# An accurate regularization technique for the backward heat conduction problem with time-dependent thermal diffusivity factor 

Milad Karimi, Fridoun Moradlou, Mojtaba Hajipour<br>Department of Mathematics, Sahand University of Technology, P.O. Box: 51335-1996, Tabriz, Iran


#### Abstract

In this work, an accurate regularization technique based on the Meyer wavelet method is developed to solve the ill-posed backward heat conduction problem with time-dependent thermal diffusivity factor in an infinite "strip". In principle, the extremely ill-posedness of the considered problem is caused by the amplified infinitely growth in the frequency components which lead to a blow-up in the representation of the solution. Using the Meyer wavelet technique, some new stable estimates are proposed in the Hölder and Logarithmic types which are optimal in the sense of given by Tautenhahn. The stability and convergence rate of the proposed regularization technique are proved. The good performance and the high-accuracy of this technique is demonstrated through various one and two dimensional examples. Numerical simulations and some comparative results are presented.


Keywords: Backward heat conduction; Meyer wavelet; Ill-posed problem; Multi-resolution analysis.

## 1 Introduction

Consider the backward heat conduction problem (BHCP) in an infinite "strip" domain as follows

$$
\left\{\begin{array}{l}
\partial_{t} u(\mathbf{x}, t)-\kappa(t) \nabla^{2} u(\mathbf{x}, t)=0, \quad(\mathbf{x}, t) \in \mathbb{R}^{n} \times[0, T)  \tag{1a}\\
u(\mathbf{x}, T)=\varphi_{T}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{n}
\end{array}\right.
$$

where $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right), \nabla^{2}=\sum_{i=1}^{n} \partial_{x_{i}}^{2}$ is an $n$-dimensional Laplace operator, $\kappa(t) \in \mathscr{C}([0, T])$ is a positive time-dependent thermal diffusivity factor and (1b) describes a final boundary value condition. The BHCP given by (11) is considered as an inverse problem in mathematical physics [1]. This problem is well-known to be extremely ill-posed in the sense of Hadamard, i.e., solution does not always exist, and when the solution exists, it do not depend continuously on the scattered data in any reasonable topology [2]. This model of the problem appears in many practical areas, such as mathematical finance, mechanics of continuous media, image processing and heat propagation in thermophysics 3-5. The nonhomogeneous type of equation (1a) is also considered as an advection-convection equation appeared in many pollution problems, particularly in groundwater pollution source identification problems [6. Due to the ill-posedness nature of the BHCP, most classical approximation techniques are not successful to find an acceptable approximate solution. To overcome this difficulty, some special regularization techniques are required.

In the past two decades, various techniques have been developed to the special cases of the BHCP given by (11). For $\kappa(t)=1$ and $\eta(x, t)=0$, the one-dimensional BHCP has been studied by some researcher e.g, John introduced in [7] a bound on the solution at $t=T$ with relaxation on the initial datum $\varphi_{T}$, Lattés and Lions [8], Showalter [9], Ames [10], Miller [11] used quasi-reversibility methods to approximate the BHCP. Moreover, the least squares schemes with Tikhonov regularization were proposed in [11, 13]. An optimal error estimate and uniqueness conditions for the one-dimensional BHCP with $\kappa(t)=1$ and $\eta(x, t)=0$ have

[^0]been studied in [14] and [15], respectively. In [16, 17], an approximate solution for the BHCP have been presented by using the Fourier truncated methods. Many numerical schemes have been also developed to solve the BHCP including Tikhonov regularization [18], fundamental solution [19], meshless [20], central difference and quasi-reversibility [21], parallel [22], quasi-reversibility [23], boundary element [24], operator marching [25], convolution regularization [26] and mollification [27] methods. The nonhomogeneous case of the BHCP has been considered by Trong et al [28,29]. By using a truncation regularization method, the one dimensional case of the BHCP with the time-dependent diffusion coefficient has been formulated in 30 . 32 . A modified quasi-reversibility method for the $n$-dimensional BHCP has been also developed in 33. The most error estimates for the BHCP presented in the literature are of the Hölder type which is not more suitable to measure with adequate accuracy. Wavelets theory as a new relatively tools is applied in engineering and mathematical sciences 34. The basis wavelets authorises us to attack problems not accessible with conventional approximate techniques 35]. This basis can be modified in a systematic way and can be applied in different regions of space with different resolutions. Therefore, wavelet methods have been introduced for solving the inverse and ill-posed parabolic partial differential equations (PDEs) [36] 39]. Recently a wavelet regularization method was proposed by the authors for solving the Helmholtz equation [36].

The main inspiration of this paper is to introduce an efficient Meyer wavelet regularization technique to solve the high-dimensional BHCP given by (11) with a positive time-dependent thermal diffusivity factor. This technique provides a regularization parameter for an appropriate multi-resolution scale space to get an optimal error estimate in the sense of given by 40]. The convergence rate of this error estimate represents in the Hölder and Logarithmic types. The main features of the new regularization technique are summarized as follows:

- The presented regularization method provides an optimal error estimate in the Logarithmic type which is more suitable to measure with high accuracy.
- This technique retrieves the solution of BHCP with smooth and non-smooth final data, satisfactory.
- The proposed technique is successful to solve the high-dimensional BHCP, accurately.

The outline of the rest of the paper is structured as follows. The ill-posedness of the BHCP is studied by Section 2. The Meyer wavelets and their properties for solving the ill-posed BHCP is described in Section 3. Section 4 provides some sharp error estimates between the approximate and exact solutions as well as the choice of the regularization parameter. Finally, the efficiency and the accuracy of the proposed technique are confirmed by solving some numerical examples in Section 5 .

## 2 The ill-posedness behavior of the problem

Here, we study the ill-posedness behavior of the BHCP. The Schwartz space and its dual are denoted by $\mathcal{S}\left(\mathbb{R}^{n}\right)$ and $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$, respectively. The Fourier transform of a function $g \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ is described by

$$
\begin{equation*}
\hat{g}(\boldsymbol{\omega}):=\frac{1}{(\sqrt{2 \pi})^{n}} \int_{\mathbb{R}^{n}} g(\mathbf{x}) e^{-i \boldsymbol{\omega} \cdot \mathbf{x}} d \mathbf{x} \tag{2}
\end{equation*}
$$

where $\boldsymbol{\omega}=\left(\omega_{1}, \cdots, \omega_{n}\right)$. While for a tempered distribution $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$, the Fourier transform is described by

$$
\langle\hat{f}, g\rangle=\langle f, \hat{g}\rangle, \quad \forall g \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)
$$

where $\langle\cdot, \cdot\rangle$ denote the inner product. For $p \geq 0, \mathbf{H}^{p}\left(\mathbb{R}^{n}\right)$ is signified the Sobolev space of all tempered distributions $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ with the following norm

$$
\begin{equation*}
\|f\|_{\mathbf{H}^{p}}:=\left(\int_{\mathbb{R}^{n}}|\hat{f}(\boldsymbol{\omega})|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{p} d \boldsymbol{\omega}\right)^{\frac{1}{2}} \tag{3}
\end{equation*}
$$

where $\|\cdot\|$ describes the Euclidian norm. It is easy to see that $\mathbf{H}^{0}\left(\mathbb{R}^{n}\right)=\mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$, and $\mathscr{L}^{2}\left(\mathbb{R}^{n}\right) \subset \mathbf{H}^{p}\left(\mathbb{R}^{n}\right)$ for $p \leq 0$. Suppose that the function $u(\cdot, t)$ satisfies in the problem given by (11) in the classical sense when $u(\cdot, t) \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$ for $0 \leq t<T$. If $u(\cdot, t) \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$ satisfies in the problem (11), then

$$
\left\{\begin{array}{lr}
\partial_{t} \hat{u}(\boldsymbol{\omega}, t)+\kappa(t) \boldsymbol{\omega}^{2} \hat{u}(\boldsymbol{\omega}, t)=0, & (\boldsymbol{\omega}, t) \in \mathbb{R}^{n} \times[0, T),  \tag{4}\\
\hat{u}(\boldsymbol{\omega}, T)=\widehat{\varphi}_{T}(\boldsymbol{\omega}), & \boldsymbol{\omega} \in \mathbb{R}^{n},
\end{array}\right.
$$

where $\hat{u}(\boldsymbol{\omega}, t)$ is the Fourier transform of $u(\cdot, t) \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$. Using a simple calculation, the solution of the problem (4) derives in the following form

$$
\begin{equation*}
\hat{u}(\boldsymbol{\omega}, t)=\widehat{\varphi}_{T}(\boldsymbol{\omega}) e^{\boldsymbol{\omega}^{2} \mu_{T}(t)} \tag{5}
\end{equation*}
$$

where $\mu_{T}(t):=\int_{t}^{T} \kappa(\nu) d \nu>0$. From the term in the right-hand side (5), the factor $e^{\omega^{2} \mu_{T}(t)}$ increase rapidly as $\|\boldsymbol{\omega}\| \rightarrow \infty$ and $t<T$. Thus the term $e^{\boldsymbol{\omega}^{2} \mu_{T}(t)}$ is the source of instability. So, the existence of a solution in $\mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$ depends on a rapid decay of $\widehat{\varphi}_{T}(\boldsymbol{\omega})$ at high frequencies. But in practice, the final data at $t=T$, is denoted by $\varphi_{T, m}(\mathbf{x})$, which often obtained on the basis of measuring of physical system. Moreover, the data $\varphi_{T, m}(\mathbf{x})$ are not accessible with absolute accuracy and does not possess such a rapid decay property in general. Hence, decay of this exact data is not likely to occur in the $\widehat{\varphi}_{T, m}(\boldsymbol{\omega})$. As a measured data $\varphi_{T, m}(\mathbf{x})$, the Fourier transform $\widehat{\varphi}_{T, m}(\boldsymbol{\omega})$ is merely belong to $\mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$. For each $t, 0<t \leq T$, a dramatically large error in calculation of the solution $u(\mathbf{x}, t)$ will be probably happen for a small perturbation in data $\varphi_{T}(\mathbf{x})$. Therefore, these perturbation of high frequencies lead to the ill-posedness of the problem (1).

We will show that how the BHCP suffers from nonexistence and instability of the solution. For that mean, suppose that the function $\varphi_{T}(\cdot)$ is exact data and $\varphi_{T, m}(\cdot)$ is measured data, corresponding exact data $\varphi_{T}(\cdot)$. We set $\varphi_{T, m}(\mathbf{x}):=\varphi_{T}(\mathbf{x})+\frac{\sin (m\|\mathbf{x}\|)}{m^{2}}$. For $0 \leq t<T$, the data error is defined as

$$
\left\|\varphi_{T, m}-\varphi_{T}\right\|_{\infty}=\sup _{\mathbf{x} \in \mathbb{R}^{n}}\left|\varphi_{T, m}(\mathbf{x})-\varphi_{T}(\mathbf{x})\right|=\sup _{\mathbf{x} \in \mathbb{R}^{n}}\left|\frac{\sin (m\|\mathbf{x}\|)}{m^{2}}\right| \leq \frac{1}{m^{2}}
$$

For $\varphi_{T, m}$, the solution of problem (1), is expressed as follows

$$
u_{m}(\mathbf{x}, t)=\frac{\sin (m\|\mathbf{x}\|) e^{m^{2} \mu_{T}(t)}}{m^{2}}+u(\mathbf{x}, t)
$$

hence

$$
\begin{aligned}
\left\|u_{m}(\cdot, t)-u(\cdot, t)\right\|_{\infty} & =\sup _{\substack{t \in[0, T) \\
\mathbf{x} \in \mathbb{R}^{n}}}\left|u_{m}(\mathbf{x}, t)-u(\mathbf{x}, t)\right| \\
& \leq \sup _{\substack{t \in[0, T) \\
\mathbf{x} \in \mathbb{R}^{n}}}\left|\frac{\sin (m\|\mathbf{x}\|) e^{m^{2} \mu_{T}(t)}}{m^{2}}\right| \leq \sup _{\substack{t \in[0, T) \\
\mathbf{x} \in \mathbb{R}^{n}}}\left|\frac{e^{m^{2} \mu_{T}(t)}}{m^{2}}\right| \leq \frac{e^{m^{2} \mu_{T}(0)}}{m^{2}}
\end{aligned}
$$

Therefore, we can derive

$$
\lim _{m \rightarrow \infty}\left\|\varphi_{T, m}-\varphi_{T}\right\|_{\infty} \leq \lim _{m \rightarrow \infty} \frac{1}{m^{2}}=0
$$

and

$$
\lim _{m \rightarrow \infty}\left\|u_{m}(\mathbf{x}, \cdot)-u(\mathbf{x}, \cdot)\right\|_{\infty} \leq \lim _{m \rightarrow \infty} \frac{e^{m^{2} \mu_{T}(0)}}{m^{2}}=\infty
$$

Consequently, the problem defined by (1) is extremely ill-posed and its approximate simulation is complicated. This ill-posedness is caused by the disturbation of high frequencies. In Figure 1 (a), we give the


Figure 1: (a) The exact solution, and (b) the unregularized solution reconstructed from $\varphi_{1, m}$ at $t=0$.
exact solution at $t=0$, that is, $u(x, y, 0)$, and the reconstructed solution $u^{\delta}(x, y, 0)$ from the noisy data $\varphi_{T, m}(x, y)$ without regularization. This figure shows that $u^{\delta}$ does not approximate the solution. Thus, some regularization procedure is necessary. From a computational analysis point of view the Figure 1 (b) shows that the problem is extremely ill-posed. Hence, it is desirable to design an efficient strategy for solving the BHCP. Using the wavelets theory, some regularization techniques are developed to overcome this type of difficulty. Unlike most other wavelets, the Meyer wavelets are special. In fact, the most important property of the Meyer wavelets is that they are compact support in the frequency domain but in the time domain there is no such property. Using correct choice of regularization parameter and applying the Meyer wavelet, we can formulate a regularized solution of problem (11) in Section 4. Therefore, this problem will become well-posed. However, we will discuss the Meyer wavelets in the next section in details.

## 3 Meyer Wavelets

Throughout of the paper, the $n$-dimensional Meyer's orthonormal scaling function denoted by $\Phi$. The onedimensional Meyer wavelet and scaling functions are respectively denoted by $\psi(x)$ and $\phi(x)$. These functions satisfy the following properties 41]:

$$
\begin{align*}
\operatorname{supp} \hat{\phi} & =\left[-\frac{4 \pi}{3}, \frac{4 \pi}{3}\right]  \tag{6}\\
\operatorname{supp} \hat{\psi} & =\left[-\frac{8 \pi}{3},-\frac{2 \pi}{3}\right] \cup\left[\frac{2 \pi}{3}, \frac{8 \pi}{3}\right] \tag{7}
\end{align*}
$$

It can be proved that the set of functions

$$
\begin{equation*}
\psi_{j, k}(x):=2^{j / 2} \psi\left(2^{j} x-k\right), \quad j, k \in \mathbb{Z} \tag{8}
\end{equation*}
$$

is an orthonormal basis of $\mathscr{L}^{2}(\mathbb{R})$ 41]. Consequently, the multi-resolution analysis (MRA) for the Meyer wavelet in $\mathscr{L}^{2}(\mathbb{R})$ is the family of all closed subspaces $\left\{V_{j}\right\}_{j \in \mathbb{Z}}$ which is produced by

$$
\begin{equation*}
V_{j}=\overline{\operatorname{span}\left\{\phi_{j, k}: k \in \mathbb{Z}\right\}}, \quad \phi_{j, k}=2^{j / 2} \phi\left(2^{j} t-k\right), \quad j, k \in \mathbb{Z} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{supp} \hat{\phi}_{j, k}=\left[-\frac{4 \pi}{3} 2^{j}, \frac{4 \pi}{3} 2^{j}\right], \quad k \in \mathbb{Z} \tag{10}
\end{equation*}
$$

Using tensor products of the spaces $V_{j}$, we can generate an $n$-dimensional MRA 42. Therefore, the function $\Phi$ given by

$$
\begin{equation*}
\Phi(\mathbf{x})=\prod_{k=1}^{n} \phi\left(x_{k}\right), \quad \mathbf{x} \in \mathbb{R}^{n} \tag{11}
\end{equation*}
$$

defines a $n$-dimensional scaling function. Moreover, the general form of the basis function $\Psi$ in the wavelet space $W_{J}$ is

$$
\begin{equation*}
\Psi(\mathbf{x})=2^{n J / 2} \psi\left(2^{J} x_{i}-k_{i}\right) \cdot \prod_{m \neq i} \Theta_{m}\left(2^{J} x_{m}-k_{m}\right), \quad \mathbf{x} \in \mathbb{R}^{n} \tag{12}
\end{equation*}
$$

where $k_{i} \in \mathbb{Z}$. Note that the functions $\phi$ or $\psi$ are corresponding to any $\Theta_{m}, m \in\{1, \cdots, n\}$. Consequently, from (6) we get that

$$
\begin{equation*}
\operatorname{supp} \hat{\Phi}=\left[-\frac{4 \pi}{3}, \frac{4 \pi}{3}\right]^{n} \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{f}(\boldsymbol{\omega})=0 \quad \text { for } \quad\|\boldsymbol{\omega}\|_{\infty} \leq \frac{2}{3} \pi 2^{J}, \quad f \in W_{J}, \quad J \in \mathbb{N} \tag{14}
\end{equation*}
$$

The orthogonal projection operators of $\mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$ onto spaces $V_{J}$ and $W_{J}$ is denoted by the following equations, respectively

$$
\begin{equation*}
\mathcal{P}_{J} f:=\sum_{k \in \mathbb{Z}^{n}}\left\langle f, \Phi_{J, k}\right\rangle \Phi_{J, k}, \quad f \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{Q}_{J} f:=\sum_{k \in \mathbb{Z}^{n}}\left\langle f, \Psi_{J, k}\right\rangle \Psi_{J, k}, \quad f \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right) \tag{16}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ denotes $\mathscr{L}^{2}$-inner product. Note that the connection between the wavelet and scale spaces simply defined by the following relation

$$
V_{J+1}=V_{J} \oplus W_{J}
$$

where the wavelet space $W_{J}$ is considered to as the orthogonal complement of $V_{J}$ in $V_{J+1}$. Let

$$
\begin{equation*}
\Lambda_{J}:=2^{J}\left[-\frac{2}{3} \pi, \frac{2}{3} \pi\right]^{n} \tag{17}
\end{equation*}
$$

Using (14), for $J \in \mathbb{N}$, we have

$$
\begin{equation*}
\widehat{\mathcal{P}_{J} f}(\boldsymbol{\omega})=0, \quad \text { for } \quad \boldsymbol{\omega} \in \Gamma_{J+1} \tag{18}
\end{equation*}
$$

where $\Gamma_{J}:=\mathbb{R}^{n} \backslash \Lambda_{J}$, and

$$
\begin{equation*}
\left(\widehat{I-\mathcal{P}_{J}}\right) f(\boldsymbol{\omega})=\widehat{\mathcal{Q}_{J}} f(\boldsymbol{\omega}), \quad \text { for } \quad \boldsymbol{\omega} \in \Lambda_{J+1} \tag{19}
\end{equation*}
$$

Let $\chi_{J}$ be the characteristic function of the cube $\Lambda_{J}$ and define the operator $M_{J}$ by the following equation

$$
\begin{equation*}
\widehat{M_{J} f}:=\left(1-\chi_{J}\right) \hat{f}, \quad J \in \mathbb{N} \tag{20}
\end{equation*}
$$

From Eq. (12), for $j \geq J$, each basis function $\Psi$ in $W_{j}$ satisfies

$$
\begin{equation*}
\hat{\Psi}(\boldsymbol{\omega})=0, \quad \boldsymbol{\omega} \in \Lambda_{J} \tag{21}
\end{equation*}
$$

and so we get

$$
\begin{array}{r}
\langle f, \Psi\rangle=\langle\hat{f}, \hat{\Psi}\rangle=\left\langle\left(1-\chi_{J} \hat{f}, \hat{\Psi}\right)=\left\langle M_{J}, \Psi\right\rangle\right. \\
\mathcal{Q}_{J}=\mathcal{Q}_{J} M_{J} \\
I-\mathcal{P}_{J}=\left(I-\mathcal{P}_{J}\right) M_{J} \tag{24}
\end{array}
$$

Many Bernestien-type inequalities are hold in [43] for the partial differential operators $\partial_{\mathbf{x}}^{r}$ where $\partial_{\mathbf{x}}^{k}:=\frac{\partial^{r}}{\partial \mathbf{x}^{r}}$.
Theorem 1. Suppose that $\left\{V_{j}\right\}_{j \in \mathbb{Z}}$ is the Meyer's MRA. Then for $J \in \mathbb{N}, q \in \mathbb{R}$ and all $\varphi \in V_{J}$, we have

$$
\begin{equation*}
\left\|\partial_{x_{i}}^{r} \varphi\right\|_{\mathbf{H}^{q}} \leq C 2^{(J-1) r}\|\varphi\|_{\mathbf{H}^{q}}, \quad i=1, \cdots, n, \quad r \in \mathbb{N} \tag{25}
\end{equation*}
$$

where $C$ is the positive constant.
Proof. From [42,43], the following inequalities are derived

$$
\begin{align*}
& \left\|\partial_{x_{i}}^{r} \varphi\right\|_{\mathbf{H}^{q}} \leq C_{0} 2^{j r}\|\varphi\|_{\mathbf{H}^{q}}, \quad r \in \mathbb{N}, \quad \varphi \in W_{j}, \quad j \geq 0, \quad i=1, \cdots, n,  \tag{26}\\
& \left\|\partial_{x_{i}}^{r} \varphi\right\|_{\mathbf{H}^{q}} \leq C_{1}\|\varphi\|_{\mathbf{H}^{q}}, \quad r \in \mathbb{N}, \quad \varphi \in V_{0}, \quad i=1, \cdots, n .
\end{align*}
$$

Because of $\mathcal{P}_{J}=\mathcal{P}_{J-1}+\mathcal{Q}_{J-1}, J \in \mathbb{N}$, for $\varphi \in V_{J} ; J \geq 0$, we have

$$
\begin{align*}
\left\|\partial_{x_{i}}^{r} \mathcal{P}_{J} \varphi\right\|_{\mathbf{H}^{q}} & =\left\|\partial_{x_{i}}^{r} \mathcal{P}_{0} \varphi\right\|_{\mathbf{H}^{q}}+\left\|\sum_{j=0}^{J-1} \partial_{x_{i}}^{r} \mathcal{Q}_{j} \varphi\right\|_{\mathbf{H}^{q}} \\
& \leq C_{0}\left\|\mathcal{P}_{0}\right\|_{\mathbf{H}^{q}}\|\varphi\|_{\mathbf{H}^{q}} 2^{J r}+C_{1} \sum_{j=0}^{J-1} 2^{j r}\left\|\mathcal{Q}_{j}\right\|_{\mathbf{H}^{q}}\|\varphi\|_{\mathbf{H}^{q}} \\
& \leq C_{0}\left\|\mathcal{P}_{0}\right\|_{\mathbf{H}^{q}}\|\varphi\|_{\mathbf{H}^{q}} 2^{J r}+C_{1} \sum_{j=0}^{J-1} 2^{J r}\left\|\mathcal{Q}_{j}\right\|_{\mathbf{H}^{q}}\|\varphi\|_{\mathbf{H}^{q}} \\
& =\left(C_{0}\left\|\mathcal{P}_{0}\right\|_{\mathbf{H}^{q}}+C_{1} \sum_{j=0}^{J-1}\left\|\mathcal{Q}_{j}\right\|_{\mathbf{H}^{q}}\right) 2^{J r}\|\varphi\|_{\mathbf{H}^{q}} \\
& =C 2^{J r}\|\varphi\|_{\mathbf{H}^{q}} \tag{27}
\end{align*}
$$

where $C:=C_{0}\left\|\mathcal{P}_{0}\right\|_{\mathbf{H}^{q}}+C_{1} \sum_{j=0}^{J-1}\left\|\mathcal{Q}_{j}\right\|_{\mathbf{H}^{q}}$. It follows from (26) and (27), that

$$
\begin{aligned}
\left\|\partial_{x_{i}}^{r} \varphi\right\|_{\mathbf{H}^{q}} & =\left\|\partial_{x_{i}}^{r} \mathcal{P}_{J} \varphi\right\|_{\mathbf{H}^{q}} \\
& \leq\left\|\partial_{x_{i}}^{r} \mathcal{P}_{J-1} \varphi\right\|_{\mathbf{H}^{q}}+\left\|\partial_{x_{i}}^{r} \mathcal{Q}_{J-1} \varphi\right\|_{\mathbf{H}^{q}} \leq C 2^{(J-1) r}\|\varphi\|_{\mathbf{H}^{q}} .
\end{aligned}
$$

The proof is complete.
Define an operator $F_{t}: \mathscr{L}^{2}\left(\mathbb{R}^{n}\right) \longrightarrow \mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$ by $\left(F_{t} \varphi_{T}\right)(\mathbf{x}):=u(\mathbf{x}, t)$. Then we have a following lemma.
Lemma 2. Suppose that $\left\{V_{j}\right\}_{j \in \mathbb{Z}}$ is the Meyer's MRA, $J \in \mathbb{N}, q \in \mathbb{R}$ and $0 \leq t<T$. Then for all $\varphi \in V_{J}$

$$
\left\|F_{t} \varphi\right\|_{\mathbf{H}^{q}} \leq C_{2} \exp \left(2^{2 J} \mu_{T}(t)\right)\|\varphi\|_{\mathbf{H}^{q}}
$$

where $C_{2}$ is the positive constant.

Proof. Let $\varphi \in V_{J}$, then we have

$$
\begin{aligned}
\left\|F_{t} \varphi\right\|_{\mathbf{H}^{q}} & =\left(\int_{\mathbb{R}^{n}}\left|\widehat{F_{t} \varphi}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\left(\int_{\mathbb{R}^{n}}\left|\widehat{\varphi}(\boldsymbol{\omega}) e^{\boldsymbol{\omega}^{2} \mu_{T}(t)}\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\left(\int_{\mathbb{R}^{n}}\left|\widehat{\varphi}(\boldsymbol{\omega}) \sum_{r=0}^{+\infty} \frac{\left(\boldsymbol{\omega}^{2} \mu_{T}(t)\right)^{r}}{r!}\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\left(\int_{\mathbb{R}^{n}}\left|\widehat{\varphi}(\boldsymbol{\omega}) \sum_{r=0}^{+\infty} \frac{\left(\mu_{T}(t)\right)^{r}}{r!} \boldsymbol{\omega}^{2 r}\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\sum_{r=0}^{+\infty} \frac{\left(\mu_{T}(t)\right)^{r}}{r!}\left(\int_{\mathbb{R}^{n}}\left|(i \boldsymbol{\omega})^{2 r} \widehat{\varphi}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\sum_{r=0}^{+\infty} \frac{\left(\mu_{T}(t)\right)^{r}}{r!}\left\|\partial_{x}^{2 r} \varphi\right\|_{\mathbf{H}^{q}} \\
& \leq C \sum_{r=0}^{+\infty} \frac{\left(\mu_{T}(t)\right)^{r}}{r!} \cdot n 2^{2(J-1) r}\|\varphi\|_{\mathbf{H}^{q}} \\
& =C_{2} \sum_{r=0}^{+\infty} \frac{\left(2^{2(J-1)} \mu_{T}(t)\right)^{r}}{r!}\|\varphi\|_{\mathbf{H}^{q}} \\
& =C_{2} \exp \left(2^{2(J-1)} \mu_{T}(t)\right)\|\varphi\|_{\mathbf{H}^{q}} \\
& \leq C_{2} \exp \left(2^{2 J} \mu_{T}(t)\right)\|\varphi\|_{\mathbf{H}^{q} .}
\end{aligned}
$$

The proof is complete.

## 4 Wavelet Regularization and Convergence Analysis

In this section, we suppose that functions $\varphi_{T}(\cdot) \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$ and $\varphi_{T, m}(\cdot)$ are exact and measured data at $t=T$ satisfying

$$
\begin{equation*}
\left\|\varphi_{T}-\varphi_{T, m}\right\|_{\mathbf{H}^{q}} \leq \delta \quad \text { for some } q \leq 0 \tag{28}
\end{equation*}
$$

In general, we know that $\varphi_{T, m}(\cdot) \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right) \subset \mathbf{H}^{q}\left(\mathbb{R}^{n}\right)$ for $q \leq 0$. The major goal of this section is to provide a sharp approximation of the exact solution $u(\cdot, t)$ for $0 \leq t<T$. To this end, we assume that $\varphi_{0}(\mathbf{x}):=u(\mathbf{x}, 0) \in \mathbf{H}^{p}\left(\mathbb{R}^{n}\right)$ for some $p \geq q$, and

$$
\begin{equation*}
\left\|\varphi_{0}\right\|_{\mathbf{H}^{p}} \leq M \tag{29}
\end{equation*}
$$

where $M$ is a positive constant. In order to find the regularization parameter $J$ and to obtain some stability estimates of the Hölder and Logarithmic types, we use the following lemma which provided in [16] for choosing a proper regularization parameter $J$.
Lemma 3. [16] Let $c \in \mathbb{R}$ and the parameters $a<1, b$ and $d$ be positive constants. Then the function $f:[0, a] \longrightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
f(\lambda)=\lambda^{b}\left(d \ln \frac{1}{\lambda}\right)^{-c} \tag{30}
\end{equation*}
$$

is invertible and

$$
\begin{equation*}
f^{-1}(\lambda)=\lambda^{\frac{1}{b}}\left(\frac{d}{b} \ln \frac{1}{\lambda}\right)^{\frac{c}{b}}(1+o(1)) \quad \text { for } \quad \lambda \rightarrow 0 \tag{31}
\end{equation*}
$$

Following theorem shows that using an appropriate $J \in \mathbb{N}, F_{t, J}:=F_{t} \mathcal{P}_{J}$ is approximation of $F_{t}$ in a stable manner, where $J$ depending on $\delta$ and $M$.

Theorem 4. For $J \in \mathbb{N}$, the problem (1) with the final data $\varphi_{T}$ in $V_{J}$ is well-posed. Suppose that $\varphi_{0}(\cdot)$ belongs to $\mathbf{H}^{p}\left(\mathbb{R}^{n}\right)$ for some $p \in \mathbb{R}$ and the inequalities (28) and (29) holds for $q \leq \min \{0, p\}$. Then the proposed method to compute $F_{t, J} \varphi_{T, m}$ is stable in the Hadamard sense. Moreover, for

$$
\begin{equation*}
\left.J^{*}:=\llbracket \frac{1}{2} \log _{2} \ln \left(\left(\frac{M}{C_{3} \delta}\right)^{\frac{1}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{-\frac{p-q}{2 \mu_{T}(0)}}\right)\right], \quad C_{3}:=\frac{C_{2}}{C_{2}+1} \tag{32}
\end{equation*}
$$

where $\llbracket a \rrbracket$ signifies the largest integer less than or equal to $a$. The following inequality is satisfied

$$
\begin{equation*}
\left\|F_{t} \varphi_{T}-F_{t, J^{*}} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} \leq\left(C_{2}+1\right)\left(C_{3} \delta\right)^{\frac{\mu_{t}(0)}{\mu_{T}(0)}} M^{1-\frac{\mu_{t}(0)}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{-\frac{p-q}{2} \frac{\mu_{T}(t)}{\mu_{T}(0)}}(1+o(1)) \tag{33}
\end{equation*}
$$

for $\delta \rightarrow 0$.
Proof. It is easy to see that

$$
\begin{align*}
\left\|F_{t} \varphi_{T}-F_{t, J} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} & \leq\left\|F_{t} \varphi_{T}-F_{t, J} \varphi_{T}\right\|_{\mathbf{H}^{q}}+\left\|F_{t, J} \varphi_{T}-F_{t, J} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} \\
& =: N_{1}+N_{2} \tag{34}
\end{align*}
$$

It follows from the Lemma 2 and the condition (28) that

$$
\begin{align*}
N_{2}=\left\|F_{t, J} \varphi_{T}-F_{t, J} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} & =\left\|F_{t} \mathcal{P}_{J}\left(\varphi_{T}-\varphi_{T, m}\right)\right\|_{\mathbf{H}^{q}} \\
& \leq C_{2} \exp \left(2^{2 J} \mu_{T}(t)\right)\left\|\mathcal{P}_{J}\left(\varphi_{T}-\varphi_{T, m}\right)\right\|_{\mathbf{H}^{q}} \\
& \leq C_{2} \exp \left(2^{2 J} \mu_{T}(t)\right) \delta \tag{35}
\end{align*}
$$

Moreovere, from (18), we have

$$
\begin{align*}
N_{1}=\left\|F_{t} \varphi_{T}-F_{t, J} \varphi_{T}\right\|_{\mathbf{H}^{q}} & =\left\|F_{t}\left(I-\mathcal{P}_{J}\right) \varphi_{T}\right\|_{\mathbf{H}^{q}} \\
& =\left(\int_{\mathbb{R}^{n}} \mid e^{\omega^{2} \mu_{T}(t)}\left(\left.\left(I \widehat{\left.-\mathcal{P}_{J}\right)} \varphi_{T}\right)(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2}\right. \\
& =\left(\int_{\Gamma_{J+1}}\left|e^{\boldsymbol{\omega}^{2} \mu_{T}(t)} \widehat{\varphi_{T}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& +\left(\int_{\Lambda_{J+1}} \mid e^{\boldsymbol{\omega}^{2} \mu_{T}(t)}\left(\left.\left(I \widehat{\left.-\mathcal{P}_{J}\right)} \varphi_{T}\right)(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2}\right. \\
& =: I_{1}+I_{2} . \tag{36}
\end{align*}
$$

We can calculate

$$
\begin{align*}
I_{1} & =\left(\int_{\Gamma_{J+1}}\left|e^{\boldsymbol{\omega}^{2} \mu_{T}(t)} \widehat{\varphi_{T}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\left(\int_{\Gamma_{J+1}}\left|e^{-\boldsymbol{\omega}^{2} \mu_{t}(0)} \widehat{\varphi_{0}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\left(\int_{\Gamma_{J+1}}\left|e^{-\boldsymbol{\omega}^{2} \mu_{t}(0)}\right|^{2}\left|\widehat{\varphi_{0}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& \leq \sup _{\boldsymbol{\omega} \in \Gamma_{J+1}} e^{-\boldsymbol{\omega}^{2} \mu_{t}(0)} \frac{1}{\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{\frac{p-q}{2}}}\left(\int_{\Gamma_{J+1}}\left|\widehat{\varphi_{0}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{p} d \boldsymbol{\omega}\right)^{1 / 2} \\
& \leq \sup _{\boldsymbol{\omega} \in \Gamma_{J+1}} e^{-\boldsymbol{\omega}^{2} \mu_{t}(0)}\|\boldsymbol{\omega}\|^{-(p-q)}\left\|\varphi_{0}\right\|_{\mathbf{H}^{p}} \\
& \leq e^{-n\left(\frac{4}{3} \pi 2^{J}\right)^{2} \mu_{t}(0)}\left(n\left(\frac{4}{3} \pi 2^{J}\right)^{2}\right)^{-\frac{p-q}{2}} M \\
& \leq e^{-2^{2 J} \mu_{t}(0)} 2^{-2 J\left(\frac{p-q}{2}\right)} M . \tag{37}
\end{align*}
$$

Due to the (19), and noting that $\mathcal{Q}_{J} \varphi \in W_{J} \subset V_{J+1}$, the integrate $I_{2}$ defined by (36) satisfies

$$
\begin{align*}
I_{2} & =\left(\int_{\Lambda_{J+1}} \mid e^{\boldsymbol{\omega}^{2} \mu_{T}(t)}\left(\left.I \widehat{\left.-\mathcal{P}_{J}\right)} \varphi_{T}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2}\right. \\
& =\left\|F_{t} \mathcal{Q}_{J} \varphi_{T}\right\|_{\mathbf{H}^{q}} \\
& \leq C_{2} \exp \left(2^{2 J} \mu_{T}(t)\right)\left\|\mathcal{Q}_{J} \varphi_{T}\right\|_{\mathbf{H}^{q}} \tag{38}
\end{align*}
$$

Noting that $\varphi_{T} \in \mathscr{L}^{2}\left(\mathbb{R}^{n}\right)$, the Parseval formula, and (15), we can derive that

$$
\begin{aligned}
\mathcal{Q}_{J} \varphi_{T} & =\sum_{K=-\infty}^{+\infty}\left\langle\varphi_{T}, \Psi_{J K}\right\rangle \Psi_{J K} \\
& =\sum_{K=-\infty}^{+\infty}\left\langle\widehat{\varphi}_{T}, \hat{\Psi}_{J K}\right\rangle \Psi_{J K} \\
& =\sum_{K=-\infty}^{+\infty}\left\langle\left(1-\chi_{J}\right) \widehat{\varphi}_{T}, \hat{\Psi}_{J K}\right\rangle \Psi_{J K} \\
& =\sum_{K=-\infty}^{+\infty}\left\langle M_{J} \varphi_{T}, \hat{\Psi}_{J K}\right\rangle \Psi_{J K} \\
& =\mathcal{Q}_{J} M_{J} \varphi_{T}
\end{aligned}
$$

So, we conclude that

$$
\begin{aligned}
\left\|\mathcal{Q}_{J} \varphi_{T}\right\|_{\mathbf{H}^{q}} & =\left\|\mathcal{Q}_{J} M_{J} \varphi_{T}\right\|_{\mathbf{H}^{q}} \\
& \leq\left\|M_{J} \varphi_{T}\right\|_{\mathbf{H}^{q}} \\
& =\left(\int_{\Lambda_{J+1}}\left|\widehat{\varphi_{T}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\left(\int_{\Lambda_{J+1}}\left|e^{-\boldsymbol{\omega}^{2} \mu_{T}(0)} \widehat{\varphi_{0}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{q} d \boldsymbol{\omega}\right)^{1 / 2} \\
& \leq \sup _{\boldsymbol{\omega} \in \Lambda_{J+1}} e^{-\boldsymbol{\omega}^{2} \mu_{T}(0)} \frac{1}{\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{\frac{p-q}{2}}}\left(\int_{\boldsymbol{\omega} \in \Lambda_{J+1}}\left|\widehat{\varphi_{0}}(\boldsymbol{\omega})\right|^{2}\left(1+\|\boldsymbol{\omega}\|^{2}\right)^{p} d \boldsymbol{\omega}\right)^{1 / 2} \\
& \leq \sup _{\boldsymbol{\omega} \in \Lambda_{J+1}} e^{-\boldsymbol{\omega}^{2} \mu_{T}(0)}\|\boldsymbol{\omega}\|^{-(p-q)}\left\|\varphi_{0}\right\|_{\mathbf{H}^{p}} \\
& \leq e^{-n\left(\frac{2}{3} \pi 2^{J}\right)^{2} \mu_{T}(0)}\left(n\left(\frac{2}{3} \pi 2^{J}\right)^{2}\right)^{-\frac{p-q}{2}} M \\
& \leq e^{-2^{2 J} \mu_{T}(0)} 2^{-2 J\left(\frac{p-q}{2}\right)} M .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
I_{2} \leq C_{2} e^{-2^{2 J} \mu_{t}(0)} 2^{-2 J\left(\frac{p-q}{2}\right)} M \tag{39}
\end{equation*}
$$

together with (37), we get

$$
\begin{equation*}
\left\|F_{t, J} \varphi_{T}-F_{t, J} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} \leq\left(C_{2}+1\right) e^{-2^{2 J} \mu_{t}(0)} 2^{-2 J\left(\frac{p-q}{2}\right)} M \tag{40}
\end{equation*}
$$

Combining (40) with (35), we obtain

$$
\begin{equation*}
\left\|F_{t} \varphi_{T}-F_{t, J} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} \leq C_{2} \exp \left(2^{2 J} \mu_{T}(t)\right) \delta+\left(C_{2}+1\right) e^{-2^{2 J} \mu_{t}(0)} 2^{-2 J\left(\frac{p-q}{2}\right)} M \tag{41}
\end{equation*}
$$

where $C_{2}$ is given by Lemma 2. Based on Lemma 3, the regularization parameter $J$ can be chosen by minimizing the right-hand side of (41). Set $e^{-2^{2 J}}:=\lambda ; \lambda \in(0,1)$ and $C_{3}:=\frac{C_{2}}{C_{2}+1}$. Thus we have

$$
\begin{equation*}
C_{3} \lambda^{-\mu_{T}(t)} \delta=\lambda^{\mu_{t}(0)}\left(\ln \frac{1}{\lambda}\right)^{-\frac{p-q}{2}} M \tag{42}
\end{equation*}
$$

this leads to

$$
\begin{equation*}
\frac{C_{3} \delta}{M}=\lambda^{\mu_{T}(0)}\left(\ln \frac{1}{\lambda}\right)^{-\frac{p-q}{2}}, \tag{43}
\end{equation*}
$$

i.e., $b=\mu_{T}(0), d=1, c=\frac{p-q}{2}$ in (30). Then by Lemma 3, we can calculate that

$$
\begin{equation*}
\lambda=\left(\frac{C_{3} \delta}{M}\right)^{\frac{1}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{\frac{p-q}{2 \mu_{T}(0)}}(1+o(1)) \quad \text { for } \quad \frac{C_{3} \delta}{M} \rightarrow 0 . \tag{44}
\end{equation*}
$$

Taking the principal part of $\lambda$, given by (44) and due to the $e^{-2^{2 J}}=\lambda$, we have

$$
\begin{equation*}
J=\frac{1}{2} \log _{2} \ln \left(\left(\frac{M}{C_{3} \delta}\right)^{\frac{1}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{-\frac{p-q}{2 \mu_{T}(0)}}\right) \tag{45}
\end{equation*}
$$

Now, summarizing above inference process, we can conclude that

$$
\begin{align*}
\left\|F_{t} \varphi_{T}-F_{t, J^{*}} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} & \leq\left(C_{2}+1\right)\left(C_{3} \delta\right)^{\frac{\mu_{t}(0)}{\mu_{T}(0)}} M^{1-\frac{\mu_{t}(0)}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{-\frac{p-q}{2} \frac{\mu_{T}(t)}{\mu_{T}(0)}} \\
& \times\left\{1+\left(\frac{\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}}{\left.\left.\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}+\ln \left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{-\frac{p-q}{2 \mu_{T}(0)}}\right)^{\frac{p-q}{2}}\right\}}\right.\right. \\
& =\left(C_{2}+1\right)\left(C_{3} \delta\right)^{\frac{\mu_{t}(0)}{\mu_{T}(0)}} M^{1-\frac{\mu_{t}(0)}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{C_{3} \delta}\right)^{-\frac{p-q}{2} \frac{\mu_{T}(t)}{\mu_{T}(0)}}(1+o(1)) \tag{46}
\end{align*}
$$

for $\delta \rightarrow 0$. Therefore, the proof is complete.
Theorem 4 suggests how to define a wavelet regularized approximation of disturbed BHCP.
Remark 5. For $p=q=0$, the inequality (33) reduces to the following $\mathscr{L}^{2}$-estimate of the Hölder type

$$
\begin{equation*}
\left\|F_{t} \varphi_{T}-F_{t, J^{*}} \varphi_{T, m}\right\|_{\mathscr{L}^{2}\left(\mathbb{R}^{n}\right)} \leq\left(C_{2}+1\right)\left(C_{3} \delta\right)^{\frac{\mu_{t}(0)}{\mu_{T}(0)}} M^{1-\frac{\mu_{t}(0)}{\mu_{T}(0)}}(1+o(1)) \quad \text { for } \quad \delta \rightarrow 0 \tag{47}
\end{equation*}
$$

Note that the inequality (47) does not guarantee the convergence of the approximate solution $F_{t, J} \varphi(\mathbf{x}, t)$ at $t=0$. For $t=0$, it provides an upper bound for error estimate by $\left(C_{2}+1\right) M$ which can not be improved in $\mathscr{L}^{2}$-scale. On the other hand, for $p-q>0$, the inequality (46) shows that the convergence of the regularization solution for $0 \leq t<T$ is faster than the approximate solution given by (47). Especially, at $t=0$ the estimate (46) becomes

$$
\begin{aligned}
\left\|F_{0} \varphi_{T}-F_{0, J^{*}} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} & =\left\|\varphi_{0}-F_{0, J^{*}} \varphi_{T, m}\right\|_{\mathbf{H}^{q}} \\
& \leq\left(C_{2}+1\right) M\left(\frac{1}{\mu_{T}(0)} \ln \frac{M}{\delta}\right)^{-\frac{p-q}{2}}(1+o(1)) \quad \text { for } \quad \delta \rightarrow 0
\end{aligned}
$$

which is a $\mathbf{H}^{q}$-estimate of the Logarithmic type.
Remark 6. The condition $p-q>0$ is not harsh. By the Sobolev imbedding theorem, the smoothness of $\varphi_{0}(\cdot):=u(\cdot, 0)$ is only slightly raised. For example, taking $q=0$ and $p=\frac{1}{2}$, then $\varphi_{0}(\cdot)$ would be in $C^{0}\left(\mathbb{R}^{n}\right)$.

Remark 7. In practice, the constants $C, C_{0}, C_{1}, C_{2}, C_{3}$ and a priori bound $M$. For $M=1$, we have

$$
J^{\dagger}:=\left[\frac{1}{2} \log _{2} \ln \left(\left(\frac{1}{\delta}\right)^{\frac{1}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{1}{\delta}\right)^{-\frac{p-q}{\mu_{T} \mu^{(0)}}}\right)\right]
$$

and we set $u_{J^{\dagger}}^{\delta}:=F_{t, J^{\dagger}} \varphi_{T, m}$, then there satisfies the estimate

$$
\left\|u(\cdot, t)-u_{J^{\dagger}}^{\delta}(\cdot, t)\right\|_{\mathscr{L}^{2}\left(\mathbb{R}^{n}\right)} \leq 2 \delta^{\frac{\mu_{t}(0)}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{1}{\delta}\right)^{-\frac{p-q}{2} \frac{\mu_{T}(t)}{\mu_{T}(0)}}(1+o(1))
$$

for $\delta \rightarrow 0$.
Note that the "optimal" or "order optimal" estimation for the upper bound of the inequality given by (33) is of the Hölder and Logarithmic forms [14,40, 44. Thus, the proposed technique is of order optimal and there is no an other efficient approximation method to approximate the solution of problem given by (11). So, the wavelet methods are useful for ill-posed problems.

## 5 Numerical Treatment

Here, we provide the numerical simulations of the proposed Meyer wavelet regularization (MWR) method for one and two dimensional cases of the BHCP with smooth and non-smooth data. The computations associated with the examples were performed using Mathematica 10.0. To derive the disturbation data, we add a random uniformly distributed perturbation to any data as follows.

$$
\begin{equation*}
\left.\left.\left.\varphi_{T, m}:=\varphi_{T}+\epsilon \operatorname{RandomReal} \text { [NormalDistribution[.],\{Length[ } \varphi_{T}\right], \text { Length }\left[\varphi_{T}\right]\right\}\right] ; \tag{48}
\end{equation*}
$$

where RandomReal [.] gives a pseudorandom real number in the range of 0 to 1 , NormalDistribution[.] represents a normal distribution with zero mean and unit standard deviation and $\epsilon$ dicates the level of noise. According to the Section 4, the function $\varphi_{T, m}$ is formulated as

$$
\begin{equation*}
u_{J^{*}}^{\delta}:=F_{t, J^{*}} \varphi_{T, m}=F_{t} \mathcal{P}_{J^{*}} \varphi_{T, m}, \tag{49}
\end{equation*}
$$

where the regularization parameter $J^{*}$ is

$$
\begin{equation*}
J^{*}:=\left[\left[\frac{1}{2} \log _{2} \ln \left(\left(\frac{1}{\delta}\right)^{\frac{1}{\mu_{T}(0)}}\left(\frac{1}{\mu_{T}(0)} \ln \frac{1}{\delta}\right)^{-\frac{p-q}{2 \mu_{T}(0)}}\right)\right] .\right. \tag{50}
\end{equation*}
$$

Here, the sequence $\left\{\varphi_{T}\left(\mathbf{x}_{i}\right)\right\}_{i=1}^{N}$ represents samples of the functions $\varphi_{T}\left(\mathbf{x}_{i}\right)$ on an equidistant grid, for an even number $N$. Though this section $J$ denotes the the number of subspace $V_{J}$.

### 5.1 One dimensional examples

Here, the numerical results of the MWR method to solve some one-dimensional cases of the BHCP with smooth and non-smooth data are illustrated.

Example 1. Consider the following BHCP with the smooth data

$$
\left\{\begin{array}{l}
\partial_{t} u(x, t)=\kappa(t) \partial_{x x} u(x, t), \quad(x, t) \in\{(x, t) \mid 0 \leq t \leq 1,0 \leq x \leq \pi\}  \tag{51}\\
u(x, T)=\exp \left(\mu_{T}(0)\right) \frac{\sin (x)}{\exp (2)}, \quad 0 \leq x \leq \pi
\end{array}\right.
$$

where $\kappa(t)=\kappa t+1$ and $\kappa$ is a positive constant. The closed form analytical solution of Example 1 is

$$
\begin{equation*}
u(x, t)=\exp \left(\mu_{t}(0)\right) \frac{\sin (x)}{\exp (2)} \tag{52}
\end{equation*}
$$

For the level of noises $\epsilon=10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$, the proposed MWR method is implemented to solve this example. For $\kappa=2$, Figure 2 illustrates the comparison between the regularization and exact solutions with different levels of noise added into the final data. Corresponding to every level of noise, a regularization parameter is chosen. Moreover, it can be seen that the approximate solution converges to the exact solution, when the magnitude of noise decreases. For $\kappa=2$, the absolute error and the relative error of the proposed MWR method for the Example 1 are also presented in Table 1. The Table 1 and Figure 2 show that for each value of $\epsilon$, the regularized solution in space $V_{3}$ is more accurate with respect to other regularized solutions for each value of $\epsilon$. This figure illustrates that the small noise level leads to an accurate the approximate solution. In 45, a regularization technique and error estimates developed for the one-dimensional BHCP given by the Example 1 when $\kappa=2$. The absolute and relative errors, derived by [45], are of order $10^{-1}$. While the absolute and relative errors of the proposed MWR method in spaces $V_{2}$ and $V_{3}$ are of order $10^{-4}$ and $10^{-3}$, respectively. Therefore, the MWR method is more accurate than the method given by 45]. The difficulty of the BHCP given by the Example 1 is stemmed from that we attempt to retrieve the initial data when the thermal diffusivity factor is a large value. To demonstrate the efficiency of the proposed MWR method, we consider the BHCP given by the Example 1 with the thermal diffusivity factor $\kappa(t)=\kappa t+1$, where $\kappa=200$


Figure 2: For $\kappa(t)=2 t+1$, the comparison of the approximate and exact solutions for Example 1 with $\epsilon=$ $10^{-2}, 10^{-3}, 10^{-4}$.


Figure 3: The comparison of the approximate and exact solutions for Example 1 with $\epsilon=10^{-2}, 10^{-3}, 10^{-4}$.
and $\kappa=400$. For these cases, the numerical simulations are illustrated by Figure 3 This figure shows that the MWR method is successful to retentive the solution for large values of the thermal diffusivity factor $\kappa$. For the level of noise $\epsilon=0.0001$, the absolute and relative errors of the proposed MWR method to solve the BHCP (51) with $\kappa=200,400$ in spaces $V_{2}$ and $V_{3}$ are of order $10^{-4}$ and $10^{-3}$, respectively. The presented numerical simulations confirm that the the proposed MWR method is an accurate and efficient regularization technique to solve the BHCP given by the Example $\mathbb{\square}$ This adopts with our theoretical results as given by Section 4

Table 1: The absolute error and the relative error of the proposed MWR method for the Example 1 defined by the Eq. (51), when $\kappa(t)=2 t+1$.

| Space | $\epsilon=0.1$ |  | $\epsilon=0.01$ |  | $\epsilon=0.001$ |  | $\epsilon=0.0001$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Absolute | Relative | Absolute | Relative | Absolute | Relative | Absolute | Relative |
| $V_{2}$ | 0.0964 | 8.2231 | 0.0082 | 0.4948 | 0.0019 | 0.0758 | $2.10 \mathrm{e}-04$ | 0.0091 |
| $V_{3}$ | 0.0862 | 2.7039 | 0.0020 | 0.2831 | 0.0010 | 0.0161 | $1.71 \mathrm{e}-04$ | 0.0063 |
| $V_{4}$ | 0.0987 | 3.3331 | 0.0090 | 0.3638 | 0.0018 | 0.0817 | $8.32 \mathrm{e}-04$ | 0.0115 |
| $V_{5}$ | 0.0990 | 7.1264 | 0.0096 | 0.4152 | 0.0021 | 0.1144 | 0.0011 | 0.0598 |
| $V_{6}$ | 0.1716 | 9.1616 | 0.0068 | 0.3714 | 0.0029 | 0.1558 | 0.0016 | 0.0867 |

Example 2. The function defined by

$$
u(x, t)=e^{-|x|}\left(\cosh \left(\mu_{t}(0)\right)+\sinh \left(\mu_{t}(0)\right)\right)
$$

is non-differentiable at $x=0$. This function is the closed form analytical solution of the following BHCP with the non-smooth data on region $\Omega=\{(x, t)|0 \leq t \leq 1,|x| \leq 10\}$

$$
\left\{\begin{array}{l}
\partial_{t} u(x, t)=\kappa(t) \partial_{x x} u(x, t), \quad(x, t) \in \Omega  \tag{53a}\\
u(x, T)=e^{-|x|}\left(\cosh \left(\mu_{T}(0)\right)+\sinh \left(\mu_{T}(0)\right)\right)
\end{array}\right.
$$

where $T=1$ and $\kappa(t)=\frac{1}{100+\exp \left(t^{2}\right)}$.
The difficulty of the Example 2 is stemmed from that we attempt to apply the non-smooth final data to retrieve a weak singular initial data. Figure 4 (a)-(d) display the comparison between the exact and reconstructed solutions from noisy data $\varphi_{T, m}$. As it is shown that for several values of the regularization parameter, the regularized solution in space $V_{3}$ is more accurate with respect to other regularized solutions for each value of $\epsilon=10^{-1}, 10^{-2}, 10^{-3}$. For $\epsilon=10^{-3}$ in the spaces $V_{2}$ and $V_{3}$, the absolute and relative errors to solve the one-dimensional case of BHCP (53) are of order $10^{-2}$. These figures show that the cut-off frequency leads to retrieve imprecise solution in spaces $V_{4}$ and $V_{5}$. Moreover, the presented approximate simulations show that the regularization parameter strategy (50) is successful. The MWR approximate solution is stable at $t=0$. This adopts with the theoretical results as given by Remark 5 ,
Example 3. Consider the governing Eq. (53a) of the Example 2 on $\Omega=\{(x, t):|x| \leq 5, t \geq 0\}$ with the following initial Cauchy data:

$$
\begin{equation*}
u(x, 0)=\chi_{[-5,5]}(x), \quad-5 \leq x \leq 5 \tag{54}
\end{equation*}
$$

where $\chi_{A}(\cdot)$ denotes the characteristic function of a set $A$. The closed form analytical solution is given by

$$
\begin{equation*}
u(x, t)=\frac{1}{2}\left(\operatorname{erf}\left(\frac{x+5}{2 \sqrt{\mu_{t}(0)}}\right)-\operatorname{erf}\left(\frac{x-5}{2 \sqrt{\mu_{t}(0)}}\right)\right) \tag{55}
\end{equation*}
$$

Consequently, this function is exact solution of the BHCP governing the Eq. (53a) and the following final data

$$
\begin{equation*}
u(x, T)=\frac{1}{2}\left(\operatorname{erf}\left(\frac{x+5}{2 \sqrt{\mu_{T}(0)}}\right)-\operatorname{erf}\left(\frac{x-5}{2 \sqrt{\mu_{T}(0)}}\right)\right) \tag{56}
\end{equation*}
$$

The complexity of the problem is stemmed from that we attempt to apply the smooth final data to reconstruct a non-smooth initial data. The comparison between the exact and the regularization solutions are depicted by Figure 5 (a)-(d), when three different levels of noise $\epsilon$ are added into final data. This figure shows that the computational effect for $|x| \leq 5$ is still rather satisfactory. We can also observe that for several values of the regularization parameter, the regularized solutions in space $V_{2}$ are more precise with respect to other regularized solutions. For $\epsilon=10^{-2}$ in the space $V_{2}$, the absolute and relative errors to solve the BHCP with discontinuous solution are of order $10^{-2}$. The computational results are in good agreement with the analytical solution.


Figure 4: The comparison of the approximate and exact solutions for Example (2), with different levels of noise $\epsilon$.

### 5.2 Two dimensional examples

Here, the proposed MWR technique is applied for two different two-dimensional problems. The computational domain is divided into $N=2^{8} \times 2^{8}$ cells.

Example 4. Consider the $n$-dimensional BHCP with smooth data defined by

$$
\left\{\begin{array}{l}
\partial_{t} u(\mathbf{x}, t)=\kappa(t) \nabla^{2} u(\mathbf{x}, t), \quad(\mathbf{x}, t) \in \mathbb{R}^{n} \times[0, T),  \tag{57a}\\
u(\mathbf{x}, T)=\frac{1}{\sqrt{1+4 \mu_{T}(0)}} \exp \left(-\frac{\|\mathbf{x}\|^{2}}{1+4 \mu_{T}(0)}\right), \quad \mathbf{x} \in \mathbb{R}^{n},
\end{array}\right.
$$

where $\kappa(t)=\frac{1}{100+\exp \left(t^{2}\right)}$. The closed analytical form solution of the BHCP (57) is

$$
u(x, t)=\frac{1}{\sqrt{1+4 \mu_{t}(0)}} \exp \left(-\frac{\|\mathbf{x}\|^{2}}{1+4 \mu_{t}(0)}\right), \quad \mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)
$$

The two-dimensional case of the BHCP defined by (57) is implemented by the proposed MWR method on the region $\Omega=\left\{(\mathbf{x}, t) \mid 0 \leq t \leq 1, \mathbf{x} \in[-10,10]^{2}\right\}$, where $\mathbf{x}=(x, y)$. The numerical simulations of the proposed method for two-dimensional BHCP (57) are depicted by Figure 6(a)-(f). This figure illustrates the comparison between the regularized solutions and the absolute errors, where the regularization parameter $J^{*}=2,4,6$ are used. We observe that in spaces $V_{4}, V_{6}$ the approximate solutions are imprecise. It may be the noise in $\varphi_{T, m}$ is not damped enough by projections $\mathcal{P}_{4}, \mathcal{P}_{6}$, and hence the high frequencies of $\widehat{\varphi}_{T, m}$ is so extremely magnified that they destroy the approximated solution. Therefore, the regularization parameter


Figure 5: The comparison of the approximate and exact solutions for Example (2) at $t=0.01$, with different levels of noise $\epsilon$.
$J^{*}=2$ is the optimal choice. For the level of noise $\epsilon=10^{-3}$, the absolute and relative errors of the proposed MWR method to solve the two-dimensional case of BHCP (57) in space $V_{2}$ are of order $10^{-2}$.

Example 5. Consider the $n$-dimensional governing Eq. (57a) with the following final data

$$
\begin{equation*}
u(\mathbf{x}, T)=\exp \left(-\|\mathbf{x}\|_{1}\right)\left(\cosh \left(n \mu_{T}(0)\right)+\sinh \left(n \mu_{T}(0)\right)\right), \quad \mathbf{x}=\left(x_{1}, \cdots, x_{n}\right) \tag{58}
\end{equation*}
$$

Then, the function

$$
u(\mathbf{x}, t)=\exp \left(-\|\mathbf{x}\|_{1}\right)\left(\cosh \left(n \mu_{t}(0)\right)+\sinh \left(n \mu_{t}(0)\right)\right)
$$

is the exact unique solution of the problem described by Eqs. (57a) and (58), where $\|\mathbf{x}\|_{1}=\sum_{r=1}^{n}\left|x_{r}\right|$.
As it is seen that the final data has no derivative at the region. For two-dimensional case, Figure 6 illustrates the comparisons between the exact solution and its regularized solution defined by the regularization parameter $J^{*}=2,3,4$ for a noise of variance $10^{-3}$. We can see that the larger regularization parameter is, the less accurate of the regularization solution is. So the computational solution in the spaces $V_{4}$ is poor, it may be the perturbation in the function $\varphi_{T, m}$ is not reduced enough by projections $\mathcal{P}_{4}$, and thus the high frequencies of $\widehat{\varphi}_{T, m}$ is so severely magnified that they destruct the approximated solution. In this example the regularization parameters $J^{*}=2$ and $J^{*}=3$ are good optimal choices. For the level of noise $\epsilon=10^{-3}$, the absolute and relative errors of the MWR method for solving the two-dimensional BHCP given by Example (5) in space $V_{3}$ is of order $10^{-2}$.


Figure 6: The approximate solutions and the absolute errors in different scale spaces for the Example (4).


Figure 7: The approximate solutions and the absolute errors in different scale spaces for the Example (5).

## 6 Conclusion

Inverse and ill-posed problems particularly in the area of the partial differential equations, have nominated the attention of many researchers due to the extremely sensitive dependence on the terminal and initial data. In
this paper, the high-dimensional BHCP with time-dependent thermal diffusivity factor in an infinite "strip" domain is studied. The main characteristic of this problem is its ill-posedness. That is to say, independence of solution on terminal data. This work incorporates two viewpoints: Firstly, in viewpoint of theoretical analysis, we have presented a new approach for a regularization scheme based on Meyer wavelet theory. According to this analyze, we have gained an optimal explicit error estimate of the Hölder and Logarithmic types as well as the regularization parameter choice criteria under a-priori bound assumption. According to the optimal error bound, one can judge whether regularization technique is ok or not. Also, some precise stable estimates between the exact solution and its approximations are provided. About the convergence rate of the error estimate, for $p-q>0$, the proposed technique demonstrates that the convergence speed of the regularization solution in the error estimate of Logarithmic type, is faster than the error estimate of the Hölder type which is one of the most important advantageous with respect to the Hölder type. Secondly, in the viewpoint of computational analysis, we have experimented some kinds of prototype smooth and non-smooth examples in one and two dimensional spaces. For instance, in Example 1 our method have been compared to another method in 45. Numerical simulations show that the MWR method is more accurate than the method given by [45]. Consequently, from these illustrated examples, it can be concluded that the proposed technique is efficient and accurate to estimate the exact solution of the BHCP. Also, we believed that the proposed method is extendable to solve the broadest spectrum of the inverse and ill-posed parabolic partial differential equations.

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[^0]:    ${ }^{*}$ Corresponding author. E-mail: moradlou@sut.ac.ir (F. Moradlou) Tel.: +98-41-33459049, Fax: +98-41-33444300 .

