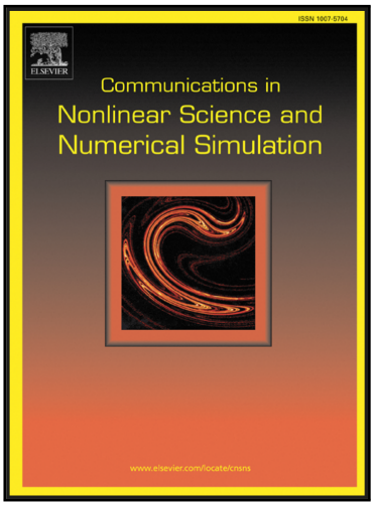

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Highlights

- A general class of nonlinear hyperbolic partial differential equations is considered
- An inverse problem technique is developed for this class of PDEs
- Suffcient conditions for applying the inverse problem method are established
- Two numerical examples are considered and implementation considerations are discussed

A collage-based approach to solving inverse problems for second-order nonlinear hyperbolic PDEs

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Abstract

A goal of many inverse problems is to find unknown parameter values, $\lambda \in \Lambda$, so that the given observed data u_{true} agrees well with the solution data produced using these parameters u_λ. Unfortunately finding u_λ in terms of the parameters of the problem may be a difficult or even impossible task. Further, the objective function may be a complicated function of the parameters $\lambda \in \Lambda$ and may require complex minimization techniques. In recent literature, the collage coding approach to solving inverse problems has emerged. This approach avoids the aforementioned difficulties by bounding the approximation error above by a more readily minimizable distance, thus making the approximation error small. The first of these methods was applied to first-order ordinary differential equations and gets its name from the "collage theorem" used in this setting to achieve an upperbound on the approximation error. A number of related ODE problems have been solved using this method and extensions thereof. More recently, collage-based methods for solving linear and nonlinear elliptic partial differential equations have been developed. In this paper we establish a collage-based method for solving inverse problems for nonlinear hyperbolic PDEs. We develop the necessary background material, discuss the complications introduced by the presence of time-dependence, establish sufficient conditions for using the collage-based approach in this setting and present examples of the theory in practice.

Keywords: inverse problems, parameter estimation, partial differential equations, nonlinear, hyperbolic, optimization

1. Introduction

A goal of many inverse problems is to find parameters λ in some parameter space Λ determining a solution u_λ so that the distance between this solution and some target solution u is minimal. That is, finding parameters that minimize the approximation error $\|u - u_\lambda\|$ in some appropriate norm. Since this is, in general, a difficult task, a collage-based approach instead bounds the approximation error above by a more readily minimizable quantity. In minimizing this new quantity, one can control the approximation error. This is similar to Tikhonov regularization in spirit, where an ill-posed problem is replaced by a well-posed problem. In order to expect any success in this effort, we must first require the existence of a unique solution to the forward problem. In the setting of ODEs a collage-based method was established in [9] for which Banach's fixed point theorem was the driving force. A number of ODE models have been treated using this method including [2, 5, 8, 12]. In the setting of elliptic PDEs, collage-based methods have been established for both linear and nonlinear second-order problems in [6, 10]. In these cases the driving force for existence and uniqueness (as well as corresponding generalized collage theorems) is the (nonlinear) Lax-Milgram representation theorem. A similar method for linear parabolic and hyperbolic problems is suggested in [7].

In this paper, we extend these methods to include inverse problems for a general class of second-order nonlinear hyperbolic PDEs. We extend the idea of linear Galerkin approximation theory to the nonlinear setting in order to establish existence and uniqueness of a weak solution to the forward problem. Following the lead of the nonlinear generalized collage method for elliptic problems, we use the hypotheses of the nonlinear Lax-Milgram representation theorem even though it does not directly apply in the time dependent setting.

The structure of this paper is as follows. In Section 2, we discuss some background theory, notation and preliminaries. In Section 3, we present the weak formulation for a

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general second-order nonlinear hyperbolic problem that will be the focus of consideration for the remainder of this paper. In Section 4, we use nonlinear Galerkin approximation theory to prove existence and uniqueness of a weak solution to the forward problem. In Section 5, we state and prove the nonlinear hyperbolic generalized collage theorem (NHGCT) and state sufficient conditions for its use. Finally, in Section 6, we present some examples of this theory in practice and provide results of numerical implementation.

2. Background

In what follows we define Ω to be an open, bounded subset of $\mathbb{R}^n$, with $\Omega_T = \Omega \times (0, T]$ where T is the maximum value of time. We define X to be an arbitrary function space, $W^{k,p}(\Omega)$ to be a Sobolev space with up to k weak spatial derivatives each in the space $L^p(\Omega)$, and $H^k(\Omega)$ to be a Hilbert space with up to k weak spatial derivatives. It is understood that all derivatives are intended in the weak sense. We use a prime notation, $'$ to denote a weak time derivative. Of particular importance in this work will be the space $W_0^{k,p}(\Omega)$ which denotes the set of functions in the space $W^{k,p}(\Omega)$ that approach zero on $\partial\Omega$.

As a result of the presence of time-dependence, it will be necessary within some of our constructions to make use of the following definition.

Definition 1. *We define functions of $\boldsymbol{x}$ and t as mappings (from the time domain $[0,T]$ to the space domain X) of functions of $\boldsymbol{x}$. That is,*

$$\tilde{u}(t) = [\tilde{u}(\boldsymbol{x})](t) := u(\boldsymbol{x}, t), \text{ for } \boldsymbol{x} \in \Omega, t \in [0, T].$$

We apply Definition 1 to all functions of $\boldsymbol{x}$ and t. The following theorems will be of use when proving and justifying our results.

Definition 2. *Given two metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is said to be Lipschitz continuous if there exists a real constant $K \geq 0$ such that, for all $x_1, x_2 \in X$,*

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

The constant K is called the Lipschitz constant.

Theorem 1. *(Riesz representation theorem [15]) Let H be a Hilbert space and $\varphi : H \to \mathbb{R}$ be a bounded linear functional. Then there is a unique $u \in H$ such that*

$$\varphi(v) = \langle u, v\rangle_H = \sum_{i=1}^{\infty} \langle u_i, v_i\rangle_H, \ \forall v \in H.$$

Theorem 2. *(Nonlinear Lax-Milgram representation theorem [15]) Assume $B : H \times H \to \mathbb{R}$ is a function such that for each $u \in H$ the functional $v \mapsto B[v, u]$ is continuous and linear on H, and $m, M > 0$ exist such that $\forall u, v, w \in H$*

(i) $m\|u - v\|_H^2 \leq B[u, u - v] - B[v, u - v]$;

(ii) $|B[u, w] - B[v, w]| \leq M\|u - v\|_H \|w\|_H$.

Finally, let $\psi : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique $u \in H$ such that

$$B[u, v] = \psi(v) \ \forall v \in H.$$

3. Weak formulation

A common method for solving PDEs, particularly those with complicated nonlinearities and time-dependence, is to build the related weak formulation and seek weak solutions. We consider the following types of nonlinear hyperbolic PDEs

$$u_{tt}(\boldsymbol{x}, t) + L[u; t] - g(u) = f(\boldsymbol{x}, t) \text{ in } \Omega_T, \tag{1}$$

$$u = 0 \text{ on } \partial\Omega \times [0, T], \tag{2}$$

$$u = h_1 \text{ on } \Omega \times \{t = 0\}, \tag{3}$$

$$u_t = h_2 \text{ on } \Omega \times \{t = 0\}, \tag{4}$$

where $g : H \to H$ is a nonlinear function of u, $f : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ is a source or sink term at each $\boldsymbol{x} \in \mathbb{R}^n$ and $t \in [0, T]$, and L is the second-order partial differential operator with dependence on t given in divergence form by

$$L[u; t] = -\sum_{i,j=1}^{n} (a_{ij}(\boldsymbol{x}, t)u_{x_i})_{x_j} + \sum_{i=1}^{n} (b_i(\boldsymbol{x}, t)u_{x_i} + c(\boldsymbol{x}, t)u). \tag{5}$$

We assume that the $n \times n$ matrix $A = (a_{ij})$ is symmetric so that $a_{ij} = a_{ji}$ for each $i, j = 1, ..., n$. We also assume that A is positive definite for each $(\boldsymbol{x}, t) \in \Omega_T$. The following definition gives a characterization of the operator L.

Definition 3. *We say that the partial differential operator $\dfrac{\partial^2}{\partial t^2} + L$ is (uniformly) hyperbolic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}, t)\xi_i\xi_j \geq \theta\|\xi\|_2^2$$

for $(\boldsymbol{x}, t) \in \Omega_T$ and all $\xi \in \mathbb{R}^n$.

In our construction, we will assume that our operator L is uniformly hyperbolic which implies that for fixed $0 \le t \le T$ and for each $\boldsymbol{x} \in \Omega$ the matrix A is positive definite with smallest eigenvalue λ greater or equal to θ.

Without loss of generality we apply the homogeneous Dirichlet boundary condition (2) recognizing that other types of boundary conditions can be accommodated with small adjustments to the construction that follows.

To construct the weak formulation associated with (1)–(4) we assume that

1. a_{ij}, b_i, $c \in L^\infty(\Omega_T)$ for each $i, j = 1, 2..., n$;
2. $\tilde{f}, g \in L^2(\Omega_T)$; and
3. $h_1, h_2 \in L^2(\Omega)$.

We fix $\tilde{v} \in C_c^\infty(\Omega)$, take the inner product of (1) with $\tilde{v}$, and integrate over Ω (applying Green's formula where applicable) to get

$$\begin{aligned}\int_\Omega \tilde{u}''\, \tilde{v}\, d\boldsymbol{x} \quad &+ \quad \int_\Omega \sum_{i,j=1}^{n} a_{ij}\tilde{u}_{x_i}\tilde{v}_{x_j}\, d\boldsymbol{x} - \int_{\partial\Omega} \sum_{i,j=1}^{n} a_{ij}\tilde{u}_{x_i}\tilde{v}\hat{n}_j\, ds \\ &+ \quad \int_\Omega \sum_{i=1}^{n} (b_i\tilde{u}_{x_i} + c\tilde{u})\, \tilde{v}\, d\boldsymbol{x} - \int_\Omega g(\tilde{u})\tilde{v}\, d\boldsymbol{x} = \int_\Omega \tilde{f}\tilde{v}\, d\boldsymbol{x}\end{aligned} \tag{6}$$

The following result enables us to state the above construction in a Sobolev space $W^{k,p}(0, T; X)$.

Theorem 3. *(Global approximation by smooth functions [3]) Let Ω be a bounded domain, $\partial\Omega$ be C^1, and suppose $\tilde{u} \in W_0^{k,p}(\Omega)$ for some $1 \le p < \infty$ and a.e. $0 \le t \le T$ and all $k > 1$. Then for a.e. $0 \le t \le T$ there exists a sequence of functions $\{\tilde{u}^m\}_{m=1}^\infty \in C_c^\infty(\bar{\Omega})$ such that $\{\tilde{u}^m\}_{m=1}^\infty$ converges (strongly) to $\tilde{u} \in W_0^{k,p}(\Omega)$, where $\bar{\Omega}$ denotes the closure of the space Ω.*

Using Theorem 3 we have that (6) holds for all $\tilde{u}, \tilde{v} \in H_0^1(\Omega)$ and *a.e.* $0 \le t \le T$. Now since $\tilde{v} \in H_0^1(\Omega)$ for *a.e.* $0 \le t \le T$ we have

$$\int_{\partial\Omega} \sum_{i,j=1}^{n} a_{ij}\tilde{u}_{x_i}\tilde{v}\hat{n}_j\, ds = 0.$$

Taking the resulting left- and right-hand sides of (6) we arrive at the time-dependent functional B given by

$$B[\tilde{u}, \tilde{v}; t] \quad = \quad -\int_\Omega \sum_{i,j=1}^{n} a_{ij}\tilde{u}_{x_i}\tilde{v}_{x_j} + (b_i\tilde{u}_{x_i} + c\tilde{u})\tilde{v}\, d\boldsymbol{x} + \int_\Omega g(\tilde{u})\tilde{v}\, d\boldsymbol{x} \tag{7}$$

and the time-dependent linear functional ψ given by

$$\psi(\tilde{v};t) = \int_{\Omega} \tilde{f}\tilde{v}\,d\boldsymbol{x} = \langle \tilde{f}, \tilde{v}\rangle_{L^2(\Omega)}, \tag{8}$$

for $\tilde{u}$, $\tilde{v} \in H_0^1(\Omega)$ and $a.e,\ 0 \leq t \leq T$. We remind the reader that the operator L used to construct the functional B is assumed to be uniformly hyperbolic in all cases.

With this development we reach the important definition of the weak solution to the time-dependent problem (1)–(4).

Definition 4. *The problem:*

$$\begin{cases} \textit{Seek } \tilde{u} \in L^2(0,T;H_0^1(\Omega)), \\ \ \textit{with } \tilde{u}' \in L^2(0,T;L^2(\Omega)) \textit{ and } \tilde{u}'' \in L^2(0,T;H^{-1}(\Omega)) \textit{ such that} \\ \text{(i) } \langle \tilde{u}'', \tilde{v}\rangle_{L^2(\Omega)} = B[\tilde{u},\tilde{v};t] + \psi(\tilde{v};t), \forall \tilde{v} \in H_0^1(\Omega) \textit{ and a.e. } 0 \leq t \leq T; \\ \text{(ii) } \tilde{u}(0) = h_1;\ \textit{and} \\ \text{(iii) } \tilde{u}'(0) = h_2, \end{cases} \tag{9}$$

is called the weak (or variational) formulation associated with the problem (1)–(4). A function $\tilde{u}$ satisfying (9) is called a weak solution of (1)–(4).

We are concerned with the existence of a unique weak solution of problem (1)–(4). The next section discusses the particulars of existence and uniqueness of weak solutions to second-order nonlinear hyperbolic problems.

4. Galerkin approximation theory

Before attempting to solve an inverse problem it is important to determine if the forward problem has a solution and if it is unique. The following is an extension of Galerkin approximation theory from weak solutions of *linear* hyperbolic PDEs (presented in [3]) to *nonlinear* hyperbolic PDEs. We make use of the background material in Section 2 and adopt the assumptions presented in Section 3. We begin by letting $w^r = w^r(\boldsymbol{x})$ for $r = 1, 2, ...$ be smooth functions such that

$$\{w^r\}_{r=1}^{\infty} \text{ is an orthogonal basis of } H_0^1(\Omega),$$

and

$$\{w^r\}_{r=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega).$$

We build a sequence of functions $\tilde{u}^m : [0,T] \to H_0^1(\Omega)$ taking the form

$$\tilde{u}^m(t) = \sum_{r=1}^{m} d^{r,m}(t) w^r(\boldsymbol{x}), \tag{10}$$

where the functions $d^{r,m}(t)$ (for $r = 1, ..., m$ and $a.e.$ $0 \leq t \leq T$) are to be chosen (if possible) to satisfy

$$\langle (\tilde{u}^m)'', w^r \rangle_{L^2(\Omega)} + \int_\Omega L[\tilde{u}^m; t] w^r \, d\boldsymbol{x} - \int_\Omega g(\tilde{u}^m) w^r \, d\boldsymbol{x} = \langle \tilde{f}, w^r \rangle_{L^2(\Omega)}, \tag{11}$$

$$d^{r,m}(0) = \langle h^1, w^r \rangle_{L^2(\Omega)}, \tag{12}$$

$$(d^{r,m})'(0) = \langle h^2, w^r \rangle_{L^2(\Omega)}. \tag{13}$$

We refer to equations (11)–(13) as the projection of the problem (1)–(4) onto the finite-dimensional subspace spanned by $\{w^r\}_{r=1}^m$. We seek a function $\tilde{u}^m$ of the form (10) that satisfies this projected problem. The following theorem gives conditions under which a unique solution to the projected problem (11)–(13) exists.

Theorem 4. *Let $g : H_0^1(\Omega) \to \mathbb{R}^n$ be locally Lipshitz in $\tilde{u}$. Then for each integer $m = 1, 2, ...$ there exists a unique function $\tilde{u}^m$ of the form (10) satisfying (11)–(13).*

Proof. Beginning with (10), differentiate with respect to t twice, multiply by w^s and integrate over Ω to get

$$\int_\Omega (\tilde{u}^m)'' w^s \, d\boldsymbol{x} = \int_\Omega \sum_{r=1}^m (d^{r,m})'' w^r w^s \, d\boldsymbol{x} = \sum_{r=1}^m (d^{r,m})'' \int_\Omega w^r w^s \, d\boldsymbol{x}.$$

Since $\{w^r\}_{r=1}^m$ is an orthonormal basis for $L^2(\Omega)$ we have that

$$\langle (\tilde{u}^m)'', w^r \rangle_{L^2(\Omega)} = (d^{r,m})''. \tag{14}$$

Furthermore, note that

$$\int_\Omega L[\tilde{u}^m; t] w^r \, d\boldsymbol{x} = \sum_{s=1}^m e^{s,r} d^{s,m}, \qquad \text{where} \qquad e^{s,r} = \int_\Omega L[w^s; t] w^r \, d\boldsymbol{x}. \tag{15}$$

Substituting (14)–(15) into (11) and rearranging gives

$$(d^{r,m})''(t) = -\sum_{s=1}^m e^{s,r} d^{s,r} + \int_\Omega g(\tilde{u}^m) w^r \, d\boldsymbol{x} + \langle \tilde{f}, w^r \rangle_{L^2(\Omega)} \tag{16}$$

for fixed $m = 1, 2, ...,$ and $r = 1, ..., m$. We see that (16) is a second-order system of ODEs. Evaluating (10) at $t = 0$ and using orthogonality, we arrive at the initial conditions

$$d^{r,m}(0) = \langle h^1, w^r \rangle_{L^2(\Omega)} \tag{17}$$

for each $r = 1, ..., m$. Similarly, differentiating (10) with respect to t, evaluating at $t = 0$ and using orthogonality, we arrive at the initial conditions

$$(d^{r,m})'(0) = \langle h^2, w^r \rangle_{L^2(\Omega)}.$$

From existence and uniqueness theory for ODEs, (16)–(17) has a unique solution provided that the right-hand side of (16) is Lipschitz continuous in $d^{r,m}$ for each fixed $m = 1, 2, ...$ and $r = 1, ..., m$. Since by hypothesis, g is locally Lipschitz in $\tilde{u}^m$ it follows that g is locally Lipschitz in $d^{r,m}$ and hence there exists a unique $d^{r,m}(t)$ satisfying (16)–(17). Thus, we can define a unique solution $\tilde{u}^m(t)$ (for fixed m) to the projected problem (11)–(13). □

For fixed m we have established the existence and uniqueness of a solution to the projected problem (11)–(13). It is our hope that by letting $m \to \infty$ that this sequence of solutions $\{\tilde{u}^m\}_{m=1}^{\infty}$ approaches a weak solution to our original problem (1)–(4). Before we can establish this we first need a couple of results. The first of these results establishes two useful bounds on the functional B.

Theorem 5. *Let $B : H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ be a functional given by*

$$\begin{aligned} B[\tilde{u}, \tilde{v}; t] &= -\int_\Omega \sum_{i,j=1}^{n} a_{ij}\tilde{u}_{x_i}\tilde{v}_{x_j} + (b_i\tilde{u}_{x_i} + c\tilde{u})\tilde{v}\, d\mathbf{x} + \int_\Omega g(\tilde{u})\tilde{v}\, d\mathbf{x} \\ &= \int (L[\tilde{u}; t]\tilde{v} - g(\tilde{u})\tilde{v})\, d\mathbf{x} \end{aligned}$$

such that for each $v \in H_0^1(\Omega)$ the functional $w \mapsto B[w, v; t]$ is bounded and linear on $H_0^1(\Omega)$. Let θ be the uniform hyperbolicity constant of the operator L, β be Poincaré's constant and define $\tilde{b} = \sum_{i=1}^{n} \|b_i\|_{L^\infty(\Omega_T)}$. If

1. *g is Lipschitz in $L^2(\Omega)$;*
2. *$\exists C_g > 0$ such that $\|g\|_{L^2(\Omega)} \leq C_g\|\tilde{u}\|_{L^2(\Omega)}$; and*
3. *$\theta - \beta\tilde{b} > 0$,*

then there exist constants $\zeta, \mu > 0$ and $\gamma \geq 0$ such that

(i) $|B[\tilde{u}, \tilde{v}; t]| \leq \zeta\|\tilde{u}\|_{H_0^1(\Omega)}\|\tilde{v}\|_{H_0^1(\Omega)}$,

(ii) $\mu\|\tilde{u}\|^2_{H_0^1(\Omega)} \leq B[\tilde{u}, \tilde{u}; t] + \gamma\|\tilde{u}\|^2_{L^2(\Omega)}$,

for a.e. $0 \leq t \leq T$.

The next result is essential for the proof of existence and uniqueness of a weak solution to (1)–(4).

Theorem 6. *Let $B : H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ be a functional given by*

$$\begin{aligned} B[\tilde{u}, \tilde{v}; t] &= -\int_\Omega \sum_{i,j=1}^{n} a_{ij} \tilde{u}_{x_i} \tilde{v}_{x_j} + (b_i \tilde{u}_{x_i} + c\tilde{u})\tilde{v}\, d\mathbf{x} + \int_\Omega g(\tilde{u})\tilde{v}\, d\mathbf{x} \\ &= \int (L[\tilde{u}; t]\tilde{v} - g(\tilde{u})\tilde{v})\, d\mathbf{x} \end{aligned}$$

such that for each $v \in H_0^1(\Omega)$ the functional $w \mapsto B[w, v; t]$ is continuous and linear on $H_0^1(\Omega)$. Then there exists a constant C, depending only on Ω, T and the coefficients in the operator L, such that

$$\begin{aligned} \max_{0\le t\le T} \left(\|\tilde{u}^m\|_{H_0^1(\Omega)} + \|(\tilde{u}^m)'\|_{L^2(\Omega)} \right) + \|(\tilde{u}^m)''\|_{L^2(0,T;H^{-1}(\Omega))} \\ \le \quad C \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|h^1\|_{L^2(\Omega)} + \|h^2\|_{L^2(\Omega)} + \|g\|_{L^2(0,T;L^2(\Omega))} \right). \end{aligned}$$

The proofs of Theorems 5 and 6 are standard in Galerkin approximation theory literature and thus are omitted here. For a detailed treatment of these proofs, please see [11]. We are now ready to state and prove a result for the existence and uniqueness of a weak solution to (1)–(4).

Theorem 7. *Let $B : H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ be a function such that for each $v \in H_0^1(\Omega)$ the functional $w \mapsto B[w, v; t]$ is continuous and linear on $H_0^1(\Omega)$. If g is Lipschitz then there exists a unique weak solution to (1)–(4).*

Proof. For existence: From Theorem 6 we have that

$$\begin{gathered} \{\tilde{u}^m\}_{m=1}^{\infty} \text{ is bounded in } L^2(0,T; H_0^1(\Omega)) \\ \{(\tilde{u}^m)'\}_{m=1}^{\infty} \text{ is bounded in } L^2(0,T; L^2(\Omega)) \\ \{(\tilde{u}^m)''\}_{m=1}^{\infty} \text{ is bounded in } L^2(0,T; H^{-1}(\Omega)). \end{gathered}$$

Since we have weak compactness, there must exist convergent subsequences $\{\tilde{u}^{m_s}\}_{s=1}^{\infty} \subset \{\tilde{u}^m\}_{m=1}^{\infty}$, $\{(\tilde{u}^{m_s})'\}_{s=1}^{\infty} \subset \{(\tilde{u}^m)'\}_{m=1}^{\infty}$, and $\{(\tilde{u}^{m_s})''\}_{s=1}^{\infty} \subset \{(\tilde{u}^m)''\}_{m=1}^{\infty}$ such that

$$\begin{cases} \tilde{u}^{m_s} \rightharpoonup \tilde{u} \text{ weakly in } H_0^1(\Omega) \\ (\tilde{u}^{m_s})' \rightharpoonup \tilde{u}' \text{ weakly in } L^2(\Omega) \\ (\tilde{u}^{m_s})'' \rightharpoonup \tilde{u}'' \text{ weakly in } H^{-1}(\Omega). \end{cases} \tag{18}$$

Fix $R \in \mathbb{N}$ and choose $\tilde{v} \in C^1([0,T]; H_0^1(\Omega))$ such that

$$\tilde{v}(t) = \sum_{r=1}^{R} d^{r,R}(t) w^r(x), \tag{19}$$

where $\{d^{r,R}\}_{r=1}^{R}$ are arbitrary smooth functions. We choose $m \ge R$, multiply (11) by $d^{r,R}(t)$, and sum over $r = 1, ..., R$ to get

$$\left\langle (\tilde{u}^m)'', \sum_{r=1}^{R} d^{r,R} w^r \right\rangle_{L^2(\Omega)} - B\left[\tilde{u}^m, \sum_{r=1}^{R} d^{r,R} w^r; t \right] = \left\langle \tilde{f}, \sum_{r=1}^{R} d^{r,R} w^r \right\rangle_{L^2(\Omega)}.$$

Applying (19) and the Riesz representation theorem we have that, for
$\tilde{v} \in L^2(0,T;H_0^1(\Omega))$

$$((\tilde{u}^m)'', \tilde{v}) - B[\tilde{u}^m, \tilde{v}; t] = \langle \tilde{f}, \tilde{v} \rangle_{L^2(\Omega)}$$

Now integrate from $t = 0$ to $t = T$ and set $m = m_s$ to get

$$\int_0^T (((\tilde{u}^{m_s})'', \tilde{v}) - B[\tilde{u}^{m_s}, \tilde{v}; t]) \, dt = \int_0^T \langle \tilde{f}, \tilde{v} \rangle_{L^2(\Omega)} \, dt. \tag{20}$$

Taking the limit as $s \to \infty$ and using (18) we have

$$\int_0^T ((\tilde{u}'', \tilde{v}) - B[\tilde{u}, \tilde{v}; t]) \, dt = \int_0^T \langle \tilde{f}, \tilde{v} \rangle_{L^2(\Omega)} \, dt, \tag{21}$$

for $\tilde{v} \in L^2(0,T;H_0^1(\Omega))$ (since functions of the form (19) are dense in this space). Hence,

$$(\tilde{u}'', v) - B[\tilde{u}, v; t] = \langle \tilde{f}, v \rangle_{L^2(\Omega)},$$

for all $v \in H_0^1(\Omega)$ and *a.e.* $0 \le t \le T$. Before continuing we need the following Lemma.

Lemma 1. *Suppose $\tilde{u} \in L^2(0,T;H_0^1(\Omega))$, with $\tilde{u}' \in L^2(0,T;H^{-1}(\Omega))$. Then*

$$\tilde{u} \in C([0,T];L^2(\Omega))$$

(after possibly being redefined on a set of measure zero).

For a proof of Lemma 1 see Theorem 2 in section 5.9.2 of [3]. Continuing, since $L^2(\Omega) \subset H^{-1}(\Omega)$ we have that
$\tilde{u} \in L^2(0,T;H_0^1(\Omega))$ and $\tilde{u}' \in L^2(0,T;H^{-1}(\Omega))$ so that Lemma 1 implies that
$\tilde{u} \in C([0,T];L^2(\Omega))$. To show that the initial conditions, $\tilde{u}(0) = h_1$ and
$\tilde{u}'(0) = h_2$ hold, choose any function $\tilde{v} \in C([0,T];H_0^1(\Omega))$ such that $\tilde{v}(T) = \tilde{v}'(T) = 0$. Integrating the first term in (21) by parts twice we have that

$$\begin{aligned} \int_0^T ((\tilde{v}'', \tilde{u}) - B[\tilde{u}, \tilde{v}; t]) \, dt = & - \langle \tilde{u}(0), \tilde{v}'(0) \rangle_{L^2(\Omega)} \\ & + \langle \tilde{u}'(0), \tilde{v}(0) \rangle_{L^2(\Omega)} + \int_0^T \langle \tilde{f}, \tilde{v} \rangle_{L^2(\Omega)} \, dt. \end{aligned} \tag{22}$$

Looking back at (20) and integrating the first term by parts twice gives

$$\begin{aligned} \int_0^T ((\tilde{v}'', \tilde{u}^{m_s}) - B[\tilde{u}^{m_s}, \tilde{v}; t]) \, dt = & - \langle \tilde{u}^{m_s}(0), \tilde{v}'(0) \rangle_{L^2(\Omega)} \\ & + \langle (\tilde{u}^{m_s})'(0), \tilde{v}(0) \rangle_{L^2(\Omega)} + \int_0^T \langle \tilde{f}, \tilde{v} \rangle_{L^2(\Omega)} \, dt. \end{aligned}$$

Letting $s \to \infty$ and since $\tilde{u}^{m_s}(0) \to h_1$ in $L^2(\Omega)$ and $(\tilde{u}^{m_s})'(0) \to h_2$ in $L^2(\Omega)$ we deduce that

$$\int_0^T ((\tilde{v}'', \tilde{u}) - B[\tilde{u}, \tilde{v}; t]) \, dt = -\langle h_1, \tilde{v}'(0) \rangle_{L^2(\Omega)} + \langle h_2, \tilde{v}(0) \rangle_{L^2(\Omega)} + \int_0^T \langle \tilde{f}, \tilde{v} \rangle_{L^2(\Omega)} \, dt. \tag{23}$$

Since $\tilde{v}(0)$ and $\tilde{v}'(0)$ are arbitrary, comparing (22) and (23) we have that $\tilde{u}(0) = h_1$ and $\tilde{u}'(0) = h_2$.

For uniqueness: Suppose that there are two weak solutions to (1)–(4), $\tilde{u}^1$ and $\tilde{u}^2$, and define $\tilde{u} = \tilde{u}^1 - \tilde{u}^2$. Then $\tilde{u}$ satisfies

$$\tilde{u}_{tt}(\boldsymbol{x}, t) + L[\tilde{u}; t] - (g(\tilde{u}^1) - g(\tilde{u}^2)) = 0 \text{ in } \Omega_T \tag{24}$$

$$\tilde{u} = 0 \text{ on } \partial\Omega \times [0, T] \tag{25}$$

$$\tilde{u} = 0 \text{ on } \Omega \times \{t = 0\} \tag{26}$$

$$\tilde{u}_t = 0 \text{ on } \Omega \times \{t = 0\}. \tag{27}$$

Fix $0 \leq t \leq T$ and set

$$\tilde{v}(t) = \begin{cases} \displaystyle\int_t^s \tilde{u}(\tau)\, d\tau, & \text{if } 0 \leq t \leq s \\ 0, & \text{if } s \leq t \leq T. \end{cases}$$

Then $\tilde{v}(t) \in H_0^1(\Omega)$ for each $0 \leq t \leq T$, and so

$$\int_0^s \left((\tilde{u}'', \tilde{v}) + \int_\Omega L[\tilde{u}; t]\tilde{v}\, d\boldsymbol{x} - \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\, d\boldsymbol{x} \right) dt = 0.$$

Since $\tilde{u}'(0) = \tilde{v}(s) = 0$, we obtain after integrating by parts in the first term above

$$\int_0^s \left(-\langle \tilde{u}', \tilde{v}' \rangle_{L^2(\Omega)} + \int_\Omega L[\tilde{u}; t]\tilde{v}\, d\boldsymbol{x} - \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\, d\boldsymbol{x} \right) dt = 0.$$

Now $\tilde{v}' = -\tilde{u}$ for $0 \leq t \leq s$, and so

$$\int_0^s \left(\langle \tilde{u}', \tilde{u} \rangle_{L^2(\Omega)} - \int_\Omega L[\tilde{v}'; t]\tilde{v}\, d\boldsymbol{x} - \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\, d\boldsymbol{x} \right) dt = 0. \tag{28}$$

Next we note that

$$\langle \tilde{u}', \tilde{u} \rangle_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2_{L^2(\Omega)}$$

and we derive that

$$\begin{aligned} -\frac{1}{2}\int_\Omega \frac{d}{dt}(L[\tilde{v}; t]\tilde{v})\, d\boldsymbol{x} = & -\frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a'_{ij}(\tilde{v})_{x_i}(\tilde{v})_{x_j} + 2a_{ij}(\tilde{v}')_{x_i}(\tilde{v}) \right) d\boldsymbol{x} \\ & -\frac{1}{2}\int_\Omega \left(\sum_{i=1}^n b'_i(\tilde{v})_{x_i}\tilde{v} + b_i(\tilde{v})_{x_i}\tilde{v}' + b_i(\tilde{v}')_{x_i}\tilde{v} \right) d\boldsymbol{x} \\ & -\frac{1}{2}\int_\Omega \left(c'\tilde{v}^2 + 2c\tilde{v}\tilde{v}' \right) d\boldsymbol{x}. \end{aligned}$$

Defining

$$P[\tilde{v}, \tilde{v}; t] = \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a'_{ij}(\tilde{v})_{x_i}(\tilde{v})_{x_j} + b'_i(\tilde{v})_{x_i}\tilde{v} + c'\tilde{v}^2 \right) d\boldsymbol{x},$$

and using the definition of the operator $L[\cdot;\cdot]$ from our PDE gives

$$-\frac{1}{2}\int_\Omega \frac{d}{dt}(L[\tilde{v};t]\tilde{v})\,d\boldsymbol{x} = -\int_\Omega L[\tilde{v}';t]\tilde{v}\,d\boldsymbol{x} \quad - \quad P[\tilde{v},\tilde{v};t] \\ -\frac{1}{2}\int_\Omega\left(\sum_{i=1}^n b_i(\tilde{v})_{x_i}\tilde{v}' - b_i(\tilde{v}')_{x_i}\tilde{v}\right)\,d\boldsymbol{x}. \tag{29}$$

Using the product rule we have that

$$\frac{1}{2}\int_{\partial\Omega}\sum_{i=1}^n b_i\tilde{v}\tilde{v}'\hat{n}_j\,d\boldsymbol{x} = \frac{1}{2}\int_\Omega\sum_{i=1}^n b_i(\tilde{v})_{x_i}\tilde{v}'\,d\boldsymbol{x} + \frac{1}{2}\int_\Omega\sum_{i=1}^n b_i\tilde{v}(\tilde{v}')_{x_i}\,d\boldsymbol{x} + \frac{1}{2}\int_\Omega\sum_{i=1}^n (b_i)_{x_i}\tilde{v}\tilde{v}'\,d\boldsymbol{x},$$

where $\hat{n}_j$ is the j^{th} component of $\hat{n}$, the unit outward normal to $\partial\Omega$. Since $\tilde{v}\in H_0^1(\Omega)$ for $a.e.$ $0\le t\le T$, the boundary integral equals zero. Rearranging gives

$$\frac{1}{2}\int_\Omega\sum_{i=1}^n b_i\tilde{v}(\tilde{v}')_{x_i}\,d\boldsymbol{x} = -\frac{1}{2}\int_\Omega\sum_{i=1}^n b_i(\tilde{v})_{x_i}\tilde{v}'\,d\boldsymbol{x} - \frac{1}{2}\int_\Omega\sum_{i=1}^n (b_i)_{x_i}\tilde{v}\tilde{v}'\,d\boldsymbol{x}.$$

Using this information in (29) we have

$$-\frac{1}{2}\int_\Omega \frac{d}{dt}(L[\tilde{v};t]\tilde{v})\,d\boldsymbol{x} = -\int_\Omega L[\tilde{v}';t]\tilde{v}\,d\boldsymbol{x} - P[\tilde{v},\tilde{v};t] + \frac{1}{2}\int_\Omega\left(\sum_{i=1}^n 2b_i(\tilde{v})_{x_i}\tilde{v}' + (b_i)_{x_i}(\tilde{v}')\tilde{v}\right)\,d\boldsymbol{x}.$$

Now since $\tilde{v}' = -\tilde{u}$ we have

$$-\frac{1}{2}\int_\Omega \frac{d}{dt}(L[\tilde{v};t]\tilde{v})\,d\boldsymbol{x} = -\int_\Omega L[\tilde{v}';t]\tilde{v}\,d\boldsymbol{x} - P[\tilde{v},\tilde{v};t] + \frac{1}{2}\int_\Omega\left(\sum_{i=1}^n 2b_i(\tilde{v})_{x_i}\tilde{u} + (b_i)_{x_i}(\tilde{u})\tilde{v}\right)\,d\boldsymbol{x}.$$

Defining

$$Q[\tilde{u},\tilde{v};t] = \frac{1}{2}\int_\Omega\left(\sum_{i=1}^n 2b_i(\tilde{v})_{x_i}\tilde{u} + (b_i)_{x_i}(\tilde{u})\tilde{v}\right)\,d\boldsymbol{x},$$

gives

$$\begin{aligned} -\frac{1}{2}\int_\Omega \frac{d}{dt}(L[\tilde{v};t]\tilde{v})\,d\boldsymbol{x} &= -\int_\Omega L[\tilde{v}';t]\tilde{v}\,d\boldsymbol{x} - P[\tilde{v},\tilde{v};t] + Q[\tilde{u},\tilde{v};t] \\ \implies -\int_\Omega L[\tilde{v}';t]\tilde{v}\,d\boldsymbol{x} &= -\frac{1}{2}\int_\Omega \frac{d}{dt}(L[\tilde{v};t]\tilde{v})\,d\boldsymbol{x} + P[\tilde{v},\tilde{v};t] - Q[\tilde{u},\tilde{v};t]. \end{aligned}$$

Substituting this information in (28) and rearranging gives

$$\begin{aligned} \frac{1}{2}\int_0^s\left(\frac{d}{dt}\left(\|\tilde{u}\|^2_{L^2(\Omega)} \quad - \quad \int_\Omega L[\tilde{v};t]\tilde{v}\,d\boldsymbol{x}\right)\right)\,dt& \\ = \int_0^s \left(Q[\tilde{v},\tilde{v};t] - P[\tilde{u},\tilde{v};t]\right)\,dt + \int_0^s\int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\,d\boldsymbol{x}\,dt,& \end{aligned}$$

for all $\tilde{u}, \tilde{v} \in H_0^1(\Omega)$. Now since $\tilde{u}(0) = \tilde{v}(s) = 0$ we have

$$\begin{aligned}
\frac{1}{2}\|\tilde{u}(s)\|^2_{L^2(\Omega)} &+ \frac{1}{2}\int_\Omega L[\tilde{v}(0);0]\tilde{v}(0)\,d\boldsymbol{x} \\
&= \int_0^s (Q[\tilde{u},\tilde{v};t] - P[\tilde{v},\tilde{v};t])\,dt + \int_0^s \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\,d\boldsymbol{x}\,dt \\
\Longrightarrow \frac{1}{2}\|\tilde{u}(s)\|^2_{L^2(\Omega)} &+ \frac{1}{2}B[\tilde{v}(0),\tilde{v}(0);t] \\
&= \int_0^s (Q[\tilde{u},\tilde{v};t] - P[\tilde{v},\tilde{v};t])\,dt + \int_0^s \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\,d\boldsymbol{x}\,dt.
\end{aligned}$$

Recall that Theorem 5 tells us that

$$\mu\|\tilde{u}\|^2_{H_0^1(\Omega)} \leq B[\tilde{u},\tilde{u};t] + \gamma\|\tilde{u}\|^2_{L^2(\Omega)},$$

for constants μ, γ and a.e. $0 \leq t \leq T$ under the hypotheses. This fact allows us to bound the second term on the left-hand side below:

$$\begin{aligned}
\frac{1}{2}\|\tilde{u}(s)\|^2_{L^2(\Omega)} &+ \frac{1}{2}\mu\|\tilde{v}(0)\|^2_{H_0^1(\Omega)} - \frac{1}{2}\gamma\|\tilde{v}(0)\|^2_{L^2(\Omega)} \\
&\leq \int_0^s (Q[\tilde{u},\tilde{v};t] - P[\tilde{v},\tilde{v};t])\,dt + \int_0^s \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\,d\boldsymbol{x}\,dt \\
\Longrightarrow \|\tilde{u}(s)\|^2_{L^2(\Omega)} + \|\tilde{v}(0)\|^2_{H_0^1(\Omega)} &\leq C\bigg(\int_0^s (Q[\tilde{u},\tilde{v};t] - P[\tilde{v},\tilde{v};t])\,dt \qquad (30)\\
&+ \int_0^s \int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\,d\boldsymbol{x}\,dt + \frac{1}{2}\gamma\|\tilde{v}(0)\|^2_{L^2(\Omega)}\bigg),
\end{aligned}$$

where $C := \dfrac{1}{2}\min\{\mu, 1\}$. Bounding the operator P above using Poincaré's inequality gives

$$P[\tilde{v},\tilde{v};t] \leq C_1\left(\|\tilde{v}\|^2_{H_0^1(\Omega)} + \|\tilde{v}\|_{H_0^1(\Omega)}\|\tilde{v}\|_{L^2(\Omega)} + \|\tilde{v}\|^2_{L^2(\Omega)}\right) \leq C_2\|\tilde{v}\|^2_{H_0^1(\Omega)}.$$

Similarly, bounding the operator Q above, we have

$$Q[\tilde{u},\tilde{v};t] \leq C_3(\|\tilde{v}\|^2_{H_0^1(\Omega)} + \|\tilde{u}\|^2_{L^2(\Omega)}). \qquad (31)$$

Finally, since g is Lipschitz with Lipschitz constant K, we have that

$$\begin{aligned}
\int_\Omega (g(\tilde{u}^1) - g(\tilde{u}^2))\tilde{v}\,d\boldsymbol{x} &\leq \int_\Omega |g(\tilde{u}^1) - g(\tilde{u}^2)||v|\,d\boldsymbol{x} \\
&\leq \|g(\tilde{u}^1) - g(\tilde{u}^2)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq K\|\tilde{u}^1 - \tilde{u}^2\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq C_4(\|\tilde{u}\|^2_{L^2(\Omega)} + \|\tilde{v}\|^2_{H_0^1(\Omega)}). \qquad (32)
\end{aligned}$$

Applying (31)–(32) in (30), we have

$$\|\tilde{u}(s)\|^2_{L^2(\Omega)} + \|\tilde{v}(0)\|^2_{H_0^1(\Omega)} \leq C_5\left(\int_0^s \|\tilde{u}(t)\|^2_{L^2(\Omega)} + \|\tilde{v}(t)\|^2_{H_0^1(\Omega)}\,dt\right) + \|\tilde{v}(0)\|^2_{L^2(\Omega)}. \qquad (33)$$

Now let us write

$$\tilde{w}(t) := \int_0^t \tilde{u}(\tau)\, d\tau$$

for *a.e.* $0 \le t \le T$ so that $\tilde{v}(t) = \tilde{w}(s) - \tilde{w}(t)$ and $\tilde{v}(0) = \tilde{w}(s)$. Then (33) becomes

$$\begin{aligned} \|\tilde{u}(s)\|^2_{L^2(\Omega)} &+ \|\tilde{w}(s)\|^2_{H^1_0(\Omega)} \\ &\le C_5\left(\int_0^s \|\tilde{w}(t) - \tilde{w}(s)\|^2_{H^1_0(\Omega)} + \|\tilde{u}(t)\|^2_{L^2(\Omega)}\, dt + \|\tilde{w}(s)\|^2_{L^2(\Omega)}\right). \qquad (34) \end{aligned}$$

But

$$\|\tilde{w}(t) - \tilde{w}(s)\|^2_{H^1_0(\Omega)} \le 2\|\tilde{w}(t)\|^2_{H^1_0(\Omega)} + 2\|\tilde{w}(s)\|^2_{H^1_0(\Omega)}$$

and

$$\|\tilde{w}(s)\|_{L^2(\Omega)} \le \int_0^s \|\tilde{u}(t)\|_{L^2(\Omega)}\, dt.$$

Therefore (34) implies

$$\|\tilde{u}(s)\|^2_{L^2(\Omega)} + (1 - 2C_5 s)\|\tilde{w}(s)\|^2_{H^1_0(\Omega)} \le 2C_5 \int_0^s \|\tilde{w}(t)\|^2_{H^1_0(\Omega)} + \|\tilde{u}(t)\|^2_{L^2(\Omega)}\, dt.$$

Choose T_1 so small that

$$1 - 2C_5 T_1 \ge \frac{1}{2}.$$

Then if $0 \le s \le T_1$, we have

$$\|\tilde{u}(s)\|^2_{L^2(\Omega)} + \|\tilde{w}(s)\|^2_{H^1_0(\Omega)} \le 4C_5 \int_0^s \|\tilde{u}(t)\|^2_{L^2(\Omega)} + \|\tilde{w}(t)\|^2_{H^1_0(\Omega)}\, dt.$$

Consequently the integral form of Gronwall's inequality implies that $\tilde{u} = 0$ on $[0, T_1]$. Applying the same argument on the intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$, etc., eventually we deduce that $\tilde{u} = 0$ on $0 \le t \le T$ and thus we have a unique weak solution to (1)–(4). $\square$

With an understanding of the existence and uniqueness theory for nonlinear hyperbolic problems, we turn our attention to solving inverse problems.

5. The nonlinear hyperbolic generalized collage theorem

In what follows we define Λ to be our parameter space and work with a family of functionals B_λ with desirable properties. We state the nonlinear hyperbolic generalized collage theorem (NHGCT).

Theorem 8. *(NHGCT) Let β be Poincaré's constant and H be a Hilbert space and $B_\lambda : \Lambda \times H \times H \times \mathbb{R} \to \mathbb{R}$ be a family of functions such that for each $\tilde{v} \in H$ and all λ the functional $\tilde{w} \mapsto B_\lambda[\tilde{w}, \tilde{v}; t]$ is continuous and linear on H, and $m_\lambda, M_\lambda, T > 0$ exist such that $\forall \tilde{u}, \tilde{v}, \tilde{w} \in H$ and a.e. $0 \le t \le T$*

(i) $m_\lambda \|\tilde{u} - \tilde{v}\|_H^2 \le B_\lambda[\tilde{u}, \tilde{u} - \tilde{v}; t] - B_\lambda[\tilde{v}, \tilde{u} - \tilde{v}; t]$; *and*

(ii) $|B_\lambda[\tilde{u}, \tilde{w}; t] - B_\lambda[\tilde{v}, \tilde{w}; t]| \le M_\lambda \|\tilde{u} - \tilde{v}\|_H \|\tilde{w}\|_H$.

Finally let $\psi : H \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a bounded, linear functional for a.e. $0 \le t \le T$. If $\tilde{u}_\lambda$ is the unique (weak) solution of the equation $\langle \tilde{u}'', \tilde{v} \rangle_{L^2} = B_\lambda[\tilde{u}, \tilde{v}; t] + \psi(\tilde{v}; t)$, then for any $\tilde{u} \in H$ such that $\langle \tilde{u}' - \tilde{u}_\lambda', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)} \Big|_{t=0}^{t=T} = 0$ we have that

$$\|\tilde{u} - \tilde{u}_\lambda\|_{L^{1,2}(0,T;L^2(\Omega))} \le \frac{1}{\tilde{m}_\lambda} F(\lambda),$$

where $\tilde{m}_\lambda = \min\left\{ \frac{m_\lambda}{\beta^2}, 1 \right\}$, $\|u\|_{L^{1,2}(0,T;L^2(\Omega))}$ denotes the sum of the norms of u and the weak time derivative of u, each on $L^2(0, T; L^2(\Omega))$. i.e. $\|u\|_{L^{1,2}(0,T;L^2(\Omega))} = \|u\|_{L^2(0,t;L^2(\Omega))} + \|u'\|_{L^2(0,t;L^2(\Omega))}$, and

$$F(\lambda) = \left(\int_0^T \left(\sup_{\substack{\tilde{v} \in L^2 \\ \|\tilde{v}\|_{L^2} = 1}} |B_\lambda[\tilde{u}, \tilde{v}; t] + \psi(\tilde{v}; t) - \langle \tilde{u}'', \tilde{v} \rangle_{L^2(\Omega)}| \right)^2 dt \right)^{\frac{1}{2}}.$$

Proof. We begin by using property (i) of B_λ

$$\begin{aligned}
m_\lambda \|\tilde{u} - \tilde{u}_\lambda\|_H^2 & \\
\le\ & B_\lambda[\tilde{u}, \tilde{u} - \tilde{u}_\lambda; t] - B_\lambda[\tilde{u}_\lambda, \tilde{u} - \tilde{u}_\lambda; t] \\
\le\ & B_\lambda[\tilde{u}, \tilde{u} - \tilde{u}_\lambda; t] + \psi(\tilde{u} - \tilde{u}_\lambda; t) - \langle \tilde{u}_\lambda'', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)} \\
\le\ & B_\lambda[\tilde{u}, \tilde{u} - \tilde{u}_\lambda; t] + \psi(\tilde{u} - \tilde{u}_\lambda; t) + \langle \tilde{u}'' - \tilde{u}_\lambda'', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)} - \langle \tilde{u}'', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)} \\
=\ & B_\lambda[\tilde{u}, \tilde{u} - \tilde{u}_\lambda; t] + \psi(\tilde{u} - \tilde{u}_\lambda; t) - \langle \tilde{u}'', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)} \\
& \qquad - \|\tilde{u}' - \tilde{u}_\lambda'\|_{L^2(\Omega)}^2 + \frac{d}{dt} \langle \tilde{u}' - \tilde{u}_\lambda', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)} \\
=\ & \|\tilde{u} - \tilde{u}_\lambda\|_{L^2(\Omega)} \left(B_\lambda\left[\tilde{u}, \frac{\tilde{u} - \tilde{u}_\lambda}{\|\tilde{u} - \tilde{u}_\lambda\|_{L^2(\Omega)}}; t \right] + \psi\left(\frac{\tilde{u} - \tilde{u}_\lambda}{\|\tilde{u} - \tilde{u}_\lambda\|_{L^2(\Omega)}}; t \right) - \left\langle \tilde{u}'', \frac{\tilde{u} - \tilde{u}_\lambda}{\|\tilde{u} - \tilde{u}_\lambda\|_{L^2(\Omega)}} \right\rangle_{L^2(\Omega)} \right) \\
& \qquad - \|\tilde{u}' - \tilde{u}_\lambda'\|_{L^2(\Omega)}^2 + \frac{d}{dt} \langle \tilde{u}' - \tilde{u}_\lambda', \tilde{u} - \tilde{u}_\lambda \rangle_{L^2(\Omega)}.
\end{aligned}$$

Letting $\tilde{v} = \dfrac{\tilde{u} - \tilde{u}_\lambda}{\|\tilde{u} - \tilde{u}_\lambda\|_{L^2(\Omega)}}$ so that $\|\tilde{v}\|_{L^2(\Omega)} = 1$ and integrating from $t = 0$ to $t = T$ gives

$$m_\lambda \|\tilde{u} - \tilde{u}_\lambda\|^2_{L^2(0,T;H)} \leq \int_0^T \|\tilde{u} - \tilde{u}_\lambda\|_{L^2(\Omega)} \sup_{\substack{\tilde{v}\in L^2 \\ \|\tilde{v}\|_{L^2}=1}} \left| B_\lambda[\tilde{u},\tilde{v};t] + \psi(\tilde{v}) - \langle \tilde{u}'',\tilde{v}\rangle_{L^2(\Omega)} \right| dt$$

$$- \int_0^T \|\tilde{u}' - \tilde{u}'_\lambda\|^2_{L^2(\Omega)}\, dt + \langle \tilde{u}' - \tilde{u}'_\lambda, \tilde{u} - \tilde{u}_\lambda\rangle_{L^2(\Omega)} \Big|_{t=0}^{t=T}.$$

Using the Cauchy-Schwarz inequality, rearranging and using the fact that $\langle \tilde{u}' - \tilde{u}'_\lambda, \tilde{u} - \tilde{u}_\lambda\rangle_{L^2(\Omega)} \Big|_{t=0}^{t=T} = 0$ we have

$$m_\lambda \|\tilde{u} - \tilde{u}_\lambda\|^2_{L^2(0,T;H)} + \int_0^T \|\tilde{u}' - \tilde{u}'_\lambda\|^2_{L^2(\Omega)}\, dt$$

$$\leq \left(\int_0^T \|\tilde{u} - \tilde{u}_\lambda\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \left(\int_0^T \left(\sup_{\substack{\tilde{v}\in L^2 \\ \|\tilde{v}\|_{L^2}=1}} \left| B_\lambda[\tilde{u},\tilde{v};t] + \psi(\tilde{v};t) - \langle \tilde{u}'',\tilde{v}\rangle_{L^2(\Omega)} \right|\right)^2 dt\right)^{\frac{1}{2}}$$

$$\Longrightarrow m_\lambda \|\tilde{u} - \tilde{u}_\lambda\|^2_{L^2(0,T;H)} + \|\tilde{u}' - \tilde{u}'_\lambda\|^2_{L^2(0,T;L^2(\Omega))}$$

$$\leq \|\tilde{u} - \tilde{u}_\lambda\|_{L^2(0,T;L^2(\Omega))} \left(\int_0^T \left(\sup_{\substack{\tilde{v}\in L^2 \\ \|\tilde{v}\|_{L^2}=1}} \left| B_\lambda[\tilde{u},\tilde{v};t] + \psi(\tilde{v};t) - \langle \tilde{u}'',\tilde{v}\rangle_{L^2(\Omega)} \right|\right)^2 dt\right)^{\frac{1}{2}}.$$

Using Poincaré's inequality to bound below on the left, and bounding above on the right gives

$$\Longrightarrow \frac{m_\lambda}{\beta^2} \|\tilde{u} - \tilde{u}_\lambda\|^2_{L^2(0,T;L^2(\Omega))} + \|\tilde{u}' - \tilde{u}'_\lambda\|^2_{L^2(0,T;L^2(\Omega))}$$

$$\leq \|\tilde{u} - \tilde{u}_\lambda\|_{L^{1,2}(0,T;L^2(\Omega))} \left(\int_0^T \left(\sup_{\substack{\tilde{v}\in L^2 \\ \|\tilde{v}\|_{L^2}=1}} \left| B_\lambda[\tilde{u},\tilde{v};t] + \psi(\tilde{v};t) - \langle \tilde{u}'',\tilde{v}\rangle_{L^2(\Omega)} \right|\right)^2 dt\right)^{\frac{1}{2}}$$

$$\Longrightarrow \frac{\min\left\{\frac{m_\lambda}{\beta^2}, 1\right\} \|\tilde{u} - \tilde{u}_\lambda\|^2_{L^{1,2}(0,T;L^2(\Omega))}}{\|\tilde{u} - \tilde{u}_\lambda\|_{L^{1,2}(0,T;L^2(\Omega))}}$$

$$\leq \left(\int_0^T \left(\sup_{\substack{\tilde{v}\in L^2 \\ \|\tilde{v}\|_{L^2}=1}} \left| B_\lambda[\tilde{u},\tilde{v};t] + \psi(\tilde{v};t) - \langle \tilde{u}'',\tilde{v}\rangle_{L^2(\Omega)} \right|\right)^2 dt\right)^{\frac{1}{2}}.$$

Simplifying and rearranging gives

$$\|\tilde{u} - \tilde{u}_\lambda\|_{L^{1,2}(0,T;L^2(\Omega))} \leq \frac{1}{\tilde{m}_\lambda} \left(\int_0^T \left(\sup_{\substack{\tilde{v}\in L^2 \\ \|\tilde{v}\|_{L^2}=1}} \left| B_\lambda[\tilde{u},\tilde{v};t] + \psi(\tilde{v};t) - \langle \tilde{u}'',\tilde{v}\rangle_{L^2(\Omega)} \right|\right)^2 dt\right)^{\frac{1}{2}},$$

where $\tilde{m}_\lambda = \min\left\{\frac{m_\lambda}{\beta^2}, 1\right\}$. $\square$

We call $F(\lambda)$ the *nonlinear hyperbolic generalized collage distance*. Theorem 8 allows us to control the approximation error by minimizing the nonlinear hyperbolic generalized collage distance provided that $\tilde{m}_\lambda = \min\left\{\frac{m_\lambda}{\beta^2}, 1\right\}$ is bounded away from 0. One way to approach this minimization problem is a penalization method, i.e.

$$\min_{\lambda \in \mathbb{R}^{\dim(\lambda)}} F(\lambda) + \sigma \max\{-m_\lambda, 0\}, \tag{35}$$

where m_λ is the coercivity constant of B_λ and $\sigma \geq 0$ is a penalty constant. This approach is reminiscent of classical regularization techniques.

In what follows, we will work on a subset of the space $H_0^1(\Omega)$ defined as

$$\tilde{H}_0^1(\Omega) := \{\tilde{u} \in H_0^1(\Omega) : \|\tilde{u}\|_{H_0^1(\Omega)} \leq \rho \text{ for } \rho > 0 \text{ and } a.e.\ 0 \leq t \leq T\},$$

where $\rho \in \mathbb{R}$ defines some fixed value for which $\tilde{u}$ remains bounded in both $H_0^1(\Omega)$ and $L^2(\Omega)$ for a.e. $0 \leq t \leq T$. We define a similar subset of $L^2(\Omega)$ as

$$\tilde{L}^2(\Omega) := \{\tilde{u} \in L^2(\Omega) : \|\tilde{u}\|_{L^2(\Omega)} \leq \rho \text{ for } \rho > 0 \text{ and } a.e.\ 0 \leq t \leq T\}.$$

A related inverse problem to the system (1)–(4) is:

given $u(\boldsymbol{x}, t)$ (possibly in the form of observational data),

$f(\boldsymbol{x}, t)$, $g(u)$, $b_i(\boldsymbol{x}, t)$, and $c(\boldsymbol{x}, t)$, find $a_{ij}(\boldsymbol{x}, t)$.

In order to apply the NHGCT we require that the following result holds.

Theorem 9. *(Sufficient conditions for using the NHGCT) Consider the problem (1)–(4) whose weak formulation consists of the functionals B_λ and ψ given in (7) and (8) respectively. Further, let $\beta > 0$ be Poincaré's constant, θ be the uniform hyperbolicity constant for $\frac{\partial^2}{\partial t^2} + L$, and define*

$$\tilde{a} = \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)}, \quad \tilde{b} = \sum_{i,j=1}^{n} \|b_i\|_{L^\infty(\Omega)}, \quad \tilde{c} = \|c\|_{L^\infty(\Omega)}$$

If

(i) *g is Lipschitz in $\tilde{L}^2(\Omega)$ with Lipschitz constant $K > 0$ and satisfies $\|g\|_{\tilde{L}^2(\Omega)} \leq C_g \|\tilde{u}\|_{\tilde{L}^2(\Omega)}$ for some constant $C_g > 0$ and a.e. $0 \leq t \leq T$,*

(ii) *$\tilde{f}$ is bounded in $\tilde{L}^2(\Omega)$ for a.e. $0 \leq t \leq T$;*

(iii) $0 < \theta - \beta\tilde{b} - \beta^2\tilde{c} - \beta^2 K$; *and*

(iv) $B_\lambda[\tilde{u}, \tilde{u} - \tilde{v}; t] - B_\lambda[\tilde{v}, \tilde{u} - \tilde{v}; t] \geq 0$

then

1. *for each* $\tilde{v} \in \tilde{H}_0^1(\Omega)$ *and a.e.* $t \in [0, T]$ *the mapping* $\tilde{w} \mapsto B_\lambda[\tilde{w}, \tilde{v}; t]$ *represents a linear, continuous functional on* $\tilde{H}_0^1(\Omega)$*;*

2. $B_\lambda : \Lambda \times \tilde{H}_0^1(\Omega) \times \tilde{H}_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ *is a mapping for which there exist* $m_\lambda, M_\lambda > 0$ *such that* $\forall \tilde{u}, \tilde{v}, \tilde{w} \in \tilde{H}_0^1(\Omega)$ *and a.e.* $t \in [0, T]$*;*

   (a) $m_\lambda \|\tilde{u} - \tilde{v}\|_{\tilde{H}_0^1(\Omega)} \leq B_\lambda[\tilde{u}, \tilde{u} - \tilde{v}; t] - B_\lambda[\tilde{v}, \tilde{u} - \tilde{v}; t]$;

   (b) $|B_\lambda[\tilde{u}, \tilde{w}; t] - B_\lambda[\tilde{v}, \tilde{w}; t]\,| \leq M_\lambda \|\tilde{u} - \tilde{v}\|_{\tilde{H}_0^1(\Omega)} \|\tilde{w}\|_{\tilde{H}_0^1(\Omega)}$; *and*

3. $\psi : \tilde{H}_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ *is a bounded linear functional on* $\tilde{H}_0^1(\Omega)$.

Proof. For 1: First to show that for each $\tilde{v} \in \tilde{H}_0^1(\Omega)$ the functional $\tilde{w} \mapsto B_\lambda[\tilde{w}, \tilde{v}]$ is linear, let $\tilde{v}^1, \tilde{v}^2 \in \tilde{H}_0^1(\Omega)$ and $\tau_1, \tau_2 \in \mathbb{R}$. Then for each $\tilde{w} \in \tilde{H}_0^1(\Omega)$, we have

$$\begin{aligned} B_\lambda[\tilde{w}, (\tau^1\tilde{v}^1 + \tau_2\tilde{v}^2); t] &= -\int_\Omega L[\tilde{w}; t](\tau_1\tilde{v}^1 + \tau_2\tilde{v}^2)_{x_j}\, d\boldsymbol{x} + \int_\Omega g(\tilde{w})(\tau_1\tilde{v}^1 + \tau_2\tilde{v}^2)\, d\boldsymbol{x} \\ &= -\tau_1 \int_\Omega L[\tilde{w}; t]\tilde{v}^1_{x_j}\, d\boldsymbol{x} - \tau_2 \int_\Omega L[\tilde{w}; t]\tilde{v}^2_{x_j}\, d\boldsymbol{x} \\ &\qquad + \tau_1 \int_\Omega g(\tilde{w})\tilde{v}^1\, d\boldsymbol{x} + \tau_2 \int_\Omega g(\tilde{w})\tilde{v}^2\, d\boldsymbol{x} \end{aligned}$$

$$\Longrightarrow B_\lambda[\tilde{w}, (\tau_1\tilde{v}^1 + \tau_2\tilde{v}^2); t] = \tau_1 B_\lambda[\tilde{w}, \tilde{v}^1; t] + \tau_2 B_\lambda[\tilde{w}, \tilde{v}^2; t].$$

To show that for each $\tilde{v} \in \tilde{H}_0^1(\Omega)$, $\tilde{w} \mapsto B_\lambda[\tilde{w}, \tilde{v}]$ is continuous we choose any $\epsilon > 0$ and suppose that for $\tilde{v}^1, \tilde{v}^2 \in \tilde{H}_0^1(\Omega)$ we have that $\|\tilde{v}^1 - \tilde{v}^2\|_{\tilde{H}_0^1(\Omega)} < \frac{\epsilon}{\zeta\rho}$, where ρ is the bound on functions in $\tilde{H}_0^1(\Omega)$. Then using linearity and Theorem 5 we have that there exists an $\zeta > 0$ such that

$$\begin{aligned} \left| B_\lambda[\tilde{w}, \tilde{v}^1; t] - B_\lambda[\tilde{w}, \tilde{v}^2; t] \right| &= \left| B_\lambda[\tilde{w}, \tilde{v}^1 - \tilde{v}^2; t] \right| \\ &\leq \zeta \|\tilde{w}\|_{\tilde{H}_0^1(\Omega)} \|\tilde{v}^1 - \tilde{v}^2\|_{\tilde{H}_0^1(\Omega)} \\ &\leq \zeta\rho \|\tilde{v}^1 - \tilde{v}^2\|_{\tilde{H}_0^1(\Omega)} \\ &< \epsilon, \end{aligned}$$

for *a.e.* $0 \leq t \leq T$.
For 2(a):

$$\theta \|\tilde{u} - \tilde{v}\|^2_{\tilde{H}_0^1(\Omega)} \leq \int_\Omega \sum_{i,j=1}^n a_{ij}(\tilde{u} - \tilde{v})_{x_i}(\tilde{u} - \tilde{v})_{x_j}\, d\boldsymbol{x}$$

$$
\begin{aligned}
&= -(B_\lambda[\tilde u,\tilde u-\tilde v;t]-B_\lambda[\tilde v,\tilde u-\tilde v;t])-\int_\Omega\sum_{i=1}^n b_i(\tilde u-\tilde v)_{x_i}(\tilde u-\tilde v)\,d\boldsymbol{x}\\
&\qquad -\int_\Omega c(\tilde u-\tilde v)^2\,d\boldsymbol{x}-\int_\Omega (g(\tilde u)-g(\tilde v))(\tilde u-\tilde v)\,d\boldsymbol{x}\\
&\le B_\lambda[\tilde u,\tilde u-\tilde v;t]-B_\lambda[\tilde v,\tilde u-\tilde v;t]+\tilde b\int_\Omega |D\tilde u-D\tilde v||\tilde u-\tilde v|\,d\boldsymbol{x}\\
&\qquad +\tilde c\int_\Omega |\tilde u-\tilde v|^2\,d\boldsymbol{x}+\int_\Omega |g(\tilde u)-g(\tilde v)||\tilde u-\tilde v|\,d\boldsymbol{x}
\end{aligned}
$$

Now using the Cauchy-Schwarz inequality and the fact that g is Lipschitz gives

$$
\begin{aligned}
\theta\|\tilde u-\tilde v\|^2_{\tilde H^1_0(\Omega)} &\le B_\lambda[\tilde u,\tilde u-\tilde v;t]-B_\lambda[\tilde v,\tilde u-\tilde v;t]+\tilde b\|\tilde u-\tilde v\|_{\tilde H^1_0(\Omega)}\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}\\
&\qquad +\tilde c\|\tilde u-\tilde v\|^2_{\tilde L^2(\Omega)}+K\|\tilde u-\tilde v\|^2_{\tilde L^2(\Omega)}.
\end{aligned}
$$

Applying Poincaré's inequality where appropriate yields

$$
\begin{aligned}
\theta\|\tilde u-\tilde v\|^2_{\tilde H^1_0(\Omega)} &\le B_\lambda[\tilde u,\tilde u-\tilde v;t]-B_\lambda[\tilde v,\tilde u-\tilde v;t]+\tilde b\beta\|\tilde u-\tilde v\|^2_{\tilde H^1_0(\Omega)}\\
&\qquad +\beta^2\tilde c\|\tilde u-\tilde v\|^2_{\tilde H^1_0(\Omega)}+\beta^2K\|\tilde u-\tilde v\|^2_{\tilde H^1_0(\Omega)}.
\end{aligned}
$$

Rearranging gives

$$
(\theta-\beta\tilde b-\beta^2\tilde c-\beta^2K)\|\tilde u-\tilde v\|^2_{\tilde H^1_0(\Omega)}\le B_\lambda[\tilde u,\tilde u-\tilde v;t]-B_\lambda[\tilde v,\tilde u-\tilde v;t].
$$

Using hypothesis (iii) gives the result.
For 2(b):

$$
\begin{aligned}
|B_\lambda[\tilde u,\tilde w;t]-B_\lambda[\tilde v,\tilde w;t]| &= \Big|-\int_\Omega\sum_{i,j=1}^n a_{ij}(\tilde u-\tilde v)_{x_i}\tilde w_{x_j}\,d\boldsymbol{x}-\int_\Omega\sum_{i=1}^n b_i(\tilde u-\tilde v)_{x_i}\tilde w\,d\boldsymbol{x}\\
&\qquad -\int_\Omega c(\tilde u-\tilde v)\tilde w\,d\boldsymbol{x}+\int_\Omega (g(\tilde u)-g(\tilde v))\tilde w\,d\boldsymbol{x}\Big|\\
&\le \int_\Omega\sum_{i,j=1}^n |a_{ij}||(\tilde u-\tilde v)_{x_i}||\tilde w_{x_j}|\,d\boldsymbol{x}+\int_\Omega\sum_{i=1}^n |b_i||(\tilde u-\tilde v)_{x_i}||\tilde w|\,d\boldsymbol{x}\\
&\qquad +\int_\Omega |c||\tilde u-\tilde v||\tilde w|\,d\boldsymbol{x}+\int_\Omega |g(\tilde u)-g(\tilde v)||\tilde w|\,d\boldsymbol{x}\\
&\le \tilde a\int_\Omega |D\tilde u-D\tilde v||D\tilde w|\,d\boldsymbol{x}+\tilde b\int_\Omega |D\tilde u-D\tilde v||\tilde w|\,d\boldsymbol{x}\\
&\qquad +\tilde c\int_\Omega |\tilde u-\tilde v||\tilde w|\,d\boldsymbol{x}+\int_\Omega |g(\tilde u)-g(\tilde v)||\tilde w|\,d\boldsymbol{x}.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality for $\tilde L^2(\Omega)$ and the fact that g is Lipschitz, we have that

$$
\begin{aligned}
\Big|B_\lambda[\tilde u,\tilde w;t]-B_\lambda[\tilde v,\tilde w;t]\Big| &\le \tilde a\|\tilde u-\tilde v\|_{\tilde H^1_0(\Omega)}\|\tilde w\|_{\tilde H^1_0(\Omega)}+\tilde b\|\tilde u-\tilde v\|_{\tilde H^1_0(\Omega)}\|\tilde w\|_{\tilde L^2(\Omega)}\\
&\qquad +\tilde c\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}\|\tilde w\|_{\tilde L^2(\Omega)}+C_g\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}\|\tilde w\|_{\tilde L^2(\Omega)}.
\end{aligned}
$$

Applying Poincaré's inequality gives

$$\left| B_\lambda[\tilde{u}, \tilde{w}; t] - B_\lambda[\tilde{v}, \tilde{w}; t] \right| \leq \left(\tilde{a} + \tilde{b}\beta + \tilde{c}\beta^2 + C_g\beta^2 \right) \|\tilde{u} - \tilde{v}\|_{\tilde{H}_0^1(\Omega)} \|\tilde{w}\|_{\tilde{H}_0^1(\Omega)}.$$

Letting $M_\lambda = \tilde{a} + \tilde{b}\beta + \tilde{c}\beta^2 + C_g\beta^2 > 0$ gives the result.
For 3: First to show that ψ is linear in $\tilde{v}$ let $\tilde{v}^1, \tilde{v}^2 \in \tilde{H}_0^1(\Omega)$ and $\tau_1, \tau_2 \in \mathbb{R}$. Then

$$\psi(\tau_1 \tilde{v}^1 + \tau_2 \tilde{v}^2; t) = \int_\Omega \tilde{f}(\tau_1 \tilde{v}^1 + \tau_2 \tilde{v}^2; t)\, d\boldsymbol{x} = \tau_1 \psi(\tilde{v}^1; t) + \tau_2 \psi(\tilde{v}^2; t).$$

Thus ψ is linear in $\tilde{v}$. Next we show that ψ is bounded in $\tilde{L}^2(\Omega)$.

$$|\psi(\tilde{v}; t)| = \left| \int_\Omega \tilde{f}\tilde{v}\, d\boldsymbol{x} \right| \leq \int_\Omega |\tilde{f}||\tilde{v}|\, d\boldsymbol{x} \leq \|\tilde{f}\|_{\tilde{L}^2(\Omega)} \|\tilde{v}\|_{\tilde{L}^2(\Omega)}.$$

Since $\tilde{f}$ is bounded in $\tilde{L}^2(\Omega)$ and functions in $\tilde{L}^2(\Omega)$ are bounded the result follows. □

Remarks:

1. This theorem can be applied to *linear* hyperbolic PDEs by letting $g = 0$.
2. Since a_{ij} is not known a priori, conditions (iii) and (iv) must be verified after the collage method process has been completed.

With sufficient conditions for using the NHGCT, we now apply this theory to a few application problems.

6. Applications

Example 1. *Consider the following nonlinear hyperbolic PDE problem*

$$u_{tt}(x,t) - (\kappa(x,t)u_x(x,t))_x + b(x,t)u_x(x,t) + c(x,t)u(x,t) - g(u) = f(x,t) \ \text{in } \Omega_T, \quad (36)$$

$$u = 0 \ \text{on } \partial\Omega \times [0,T], \quad (37)$$

$$u = h_1(x) \ \text{on } \Omega \times \{t = 0\}, \quad (38)$$

$$u_t = h_2(x) \ \text{on } \Omega \times \{t = 0\}, \quad (39)$$

for $\Omega_T = \Omega \times (0,T]$ where $\Omega = (0,1)$ and $T = 10$. We discretize the problem with respect to time, thus performing collage coding on multiple time steps, each one at a different fixed value of t. We simulate observational data by sampling

$$u(x,t) = 4x(1-x)(1+t)$$

at N uniformly distributed nodes on $\Omega = (0,1)$ (at each fixed time step). To account for experimental error, we add noise that is normally distributed with mean 0 and standard deviation ϵ to these data points. Suppose further that we are given that

$$b(x,t) = x^2 - 4 \qquad c(x,t) = x \qquad g(u) = u^3 - 3u^2 + 4u$$

and $f:\mathbb{R}^n\times[0,T)\to\mathbb{R}$ is chosen so that the true value of $\kappa(x,t)$ is given by

$$\kappa_{\text{true}}(x,t)=(3\sin(x)+5)(5+t).$$

We recognize that in practice such information about the true value of $\kappa(x,t)$ is unknown and thus we use the knowledge of $\kappa_{\text{true}}(x,t)$ only for comparison purposes at the end of the generalized collage coding process to check our accuracy. Assuming a fifth-degree polynomial representation of $\kappa_{\text{collage}}(x,t)$,

$$\kappa(x,t_\tau)=\sum_{i=0}^{5}K_{i\tau}x^i$$

we must choose the coefficients $K_{i\tau}$ for $i=0,...,5,$ and $\tau=0,...,T$ so that the non-linear hyperbolic generalized collage distance is minimal. Following the development in Section 3, we build the weak formulation associated with (36)–(39) working in the infinite-dimensional space $\tilde{H}_0^1(\Omega)$ as suggested by the homogeneous Dirichlet boundary condition (37). The functionals B_λ and ψ are given in (7) and (8), respectively. Before continuing, we check that each of the sufficient conditions for using the NHGCT are satisfied.

For (i): g is Lipschitz in $\tilde{L}^2(\Omega)$

$$\begin{aligned}
\|g(\tilde u)-g(\tilde v)\|_{\tilde L^2(\Omega)} &= \|\tilde u^3-3\tilde u^2+4\tilde u-\tilde v^3+3\tilde v^2-4\tilde v\|_{\tilde L^2(\Omega)}\\
&\le \|\tilde u^3-\tilde v^3\|_{\tilde L^2(\Omega)}+3\|\tilde u^2-\tilde v^2\|_{\tilde L^2(\Omega)}+4\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}\\
&\le \|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}\|\tilde u^2+\tilde u\tilde v+\tilde v^2\|_{\tilde L^2(\Omega)}\\
&\qquad +3\|\tilde u+\tilde v\|_{\tilde L^2(\Omega)}\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}+4\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}\\
&\le (3\rho^2+6\rho+4)\|\tilde u-\tilde v\|_{\tilde L^2(\Omega)}.
\end{aligned}$$

Defining $K=3\rho^2+6\rho+4$ gives the result. Next, we show that there exists $C_g>0$ such that $\|g\|_{\tilde L^2(\Omega)}\le C_g\|\tilde u\|_{\tilde L^2(\Omega)}$.

$$\|g\|_{\tilde L^2(\Omega)}=\|\tilde u^3-3\tilde u^2+4\tilde u\|_{\tilde L^2(\Omega)}=\|\tilde u(\tilde u^2-3\tilde u+4)\|_{\tilde L^2(\Omega)}\le(\rho^2+3\rho+4|\Omega|^{\frac12})\|\tilde u\|_{\tilde L^2(\Omega)}.$$

Defining $C_g=\rho^2+3\rho+4|\Omega|^{\frac12}$ gives the result.

For (ii): $\tilde f$ is bounded in $\tilde L^2(\Omega)$ for a.e. $0\le t\le T$ Recall that f was chosen so that the remaining functional values of the problem were as stated above. By plugging these values into the left-hand side of the PDE, we arrive at an expression for f. We then compute the norm on $L^2(0,T;\tilde L^2(\Omega))$ of the expression for f. Using mathematical software we compute $\|\tilde f\|_{L^2(0,T;\tilde L^2(\Omega))}\approx 930.89$.

Finally for (iii), $\theta-\beta b-\beta^2c-\beta^2K>0$, we compute each quantity separately: First, we recall that θ is the uniform hyperbolicity constant which in one-dimension, is equivalent to taking

$$\theta=\inf_{(\boldsymbol{x},t)\in\Omega_T}\kappa(x,t).$$

Thus,

$$\theta = \inf_{(\boldsymbol{x},t)\in\Omega_T} (3\sin(x)+5)(5+t) = 25.$$

Now

$$\tilde{b} = \sup_{(\boldsymbol{x},t)\in\Omega_T} |x^2-4| = 4 \qquad \tilde{c} = \sup_{(\boldsymbol{x},t)\in\Omega_T} |x| = 1 \qquad \textit{and} \qquad \beta = \mathrm{diam}(\Omega) = 1.$$

So we calculate

$$\theta - \beta\tilde{b} - \beta^2\tilde{c} - \beta^2 K = 25 - (1)(4) - (1)^2(1) - (1)^2(3\rho^2+6\rho+4) = 16 - 3\rho^2 - 6\rho$$

Thus we must restrict the spaces $\tilde{H}_0^1(\Omega)$ *and* $\tilde{L}^2(\Omega)$ *so that*

$$\rho \in \left(0, \frac{-3+\sqrt{57}}{3}\right) \approx (0, 1.5166).$$

Having verified the sufficient conditions for using the NHGCT we proceed with implementation of the example.

To compute the nonlinear hyperbolic generalized collage distance (at each time step τ*), we approximate* $\tilde{H}_0^1(\Omega)$ *by a finite-dimensional subspace* V_N *of* $\tilde{H}_0^1(\Omega)$*. Partitioning the space* Ω *at* N *(uniformly distributed) points we let*

$$x_i = \frac{i}{N+1},$$

for $i = 0, ..., N+1$*. At each of these partition points, we define the hat basis for the subspace* V_N *by*

$$\xi_i(x) = \begin{cases} (N+1)(x - x_{i-1}), & x_{i-1} \le x \\ (N+1)(x_{i+1} - x), & x \le x_{i+1} \\ 0, & \textit{otherwise}, \end{cases}$$

for $i = 1, ..., N$*. We require a value of zero on the boundary of* Ω *and thus we define* $\xi_0(x) = 0 = \xi_{N+1}(x)$*. Using the hat basis functions as our test functions* $v(x)$ *and representing the target function,* $u(x)$ *in terms of the hat basis we must minimize the nonlinear hyperbolic generalized collage distance at each time step. We use a centred difference approximation to the second time derivative of* u *where possible, defining*

$$u''(x_i, t_\tau) = \frac{u(x_i, t_{\tau+2}) - 2u(x_i, t_\tau) + u(x_i, t_{\tau-2})}{4\Delta t^2}$$

and forward or backward difference approximations where necessary, given by

$$u''(x_i, t_\tau) = \frac{u(x_i, t_{\tau+2}) - 2u(x_i, t_{\tau+1}) + u(x_i, t_\tau)}{\Delta t^2}$$

$$u''(x_i, t_\tau) = \frac{u(x_i, t_\tau) - 2u(x_i, t_{\tau-1}) + u(x_i, t_{\tau-2})}{\Delta t^2},$$

respectively, where Δt *denotes the distance between consecutive time steps. The results for various numbers of sample points* N*,* $T = 10$ *with* $\Delta t = 1$*, and various amplitudes of noise* ϵ *are presented in Table 1. We report the average (over all time steps) of the* *approximation error in κ, denoted by $\frac{1}{T+1}\|\kappa_{\text{true}} - \kappa_{\text{collage}}\|_{\tilde{L}^2(\Omega)}$, the average generalized collage distance and the average approximation error in the solution u. As we expect, increasing the number of grid points (thus increasing the amount of given information) produces smaller errors. Looking at the results for different amplitudes of error ϵ, we see that the method responds well to error. We see that our errors indeed increase as ϵ increases. In an attempt to improve our results, we increase the degree of κ_{collage} in order to better approximate its true sinusoidal value. With this idea in mind, Table 2 presents the average approximation errors in κ and u for various degrees of κ_{collage}, $N = 10$ and $T = 10$ with $\Delta t = 1$. We see that as we increase the degree of κ_{collage} we indeed see a decrease in error. There is a limit to the success of this tactic for reducing error, as we require more grid points in order to avoid underdetermination of the problem. Finally, we recognize that error has been introduced by the approximation of the second time derivative in our collage distance. Table 3 presents results achieved by changing Δt. We see that our errors do decrease as we decrease the distance between successive time steps. However, in order to attain more significant decreases in error, a more substantial decrease in Δt is required. Computationally, this is an expensive choice for reducing error.*

N	ϵ	$\frac{1}{T}\|\kappa_{\text{true}} - \kappa_{\text{collage}}\|_{L^2(\Omega)}$	$\frac{1}{T}F_N(\lambda)$
10	0	0.72294×10^{-1}	0.67476×10^{-3}
	0.01	0.73041×10^{-1}	0.86716×10^{-2}
	0.10	0.27778	0.15821
20	0	0.35864×10^{-1}	0.19413×10^{-3}
	0.01	0.36737×10^{-1}	0.21619×10^{-2}
	0.10	0.21399	0.85396×10^{-1}
30	0	0.23975×10^{-2}	0.33264×10^{-4}
	0.01	0.26565×10^{-1}	0.43452×10^{-3}
	0.10	0.14593	0.20782×10^{-1}
40	0	0.17897×10^{-2}	0.31220×10^{-4}
	0.01	0.23814×10^{-1}	0.15861×10^{-3}
	0.10	0.10803	0.40696×10^{-2}

Table 1: Results of the generalized collage coding process on (36)–(39) for $T = 10$ with $\Delta t = 1$, various numbers of data points N, and levels of Gaussian noise added ϵ.

Degree of κ_{collage}	$\frac{1}{T}\lVert\kappa_{\text{true}} - \kappa_{\text{collage}}\rVert_{L^2(\Omega)}$	$\frac{1}{T}F_N(\lambda)$
2	0.83275	0.36728
3	0.21766	0.15066×10^{-1}
4	0.96592×10^{-1}	0.10830×10^{-2}
5	0.72294×10^{-1}	0.19413×10^{-3}
6	0.18796×10^{-1}	0.82307×10^{-5}
7	0.48871×10^{-2}	0.65379×10^{-9}
8	0.12706×10^{-2}	0.43307×10^{-12}
9	0.33037×10^{-3}	0.23085×10^{-12}
10	0.85895×10^{-4}	0.50596×10^{-13}

Table 2: Results for various degrees of κ_{collage} for $\Delta t = 1$ with $T = 10$ and $N = 10$ (and no noise added).

Δt	$\frac{1}{T}\lVert\kappa_{\text{true}} - \kappa_{\text{collage}}\rVert_{L^2(\Omega)}$	$\frac{1}{T}F_N(\lambda)$
0.01	0.32637×10^{-1}	0.30714×10^{-3}
0.025	0.32688×10^{-1}	0.48635×10^{-3}
0.05	0.32768×10^{-1}	0.68950×10^{-3}
0.075	0.32828×10^{-1}	0.84240×10^{-3}
0.1	0.32901×10^{-1}	0.97993×10^{-3}
0.25	0.33425×10^{-1}	0.15723×10^{-3}
0.5	0.34234×10^{-1}	0.22775×10^{-3}
0.75	0.34533×10^{-1}	0.27199×10^{-3}
1	0.35864×10^{-1}	0.67476×10^{-3}

Table 3: Results for various values of Δt with $T = 10$ and $N = 20$.

Example 2. *Consider the following nonlinear hyperbolic PDE problem*

$$u_{tt}(x,y,t) - \nabla \cdot (\kappa(x,y,t)\nabla u(x,y,t)) + \boldsymbol{b}(x,y,t) \cdot \nabla u(x,y,t) + c(x,y,t)u(x,y,t) - g(u) = f(x,y,t) \text{ in } \Omega_T, \quad (40)$$

$$u = 0 \text{ on } \partial\Omega \times [0,T], \quad (41)$$

$$u = h_1(x,y) \text{ on } \Omega \times \{t=0\}, \quad (42)$$

$$u_t = h_2(x,y) \text{ on } \Omega \times \{t=0\}, \quad (43)$$

where $\Omega_T = \Omega \times (0,T]$, $\Omega = (0,1) \times (0,1)$ and $T = 5$. (40)–(43) is the two-dimensional equivalent to the problem studied in Example 1. We follow a similar technique to that of Example 1, this time defining

$$\boldsymbol{b}(x,y,t) = \begin{pmatrix} xy \\ x^2 \end{pmatrix} \qquad c(x,y,t) = x^2 y \qquad g(u) = u(1-u)$$

and that $f : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ is chosen so that the true value of $\kappa(x,y,t)$ is given by

$$\kappa_{\text{true}}(x,y,t) = \begin{bmatrix} (10+3x+2y+xy)(1+t) & 0 \\ 0 & (14+x+y+2x^2)(1+t) \end{bmatrix}.$$

The choice of a diagonal κ is consistent with many examples in the literature, particularly those in biological modelling. Further, this greatly reduces computing time since fewer parameters need to be recovered in the diagonal case. A more rigorous discussion is required to extend this work to the non-diagonal case.

To generate data values, we sample

$$u(x,y,t) = \frac{1}{5}\sin(\pi x)\sin(\pi y)(1+t)$$

at $N \times N$ uniformly distributed nodes on $\Omega = (0,1) \times (0,1)$ (at each fixed time step). To account for experimental error, we add normally distributed noise with mean 0 and standard deviation ϵ to these data points. We assume a componentwise polynomial representation of $\kappa(x,y,t)$,

$$\kappa(x,y,t_\tau) = \begin{bmatrix} \displaystyle\sum_{i=0}^{2}\sum_{j=0}^{1} K^1_{ij\tau} x^i y^j & 0 \\ 0 & \displaystyle\sum_{i=0}^{1}\sum_{j=0}^{1} K^2_{ij\tau} x^i y^j \end{bmatrix},$$

we must choose the coefficients $K^1_{ij\tau}$ and $K^2_{ij\tau}$ so that the nonlinear hyperbolic generalized collage distance is minimal. Again following the development in Section 3 and working in the infinite-dimensional space $\tilde{H}^1_0(\Omega)$ as suggested by the homogeneous Dirichlet boundary condition (41) the functionals B_λ and ψ associated with (40)–(43) are given by (7) and (8), respectively. Before continuing, we check that each of the sufficient conditions for using the NHGCT are satisfied.

For (i): $\bar{g}$ is Lipschitz in $\tilde{L}^2(\Omega)$

$$\begin{aligned}
\|\bar{g}(\tilde{u}) - \bar{g}(\tilde{v})\|_{\tilde{L}^2(\Omega)} &= \|\tilde{u}(1-\tilde{u} - \tilde{v}(1-\tilde{v})\|_{\tilde{L}^2(\Omega)} \\
&\leq \|\tilde{u} - \tilde{v}\|_{\tilde{L}^2(\Omega)} + \|\tilde{u}^2 - \tilde{v}^2\|_{\tilde{L}^2(\Omega)} \\
&= \|\tilde{u} - \tilde{v}\|_{\tilde{L}^2(\Omega)} + \|\tilde{u} + \tilde{v}\|_{\tilde{L}^2(\Omega)}\|\tilde{u} - \tilde{v}\|_{\tilde{L}^2(\Omega)} \\
&= (1 + \|\tilde{u} + \tilde{v}\|_{\tilde{L}^2(\Omega)})\|\tilde{u} - \tilde{v}\|_{\tilde{L}^2(\Omega)} \\
&\leq (2\rho + 1)\|\tilde{u} - \tilde{v}\|_{\tilde{L}^2(\Omega)}.
\end{aligned}$$

Defining $K = 2\rho + 1$ gives the result. Next, we show that there exists $C_g > 0$ such that $\|\bar{g}\|_{\tilde{L}^2(\Omega)} \leq C_g \|\tilde{u}\|_{\tilde{L}^2(\Omega)}$.

$$\|\bar{g}\|_{\tilde{L}^2(\Omega)} = \|\tilde{u}(1-\tilde{u})\|_{\tilde{L}^2(\Omega)} = \|1 - \tilde{u}\|_{\tilde{L}^2(\Omega)}\|\tilde{u}\|_{\tilde{L}^2(\Omega)} \leq (\rho + |\Omega|^{\frac{1}{2}})\|\tilde{u}\|_{\tilde{L}^2(\Omega)}.$$

Defining $C_g = \rho + |\Omega|^{\frac{1}{2}}$ gives the result.

For (ii): $\bar{f}$ is bounded in $\tilde{L}^2(\Omega)$ for a.e. $0 \leq t \leq T$ Once again, recall that f was chosen so that the remaining functional values of the problem were as stated above. By plugging these values into the left-hand side of the PDE, we arrive at an expression for f. We then compute the norm on $L^2(0,T;\tilde{L}^2(\Omega))$ of the expression for f. Using mathematical software we compute $\|\bar{f}\|_{L^2(0,T;\tilde{L}^2(\Omega))} \approx 1105.6$.

Finally for (iii), $\theta - \beta b - \beta^2 c - \beta^2 K > 0$, we compute each quantity separately. First, we recall that θ is the uniform hyperbolicity constant of the operator L which in two-dimensions, is less or equal to the smallest eigenvalue of the matrix $\kappa(x,y,t)$. Since the matrix $\kappa(x,y,t)$ is diagonal, we will take

$$\begin{aligned}
\theta &= \min\left\{\inf_{(\boldsymbol{x},t)\in\Omega_T} \kappa_{11}(x,y,t), \inf_{(\boldsymbol{x},t)\in\Omega_T} \kappa_{22}(x,y,t)\right\} \\
&= \min\left\{\inf_{(\boldsymbol{x},t)\in\Omega_T} (10 + 3x + 2y + xy)(1+t), \inf_{(\boldsymbol{x},t)\in\Omega_T} (14 + x + y + 2x^2)(1+t)\right\} \\
&= \min\{10, 14\} \\
&= 10.
\end{aligned}$$

Now

$$\tilde{b} = \sup_{(\boldsymbol{x},t)\in\Omega_T} |xy| + \sup_{(\boldsymbol{x},t)\in\Omega_T} |x^2| = 2 \qquad \tilde{c} = \sup_{(\boldsymbol{x},t)\in\Omega_T} x^2 y = 1 \qquad \textit{and} \qquad \beta = \operatorname{diam}(\Omega) = \sqrt{2}.$$

So we calculate

$$\theta - \beta\tilde{b} - \beta^2\tilde{c} - \beta^2 K = 10 - (\sqrt{2})(2) - (\sqrt{2})^2(1) - (\sqrt{2})^2(2\rho + 1) = 6 - 2\sqrt{2} - 4\rho$$

Thus we must restrict the spaces $\tilde{H}_0^1(\Omega)$ and $\tilde{L}^2(\Omega)$ so that $\rho < \frac{3}{2} - \frac{1}{\sqrt{2}} \approx 0.79289$.

To compute the nonlinear hyperbolic generalized collage distance (at each time step τ), we approximate $\tilde{H}_0^1(\Omega)$ by a finite-dimensional subspace V_N of $\tilde{H}_0^1(\Omega)$. Partitioning the space Ω at $N \times N$ (uniformly distributed) points we let

$$(x_i, y_j) = \left(\frac{i}{N+1}, \frac{j}{N+1}\right),$$

for $i, j = 0, ..., N+1$. A common choice for a basis for two spatial dimensions is hexagonal-based pyramids. For each partition point in Ω, the corresponding basis function $\xi_{ij}(x, y)$ has height 1 at (x_i, y_j) and 0 at neighbouring partition points that form a hexagon. Neighbouring points are then joined with (x_i, y_j) by planes. We require a value of zero on the boundary of Ω and thus we define all basis functions on $\partial\Omega$ to be zero. We use these basis functions as our test functions $v(x, y)$ and to represent the target function, $u(x, y, t)$ in terms of this basis. To deal with the time derivatives present in the nonlinear hyperbolic generalized collage distance we again use a centred difference approximation where possible and forward or backward difference approximations where necessary.

Table 4 presents the results of generalized collage coding for various numbers of grid points and amounts of Gaussian noise for $T = 5$ with $\Delta t = 1$.

N	ϵ	$\frac{1}{T}\|\kappa_{\text{true}} - \kappa_{\text{collage}}\|_{L^2(\Omega)}$	$\frac{1}{T}F_N(\lambda)$
10	0	0.18920×10^{-1}	0.13097×10^{-2}
	0.01	0.18942×10^{-1}	0.46764×10^{-2}
	0.10	0.68203	0.44894×10^{-1}
20	0	0.23416×10^{-2}	0.12229×10^{-3}
	0.01	0.23882×10^{-2}	0.40248×10^{-2}
	0.10	0.13038	0.36820×10^{-1}
30	0	0.70896×10^{-3}	0.27722×10^{-4}
	0.01	0.18355×10^{-2}	0.31594×10^{-2}
	0.10	0.18994×10^{-1}	0.31476×10^{-1}
40	0	0.29932×10^{-3}	0.98436×10^{-5}
	0.01	0.77294×10^{-3}	0.30382×10^{-2}
	0.10	0.10210×10^{-1}	0.29926×10^{-1}

Table 4: Results of the generalized collage coding process on (40)–(43) for $T = 5$ with $\Delta t = 1$, various numbers of data points N, and levels of Gaussian noise added ϵ.

As in Example 1, we see that increasing the number of grid points improves our results. One explanation for this improvement lies in functional forms. In Example 1 error was introduced because the functional forms of κ_{true} and κ_{collage} differed. Conversely, in this example, both functional forms are the same (polynomial) and the method performs exceptionally well. We also see that increasing the amount of Gaussian noise in our data increases error (as expected). While the method tolerates low amplitudes of Gaussian noise well, it seems to struggle with higher levels of noise. One way to combat this issue and achieve better results by decreasing the step size between successive time steps as was noted in the discussion of Example 1.

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