these special cases based on the Green function, one can refer to the papers by Mainardi et al. (2001, 2005). For  $\nu=0$  the fractional derivative  ${}_{0}D_{0+}^{\mu,\nu}$  reduces to Riemann-Liouville fractional derivative operator  ${}_{0}^{RL}D_{t}^{\mu}$ , defined by (A3), and the theorem yields

Corollary 3.3. Consider an extended fractional reaction-diffusion model

$${}_{0}^{RL}D_{t}^{\mu}N(x,t) = \eta_{x}D_{\theta}^{\alpha}N(x,t) + \phi(x,t)$$
(3.25)

where  $\eta, t > 0, x \in R; \alpha, \theta, \mu$  are real parameters with the constraints

$$0 < \alpha \le 2, |\theta| \le \min(\alpha, 2 - \alpha), \ 0 < \alpha \le 2 \tag{3.26}$$

where the Riemann-Liouville operator of order  $\mu$  defined by (A3) has the initial conditions

$${}_{0}^{RL}D_{t}^{\mu-1}N(x,0) = N_{0}(x); \ {}_{0}^{RL}D_{t}^{\mu-2}N(x,0) = 0, 0 < \mu \le 2, \lim_{|x| \to \pm \infty} N(x,t) = 0.$$
 (3.27)

and  $_xD_\theta^\alpha$  is the Riesz-Feller space-fractional derivative or order  $\alpha$  and asymmetry  $\theta$  defined by (A11),  $\eta$  is a diffusion constant and  $\phi(x,t)$  is a nonlinear function. Then for the solution of (3.25) subject to the above constraints, there holds the formula

$$N(x,t) = \int_0^x G_5(x-\tau,t)N_0(\tau)d\tau + \int_0^t (t-\xi)^{\beta-1} \int_0^x G_6(x-\tau,t-\xi)\phi(\tau,\xi)d\tau d\xi$$
 (3.28)

where

$$G_5(x,t) = \frac{t^{\mu-1}}{\alpha |x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,1),(1,\rho)} \right], \quad \alpha > 0$$
(3.29)

and

$$G_6(x,t) = \frac{1}{\alpha |x|} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,1),(1,\rho)} \right]$$
(3.30)

for  $\alpha > 0, \Re(\mu) > 0, \rho + \frac{\alpha - \theta}{2\alpha}$ .

#### 4. Fractional order moments

In this section we will calculate the fractional order moments defined by

$$\langle |x(t)|^{\delta} \rangle = \int_{-\infty}^{\infty} |x|^{\delta} N(x,t) dx$$
 (4.1)

Using the result (3.9) and the following definition of the Mellin transform

$$M\{f(t); s\} = \int_0^\infty t^{s-1} f(t) dt$$
 (4.2)

we find

$$\langle |x|^{\delta} \rangle = \int_{-\infty}^{\infty} |x|^{\delta} N(x,t) dx$$
 (4.3)

$$= \frac{2t^{\mu+\nu(1-\mu)-1}}{\alpha} \int_0^\infty x^{\delta-1} H_{3,3}^{2,1} \left[ \frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\mu}{\alpha}}} \Big|_{(1,1),(1,\frac{1}{\alpha}),(1,\rho)}^{(1,\frac{1}{\alpha}),(\mu+\nu(1-\mu),\frac{\mu}{\alpha}),(1,\rho)} \right] dx \tag{4.4}$$

The following formula gives the Mellin transform of the H-function (Mathai et al., 2010):

$$\int_{0}^{\infty} x^{\delta - 1} H_{p,q}^{m,n} \left[ ax \Big|_{(b_{q},B_{q})}^{(a_{p},A_{p})} \right] dx = a^{-\delta} \frac{\{ \prod_{j=1}^{m} \Gamma(b_{j} + B_{j}\delta) \} \{ \prod_{j=1}^{m} \Gamma(1 - a_{j} - A_{j}\delta) \}}{\{ \prod_{j=m+1}^{q} \Gamma(1 - b_{j} - B_{j}\delta) \} \{ \prod_{j=n+1}^{p} \Gamma(a_{j} + A_{j}\delta) \}}$$
(4.5)

where  $-\min_{1\leq j\leq m}\Re(\frac{b_j}{B_j})<\Re(\delta)<\max_{1\leq j\leq n}\Re(\frac{1-a_j}{A_j}), |\arg a|<\frac{1}{2}\pi\theta, \ \theta=\sum_{j=1}^m B_j-\sum_{j=m+1}^q B_j+\sum_{j=1}^n A_j-\sum_{j=n+1}^p A_j>0.$  Applying the above formula to evaluate the integral in (4.5), we see that

$$\langle |x(t)|^{\delta} \rangle = \frac{2}{\alpha} \eta^{\frac{\delta}{\alpha}} t^{\mu + \nu(1-\mu) + \mu(\frac{\delta}{\alpha}) - 1} \frac{\Gamma(-\frac{\delta}{\alpha})\Gamma(1+\delta)\Gamma(1+\frac{\delta}{\alpha})}{\Gamma(-\rho\delta)\Gamma(\mu + \nu(1-\mu) + \frac{\mu\delta}{\alpha})\Gamma(1+\rho\delta)}$$
(4.6)

for  $-\min\{\alpha, 1\} < \Re(\delta) < 0$ .

#### 5. Computational representations of the solution (3.9)

In this section we will derive the computational representation of the fundamental solution (3.9), which can be expressed in terms of the Mellin-Barnes type integral as

$$N(x,t) = \frac{1}{\pi x} \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi i} \int_{L} \frac{\Gamma(s)\Gamma(1-s)\Gamma(1-s\alpha)}{\Gamma(\mu+\nu(1-\mu)-s\mu)} \sin\left[\frac{s\pi}{2\alpha}(\theta-\alpha)\right] \left[\frac{x^{\alpha}}{\eta t^{\mu}}\right]^{s} ds.$$
 (5.1)

Let us assume that the poles of the gamma functions in the integrand of (5.1) are all simple. Now, evaluating the sum of residues in ascending powers of x by calculating the residues at the poles of  $\Gamma(1-s)$  at the points  $s = 1 + n, n \in N_0$  and  $\Gamma(1-s\alpha)$  at the points  $s = (1+n)/\alpha$ ,  $n \in N_0$  we obtain the following representation of the fundamental solution (3.12) in terms of two convergent series in ascending powers of x

$$N(x,t) = \frac{x^{\alpha-1}t^{\mu+\nu(1-\mu)-1}}{\pi(\eta t^{\mu})} \sum_{n=0}^{\infty} \frac{\Gamma[1-\alpha(1+n)]}{\Gamma[\mu+\nu(1-\mu)-(1+n)\mu]} \sin\left[\frac{(1+n)}{2\alpha}(\theta-\alpha)\right] \left[\frac{-x^{\alpha}}{\eta t^{\mu}}\right]^{n} + \frac{t^{\mu+\nu(1-\mu)-1}}{\pi\alpha\eta^{\frac{1}{\alpha}}t^{\frac{\mu}{\alpha}}} \sum_{n=0}^{\infty} \frac{\Gamma[(1+n)/\alpha]\Gamma[1-(1+n)/\alpha]}{n!\Gamma[\mu+\nu(1-\mu)-(1+n)(\mu/\alpha)]} \sin\left[\frac{(1+n)\pi}{2\alpha^{2}}(\theta-\alpha)\right] \left[\frac{-x}{\eta^{\frac{1}{\alpha}}t^{\frac{\mu}{\alpha}}}\right]^{n}$$
(5.2)

where  $\left|\frac{x^{\alpha}}{\eta t^{\mu}}\right| < 1$ .

For  $\nu=1$  the time fractional derivative  $D_{0_+}^{\mu,\nu}$  becomes Caputo derivative  ${}_0^C D_t^{\mu}$  and we obtain the following simplified form of the result given by Mainardi et al. (2001):

$$N(x,t) = \frac{x^{\alpha-1}}{\pi(\eta t^{\mu})} \sum_{n=0}^{\infty} \frac{\Gamma[1-\alpha(1+n)]}{\Gamma[1-\mu(1+n)]} \sin[\frac{(1+n)\pi}{2\alpha}(\theta-\alpha)] [\frac{-x^{\alpha}}{\eta t^{\mu}}]^{n} + \frac{1}{\pi\alpha(\eta^{\frac{1}{\alpha}}t^{\frac{\mu}{\alpha}})} \sum_{n=0}^{\infty} \frac{\Gamma[(1+n)/\alpha]}{n!\Gamma[1-(1+n)(\mu/\alpha)]} \sin[\frac{(1+n)\pi}{2\alpha^{2}}(\theta-\alpha)] [\frac{-x}{\eta^{\frac{1}{\alpha}}t^{\frac{\mu}{\alpha}}}]^{n}$$
(5.3)

Further, if we calculate the residues at the poles of  $\Gamma(s)$  at the points  $s=-n, n \in N_0$  it gives

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\pi x} \sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha n)}{\Gamma(\mu+\nu(1-\mu)+n\mu)} \sin\left[\frac{-n\pi}{2\alpha}(\theta-\alpha)\right] \left[\frac{-x^{\alpha}}{\eta t^{\mu}}\right]^{-n}, \ 1 < x < \infty.$$
 (5.4)

For  $\nu = 1$  the above result simplifies to the following series representation:

$$N(x,t) = \frac{1}{x\pi} \sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha n)}{\Gamma(1+\mu n)} \sin[\frac{-n\pi}{2\alpha}(\theta-\alpha)] [-\frac{x^{\alpha}}{\eta t^{\mu}}]^{-n}, \ 1 < x < \infty.$$
 (5.5)

Finally, from (5.5) it follows that  $N(x,t) \sim \frac{1}{|x|}$  for large |x|.

## 6. Finite number of Riesz-Feller derivatives

Following a similar procedure it is not difficult to establish the following

**Theorem 6.1.** Consider an unified fractional reaction-diffusion model

$${}_{0}D_{t}^{\mu,\nu}N(x,t) = \sum_{j=1}^{m} \eta_{j} {}_{x}D_{\theta_{j}}^{\alpha_{j}}N(x,t) + \phi(x,t)$$

$$(6.1)$$

where  $\eta_j, t > 0, x \in R; \alpha_j, \theta_j, j = 1, ..., m, \mu, \nu$  are real parameters with the constraints  $0 < \alpha_j \le 2, |\theta_j| \le \min\{\alpha_j, 2 - \alpha_j\}, 0D_t^{\mu,\nu}$  is the generalized Riemann-Liouville fractional derivative operator defined by (A5) with the initial conditions

$$[I_{0_{+}}^{(1-\nu)(1-\mu),0}]N(x,0_{+}) = N_{0}(x); \ 0 < \mu < 1, \ 0 \le \nu \le 1$$
(6.2)

involving the Riemann-Liouville fractional integral of order  $(1-\nu)(1-\mu)$  evaluated for  $t\to 0_+$  and  $\lim_{|x|\to\infty}N(x,t)=0$ . Here  $_xD^{\alpha_j}_{\theta_j},j=1,...,m$  are the Riesz-Feller space fractional derivatives of orders  $\alpha_j,j=1,...,m$  and asymmetry  $\theta_j,j=1,...,m$  respectively, defined by (A10),  $\phi(x,t)$  is a nonlinear function. Then for the solution of (6.1), subject to the constraints above, there holds the formula

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} N_0^*(k) E_{\mu,\mu+\nu(1-\mu)}(-t^{\mu} \sum_{j=1}^m \eta \ \psi_{\alpha_j}^{\theta_j}(k)) \exp(-ikx) dk$$
$$+ \frac{1}{2\pi} \int_0^t \xi^{\mu-1} \int_{-\infty}^{\infty} \phi^*(k,t-\xi) E_{\mu,\mu}(-t^{\mu} \sum_{j=1}^m \eta_j \ \psi_{\alpha_j}^{\theta_j}(k)) \exp(-ikx) dk \ d\xi.$$
(6.3)

Some special cases of Theorem 6.1 are deduced below:

(i): If we set  $\theta_1 = ... = \theta_m = 0$  then by virtue of the identity (A13) we arrive at the following corollary associated with Riesz space fractional derivative:

Corollary 6.1. Consider the extended reaction-diffusion model

$${}_{0}D_{t}^{\mu,\nu}N(x,t) = \sum_{j=1}^{m} \eta_{j} {}_{x}D_{0}^{\alpha_{j}}N(x,t) + \phi(x,t)$$

$$(6.4)$$

where  $\eta_j > 0, j = 1, ..., m, t > 0, x \in R, \alpha_j, j = 1, ..., m, \mu, \nu$  are real parameters with the constraints  $0 < \alpha_j \le 2$ ,  $|\theta_j| \le \min(\alpha_j, 2 - \alpha_j)$ .  ${}_0D_t^{\mu,\nu}$  is the generalized Riemann-Liouville fractional derivative operator defined by (A7) with the initial conditions

$$[I_{0_{+}}^{(1-\nu)(1-\mu),0}]N(x,0_{+}) = N_{0}(x); \ 0 < \mu < 1, \ 0 \le \nu \le 1$$

$$(6.5)$$

involving the Riemann-Liouville fractional integral of order  $(1 - \nu)(1 - \mu)$  evaluated for  $t \to 0_+$  and  $\lim_{|x| \to \infty} N(x,t) = 0$ . Then for the solution of (6.4) there holds the formula (6.3) with  $\psi_{\alpha_j}^{\theta_j}(k)$  replaced by  $|k|^{\alpha_j}$ , j = 1, ..., m.

(ii): If we further take  $\nu=1$  in the above Corollary 6.1 then the operator  $[I_{0_+}^{(1-\nu)(1-\mu),0}]$  reduces to Caputo operator  ${}_0^CD_t^\mu$  defined in (A5) and we arrive at the following result:

Corollary 6.2. Consider the extended reaction-diffusion model

$${}_{0}^{C}D_{t}^{\mu}N(x,t) = \sum_{j=1}^{m} \eta_{j} {}_{x}D_{0}^{\alpha_{j}}N(x,t) + \phi(x,t)$$

$$(6.6)$$

where all the quantities are as defined above, with the initial condition  $N(x,0_+)=N_0(x);\ 0<\alpha_j\leq 2,\ 0<\mu<1,\ \lim_{|x|\to\infty}N(x,t)=0.\ _xD_0^{\alpha_j}, j=1,...,m$  are the Riesz space fractional derivatives of order  $\alpha_j,j=1,...,m$  defined by (A14),  $\phi(x,t)$  is a nonlinear function. Then for the solution of (6.6) there holds the formula

$$N(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} N_0^*(k) E_{\mu,1}(-t^{\mu} \sum_{j=1}^m \eta_j |k|^{\alpha_j}) \exp(-ikx) dk$$
$$+ \frac{1}{2\pi} \int_0^t \xi^{\mu-1} \int_{-\infty}^{\infty} \phi^*(k,t-\xi) E_{\mu,\mu}(-t^{\mu} \sum_{j=1}^m \eta_j |k|^{\alpha_j}) \exp(-ikx) dk \ d\xi$$
(6.7)

For m=1 the result (6.7) reduces to one given by Tomovski et al. (2011).

(iii): If we set  $\nu = 0$  then the Hilfer (2000) fractional operator defined by (A7) reduces to Riemann-Liouville operator defined by (A3) and we arrive at the following

Corollary 6.3. Consider an extended fractional reaction-diffusion model

$${}_{0}^{RL}D_{t}^{\mu}N(x,t) = \sum_{j=1}^{m} \eta_{j} {}_{x}D_{\theta_{j}}^{\alpha_{j}}N(x,t) + \phi(x,t)$$

$$(6.8)$$

where the parameters and restrictions as defined before and with the initial conditions

$${}^{RL}_{0}D^{\mu-1}_{t}N(x,0) = N_{0}(x); \quad {}^{RL}_{0}D^{\mu-2}_{t}N(x,0) = 0, \quad 0 < \mu \le 2, \lim_{|x| \to \infty} N(x,t) = 0.$$
 (6.9)

Then for the solution of (6.8) there holds the formula:

$$N(x,t) = \frac{t^{\mu-1}}{2\pi} \int_{-\infty}^{\infty} N_0^*(k) E_{\mu,\mu}(-t^{\mu} \sum_{j=1}^m \eta_j \ \psi_{\alpha_j}^{\theta_j}(k)) \exp(-ikx) dk$$
$$+ \frac{1}{2\pi} \int_0^t \xi^{\mu-1} \int_{-\infty}^{\infty} \phi^*(k,t-\xi) E_{\mu,\mu}(-t^{\mu} \sum_{j=1}^m \eta_j \ \psi_{\alpha_j}^{\theta_j}) \exp(-ikx) dk \ d\xi.$$
(6.10)

(iv): Finally, if we set  $\theta_j = 0, j = 1, ..., m$  in Corollary 6.3 then the Riesz-Feller derivatives reduce to Riesz space fractional derivative and we arrive at the following

Corollary 6.4. Consider an extended fractional reaction-diffusion model

$${}_{0}^{RL}D_{t}^{\mu}N(x,t) = \sum_{j=1}^{m} \eta_{j} {}_{x}D_{0}^{\alpha_{j}}N(x,t) + \phi(x,t)$$

$$(6.11)$$

with the parameters and conditions on them as defined before and with the initial conditions as in (6.7) then for the solution of (6.11) there holds the formula

$$N(x,t) = \frac{t^{\mu-1}}{2\pi} \int_{-\infty}^{\infty} N_0^*(k) E_{\mu,\mu}(-t^{\mu} \sum_{j=1}^m \eta_j |k|^{\alpha_j}) \exp(-ikx) dk$$
$$+ \frac{1}{2\pi} \int_0^t \xi^{\mu-1} \int_{-\infty}^{\infty} \phi^*(k,t-\xi) E_{\mu,\mu}(-t^{\mu} \sum_{j=1}^m \eta_j |k|^{\alpha_j}) \exp(-ikx) dk d\xi.$$
(6.13)

#### 7. Conclusions

In this paper, the authors have presented an extension of the fundamental solution of space-time fractional diffusion given by Mainardi-Luchko-Pagnini (2001) by using the fractional order derivative operator defined by Hilfer (2000). The fundamental solution of the equation (2.1) is obtained in terms of H-function in closed and computable forms. Computational representations and fractional moments of the solutions are also obtained which will enhance the utility of the derived results in practical problems. Solutions of unified reaction-diffusion models associated with a finite number of Riesz-Feller space fractional derivatives are also investigated.

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# Appendix A. Mathematical preliminaries

A generalization of the Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}, \ \alpha \in C, \Re(\alpha) > 0$$
(A1)

was introduced by Wiman (1905) in the generalized form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \ \Re(\alpha) > 0, \Re(\beta) > 0.$$
(A2)

The main results of these functions are available in the handbook of Erdélyi, et al. (1955, Section 18.1) and the monographs of Dzherbashyan (1966, 1993). The left-sided Riemann-Liouville fractional integral of order  $\nu$  is defined by Miller and Ross (1993, p.45), Samko et al. (1990), Kilbas et al. (2006) as

$${}_{0}^{RL}D_{t}^{-\nu}N(x,t) = I_{0}^{\nu}N(x,t) = \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-u)^{\nu-1}N(x,u)\mathrm{d}u, \ t > 0, \Re(\nu) > 0. \tag{A3}$$

The left-sided Riemann-Liouville fractional derivative of order  $\alpha$  is defined as

$${}_{0}^{RL}D_{t}^{\mu}N(x,t) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\right]^{n}(I_{0}^{n-\mu}N(x,t)), \ \Re(\mu) > 0, n = \left[\Re(\mu)\right] + 1 \tag{A4}$$

where [x] represents the greatest integer in the real number x. Caputo derivative (Caputo, 1969) is defined in the form

$${}_{0}^{C}D_{t}^{\alpha}f(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} \frac{f^{(m)}(x,t)}{(t-\tau)^{\alpha+1-m}} d\tau, \ m-1 < \Re(\alpha) < m, \ m \in N$$
 (A5)

$$= \frac{\partial^m f(x,t)}{\partial t^m}, \text{ for } \alpha = m \tag{A6}$$

where  $\frac{\partial^m}{\partial t^m} f(x,t)$  is the *m*-th partial derivative of f(x,t) with respect to t. When there is no confusion, then the Caputo operator  ${}_0^C D_t^{\alpha}$  will be simply denoted by  ${}_0 D_t^{\alpha}$ .

A generalization of the Riemann-Liouville fractional derivative operator (A4) as well as Caputo fractional derivative operator (A5) is given by Hilfer (2000) by introducing a left-sided fractional derivative operator of two parameters of order  $0 < \mu < 1$  and type  $0 \le \nu \le 1$  in the form

$${}_{0}D_{a_{+}}^{\mu,\nu}N(x,t) = \left[I_{a_{+}}^{\nu(1-\mu)}\frac{\partial}{\partial x}\left(I_{a_{+}}^{(1-\nu)(1-\mu)}N(x,t)\right)\right]. \tag{A7}$$

For  $\nu = 0$ , (A7) reduces to the classical Riemann-Liouville fractional derivative operator. On the other hand for  $\nu = 1$  it yields the Caputo fractional derivative operator defined by (A5). The Laplace transform formula for this operator is given by Hilfer (2000).

$$L[{}_{0}D^{\mu,\nu}_{0_{+}}N(x,t);s] = s^{\mu}\tilde{N}(x,s) - s^{\nu(\mu-1)}I^{(1-\nu)(1-\mu)}_{0_{+}}N(x,0_{+}), \ 0 < \mu < 1 \tag{A8}$$

where the initial value term  $I_{0_+}^{(1-\nu)(1-\mu)}N(x,0_+)$  involves the Riemann-Liouville fractional integral operator of order  $(1-\nu)(1-\mu)$  evaluated in the limit as  $t\to 0_+$ , it being understood that the integral

$$\tilde{N}(x,s) = L\{N(x,t); s\} = \int_0^\infty e^{-st} N(x,t) dt,$$
 (A9)

where  $\Re(s) > 0$ , exists.

**Note A1.** The derivative defined by (A7) also occurs in recent papers by Hilfer (2003, 2009), Srivastava et al. (2009), Tomovski et al. (2011) and Saxena et al. (2010).

Following Feller (1952, 1971), it is conventional to define the Riesz-Feller space fractional derivative of order  $\alpha$  and skewness  $\theta$  in terms of its Fourier transform as

$$F\{_x D_\theta^\alpha f(x); k\} = -\psi_\alpha^\theta(k) f^*(k), \tag{A10}$$

where

$$\psi_{\alpha}^{\theta}(k) = |k|^{\alpha} \exp[i(sign\ k)\frac{\theta\pi}{2}], \ 0 < \alpha \le 2. \ |\theta| \le \min\{\alpha, 2 - \alpha\}. \tag{A11}$$

When  $\theta = 0$  we have a symmetric operator with respect to x, that can be interpreted as

$$_{x}D_{\theta}^{\alpha} = -\left[-\frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}\right]^{\frac{\alpha}{2}}.\tag{A12}$$

This can be formally deduced by writing  $-(k)^{\alpha} = -(k^2)^{\frac{\alpha}{2}}$ . For  $\theta = 0$  we also have

$$F\{_x D_0^{\alpha} f(x); k\} = -|k|^{\alpha} f^*(k). \tag{A13}$$

For  $0 < \alpha \le 2$  and  $|\theta| \le \min\{\alpha, 2 - \alpha\}$  the Riesz-Feller derivative can be shown to possess the following integral representation in x domain:

$${}_{x}D_{\theta}^{\alpha}f(x) = \frac{\Gamma(1+\alpha)}{\pi} \left\{ \sin[(\alpha+\theta)\frac{\pi}{2}] \int_{0}^{\infty} \frac{f(x+\xi) - f(x)}{\xi^{1+\alpha}} d\xi + \sin[(\alpha-\theta)\frac{\pi}{2}] \int_{0}^{\infty} \frac{f(x-\xi) - f(x)}{\xi^{1+\alpha}} d\xi \right\}.$$

For  $\theta=0$ , the Riesz-Feller fractional derivative becomes the Riesz fractional derivative of order  $\alpha$  for  $1<\alpha\leq 2$  defined by analytic continuation in the whole range  $0<\alpha\leq 2$ ,  $\alpha\neq 1$ , see Gorenflo and Mainardi (1999), as

$$_{x}D_{0}^{\alpha} = -\lambda [I_{+}^{-\alpha} - I_{-}^{-\alpha}] \tag{A14}$$

where

$$\lambda = \frac{1}{2\cos(\alpha\pi/2)}; \quad I_{\pm}^{-\alpha} = \frac{d^2}{dx^2}I_{\pm}^{2-\alpha}.$$
 (A15)

The Weyl fractional integral operators are defined in the monograph by Samko et al. (1990) as

$$(I_{+}^{\beta}N)(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} (x - \xi)^{\beta - 1} N(\xi) d\xi, \ \beta > 0; \tag{A16}$$

$$(I_{-}^{\beta}N)(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} (\xi - x)^{\beta - 1} N(\xi) d\xi, \ \beta > 0.$$
 (A17)

**Note A2**. We note that  $_xD_0^{\alpha}$  is a pseudo differential operator. In particular, we have

$$_{x}D_{0}^{2} = \frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}, \text{ but } _{x}D_{0}^{1} \neq \frac{\mathrm{d}}{\mathrm{d}x}.$$
 (A18)