THE WRONSKIAN SOLUTION OF THE CONSTRAINED DISCRETE KP HIERARCHY

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Abstract. From the constrained discrete KP (cdKP) hierarchy, the Ablowitz-Ladik lattice has been derived. By means of the gauge transformation, the Wronskian solution of the Ablowitz-Ladik lattice have been given. The u_1 of the cdKP hierarchy is a Y-type soliton solution for odd times of the gauge transformation, but it becomes a dark-bright soliton solution for even times of the gauge transformation. The role of the discrete variable n in the profile of the u_1 is discussed.

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1. Introduction

In the past few years, lots of attention have been given to the study of Kadomtsev-Petviashvili (KP) hierarchy [1, 2] in the field of integrable systems. The Lax pairs, Hamiltonian structures, symmetries and conservation laws, the N-soliton, tau function, the gauge transformation, reductions etc. of the KP hierarchy and its sub-hierarchies have been discussed. There are several sub-hierarchies of the KP by considering different reduction conditions on the Lax operator L. One of them is called constrained KP (cKP) hierarchy [3, 4, 5] by setting the Lax operator as $L = \partial + \sum_{i=1}^{m} \Phi_i \partial^{-i} \Psi_i$. The cKP hierarchy contains a large number of interesting soliton equations. The basic idea of this procedure is so-called symmetry constraint [3, 4, 5]. The negative part of the Lax operator of the constrained KP, i.e. $\sum_{i=1}^{m} \Phi_i \partial^{-i} \Psi_i$, is a generator [2] of the additional symmetry [6] of the KP hierarchy. And the additional symmetry of BKP hierarchy and CKP hierarchy have been given [7, 8]. Very recently, by a further modification of the additional flows, the additional symmetries of the constrained BKP and constrained CKP hierarchies are given in references [9, 10].

It is well known that a continuous integrable system has a discrete analogue in general. The famous 3-dimensional difference equation is known to provide a canonical integrable discretization for most important types of soliton equations. There are several different kinds of the discrete hierarchies including differential-difference KP (dKP) hierarchy [11, 12], semi-discrete

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integrable systems, full discrete equations and so on. The differential-difference KP hierarchy, defined by the difference operator Δ, is one interesting object of the discrete integrable systems. Note that, the additional symmetry of dKP hierarchy and it's Sato Bäcklund transformations have been given in reference [13]. Moreover, gauge transformation is one kind of powerful method to construct the solutions of the integrable systems for both the continuous KP hierarchy [14, 15, 16, 17, 18, 19, 20, 21, 22] and the dKP hierarchy [23, 24]. It is discussed to reduce the gauge transformation of the dKP hierarchy to the constrained discrete KP(cdKP) hierarchy [25]. And the algebraic structure of the additional symmetry of the cdKP hierarchy also has been found [26], which is same for the cKP hierarchy [8].

A crucial observation [12] about the KP hierarchy and the dKP hierarchy is that the τ function of the discrete KP hierarchy can be constructed by shift of the t of the τ function of the continuous KP hierarchy. It is an interesting question to find any other difference among the two hierarchies. In this direction, the correspondence between the solutions of discrete and continuous hierarchy can be used to explore the difference between them. In particular, a key step is to demonstrate how the discrete variable n affects the profile of the solutions of the dKP hierarchy.

The purpose of this paper is to find the the correspondence between the solutions of the KP hierarchy and the dKP hierarchy by means of the multi-channel gauge transformation. The paper is organized as follows. Some basic results of the dKP hierarchy and the cdKP hierarchy are summarized in Section 2. The main theorem about the solution of cdKP hierarchy are give in Section 3. An example is give in section 4. We find that the odd kinds of gauge transformation of cdKP hierarchy can change to a new profile of solution of the cdKP hierarchy. Section 5 is devoted to conclusions and discussions.

2. THE CDKP HIERARCHY

Let L be a general first-order pseudo difference operator(PDO)

$$L(n) = \Delta + \sum_{i=1}^{\infty} u_i(n)\Delta^{-i}, \tag{2.1}$$

the cdKP hierarchy [26] is defined by the following Lax equation

$$\frac{\partial L}{\partial t_l} = [B_l, L], B_l := (L^l)_+, l = 1, 2, \cdots, \tag{2.2}$$

associated with a constrained Lax operator

$$L^l_- = \sum_{i=1}^{m} q_i(t)\Delta^{-1}r_i(t), \tag{2.3}$$

which is m-components Lax operator of the cdKP hierarchy. It has relation between the dynamical variables q_i, r_i and u_i. Specially, $u_1 = q_1\Lambda^{-1}(r_1)$, where $\Delta = \Lambda - I$. The eigenfunction and adjoint eigenfunction $q_i(t), r_i(t)$ are important dynamical variables in the cdKP hierarchy.

It can be checked that the Lax equation (2.2) is consistent with the evolution equations of the eigenfunction (or adjoint eigenfunction)

$$\begin{cases} q_{i,t_m} = B_m q_i, \\ r_{i,t_m} = -B_m^* r_i, \quad B_m = (L^m)_+, \forall m \in N. \end{cases} \tag{2.4}$$

Therefore the cdKP hierarchy in eq.(2.2) is well defined.

From the Lax equation (2.2), we get the first nontrival t_2 flow equations of the cdKP hierarchy for $m = 1, l = 2$ as

$$\begin{cases} q_{1,t_2} = \Delta^2 q_1 + 2q_1^2 r_1 = q_1(n+2) - 2q_1(n+1) + q_1(n) + 2q_1^2 r_1, \\ r_{1,t_2} = -\Delta^{*2} r_1 + 2q_1 r_1^2 = r_1(n) - 2r_1(n-1) + r_1(n-2) + 2q_1(n) r_1(n)^2. \end{cases} \tag{2.5}$$

It is nothing but the Ablowitz-Ladik lattice [27]. It can be reduced to the discrete non-linear Schrödinger (DNLS) equation [28] by letting $r_1 = q_1^*$ and a scaling transformation $t_2 = it_2$.

The Lax operator in eq.(2.3) can be generated by the dressing action

$$L = W \circ \Delta \circ W^{-1}, \tag{2.6}$$

with a dressing operator

$$W(n;t) = 1 + \sum_{j=1}^{\infty} w_j(n;t) \Delta^{-j}. \tag{2.7}$$

Further the flow equation (2.2) is equivalent to the so-called Sato equation,

$$\partial_{t_l} W = -(L^l)_- \circ W. \tag{2.8}$$

Denote the exponential function as following

$$Exp(n;t,z) = (1+z)^n exp(\sum_{i=1}^{\infty} t_i z^i) = exp(\sum_{i=1}^{\infty}(t_i + n\frac{(-1)^{i-1}}{i})z^i), \tag{2.9}$$

then

$$\Delta Exp(n;t,z) = zExp(n;t,z), \Delta^* Exp^{-1}(n;t,z) = zExp^{-1}(n;t,z). \tag{2.10}$$

There are the wave function $w(n;t,z)$ and the adjoint wave function $w^*(n;t,z)$ for the dKP hierarchy as the following forms:

$$w(n;t,z) = W(n;t)Exp(n;t,z) = (1+\frac{w_1(n;t)}{z}+\frac{w_2(n;t)}{z^2}+\cdots)exp(\sum_{i=1}^{\infty}(t_i+n\frac{(-1)^{i-1}}{i})z^i) \tag{2.11}$$

and

$$\begin{aligned} w^*(n;t,z) &= (W^{-1}(n-1;t))^* Exp^{-1}(n;t,z) \\ &= (1 + \frac{w_1^*(n;t)}{z} + \frac{w_2^*(n;t)}{z^2} + \cdots)exp(\sum_{i=1}^{\infty} -(t_i + n\frac{(-1)^{i-1}}{i})z^i). \end{aligned} \tag{2.12}$$

There also exists a τ function $\tau(n;t)$ for the dKP hierarchy [12] such that the wave function is expressed by

$$w(n;t,z) = \frac{\tau(n,t-[z^{-1}])}{\tau(n,t)} Exp(n;t,z), \tag{2.13}$$

and the adjoint wave function is expressed by

$$w^*(n;t,z) = \frac{\tau(n,t+[z^{-1}])}{\tau(n,t)} Exp^{-1}(n;t,z), \tag{2.14}$$

where $[z] = (z, z^2/2, x^3/3, \cdots)$.

The difference $\Delta-$Wronskian [24]

$$\tau_\Delta(n) = W_m^\Delta(q_1, q_2, \ldots, q_m) = \begin{vmatrix} q_1 & q_2 & \cdots & q_m \\ \Delta q_1 & \Delta q_2 & \cdots & \Delta q_m \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{m-1} q_1 & \Delta^{m-1} q_2 & \cdots & \Delta^{m-1} q_m \end{vmatrix} \tag{2.15}$$

is a τ function of dKP hierarchy. In this section, we will reduce $\tau_\Delta(n)$ in (2.15) to a τ function of the constrained discrete KP hierarchy.

Now we consider a chain of gauge transformation operator of multi-channel difference type T_d [19, 21, 25] starting from the initial m-component Lax operator $L^{(0)} = L = L_+ + \sum_{i=1}^m q_i(t)\Delta^{-1} r_i(t)$,

$$L^{[0]} \xrightarrow{T_d^{[1]}(q_1^{[0]})} L^{[1]} \xrightarrow{T_d^{[2]}(q_2^{[1]})} L^{[2]} \to \cdots \to L^{[n-1]} \xrightarrow{T_d^{[n]}(q_n^{[n-1]})} L^{[n]}. \tag{2.16}$$

Here the index i in the gauge transformation operator $T_d^{[i]}(q_j^{[j-1]})$ $(j > i)$means the i-th gauge transformation, and $q_j^{[j-1]}$ (or $r_j^{[j-1]}$) is transformed by $(j-1)$-steps gauge transformations from q_j (or r_j), $L^{[k]}$ is transformed by k-steps gauge transformations from the initial Lax operator L.

Now we firstly consider successive gauge transformations in (2.16). We define the operator as

$$T_m = T_d^{[m]}(q_m^{[m-1]}) \circ \cdots \circ T_d^{[2]}(q_2^{[1]}) \circ T_d^{[1]}(q_1^{[0]}), \tag{2.17}$$

in which

$$q_i^{[j]} = T_d^{[j]}(q_j^{[j-1]}) \circ \cdots \circ T_d^{[2]}(q_2^{[1]}) \circ T_d^{[1]}(q_1^{[0]}) q_i, i, j = 1, \cdots, m; \tag{2.18}$$

$$r_k^{[j]} = ((T_d^{[j]})^{-1})^*(q_j^{[j-1]}) \circ \cdots \circ ((T_d^{[2]})^{-1})^*(q_2^{[1]}) \circ ((T_d^{[1]})^{-1})^*(q_1^{[0]}) r_k, j, k = 1, \cdots, m. \tag{2.19}$$

It means that $q_i^{[0]} = q_i$, $r_i^{[0]} = r_i$. We shall find another criterion for the Wronskian entries $f_1, f_2, \cdots, f_n$ leading to cdKP flows. The following theorem can be easily got from the Ref. [25].

Theorem 2.1. *The gauge transformation operator T_m and T_m^{-1} have the following determinant representation:*

$$T_m = T_d^{[m]}(q_m^{[m-1]}) \circ \cdots \circ T_d^{[2]}(q_2^{[1]}) \circ T_d^{[1]}(q_1^{[0]})$$
$$= \frac{1}{W_m^{\Delta}(q_1,q_2,\ldots,q_m)} \begin{vmatrix} q_1 & q_2 & \cdots & q_m & 1 \\ \Delta q_1 & \Delta q_2 & \cdots & \Delta q_m & \Delta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta^{m-1} q_1 & \Delta^{m-1} q_2 & \cdots & \Delta^{m-1} q_m & \Delta^{m-1} \\ \Delta^m q_1 & \Delta^m q_2 & \cdots & \Delta^m q_m & \Delta^m \end{vmatrix}, \tag{2.20}$$

and

$$T_m^{-1} = \begin{vmatrix} q_1 \circ \Delta^{-1} & \Lambda(q_1) & \Lambda(\Delta q_1) & \cdots & \Lambda(\Delta^{m-2} q_1) \\ q_2 \circ \Delta^{-1} & \Lambda(q_2) & \Lambda(\Delta q_2) & \cdots & \Lambda(\Delta^{m-2} q_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_m \circ \Delta^{-1} & \Lambda(q_m) & \Lambda(\Delta q_m) & \cdots & \Lambda(\Delta^{m-2} q_m) \end{vmatrix} \frac{(-1)^{m-1}}{\Lambda(W_m^{\Delta}(q_1,q_2,\ldots,q_m))}$$
$$= \sum_{i=1}^{m} \phi_i \circ \Delta^{-1} b_i \tag{2.21}$$

with

$$b_i = (-1)^{m+i} \Lambda\left(\frac{W_m^{\Delta}(q_1,q_2,\ldots,q_{i-1},\hat{i},q_{i+1},\ldots,q_m)}{W_m^{\Delta}(q_1,q_2,\ldots,q_{i-1},q_i,q_{i+1},\ldots,q_m)}\right). \tag{2.22}$$

Here $\hat{i}$ means that the column containing q_i is delete from $W_m^{\Delta}(q_1,q_2,\ldots,q_{i-1},q_i,q_{i+1},\ldots,q_m)$ and the last row is also deleted. Here the determinant of T_m is expanded by the last column and collecting all sub-determinants on the left side of the Δ^i with the action " $\circ$ ". And T_m^{-1} is expanded by the first column and all the sub-determinants are on the right side with the action " $\circ$ ".

3. Wronskian solution of constrained discrete KP hierarchy

Similar to reference [29], it has [26]

$$(K \circ q \circ \Delta^{-1} \circ r)_- = K(q) \circ \Delta^{-1} \circ r, (q \circ \Delta^{-1} \circ r \circ K)_- = -q \circ \Delta^{-1} \circ K^*(r), \tag{3.1}$$

for a pure-difference operator K and two arbitrary smooth functions (q, r).

An important fact is that there exist two m-th order Δ-differential operators

$$A = \Delta^m + a_{m-1}\Delta^{m-1} + \cdots + a_0, B = \Delta^m + b_{m-1}\Delta^{m-1} + \cdots + b_0, \tag{3.2}$$

such that AL^l and L^lB are differential operators. From $(AL^l)_- = 0$ and $(L^lB)_- = 0$, we get that A and B annihilate the functions q^i and r_j , i.e., $A(q_1) = \cdots = A(q_m) = 0$, $B^*(r_1) = \cdots = B^*(r_m) = 0$, that implies $q_i \in Ker(A)$, $r_i \in Ker(B^*)$, $i = 1, \ldots, m$. The dimension of $Ker(A)$ is m.

The following theorem provides a criterion for reducing the Δ-Wronskian τ function in (2.15) of dKP hierarchy to the Δ-Wronskian τ function of the cdKP hierarchy defined by (2.2).

Theorem 3.1. *The constrained discrete KP hierarchy has a solution $L = (L)_+ + \sum_{j=1}^{m} f_j \circ \Delta \circ g_j$ generated by the τ function $\tau_\Delta(n) = W_m^\Delta(f_1, \cdots, f_m) \neq 0$ satisfies the k-constrained with some suitable functions $q_1, q_2, \cdots, q_M$ and $r_1, r_2, \cdots, r_M$ if and only if*

$$W^\Delta_{m+M+1}(f_1, \cdots, f_m, \Delta^k f_{i_1}, \cdots, \Delta^k f_{i_{M+1}}) = 0 \tag{3.3}$$

for all $(M + 1)$ indices $1 \leq i_1 < i_2 < \cdots < i_{M+1} \leq m$, which is equivalent to

$$W^\Delta_{M+1}\Big(\frac{W^\Delta_{m+1}(f_1, \ldots, f_m, \Delta^k f_{i_1})}{W^\Delta_m(f_1, \ldots, f_m)}, \frac{W^\Delta_{m+1}(f_1, \ldots, f_m, \Delta^k f_{i_2})}{W^\Delta_m(f_1, \ldots, f_m)}, \cdots, \frac{W^\Delta_{m+1}(f_1, \ldots, f_m, \Delta^k f_{i_{M+1}})}{W^\Delta_m(f_1, \ldots, f_m)}\Big) = 0. \tag{3.4}$$

Here f_i satisfied linear $\Delta-$difference equations

$$\frac{\partial f_i}{\partial t_k} = \Delta^k f_i, \quad i = 1, 2, \cdots, m; k = 1, 2, \cdots. \tag{3.5}$$

Proof. Similar to case of KP hierarchy [30] and q-KP hierarchy [31], there has the following Wronskian identity

$$W^\Delta_{m+M+1}\Big(\frac{W^\Delta_{m+1}(f_1, \ldots, f_m, \Delta^k f_{i_1})}{W^\Delta_m(f_1, \ldots, f_m)}, \frac{W^\Delta_{m+1}(f_1, \ldots, f_m, \Delta^k f_{i_2})}{W^\Delta_m(f_1, \ldots, f_m)}, \cdots, \frac{W^\Delta_{m+1}(f_1, \ldots, f_m, \Delta^k f_{i_{M+1}})}{W^\Delta_m(f_1, \ldots, f_m)}\Big)$$
$$= W_m(f_1, \cdots, f_m, \Delta^k f_{i_1}, \cdots, \Delta^k f_{i_{M+1}}) \tag{3.6}$$

which provides the equivalence between (3.3) and (3.4).

Using Theorem 2.1 and the relation of operator identities in (3.1), one finds

$$(L^k)_- = (W \circ \Delta^k \circ W^{-1})_- = \sum_{j=1}^{m} W(\Delta^k f_j)\Delta^{-1} g_j \tag{3.7}$$

As it was pointed out in the beginning of this section, there exists an m-th order differential operator A such that AL^k is a difference operator. Application to $W(f_j) = 0$ yields

$$0 = AL^k(W(f_j)) = AW \circ \Delta^k(f_j) = AW(\partial_{t_k}(f_j)). \tag{3.8}$$

So,

$$W(\partial_{t_k}(f_j)) = \frac{W^\Delta_m(f_1, \cdots, f_m, \Delta^k f_j)}{W^\Delta_m(f_1, \cdots, f_m)} \in Ker(A). \tag{3.9}$$

Since the kernel of A has dimension m, at most m of these functions $\Delta^k f_j$ can be linearly independent. So, (3.4) is deduced.

Conversely, if (3.4) holds, then there exists one M-component of cdKP $(M < m)$ constrained from (3.7). The equation (3.4) implies that at most M of functions $W(\Delta^k(f_j))$ are linearly independent, here f_j satisfy (3.5). Then we can find suitable M functions $q_1, q_2, \ldots, q_M$, which are linearly independent, to express functions $W(\Delta^k(f_j))$ as

$$W(\partial_{t_k}(f_j)) = \frac{W^\Delta_m(f_1, \cdots, f_m, \Delta^k f_j)}{W^\Delta_m(f_1, \cdots, f_m)} = \sum_{i=1}^{M} c_{ij} q_i, j = 1, \ldots, m. \tag{3.10}$$

with some constant c_{ij}. Taking it back into the (3.7), it becomes

$$(L^k)_- = \sum_{j=1}^{m}(\sum_{i=1}^{M} c_{ij}q_i) \circ \Delta^{-1} \circ g_j = \sum_{i=1}^{M} q_i \circ \Delta^{-1} \circ (\sum_{j=1}^{m} c_{ij}g_j) = \sum_{i=1}^{M} q_i \circ \Delta^{-1} \circ r_i, \quad (3.11)$$

then a m-component cdKP hierarchy is reduced to a M-component cdKP hierarchy. □

Remark: This theorem is a difference version of the corresponding theorem of the Ref. [30].

The Wronskian solution of the cdKP hierarchy can be got by the Theorem 3.1 under the gauge transformation. If the initial Lax operator of the constrained discrete KP hierarchy is a "free" operator Δ, then $L = \Delta$ means that the initial τ function τ_Δ is 1.

4. Example of reducing dKP hierarchy to cdKP hierarchy

In this section, we use the method in Theorem 3.1 to find the solution of the multi-component cdKP hierarchy. We discuss the cdKP hierarchy generated by $T_i\mid_{i=2}$, possesses a τ function

$$\tau_\Delta^{(2)} = W_2^\Delta(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ \Delta f_1 & \Delta f_2 \end{vmatrix} = f_1 \circ \Delta f_2 - f_2 \circ \Delta f_1, \quad (4.1)$$

with

$$f_1 = f_{11}(z_1, n, t) + f_{12}(z, n, t), f_2 = f_{21}(z_2, n, t) + f_{22}(z_3, n, t). \quad (4.2)$$

Here

$$f_{11}(z_1, n, t) = (1 + z_1)^n e^{\xi_1}, f_{12}(z, n, t) = (1 + z)^n e^{\xi},$$
$$f_{21}(z_2, n, t) = (1 + z_2)^n e^{\xi_2}, f_{22}(z_3, n, t) = (1 + z_3)^n e^{\xi_3},$$

where $\xi_i = c_i + z_i t_1 + z_i^2 t_2 + z_i^3 t_3, i = 1, 2, 3$ and $\xi = d + z t_1 + z^2 t_2 + z^3 t_3$, c_i and d are arbitrary constants. These functions f_1 and f_2 satisfy linear equations (3.5) for $k = 1, 2, 3$. By (3.5), the cdKP hierarchy generated by T_2 is in the form of

$$\begin{aligned} L^l &= (L^l)_+ + (T_2(\Delta^k f_1)) \circ \Delta^{-1} \circ g_1 + (T_2(\Delta^k f_2)) \circ \Delta^{-1} \circ g_2 & (4.3)\\ &\overset{constraint}{=\!=\!=\!=} (L^l)_+ + q_1 \circ \Delta^{-1} \circ r_1, & (4.4)\end{aligned}$$

where q_1, r_1 are undetermined, which can be expressed by f_1 and f_2 as follows. $\tau_\Delta^{(2)}$ possesses a form as

$$\begin{aligned} \tau_\Delta^{(2)} &= W_2^\Delta(f_1, f_2) \\ &= (z_2 - z_1)(1 + z_1)^n (1 + z_2)^n e^{\xi_1+\xi_2} + (z_2 - z)(1 + z)^n (1 + z_2)^n e^{\xi+\xi_2} \\ &\quad +(z_3 - z_1)(1 + z_1)^n (1 + z_3)^n e^{\xi_1+\xi_3} + (z_3 - z)(1 + z)^n (1 + z_3)^n e^{\xi+\xi_3}. \quad (4.5)\end{aligned}$$

According to (3.3) in Theorem 3.1, the restriction for f_1 and f_2 to reduce (4.3) to (4.4) is given by

$$\begin{aligned} 0 &= W_4^\Delta(f_1, f_2, f_1^{(k)}, f_2^{(k)}) \\ &= (z^k - z_1^k)(z_2^k - z_3^k) V(z_1, z_2, z_3, z) F(n; z_i, t), \quad (4.6)\end{aligned}$$

with Vandermonde determinant

$$V(z_1,z_2,z_3,z)=\begin{vmatrix} 1 & 1 & 1 & 1\\ z_1 & z_2 & z_3 & z\\ z_1^2 & z_2^2 & z_3^2 & z^2\\ z_1^3 & z_2^3 & z_3^3 & z^3 \end{vmatrix}, \tag{4.7}$$

and

$$F(n;z_i,t)=(1+z_1)^n(1+z_2)^n(1+z_3)^n(1+z)^n e^{\xi_1+\xi_2+\xi_3+\xi}. \tag{4.8}$$

Obviously, f_1 and f_2 satisfy (4.6) by setting $z=z_2$ and $d=c_2$. Then the τ function of a single component k-constrained cdKP hierarchy defined by

$$\begin{aligned}\tau^{\Delta}_{cdKP} \;=\;& (z_2-z_1)(1+z_1)^n(1+z_2)^n e^{\xi_1+\xi_2}+(z_3-z_1)(1+z_1)^n(1+z_3)^n e^{\xi_1+\xi_3}\\ &+(z_3-z_2)(1+z_2)^n(1+z_3)^n e^{\xi_2+\xi_3},\end{aligned} \tag{4.9}$$

which is deduced by (4.5) with $\xi_2=\xi$. It means that we indeed reduced the τ function $\tau^{(2)}_{\Delta}$ in (4.5) of the dKP hierarchy to the τ function τ^{Δ}_{cdKP} of the 1-component cdKP hierarchy.

We would like to get the explicit forms of q_1 and r_1 of cdKP hierarchy in (4.4). With the determinant representation of T_2 and T_2^{-1}, one can have

$$f_1^{\Delta}\triangleq T_2(\Delta^k f_1)=\frac{(z_1^k-z_2^k)V(z_1,z_2,z_3)(1+z_1)^n(1+z_2)^n(1+z_3)^n e^{\xi_1+\xi_2+\xi_3}}{\tau^{\Delta}_{cdKP}} \tag{4.10a}$$

$$f_2^{\Delta}\triangleq T_2(\Delta^k f_2)=\frac{(z_3^k-z_2^k)V(z_1,z_2,z_3)(1+z_1)^n(1+z_2)^n(1+z_3)^n e^{\xi_1+\xi_2+\xi_3}}{\tau^{\Delta}_{cdKP}} \tag{4.10b}$$

$$g_1^{\Delta}\triangleq (T_2^{-1})^*(\Delta^k g_1)=-\Lambda(\frac{f_2}{W_2^{\Delta}(f_1,f_2)})=-\Lambda(\frac{(1+z_2)^n e^{\xi_2}+(1+z_3)^n e^{\xi_3}}{\tau^{\Delta}_{cdKP}}) \tag{4.10c}$$

$$g_2^{\Delta}\triangleq (T_2^{-1})^*(\Delta^k g_1)=\Lambda(\frac{f_1}{W_2^{\Delta}(f_1,f_2)})=\Lambda(\frac{(1+z_1)^n e^{\xi_1}+(1+z_2)^n e^{\xi_2}}{\tau^{\Delta}_{cdKP}}) \tag{4.10d}$$

with the Vandermonde determinant

$$V(z_1,z_2,z_3)=\begin{vmatrix} 1 & 1 & 1\\ z_1 & z_2 & z_3\\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix}. \tag{4.11}$$

It is clearly

$$(z_3^k-z_2^k)f_1^{\Delta}=(z_1^k-z_2^k)f_2^{\Delta}.$$

So the q_1 in (4.4) is

$$\begin{aligned}q_1 &\triangleq (z_3^k-z_2^k)f_1^{\Delta}=(z_1^k-z_2^k)f_2^{\Delta}\\ &=\frac{(z_3^k-z_2^k)(z_1^k-z_2^k)(z_2-z_1)(z_3-z_1)(z_3-z_2)(1+z_1)^n(1+z_2)^n(1+z_3)^n e^{\xi_1+\xi_2+\xi_3}}{\tau^{\Delta}_{cdKP}}.\end{aligned} \tag{4.12}$$

And the (4.3) is reduced to

$$\begin{aligned}(L^l)_- &= f_1^\Delta \circ \Delta^{-1} \circ g_1^\Delta + f_2^\Delta \circ \Delta^{-1} \circ g_2^\Delta \\ &= (z_3^k - z_2^k) f_1^\Delta \circ \Delta^{-1} \circ \frac{g_1^\Delta}{(z_3^k - z_2^k)} + (z_1^k - z_2^k) f_2^\Delta \circ \Delta^{-1} \circ \frac{g_2^\Delta}{z_1^k - z_2^k} \\ &= q_1 \circ \Delta^{-1} \circ r_1, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} r_1 &\triangleq \frac{g_1^\Delta}{(z_3^k - z_2^k)} + \frac{g_2^\Delta}{z_1^k - z_2^k} \\ &= \Lambda\left(\frac{(z_3^k - z_2^k)(1+z_1)^n e^{\xi_1} + (z_3^k - z_1^k)(1+z_2)^n e^{\xi_2} + (z_2^k - z_1^k)(1+z_3)^n e^{\xi_3}}{(z_3^k - z_2^k)(z_1^k - z_2^k)\tau_{cdKP}^\Delta}\right). \end{aligned} \tag{4.14}$$

For simplicity, denote $t_1 = x; t_2 = y$. In particular, choosing $z_1 = z$, $z_2 = 0$, $z_3 = -z$, $c_1 = c$, $c_2 = 0$, $c_3 = -c$ then $\xi_2 = 0$, $\xi_3 = -\xi_1$ and

$$q_1 = \frac{(-1)^{k+2} z^{2k+2} (1+z)^n (1-z)^n e^{2z^2 y}}{(1+z)^n (1-z)^n + \frac{(1+z)^n e^\eta + (1-z)^n e^{-\eta}}{2} e^{z^2 y}}. \tag{4.15}$$

Base on above choice,

$$r_1 = -\frac{1}{z^{k+1}} \Lambda\left(\frac{\frac{[(1+z)^n e^\eta + (1-z)^n e^{-\eta}]}{2} + e^{-z^2 y}}{\frac{[(1+z)^n e^\eta + (1-z)^n e^{-\eta}]}{2} + (1-z^2)^n e^{z^2 y}}\right) \tag{4.16}$$

and if k is odd, and

$$r_1 = -\frac{1}{z^{k+1}} \Lambda\left(\frac{\frac{(1+z)^n e^\eta - (1-z)^n e^{-\eta}}{2}}{\frac{(1+z)^n e^\eta + (1-z)^n e^{-\eta}}{2} + (1-z^2)^n e^{z^2 y}}\right) \tag{4.17}$$

if k is even. Here $\eta \triangleq c + zx + z^3 t_3$.

So the dynamical variable $u_1 = q_1 \Lambda^{-1}(r_1)$ of the Lax operator L of the cdKP hierarchy

$$u_1 = 2z^{k+1} \frac{(1 - (-1)^k) + e^{-(c+zx+z^3 t_3) + z^2 y}(1-z)^n - e^{(c+zx+z^3 t_3) + z^2 y}(-1)^k (1+z)^n}{(1-z^2)^n \left(\frac{e^{c+zx+z^3 t_3}}{(1-z)^n} + 2e^{z^2 y} + \frac{e^{-(c+zx+z^3 t_3)}}{(1+z)^n}\right)^2}. \tag{4.18}$$

An example is

$$u_1 = 2z^{k+1} \frac{(1 - (-1)^k) + e^{-zx+z^2 y}(1-z)^n - e^{zx+z^2 y}(-1)^k (1+z)^n}{(1-z^2)^n \left(\frac{e^{zx}}{(1-z)^n} + 2e^{z^2 y} + \frac{e^{-zx}}{(1+z)^n}\right)^2}, \tag{4.19}$$

by setting $t_3 = 0, c = 0$. For this case

$$u_1 = 2z^{k+1}\frac{2 + e^{-zx+z^2y}(1-z)^n + e^{zx+z^2y}(1+z)^n}{(1-z^2)^n(\frac{e^{zx}}{(1-z)^n} + 2e^{z^2y} + \frac{e^{-zx}}{(1+z)^n})^2}, k \text{ is odd}, \tag{4.20}$$

$$u_1 = 2z^{k+1}\frac{e^{-zx+z^2y}(1-z)^n - e^{zx+z^2y}(1+z)^n}{(1-z^2)^n(\frac{e^{zx}}{(1-z)^n} + 2e^{z^2y} + \frac{e^{-zx}}{(1+z)^n})^2}, k \text{ is even}. \tag{4.21}$$

Remark: Actually, (4.6) also can be satisfied by other two choices $z = z_1$ or $z = z_3$. But u_1 (4.18) will be only one-soliton solution because the $f_1^{\Delta} = 0$ in (4.10a) or $f_2^{\Delta} = 0$ in (4.10b) separately.

The graph of $q_1 = q_1(x, y, n), r_1 = r_1(x, y, n), u_1 = u_1(x, y, n)$ were plotted in below for fixed $k = 1, 2$. We shall discuss the function of the gauge transformation for the the cdKP hierarchy to emphasize two sides about the discrete variable n of it and the times variable k of the gauge transformation of it. The profile of q_1, r_1, u_1 are plotted according to the value of discrete variable n from 0 to 2 and the value of time k of the gauge transformations from 1 to 2. The five conditions of the profile of q_1, r_1, u_1 are $\{n = 0, k = 1\}$,$\{n = 1, k = 1\}$, $\{n = 2, k = 1\}$, $\{n = 0, k = 2\}$, $\{n = 1, k = 2\}$ and $\{n = 2, k = 2\}$ as following.

(1).The profile of q_1 are plotted with $k = 1, n = 0, 1, 2$ in Figure 1, Figure 2 and Figure 3 respectively.

(2).The profile of r_1 are plotted with $k = 1, n = 0, 1, 2$ in Figure 4, Figure 5 and Figure 6 respectively.

(3).The Y-type soliton profile of u_1 are plotted with $k = 1, n = 0, 1, 2$ in Figure 7 , Figure 8 and Figure 9 respectively.

(4).The bright-dark soliton profile of u_1 are plotted with $k = 2$ and $n = 0, 1, 2$ in Figure 10, Figure 11 and Figure 12 respectively.

From the graphs of the solution of cdKP hierarchy, it can be found that:

(1) The profile of the solution q_1 of the cdKP hierarchy is decreasing to the one of the classical KP hierarchy in Ref. [30] when $n \to 0$ see Figure 1 ($k = 1$). For r_1, u_1 of the cdKP hierarchy, the profile of its are also decreasing the analogues of the classical KP hierarchy (see Figure 4 and Figure 7).

(2)When the times k of gauge transformation is an odd number, the profiles of u_1 become the Y-type soliton, see Figure 7, Figure 8 and Figure 9.

(3)When the times k of gauge transformation is an even number, the profiles of u_1 become bright-dark soliton, see Figure 10, Figure 11 and Figure 12.

For the end of showing more detail about dependence of u_1, q_1 on n, it is necessary to define n-effect quantity $\Delta u_1(z, x, y, n) = u_1(z, x, y, n) - u_1(z, x, y, n = 0) = u_1(n) - u_1(0)$ for fixed $z = 0.5$. Figure 13 are plotted for the $\Delta u_1(z, x, y, n)$ where $n = 1, 2, 3$ respectively, which shows the dependence of u_1 on n. It was obviously they are decreasing to almost zero when n goes from 3 to 1 with fixed $z = 0.5$. They also demonstrate that discretization of the cdKP hierarchy keeps the profile of the soliton though it has discrete variable n. These figures give us again an opportunity to observe the role of discrete variable n in the Wronskian solution of the cdKP hierarchy.

5. Conclusions

In this paper, the Wronskian solutions of the equation in the cdKP hierarchy have been given by means of the multi-channel gauge transformation. Based on the results of our previous papers [25, 26], Theorem 3.1 provides a necessary and sufficient condition of the k-constrained discrete KP hierarchy with m components. As an example, the reduction from 2-cdKP hierarchy to 1-cdKP hierarchy is presented. It can be found that the profiles of solution u_1 of cdKP hierarchy can be the Y-type solition by the odd number times gauge transformation, but the solution of cdKP hierarchy becomes bright-dark solition by even times gauge transformation. From these profiles, it can be find that the solution u_1 of the cdKP hierarchy is decreasing to the analogues of the classical KP hierarchy when $n \to 0$.

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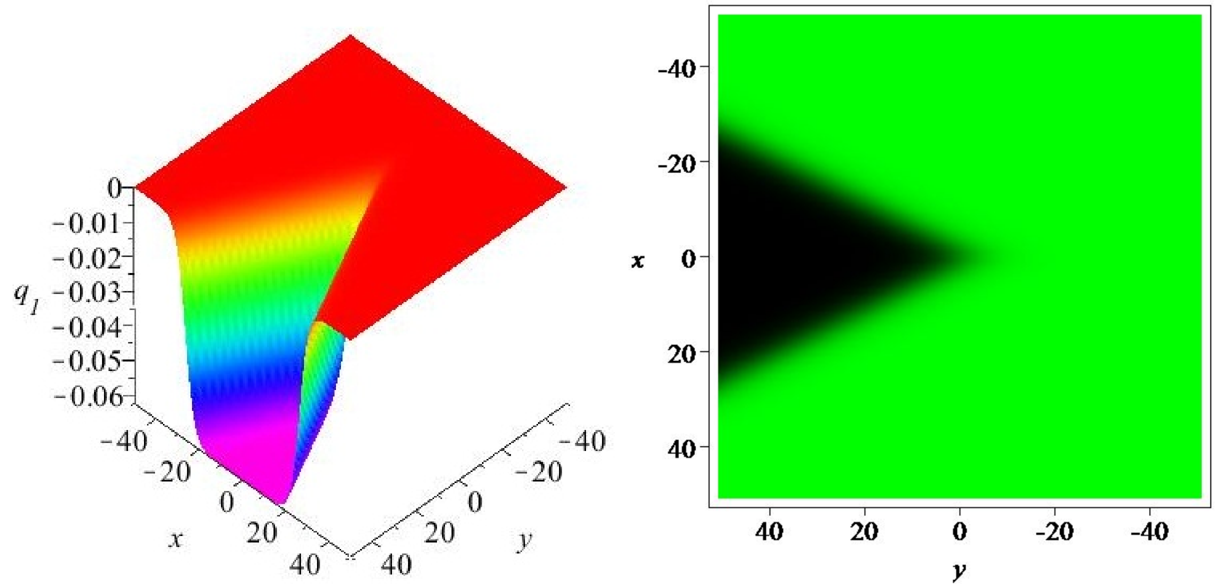

FIGURE 1. The profile of the solution q_1(4.15) of equation (2.5) (left) and its density plot (right)with $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 0$.

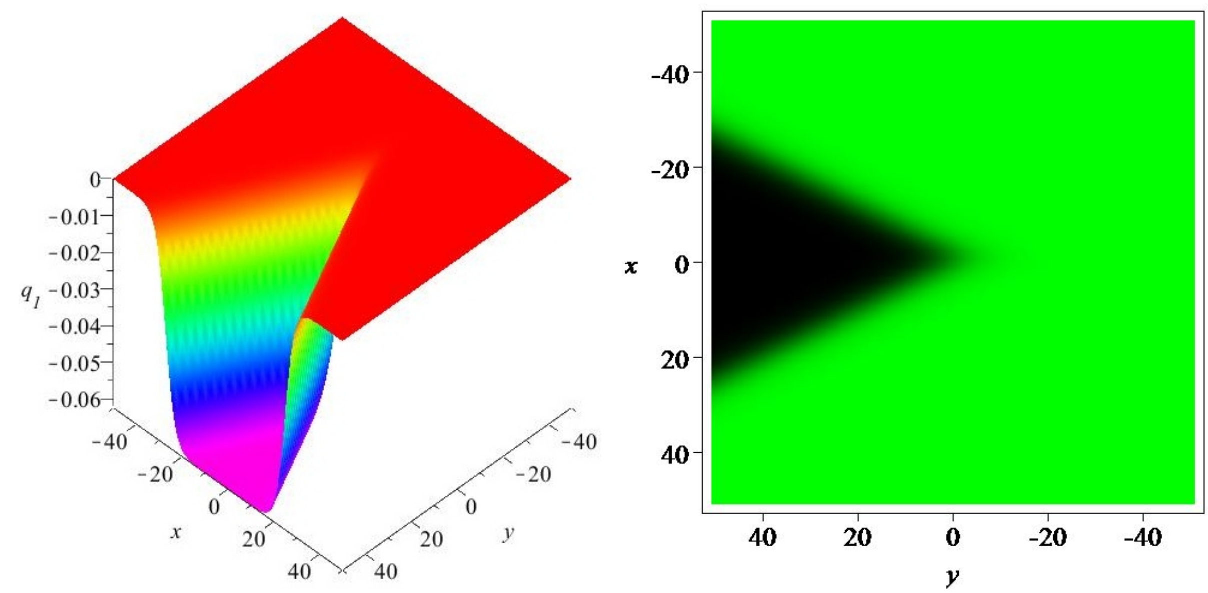

FIGURE 2. The profile of the solution of q_1(4.15) of equation (2.5) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 1$.

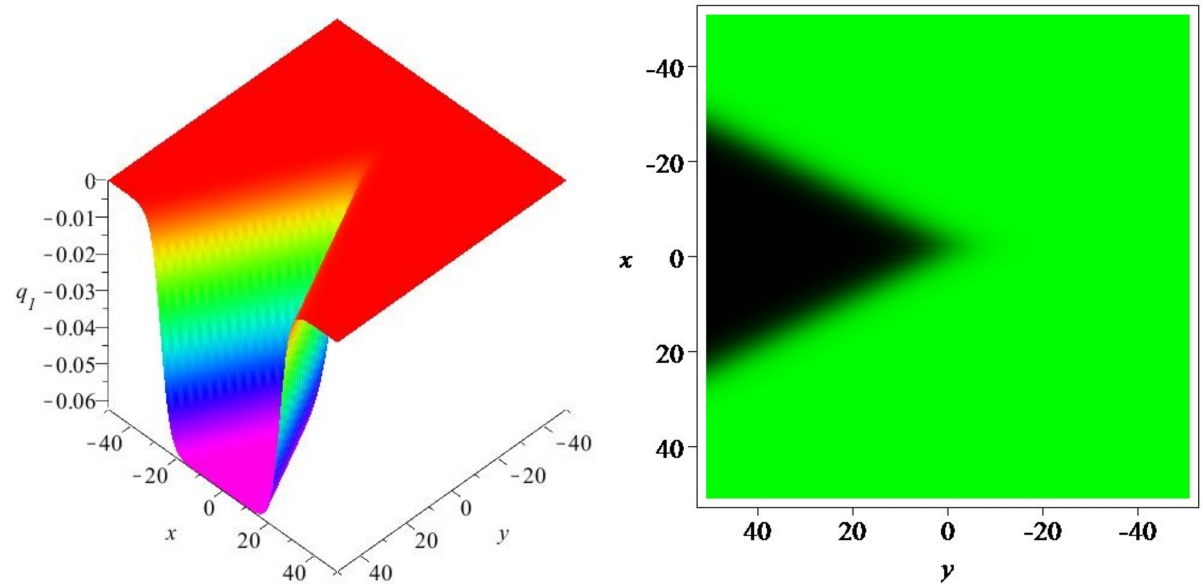

FIGURE 3. The profile of the solution of q_1(4.15) of equation (2.5) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 2$.

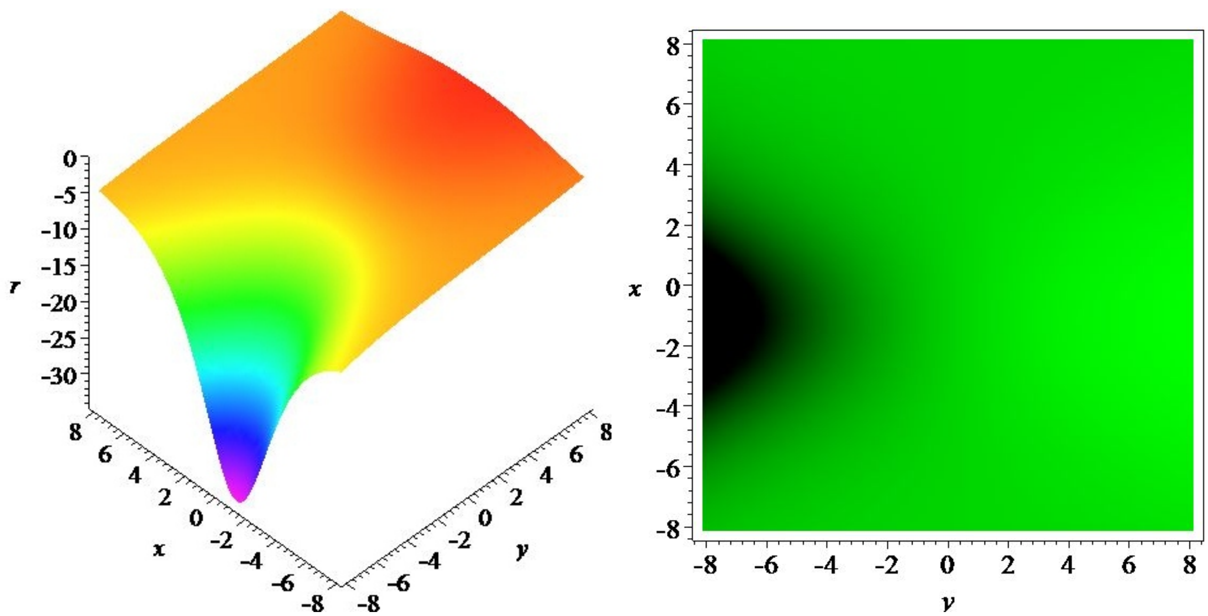

FIGURE 4. The profile of the solution r_1 (4.16) (left) of equation (2.5) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 0$. The vertical axis r denotes the r_1.

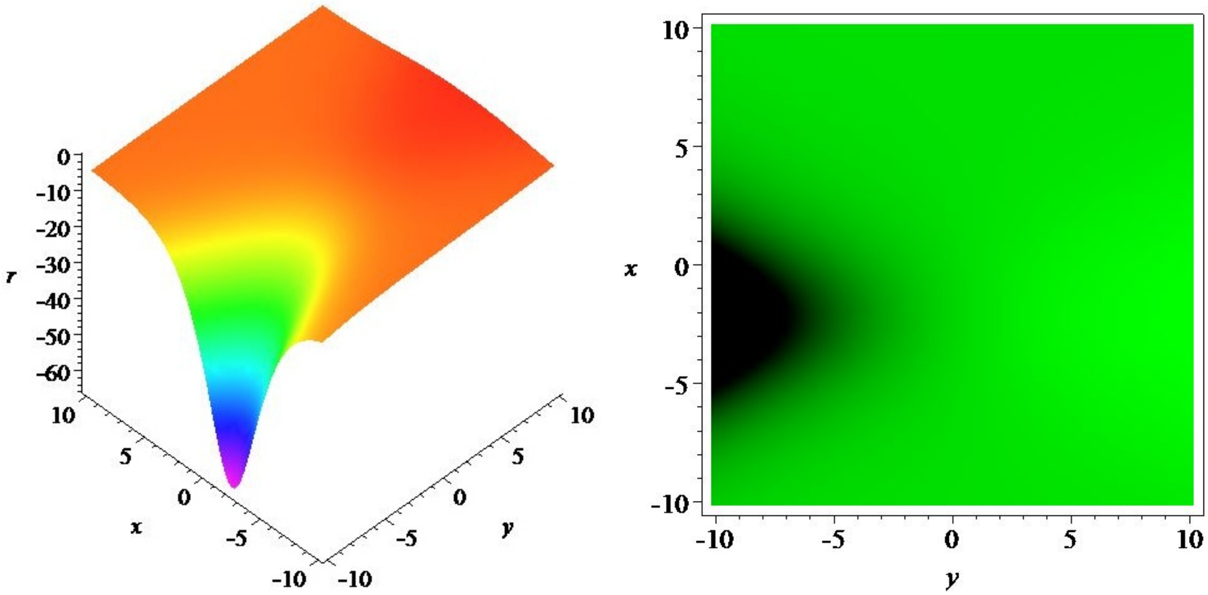

FIGURE 5. The profile of r_1 (4.16) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 1$. The vertical axis r denotes the r_1.

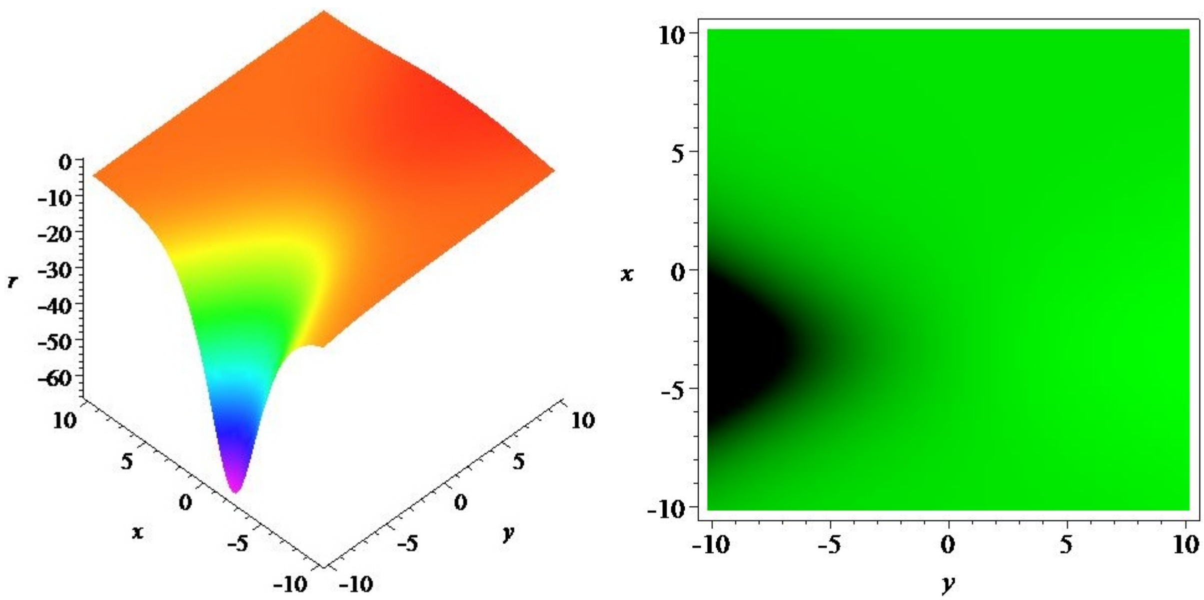

FIGURE 6. The profile of r_1 (4.16) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 2$. The vertical axis r denotes the r_1.

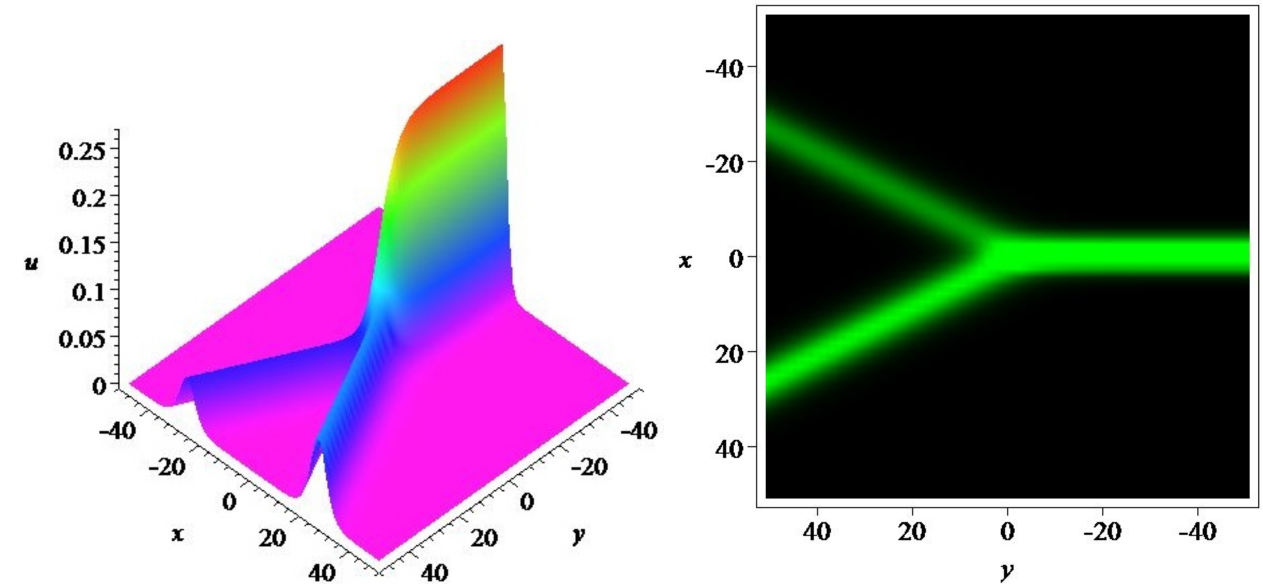

FIGURE 7. The profile of type Y soliton of u_1 (4.20) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 0$. The vertical axis u denotes the u_1.

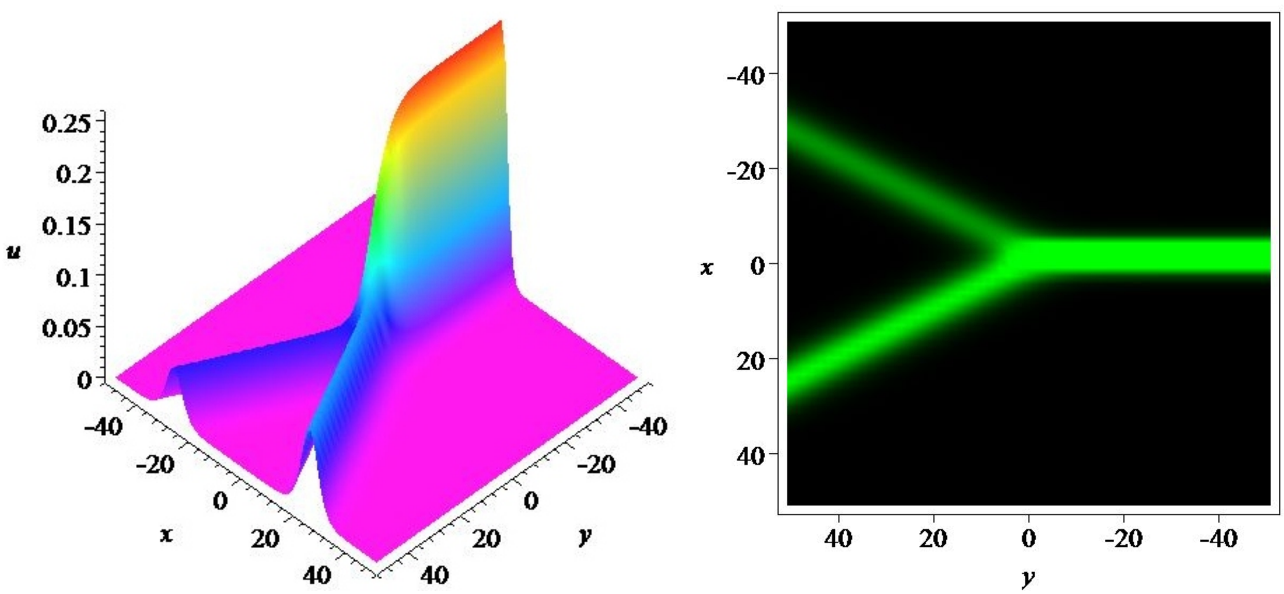

FIGURE 8. The profile of type Y soliton of u_1 (4.20) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 1$. The vertical axis u denotes the u_1.

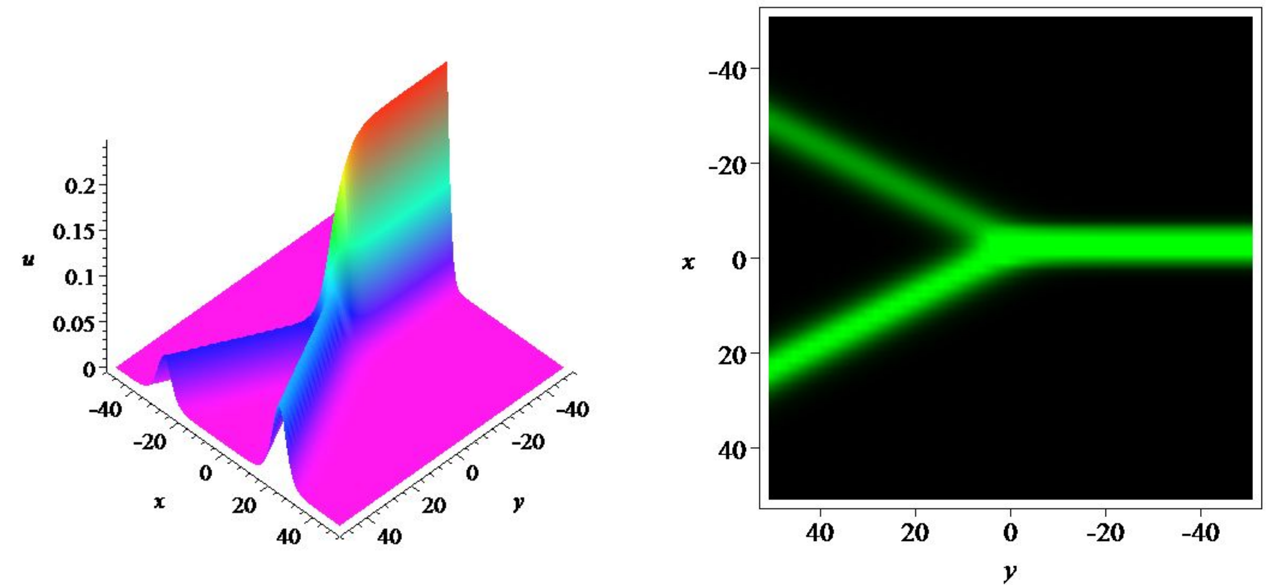

FIGURE 9. The profile of type Y soliton of u_1 (4.20) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 1$ and $n = 2$. The vertical axis u denotes the u_1.

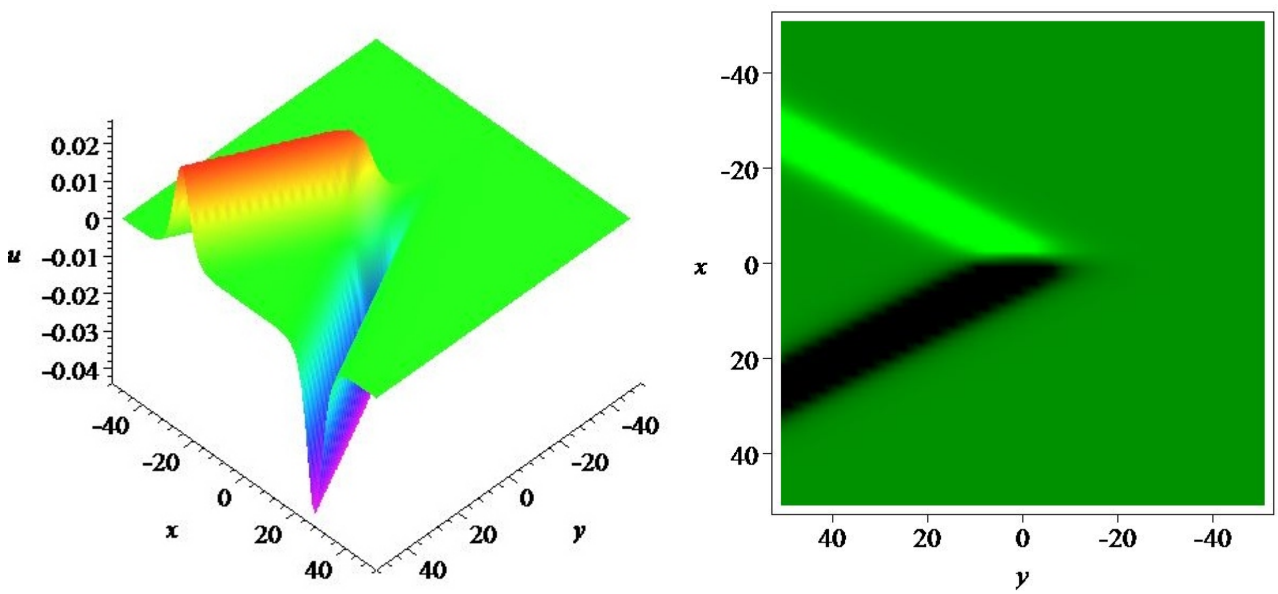

FIGURE 10. The profile of bright-dark type soliton of u_1 (4.21) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 2$ and $n = 0$. The vertical axis u denotes the u_1.

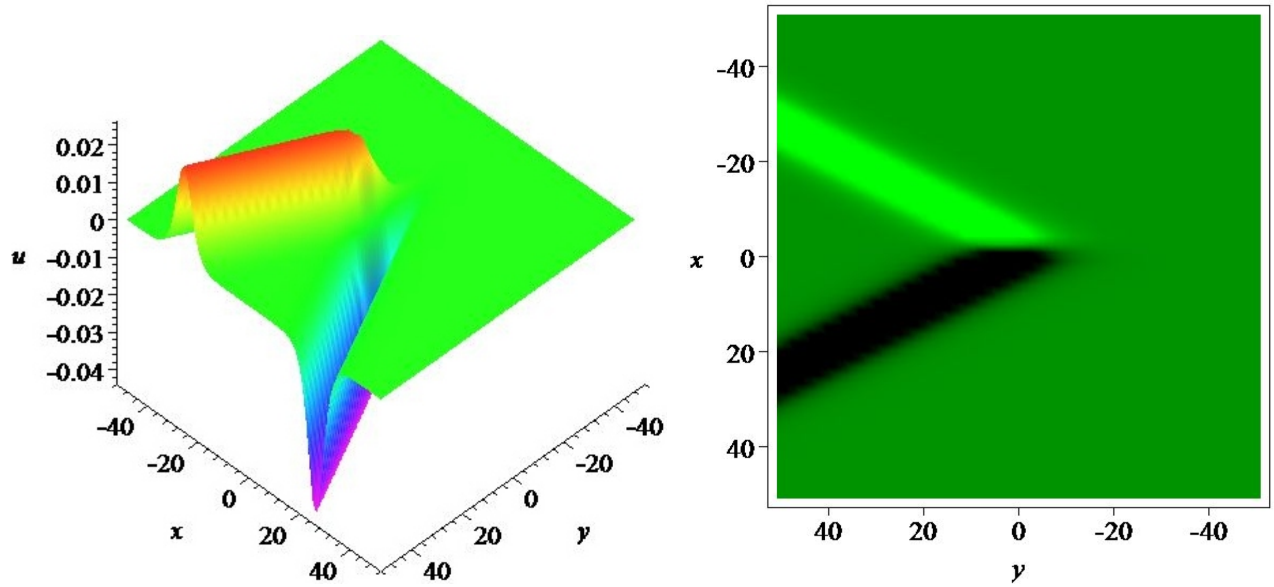

FIGURE 11. The profile of bright-dark type soliton of u_1 (4.21) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 2$ and $n = 1$. The vertical axis u denotes the u_1.

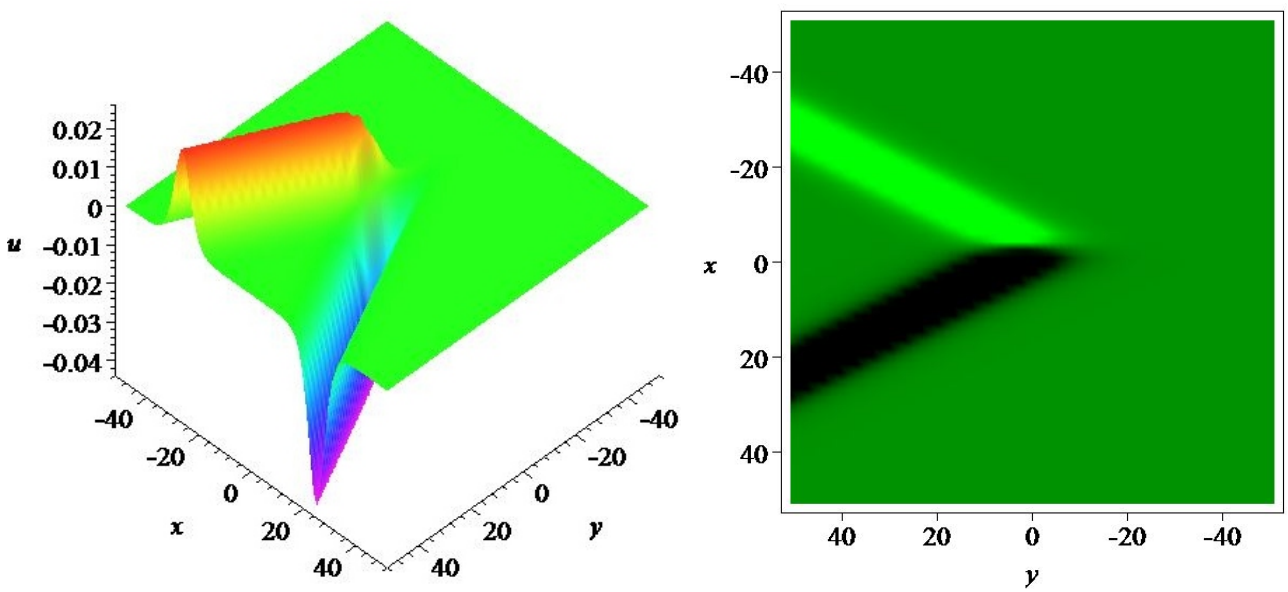

FIGURE 12. The profile of bright-dark soliton of u_1 (4.21) (left) and its density plot (right) with parameters $c = 0, t_3 = 0, z = 0.5, k = 2$ and $n = 2$. The vertical axis u denotes the u_1.

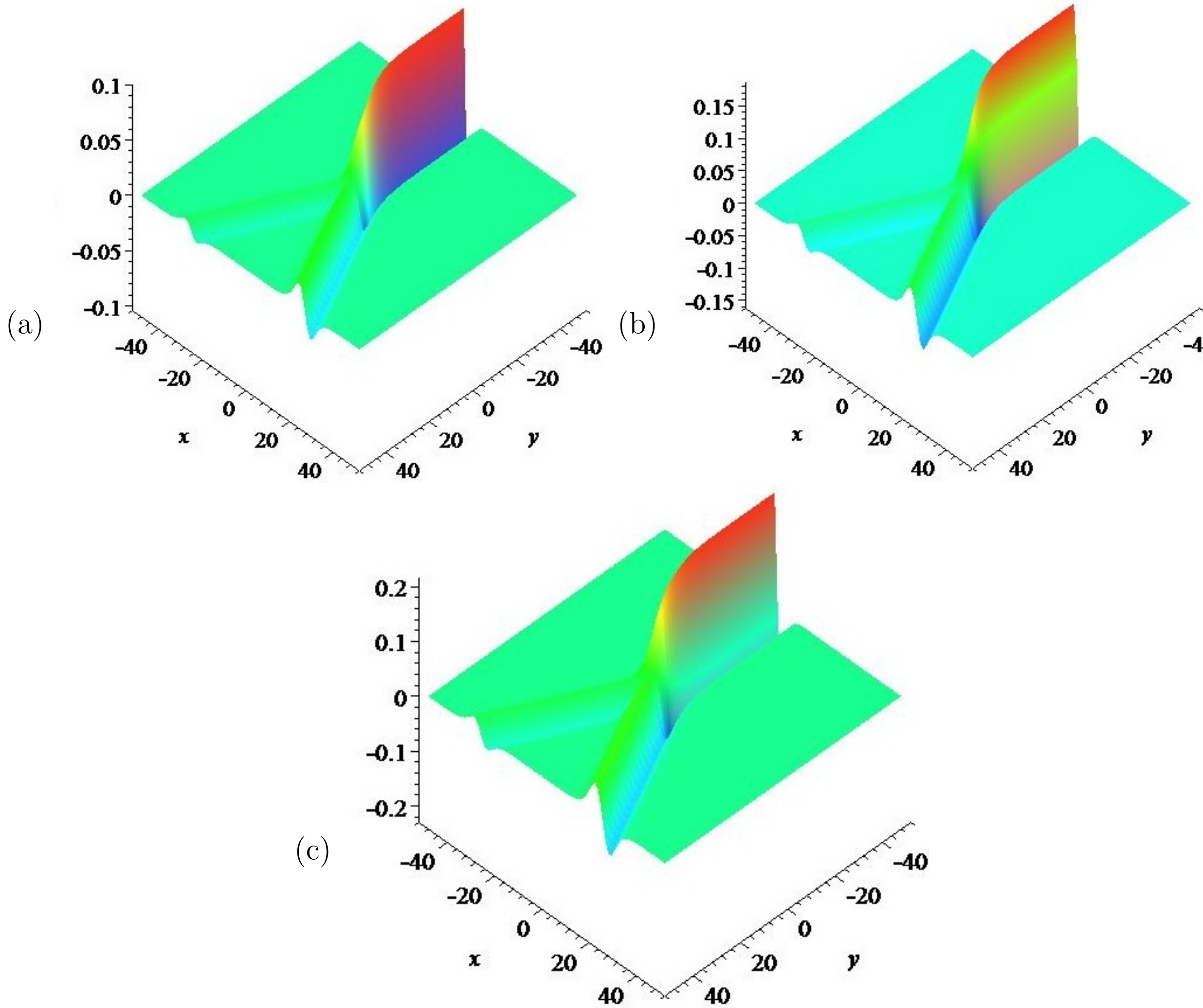

FIGURE 13. $\Delta u_1(z,x,y,n)$ with $c=0, t_3=0, z=0.5, k=1$ and $n=1$ in (a), 2 in (b) and 3 in (c).