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Abstract

In this paper, the linear sigma model is studied using a method for finding analytical solutions based on Padé approximants. Using the solutions of two and three traveling waves in 1+3 dimensions we found, we are able to show a solution that is valid for an arbitrary number of bosons and traveling waves.

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1. Introduction

Field theory is a well established way to describe interactions among particles, but usually it is very challenging to find analytical classical solutions in 1+3 dimensions. In this paper, we will study the bosonic interaction of the linear sigma model, which was proposed in [1] as an model to describe the pion-nucleon interaction. The model we are considering possess N_{ϕ} bosonic fields whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \Phi)^2 - \frac{1}{2} m^2 (\Phi)^2 - \frac{\lambda}{4} ((\Phi)^2)^2$$

where $(\Phi)^2 \equiv \Phi \cdot \Phi$ and $\Phi = (\phi_1, \phi_2, ..., \phi_{N_{\phi}})$. The Euler-Lagrange equation for this Lagrangian yields the system of equations

$$\phi_{i,tt} - \nabla \phi_i + m^2 \phi_i + \lambda \left(\phi_i^3 + \phi_i \sum_{j=1}^{N_\phi - 1} \phi_{i+j}^2 \right) = 0, \qquad i = 1, \dots, N_\phi, \tag{1}$$

where we use the notation $\phi_i = \phi_{i+N_{\phi}}$ and the metric (+ - - -). The purpose of this paper is to seek classical solutions for this system in the form of traveling waves. In order to achieve that, we will employ the algorithm based on Padé approximants presented in [19, 20]. This method has the advantage of needing less undetermined constants than other methods in the literature [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], such that we can earn efficiency in the processing of data.

We briefly recall the algorithm from [20] in section (2) and we apply it to the linear sigma model in section (3). Using the solutions for two and three traveling waves we found, we are able to show a solution for an arbitrary number of traveling waves and

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bosons in section (3.3). Section (4) is devoted to show a graphical illustration of the analytical solution we found. Finally, in section 5, we compare the Padé approximant approach with the multiple exp-method in order to see the benefits of each method.

2. Review of the algorithm

Consider a system of N_e equation in D dimensions

$$E_k(x^{\mu}, \phi_i, \partial_{\mu}\phi_i, ...; \mathcal{S}_0) = 0, \qquad k = 1, ..., N_e.$$
(2)

where S_0 is the set of all parameters of the model. Now suppose there is at least one solution of this system which can be written in terms of N_{ρ} functions we know the first derivative, i.e.

$$\phi_i(x^{\mu}) = \hat{\phi}_i(\rho_1, ..., \rho_{N_{\rho}}), \qquad i = 1, ..., N_e$$

$$\partial_{\mu}\rho_k = \mathcal{F}_{\mu,k}(\rho_1, ..., \rho_{N_{\rho}}; \mathcal{S}_1), \qquad \mu = 1, ..., D \qquad k = 1, ..., N_{\rho}$$

where $\mathcal{F}_{\mu,k}(\rho_1, ..., \rho_{N_{\rho}}; \mathcal{S}_1)$ is determined by the functions we choose for ρ_k and \mathcal{S}_1 is the set of constants introduced by these functions. This ansatz transforms the system (2) as

$$E_k(x^{\mu}, \phi_i, \partial_{\mu}\phi_i, ...; \mathcal{S}_0) = \hat{E}_k(\rho_k, \hat{\phi}_i, \partial_k \hat{\phi}_i, ...; \mathcal{S}_0 \times \mathcal{S}_1) = 0, \qquad k = 1, ..., N_e.$$
(3)

Suppose there is a solution that is regular when $\rho_k \to 0$ such that we can calculate the multivariate Taylor expansion at $\rho = 0$, i. e.

$$\hat{\phi}_i(\rho) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_{N_\rho}=0}^{\infty} c_{j_1,j_2,\dots,j_{N_\rho}}^{(i)} \prod_{d=1}^{N_\rho} \rho_d^{j_d}, \qquad i = 1,\dots,N_e$$
(4)

In the expansion (4), we may also impose a set of constraints $\psi_i(\mathcal{S}_0 \times \mathcal{S}_1) = 0$ on the undetermined constants such that the series does not truncate in the vacuum case. We will call \mathcal{S}_2 the set of all undetermined $c_{j_1,j_2,\ldots,j_{N_{\rho}}}^{(i)}$ and $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1 \times \mathcal{S}_2$ the set of all constants that remains undetermined. Let us do the scaling transformation $\rho \to \xi \rho$ such that we can rearrange the Taylor expansion as

$$\hat{\phi}_{i}(\xi\rho) = \sum_{n=0}^{L_{i}+M_{i}} a_{i;n}(\rho)\xi^{n} + \mathcal{O}(\xi^{L_{i}+M_{i}+1}), \qquad i = 1, \dots, N_{e},$$
$$a_{i;n}(\rho) = \sum_{j_{1}=0}^{n} \sum_{j_{2}=0}^{n-j_{1}} \dots \sum_{j_{N_{\rho}-1}=0}^{n-\sum_{r=1}^{N_{\rho}-2} j_{r}} c_{j_{1},j_{2},\dots,j_{N_{\rho}-1},n-\sum_{r=1}^{N_{\rho}-1} j_{r}} \left(\prod_{d=1}^{N_{\rho}-1} \rho_{d}^{j_{d}}\right) \rho_{N_{\rho}}^{n-\sum_{r=1}^{N_{\rho}-1} j_{r}}.$$

The coefficients of the expansion (4) need be calculated until we determine all $a_{i;n}(\rho)$ up to order $L_i + M_i$. Now, for a chosen L_i and M_i , we can calculate the Padé approximant for the field using ξ as variable, i.e.

$$\hat{\phi}_{i}(\xi\rho) = \frac{P_{\rho,L_{i}}^{(i)}(\xi;\mathcal{S})}{Q_{\rho,M_{i}}^{(i)}(\xi;\mathcal{S})} + \mathcal{O}(\xi^{L_{i}+M_{i}+1}).$$
(5)

The Padé approximants consist in an approximation of a function as a ratio of two polynomials given by

$$[L_i/M_i]_{\rho}^{(i)}(\xi;\mathcal{S}) \equiv \frac{P_{\rho,L_i}^{(i)}(\xi;\mathcal{S})}{Q_{\rho,M_i}^{(i)}(\xi;\mathcal{S})} = \frac{\sum_{j=0}^{L_i} p_j^{(i)}(\rho;\mathcal{S})\xi^j}{1 + \sum_{j=1}^{M_i} q_j^{(i)}(\rho;\mathcal{S})\xi^j}$$

whose coefficient are calculated by the system of equations

$$\sum_{r+s=j} a_{i;r}(\rho) q_s^{(i)}(\rho; \mathcal{S}) - p_j^{(i)}(\rho; \mathcal{S}) = 0, \qquad j = 0, 1, \dots, L_i + M_i.$$

Now we would like to find the subset $\hat{S} \subset S$ which makes the approximation (5) become an exact solution for the system (2) when $\xi = 1$, i.e.

$$\hat{\phi}_{i}(\rho) = \frac{P_{\rho,L_{i}}^{(i)}(\xi;\hat{\mathcal{S}})}{Q_{\rho,M_{i}}^{(i)}(\xi;\hat{\mathcal{S}})}\Big|_{\xi=1}.$$
(6)

Substituting (6) in (3), we can rewrite the system as

$$\hat{E}_{k}(\rho_{k},\hat{\phi}_{i},\partial_{k}\hat{\phi}_{i},...;\mathcal{S}_{0}\times\mathcal{S}_{1}) = \frac{\sum_{n=0}^{\Lambda}\hat{E}_{k;n}(\hat{S})}{D_{k}(\hat{S})} = 0,$$

$$\hat{E}_{k;n}(\hat{S}) = \sum_{j_{1}=0}^{n}\sum_{j_{2}=0}^{n-j_{1}}...\sum_{j_{N_{\rho}-1}=0}^{n-\sum_{r=1}^{N_{\rho}-2}j_{r}}\tilde{E}_{k;j_{1},j_{2},...,j_{N_{\rho}-1},n-\sum_{r=1}^{N_{\rho}-1}j_{r}}\left(\prod_{d=1}^{N_{\rho}-1}\rho_{d}^{j_{d}}\right)\rho_{N_{\rho}}^{n-\sum_{r=1}^{N_{\rho}-1}j_{r}}.$$
(7)

The coefficients for all powers of ρ_k yield the set of algebraic equations

$$\tilde{E}_{k;j_1,j_2,...,j_{N_{\rho}}}(\hat{S}) = 0, \quad k = 1,...,N_e, \quad n = 0,...,\Lambda, \quad j_l = 0,...,n - \sum_{r=1}^{l-1} j_r$$
(8a)

$$D_k(\hat{S}) \neq 0, \qquad k = 1, \dots, N_e \tag{8b}$$

whose solution will determine the subset of constants \hat{S} . After determine \hat{S} , we substitute its elements back into (6). Solutions of (8a) which simplify (6) to a vacuum solution will be omitted through the paper.

3. Application to the linear sigma model

In this paper, let us consider a ansatz with N_{ρ} traveling waves that obey a structure of exponential functions, i.e.

$$\phi_i(x^{\mu}) = \hat{\phi}_i(\rho_1, ..., \rho_{N_{\rho}}), \qquad \partial_{\mu}\rho_j = k_{j\mu}\rho_j, \qquad i = 1, ..., N_{\phi}, \qquad j = 1, ..., N_{\rho}.$$
(9)

Substituting the ansatz (9) into system (1), we map the original system in 1+3 dimensions to a system in N_{ρ} dimensions with the form

$$\sum_{p,q=1}^{N_{\rho}} k_{p\mu} k_{q}^{\mu} \Big(\rho_{p} \rho_{q} \hat{\phi}_{i,\rho_{p}\rho_{q}} + \delta_{pq} \rho_{p} \hat{\phi}_{i,\rho_{p}} \Big) + m^{2} \hat{\phi}_{i} + \lambda \Big(\hat{\phi}_{i}^{3} + \hat{\phi}_{i} \sum_{j=1}^{N_{\phi}-1} \hat{\phi}_{i+j}^{2} \Big) = 0, \qquad i = 1, \dots, N_{\phi}$$

$$\tag{10}$$

In the subsections below, we will consider the cases with $N_{\phi} = N_{\rho} = 2$ and $N_{\phi} = N_{\rho} = 3$. With these solution, we are able to show a solution for a general case with an arbitrary number of fields and traveling waves in subsection 3.3. In both cases, we will employ Padé approximants with $L_i = M_i = 2$ for all *i*.

3.1. Case with $N_{\phi} = 2$ and $N_{\rho} = 2$

In order to calculate a nonzero Taylor expansion, we will impose the constraints

$$k_{1\mu}k_1^{\mu} + m^2 = 0, \qquad k_{2\mu}k_2^{\mu} + m^2 = 0.$$

With these constraints, we can eliminate k_{10} and k_{20} . However, for simplify the calculation, we will eliminate only the quadratic terms of k_{10} and k_{20} at this point. Let us consider the Taylor expansions for the fields $\hat{\phi}_1 = \sum_{i,j=0} c_{ij} \rho_1^i \rho_2^j$ and $\hat{\phi}_2 = \sum_{i,j=0} d_{ij} \rho_1^i \rho_2^j$, such that after the rescaling $\rho \to \xi \rho$ we have

$$\begin{aligned} \hat{\phi}_{1}(\xi\rho) &= \xi(c_{10}\rho_{1}+c_{01}\rho_{2})+\xi^{3} \left(\frac{c_{10}(c_{10}^{2}+d_{10}^{2})\lambda\rho_{1}^{3}}{8m^{2}}-\frac{(3c_{01}c_{10}^{2}+2c_{10}d_{10}d_{01}+c_{01}d_{10}^{2})\lambda\rho_{1}^{2}\rho_{2}}{4(k_{10}k_{20}-k_{11}k_{21}-k_{12}k_{22}-k_{13}k_{23}-m^{2})} \\ &- \frac{(3c_{01}^{2}c_{10}+2c_{01}d_{10}d_{01}+c_{10}d_{01}^{2})\lambda\rho_{1}\rho_{2}^{2}}{4(k_{10}k_{20}-k_{11}k_{21}-k_{12}k_{22}-k_{13}k_{23}-m^{2})} + \frac{c_{01}(c_{01}^{2}+d_{01}^{2})\lambda\rho_{2}^{3}}{8m^{2}}\right) + O(\xi^{5}) \\ \hat{\phi}_{2}(\xi\rho) &= \xi(d_{10}\rho_{1}+d_{01}\rho_{2}) + \xi^{3} \left(\frac{d_{10}(c_{10}^{2}+d_{10}^{2})\lambda\rho_{1}^{3}}{8m^{2}} - \frac{(3d_{01}d_{10}^{2}+2d_{10}c_{10}c_{01}+d_{01}c_{10}^{2})\lambda\rho_{1}^{2}\rho_{2}}{4(k_{10}k_{20}-k_{11}k_{21}-k_{12}k_{22}-k_{13}k_{23}-m^{2})} \\ &- \frac{(3d_{01}^{2}d_{10}+2d_{01}c_{10}c_{01}+d_{10}c_{01}^{2})\lambda\rho_{1}\rho_{2}^{2}}{4(k_{10}k_{20}-k_{11}k_{21}-k_{12}k_{22}-k_{13}k_{23}-m^{2})} + \frac{d_{01}(c_{01}^{2}+d_{01}^{2})\lambda\rho_{2}^{3}}{8m^{2}}\right) + O(\xi^{5}) \end{aligned}$$

Considering the expansion until order 4 for the auxiliary parameter ξ , we can calculate the Padé approximants $[2/2]_{\rho}^{(i)}(\xi; S)$ as

$$\frac{P_{\rho,2}^{(1)}(\xi;S)}{Q_{\rho,2}^{(1)}(\xi;S)}\Big|_{\xi=1} = 8m^{2}(-k_{10}k_{20}+k_{11}k_{21}+k_{12}k_{22}+k_{13}k_{23}+m^{2})(c_{10}\rho_{1}+c_{01}\rho_{2})^{2} / \left[c_{10}(k_{10}k_{20}-k_{11}k_{21}-k_{12}k_{22}-k_{13}k_{23}-m^{2})(-8m^{2}+(c_{10}^{2}+d_{10}^{2})\lambda\rho_{1}^{2})\rho_{1} + 2m^{2}(4c_{01}(-k_{10}k_{20}+k_{11}k_{21}+k_{12}k_{22}+k_{13}k_{23}+m^{2}) - (2c_{10}d_{01}d_{10} + c_{01}(3c_{10}^{2}+d_{10}^{2}))\lambda\rho_{1}^{2})\rho_{2} - 2(3c_{01}^{2}c_{10}+c_{10}d_{01}^{2}+2c_{01}d_{01}d_{10})m^{2}\lambda\rho_{1}\rho_{2}^{2} - c_{01}(c_{01}^{2}+d_{01}^{2})(-k_{10}k_{20}+k_{11}k_{21}+k_{12}k_{22}+k_{13}k_{23}+m^{2})\lambda\rho_{2}^{3}\right]$$

$$\begin{aligned} \frac{P_{\rho,2}^{(2)}(\xi;\mathcal{S})}{Q_{\rho,2}^{(2)}(\xi;\mathcal{S})}\Big|_{\xi=1} &= 8m^2(-k_{10}k_{20}+k_{11}k_{21}+k_{12}k_{22}+k_{13}k_{23}+m^2)(d_{10}\rho_1+d_{01}\rho_2)^2 \Big/\\ & \left[d_{10}(k_{10}k_{20}-k_{11}k_{21}-k_{12}k_{22}-k_{13}k_{23}-m^2)(-8m^2+(c_{10}^2+d_{10}^2)\lambda\rho_1^2)\rho_1 + 2m^2(4d_{01}(-k_{10}k_{20}+k_{11}k_{21}+k_{12}k_{22}+k_{13}k_{23}+m^2) - (2d_{10}c_{01}c_{10}+d_{01}(3d_{10}^2+c_{10}^2))\lambda\rho_1^2)\rho_2 - 2(3d_{01}^2d_{10}+d_{10}c_{01}^2+2d_{01}c_{01}c_{10})m^2\lambda\rho_1\rho_2^2 \\ & - d_{01}(c_{01}^2+d_{01}^2)(-k_{10}k_{20}+k_{11}k_{21}+k_{12}k_{22}+k_{13}k_{23}+m^2)\lambda\rho_2^3 \Big] \end{aligned}$$

where $S = \{m, \lambda, k_{10}, k_{11}, k_{12}, k_{13}, k_{20}, k_{21}, k_{22}, k_{23}, c_{10}, c_{01}, d_{10}, d_{01}\}$. Following step (6) of the algorithm, we substitute these Padé approximants into the system (10) with $N_{\phi} = N_{\rho} = 2$ and it yields a set of 66 algebraic equations that need be solved in order to determine the subset $\hat{S} \subset S$. We will not write the 66 algebraic equations, but its nontrivial solutions together with the constraints imposed on the Taylor expansion can be summarized by the following conditions on the constants $k_{i\mu}$:

$$k_{1\mu}k_1^{\mu} + m^2 = 0, \qquad k_{2\mu}k_2^{\mu} + m^2 = 0, \qquad k_{1\mu}k_2^{\mu} + m^2 = 0$$
 (11)

Finally, substituting this relations on the ansatz

$$\hat{\phi}_1 = \frac{P_{\rho,2}^{(1)}(\xi;\hat{\mathcal{S}})}{Q_{\rho,2}^{(1)}(\xi;\hat{\mathcal{S}})}\Big|_{\xi=1}, \qquad \hat{\phi}_2 = \frac{P_{\rho,2}^{(2)}(\xi;\hat{\mathcal{S}})}{Q_{\rho,2}^{(2)}(\xi;\hat{\mathcal{S}})}\Big|_{\xi=1},$$

we have the solution

$$\hat{\phi}_1 = \frac{8m^2(c_{10}\rho_1 + c_{01}\rho_2)}{8m^2 - \lambda[(c_{10}^2 + d_{10}^2)\rho_1^2 + 2(c_{10}c_{01} + d_{10}d_{01})\rho_1\rho_2 + (c_{01}^2 + d_{01}^2)\rho_2^2]}$$
$$\hat{\phi}_2 = \frac{8m^2(d_{10}\rho_1 + d_{01}\rho_2)}{8m^2 - \lambda[(c_{10}^2 + d_{10}^2)\rho_1^2 + 2(c_{10}c_{01} + d_{10}d_{01})\rho_1\rho_2 + (c_{01}^2 + d_{01}^2)\rho_2^2]}$$

3.2. Case with $N_{\phi} = 3$ and $N_{\rho} = 3$

In this section, we will exploit the linear sigma model with $N_{\phi} = N_{\rho} = 3$. As we are dealing with a system of 3 field in 1+3 dimensions, the processing of the algebraic data increases considerably. So we will extend the information we found for $N_{\phi} = N_{\rho} = 2$ in order to simplify the calculation and consider the constraints

$$k_{1\mu}k_1^{\mu} + m^2 = 0, \qquad k_{2\mu}k_2^{\mu} + m^2 = 0, \qquad k_{3\mu}k_3^{\mu} + m^2 = 0,$$
 (12)

$$k_{1\mu}k_2^{\mu} + m^2 = 0, \qquad k_{1\mu}k_3^{\mu} + m^2 = 0, \qquad k_{2\mu}k_3^{\mu} + m^2 = 0$$
 (13)

on the Taylor expansions $\hat{\phi}_1 = \sum_{i,j,k=0} c_{ijk} \rho_1^i \rho_2^j \rho_3^k$, $\hat{\phi}_2 = \sum_{i,j,k=0} d_{ijk} \rho_1^i \rho_2^j \rho_3^k$ and $\hat{\phi}_3 = \sum_{i,j,k=0} e_{ijk} \rho_1^i \rho_2^j \rho_3^k$. Although we only need constraint (12) to avoid the vacuum solution, the constraint (13) simplify considerably the calculation as we will see. The expansions are

$$\hat{\phi}_{1}(\xi\rho) = \xi(c_{100}\rho_{1} + c_{010}\rho_{2} + c_{001}\rho_{3}) + \frac{\xi^{3}\lambda}{8m^{2}} \Big[(c_{100}\rho_{1} + c_{010}\rho_{2} + c_{001}\rho_{3})((c_{100}^{2} + d_{100}^{2} + e_{100}^{2})\rho_{1}^{2} \\ + (c_{010}^{2} + d_{010}^{2} + e_{010}^{2})\rho_{2}^{2} + (c_{001}^{2} + d_{001}^{2} + e_{001}^{2})\rho_{3}^{2} + 2(c_{100}c_{010} + d_{100}d_{010} + e_{100}e_{010})\rho_{1}\rho_{2} \\ + 2(c_{100}c_{001} + d_{100}d_{001} + e_{100}e_{001})\rho_{1}\rho_{3} + 2(c_{010}c_{001} + d_{010}d_{001} + e_{010}e_{001})\rho_{2}\rho_{3} \Big] + O(\xi^{5}) \\ \hat{\phi}_{2}(\xi\rho) = \xi(d_{100}\rho_{1} + d_{010}\rho_{2} + d_{001}\rho_{3}) + \frac{\xi^{3}\lambda}{8m^{2}} \Big[(d_{100}\rho_{1} + d_{010}\rho_{2} + d_{001}\rho_{3})((c_{100}^{2} + d_{100}^{2} + e_{100}^{2})\rho_{1}^{2} \\ + (c_{010}^{2} + d_{010}^{2} + e_{010}^{2})\rho_{2}^{2} + (c_{001}^{2} + d_{001}^{2} + e_{001}^{2})\rho_{3}^{2} + 2(c_{100}c_{010} + d_{100}d_{010} + e_{100}e_{010})\rho_{1}\rho_{2} \\ + 2(c_{100}c_{001} + d_{100}d_{001} + e_{100}e_{001})\rho_{1}\rho_{3} + 2(c_{010}c_{001} + d_{010}d_{001} + e_{010}e_{001})\rho_{2}\rho_{3} \Big] + O(\xi^{5})$$

$$\hat{\phi}_{3}(\xi\rho) = \xi(e_{100}\rho_{1} + e_{010}\rho_{2} + e_{001}\rho_{3}) + \frac{\xi^{3}\lambda}{8m^{2}} \Big[(e_{100}\rho_{1} + e_{010}\rho_{2} + e_{001}\rho_{3})((c_{100}^{2} + d_{100}^{2} + e_{100}^{2})\rho_{1}^{2} \\ + (c_{010}^{2} + d_{010}^{2} + e_{010}^{2})\rho_{2}^{2} + (c_{001}^{2} + d_{001}^{2} + e_{001}^{2})\rho_{3}^{2} + 2(c_{100}c_{010} + d_{100}d_{010} + e_{100}e_{010})\rho_{1}\rho_{2} \\ + 2(c_{100}c_{001} + d_{100}d_{001} + e_{100}e_{001})\rho_{1}\rho_{3} + 2(c_{010}c_{001} + d_{010}d_{001} + e_{010}e_{001})\rho_{2}\rho_{3} \Big] + O(\xi^{5})$$

Considering the expansion until order 4 for the auxiliary parameter ξ , we can calculate the ansatz $\hat{\phi}_i = [2/2]^{(i)}_{\rho}(\xi; S)$ as

$$\hat{\phi}_{1} = \frac{8m^{2}(c_{100}\rho_{1} + c_{010}\rho_{2} + c_{001}\rho_{3})}{\left| \frac{8m^{2} - \lambda[(c_{100}^{2} + d_{100}^{2} + e_{100}^{2})\rho_{1}^{2} + (c_{010}^{2} + d_{010}^{2} + e_{010}^{2})\rho_{2}^{2} + (c_{001}^{2} + d_{001}^{2} + e_{001}^{2})\rho_{3}^{2} + 2(c_{100}c_{010} + d_{100}d_{010} + e_{100}e_{010})\rho_{1}\rho_{2} + 2(c_{100}c_{001} + d_{100}d_{001} + e_{100}e_{001})\rho_{1}\rho_{3} + 2(c_{010}c_{001} + d_{010}d_{001} + e_{010}e_{001})\rho_{2}\rho_{3} \right|$$

$$\hat{\phi}_{2} = 8m^{2}(d_{100}\rho_{1} + d_{010}\rho_{2} + d_{001}\rho_{3}) \Big/ \Big[8m^{2} - \lambda [(c_{100}^{2} + d_{100}^{2} + e_{100}^{2})\rho_{1}^{2} \\ + (c_{010}^{2} + d_{010}^{2} + e_{010}^{2})\rho_{2}^{2} + (c_{001}^{2} + d_{001}^{2} + e_{001}^{2})\rho_{3}^{2} + 2(c_{100}c_{010} + d_{100}d_{010} + e_{100}e_{010})\rho_{1}\rho_{2} \\ + 2(c_{100}c_{001} + d_{100}d_{001} + e_{100}e_{001})\rho_{1}\rho_{3} + 2(c_{010}c_{001} + d_{010}d_{001} + e_{010}e_{001})\rho_{2}\rho_{3} \Big]$$

$$\hat{\phi}_{3} = 8m^{2}(e_{100}\rho_{1} + e_{010}\rho_{2} + e_{001}\rho_{3}) \Big/ \Big[8m^{2} - \lambda [(c_{100}^{2} + d_{100}^{2} + e_{100}^{2})\rho_{1}^{2} \\ + (c_{010}^{2} + d_{010}^{2} + e_{010}^{2})\rho_{2}^{2} + (c_{001}^{2} + d_{001}^{2} + e_{001}^{2})\rho_{3}^{2} + 2(c_{100}c_{010} + d_{100}d_{010} + e_{100}e_{010})\rho_{1}\rho_{2} \\ + 2(c_{100}c_{001} + d_{100}d_{001} + e_{100}e_{001})\rho_{1}\rho_{3} + 2(c_{010}c_{001} + d_{010}d_{001} + e_{010}e_{001})\rho_{2}\rho_{3} \Big]$$

Substituting the above expressions into the system (10) with $N_{\phi} = N_{\rho} = 3$, we can check that these ansatz already are a exact solution for the linear sigma model.

3.3. General case

With the information we gathered in the previous sections, we can seek a solution for the general case with N_{ϕ} and N_{ρ} arbitrary. Let us consider for a moment the following expressions:

$$\hat{\phi}_{i} = \frac{8m^{2} \sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\delta_{2j},...,\delta_{N_{\rho}j}}^{(i)} \rho_{j}}{8m^{2} - \lambda \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\delta_{2j},...,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}}, \qquad i = 1,...,N_{\phi}$$
(14)

$$k_{i\mu}k_j^{\mu} + m^2 = 0, \qquad i, j = 1, ..., N_{\rho}$$
 (15)

On the one hand, if we substitute these expressions in the kinematic part of the model (10), we have

$$\begin{split} \sum_{p,q=1}^{N_{\rho}} k_{p\mu} k_{q}^{\mu} \bigg(\rho_{p} \rho_{q} \hat{\phi}_{i,\rho_{p}\rho_{q}} + \delta_{pq} \rho_{p} \hat{\phi}_{i,\rho_{p}} \bigg) &= -m^{2} \bigg(\frac{8m^{2} \sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(i)} \rho_{j}}{8m^{2} - \lambda \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}} \\ &+ \frac{8\lambda (8m^{2})^{2} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(i)} \rho_{j}) \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}}{[8m^{2} - \lambda \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}]^{3}} \bigg). \end{split}$$

On the other, if we substitute (14) on the potential part, we have

$$\begin{split} m^{2}\hat{\phi}_{i} + \lambda \bigg(\hat{\phi}_{i}^{3} + \hat{\phi}_{i} \sum_{j=1}^{N_{\phi}-1} \hat{\phi}_{i+j}^{2} \bigg) &= m^{2} \bigg(\frac{8m^{2} \sum_{j=1}^{N_{\phi}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(i)} \rho_{j}}{8m^{2} - \lambda \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\delta_{2j},\dots,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}} \\ &+ \frac{8\lambda (8m^{2})^{2} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(i)} \rho_{j}) \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}}{[8m^{2} - \lambda \sum_{p=1}^{N_{\phi}} (\sum_{j=1}^{N_{\rho}} c_{\delta_{1j},\dots,\delta_{N_{\rho}j}}^{(p)} \rho_{j})^{2}]^{3}} \bigg). \end{split}$$

Therefore, we can easily check that (14) with condition (15) is a solution for the general case. Observe that we have $\frac{1}{2}N_{\rho}(N_{\rho}+1)$ algebraic equations in (15) and DN_{ρ} constantes $k_{i\mu}$ if we are in a D-dimensional space. Hence, for the system (15) be solvable, we need have $N_{\rho} \leq 2D - 1$ different ρ_j .

4. Graphical Illustration

In order to illustrate the analytical solution (14), let us consider the case $N_{\phi} = 2$ and $N_{\rho} = 4$ with

$$\begin{aligned} &(k_{10}, k_{11}, k_{12}, k_{13}) &= (2, -1, 2, 0) \\ &(k_{20}, k_{21}, k_{22}, k_{23}) &= (-2, -1, -2, 0) \\ &(k_{30}, k_{31}, k_{32}, k_{33}) &= (1/2, -1, 1/2, 0) \\ &(k_{40}, k_{41}, k_{42}, k_{43}) &= (-1/2, -1, -1/2, 0). \end{aligned}$$

Substituting these parameters in (14) and rewriting the expression in the x^{μ} coordinates using the structure of exponential functions, we have

$$\begin{aligned} \hat{\phi}_{1} &= 8m^{2} \left(c_{1000}^{(1)} e^{2t - x + 2y} + c_{0100}^{(1)} e^{-2t - x - 2y} + c_{0010}^{(1)} e^{\frac{1}{2}t - x + \frac{1}{2}y} + c_{0001}^{(1)} e^{-\frac{1}{2}t - x - \frac{1}{2}y} \right) \\ &= \left[8m^{2} - \lambda \left[\left(c_{1000}^{(1)} e^{2t - x + 2y} + c_{0100}^{(1)} e^{-2t - x - 2y} + c_{0010}^{(1)} e^{\frac{1}{2}t - x + \frac{1}{2}y} + c_{0001}^{(1)} e^{-\frac{1}{2}t - x - \frac{1}{2}y} \right)^{2} \right] \\ &+ \left(c_{1000}^{(2)} e^{2t - x + 2y} + c_{0100}^{(2)} e^{-2t - x - 2y} + c_{0010}^{(2)} e^{\frac{1}{2}t - x + \frac{1}{2}y} + c_{0001}^{(2)} e^{-\frac{1}{2}t - x - \frac{1}{2}y} \right)^{2} \end{aligned}$$
(16)
$$\hat{\phi}_{2} &= 8m^{2} \left(c_{1000}^{(2)} e^{2t - x + 2y} + c_{0100}^{(2)} e^{-2t - x - 2y} + c_{0010}^{(2)} e^{\frac{1}{2}t - x + \frac{1}{2}y} + c_{0001}^{(2)} e^{-\frac{1}{2}t - x - \frac{1}{2}y} \right) \\ &= \left[8m^{2} - \lambda \left[\left(c_{1000}^{(1)} e^{2t - x + 2y} + c_{0100}^{(1)} e^{-2t - x - 2y} + c_{0010}^{(2)} e^{\frac{1}{2}t - x + \frac{1}{2}y} + c_{0001}^{(2)} e^{-\frac{1}{2}t - x - \frac{1}{2}y} \right) \right] \\ &+ \left(c_{1000}^{(2)} e^{2t - x + 2y} + c_{0100}^{(2)} e^{-2t - x - 2y} + c_{0010}^{(2)} e^{\frac{1}{2}t - x + \frac{1}{2}y} + c_{0001}^{(2)} e^{-\frac{1}{2}t - x - \frac{1}{2}y} \right)^{2} \end{aligned}$$
(17)

Observe that these solutions will present singularities if λ is positive and the chosen values for $k_{i\mu}$ are real, therefore, we will consider only the case $\lambda < 0$. For illustration purpose, let us consider m = 1 and $\lambda = -1$. As we have chosen $k_{i3} = 0$, we will display the solution as a sequence of three bi-dimensional graphics at different times. In figure 1, we show the graphic for the following combination of the arbitrary constants:

$$(c_{1000}^{(1)}, c_{0100}^{(1)}, c_{0010}^{(1)}, c_{0001}^{(1)}) = \sqrt{8}(0, 0, 1, 0)$$
(18)

$$(c_{1000}^{(2)}, c_{0100}^{(2)}, c_{0010}^{(2)}, c_{0001}^{(2)}) = \sqrt{8}(1, -1, 1, -1).$$
(19)



Figure 1: In these figures, we illustrate the solutions (16) and (17) for the linear sigma model with two fields, four traveling-wave and the parameters (18) and (19) in a succession of three graphics at different time for each field. The z-coordinate is omitted because we chose null components for k_{j3} .

This is one of the many combinations we can obtain from the analytical solution, but it is interesting to notice a very localized pattern for ϕ_1 produced by the combination of four traveling-waves that resembles a particle. A phenomenological study of this solution will be considered in a future work.

5. Efficiency in the processing of data

In this section, we will compare the algorithm used in this paper with the multiple exp-function method [14] in order to see the benefit of each method. The method in [14] was the first method to seek multi-wave solutions using a direct approach. On one hand, if we would like to use the multiple exp-method in the case of subsection 3.1, we would use the ansatz

$$\phi_i = \frac{\sum_{m,n=0}^{M} p_{i,m,n} \rho_1^m \rho_2^n}{\sum_{m,n=0}^{N} q_{i,m,n} \rho_1^m \rho_2^n}, \qquad \partial_\mu \rho_j = k_{j\mu} \rho_j, \qquad i, j = 1, 2$$
(20)

and the solution would be found for M = 1 and N = 2. Substituting this ansats in equation (10) with $N_{\phi} = N_{\rho} = 2$ and forming a system of algebraic equations with the coefficients of all powers of ρ_k , we have a system with 34 variables (4 $p_{i,m,n}$ and 9 $q_{i,m,n}$ for each field and 4 $k_{j\mu}$ for each wave). If we consider the case of subsection 3.2 without any previous assumption, the number of variables jump to 117 (8 $p_{i,m,n}$ and 27 $q_{i,m,n}$ for each field and 4 $k_{j\mu}$ for each wave). Such systems can be quite complicated depending of the model.

On the other hand, the algorithm based on Padé approximants we have used splits the process of find the undetermined coefficients in two stages. Let us dig the calculation of the Taylor expansion in the case of subsection 3.1. The algebraic equation used to determine the firsts elements of the expansion are:

$$\begin{split} \hat{E}_{1;00} &= c_{00}(m^2 + \lambda(c_{00}^2 + d_{00}^2)) = 0 \\ \hat{E}_{2;00} &= d_{00}(m^2 + \lambda(c_{00}^2 + d_{00}^2)) = 0 \\ \hat{E}_{1;10} &= 2\lambda c_{00} d_{00} d_{10} + c_{10}(k_{1\mu}k_1^{\mu} + m^2 + \lambda(3c_{00}^2 + d_{00}^2)) = 0 \\ \hat{E}_{2;10} &= 2\lambda c_{00} d_{00} c_{10} + d_{10}(k_{1\mu}k_1^{\mu} + m^2 + \lambda(c_{00}^2 + 3d_{00}^2)) = 0 \\ \hat{E}_{1;01} &= 2\lambda c_{00} d_{00} d_{01} + c_{01}(k_{2\mu}k_2^{\mu} + m^2 + \lambda(3c_{00}^2 + d_{00}^2)) = 0 \\ \hat{E}_{2;01} &= 2\lambda c_{00} d_{00} c_{01} + d_{01}(k_{2\mu}k_2^{\mu} + m^2 + \lambda(c_{00}^2 + 3d_{00}^2)) = 0 \end{split}$$

In these system, we impose $c_{00} = d_{00} = 0$ because we would like that the fields be null at infinite (similarly, this physical constraint could be imposed on ansats (20) for simplify the solution of the algebraic system). So, the above system yields

 $c_{00} = d_{00} = 0,$ $k_{1\mu}k_1^{\mu} + m^2 = 0,$ $k_{2\mu}k_2^{\mu} + m^2 = 0,$ $c_{10}, c_{01}, d_{10}, d_{01}$ arbitrary.

For calculate the Padé approximants $[2/2]^{(i)}_{\rho}(\xi; \mathcal{S})$, we still need to find c_{mn} and d_{mn} for $m + n \leq 4$; however, the equations that yield these coefficients are linear and ease to solve. After we have calculated the Taylor expansion until the order we need, we stay with only 10 variables to be determined in the second stage of the algorithm $(c_{10}, c_{01}, d_{10}, d_{01} \text{ and } 3 k_{j\mu}$ for each wave).

Therefore, we can see that the algorithm based on Padé approximants can organize and simplify the processing of data by transforming part of procedure in a linear system. However, this algorithm has a disadvantage. If we deal with a model that has a singularity at the origin in the ρ_k variable, the multiple exp-method may perform better. The solution for such problem in the Padé approach is redefine the variables ρ_k as

$$\rho_j = e^{k_{j\mu}x^{\mu}} \rightarrow \rho_j = e^{k_{j\mu}x^{\mu}} - \alpha, \quad \partial_{\mu}\rho_j = k_{j\mu}\rho_j \rightarrow \partial_{\mu}\rho_j = k_{j\mu}(\rho_j + \alpha), \quad \alpha = \text{constant}$$

in order to avoid the singularity, but this could complicate the equation to be solved.

6. Conclusion

In this paper, it was applied a method based on the Padé approximants in order to obtain traveling wave solutions for the linear sigma model. With the solutions found for two and three bosonic fields, we were able to write a solution for an arbitrary number of bosons and traveling waves. The results of the current work show that the method developed in [20] is robust and can be used to find explicit solutions in specific problems in classical field theory.

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