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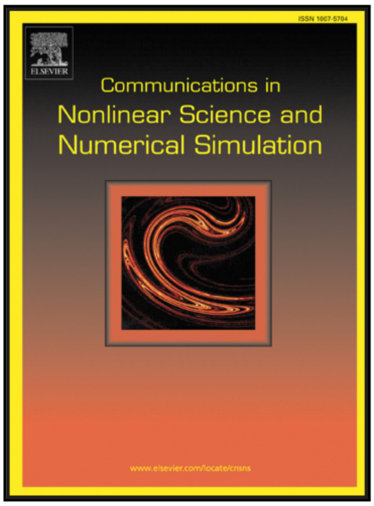


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## Highlights

- We characterize the analytic integrability of nilpotent singular points.
- Our characterization uses the existence of an inverse integrating factor.
- The results are applied to some nilpotent families of systems.

# Integrability of planar nilpotent differential systems through the existence of an inverse integrating factor

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**Abstract**

In this work is characterized the analytic integrability problem around a nilpotent singularity for differential systems in the plane under generic conditions. The analytic integrability problem is characterized via the existence of a formal inverse integrating factor. The relation between the analytic integrability and the existence of an algebraic inverse integrating factor is also given.

## 1 Introduction and statement of the main result

In this paper we are interested on the study of the analytic integrability for differential systems in the plane i.e., for differential systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.1}$$

where $P$ and $Q$ are analytic in a neighborhood of the origin and coprime. For such differential systems (1.1) with non-null linear part we have the following cases in function of their eigenvalues: if $\lambda_1\lambda_2 \neq 0$ we have either a saddle, a node, a focus, or a center type singular point. If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ we have a saddle-node. Finally if $\lambda_1 = \lambda_2 = 0$ we have a nilpotent singular point. The nodes, focus and saddle-nodes are not analytically integrable. If $\lambda_1$ and $\lambda_2$ are pure imaginary eigenvalues we say that we have a linear part of center type and the system is analytically integrable if, and only if, it is a center and if, and only if, it is orbitally linearizable, see [3, 23, 26]. For a saddle singular point, i.e., $\lambda_1 < 0 < \lambda_2$ if $\lambda_1/\lambda_2 \notin \mathbb{Q}$ then system (1.1), although it is linearizable, is not analytically integrable around the singular point. If $\lambda_1/\lambda_2 = -p/q \in \mathbb{Q}$ then we have a *resonant saddle*. A resonant saddle has an analytic first integral around the singular point if, and only if, it is orbitally linearizable, see for instance [17, 22, 28] and references therein.

The analytic integrability problem for a nilpotent singularity has been recently theoretically characterized in [11, Theorem 1.2], see also below. In this case the system is analytically integrable, if and only if, it is formally orbitally equivalent to the first quasi-homogeneous component under the generic condition that the origin of this first component is an isolated singularity.

During the last decades the inverse integrating factor has been used to characterize integrability. The relationship of the inverse integrating factor and the center problem has been also studied by several authors, see [5, 8, 9, 10, 13, 20, 25]. For instance a system with a linear part of center type is analytically integrable if, and only if, there exists an inverse integrating factor of the form $V = V_0 + \cdots$ with $V_0$ is a constant different form zero and where the dots indicate higher order terms, see [27].

The aim of this paper is to characterize the integrability of a nilpotent differential system in terms of the existence of a formal inverse integrating factor. We recall that a non-null $\mathcal{C}^1$ class function $V$ is an inverse integrating factor of $\mathbf{F}$ on $U$ if satisfies the linear partial differential equation $\nabla V \cdot \mathbf{F} = \operatorname{div}(\mathbf{F})\, V$, being $\operatorname{div}(\mathbf{F}) := \partial P/\partial x + \partial Q/\partial y$ the divergence of $\mathbf{F}$. We say that $V$ is a formal inverse integrating factor of $\mathbf{F}$ if $V \in \mathbb{C}[[x,y]]$ where $\mathbb{C}[[x,y]]$ is the algebra of the power series in $x$ and $y$ with coefficients in $\mathbb{C}$, convergent or not.

In order to present the results first we need some notation. A scalar polynomial $f$ is quasi-homogeneous of type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$ and degree $k$ if $f(\varepsilon^{t_1} x, \varepsilon^{t_2} y) = \varepsilon^k f(x,y)$. The vector space of quasi-homogeneous scalar polynomials of type $\mathbf{t}$ and degree $k$ is denoted by $\mathcal{P}_k^{\mathbf{t}}$. A polynomial vector field $\mathbf{F} = (P,Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree $k$ if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of polynomial quasi-homogeneous vector fields of type $\mathbf{t}$ and degree $k$ is denoted by $\mathcal{Q}_k^{\mathbf{t}}$. Given an analytic vector field $\mathbf{F}$, we can write it as a quasi-homogeneous expansion corresponding to a fixed type $\mathbf{t}$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots = \sum_{j \geq r} \mathbf{F}_j, \tag{1.2}$$

where $\mathbf{x} \in \mathbb{R}^2$, $r \in \mathbb{Z}$ and $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ i.e., each term $\mathbf{F}_j$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree $j$. Any $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ can be uniquely written as

$$\mathbf{F}_j = \mathbf{X}_{h_j} + \mu_j \mathbf{D}_0, \tag{1.3}$$

where $\mu_j = \frac{1}{r+|\mathbf{t}|} \operatorname{div}(\mathbf{F}_j) \in \mathcal{P}_j^{\mathbf{t}}$, $h_j = \frac{1}{r+|\mathbf{t}|} \mathbf{D}_0 \wedge \mathbf{F}_j \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, and $\mathbf{X}_{h_j} = (-\partial h_j/\partial y, \partial h_j/\partial x)^T$ is the Hamiltonian vector field with Hamiltonian function $h_j$ (see [2, Prop.2.7]).

Notice that the condition of polynomial integrability of the first quasi-homogeneous component is a necessary condition in order that $\mathbf{F}$ is analytically integrable, as the following lemma establishes.

**Lemma 1.1** *Let $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$ be a vector field, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$. If $\mathbf{F}$ is analytically integrable then $\mathbf{F}_r$ is polynomially integrable.*

Hence without loss of generality we can take from now on that the first quasi-homogeneous component of the vector field is polynomially integrable. A necessary and sufficient condition on the polynomial integrability of a quasi-homogeneous vector field is given in [1]. The main result of [11] for nilpotent singular points is the following.

**Theorem 1.2 ([11])** *Let $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ be a nilpotent vector field such that the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is isolated and $\mathbf{F}_r$ is polynomially integrable, then $\mathbf{F}$ is analytically integrable if, and only if, it is formally orbitally equivalent to $\mathbf{F}_r$.*

The following result is the main result of this work and provides the relationship between the analytic integrability of a nilpotent singular point such that the origin of the first quasi-homogeneous component is an isolated singularity and the existence of a formal inverse integrating factor defined in a neighborhood of the origin. Note that the first quasi-homogeneous component of the vector field we are studying has a non-null dissipative part and therefore the techniques used in the Reeb's theorem are not applicable, see [27]. Specifically the first quasi-homogeneous component is not equivalent to an exact differential form $dH$ with first integral $H$ having simple factors.

**Theorem 1.3** *Let $\mathbf{F} = \sum_{j\geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ be a vector field with non-null linear part such that $\mathbf{F}_r = \mathbf{X}_h + \mu_r \mathbf{D}_0$ is polynomially integrable and the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is an isolated singular point. Then $\mathbf{F}$ is analytically integrable if, and only if, there exists $V = h + \cdots$ a formal inverse integrating factor of $\mathbf{F}$.*

The proof of Theorem 1.3 is given in Section 2.

**Remark.** The above result can be applied to differential systems with a nilpotent singular point, with a linear type center singular point or with a resonant saddle singular point. In this sense it is a generalization of the well-known Reeb's theorem, see [19, 27].

The equivalence between analytic integrability and the existence of a formal inverse integrating factor is not true if $V$ does not begin with $h$. For instance, the system

$$\dot{x} = y + x^2(x^4 - 2y^2), \qquad \dot{y} = x^3 + 2xy(x^4 - 2y^2),$$

is not analytically integrable but $V = (x^4 - 2y^2)^2 = h^2/16$ is a polynomial inverse integrating factor of it, see [8, pag 870].

From Theorem 1.3, we have the following corollary for the resonant saddle case.

**Corollary 1.4** *System $\dot{x} = -\lambda_1 x + \cdots$, $\dot{y} = \lambda_2 y + \cdots$ with $\lambda_1, \lambda_2 \in \mathbb{N}$ is analytically integrable if, and only if, there exists a formal inverse integrating factor of the form $V = xy + \cdots$.*

This condition is necessary and sufficient. For instance system $\dot{x} = -x + x^2 y$, $\dot{y} = y + xy^2$ has the inverse integrating factor $V = h^2 = x^2y^2$. However, it is easy to prove that this system is not analytically integrable.

Now, we are going to provide a sufficient condition to have analytic integrability for a nilpotent vector field. To state this result we first need a normal form for the first quasi-homogeneous component of the vector field.

Any planar nilpotent vector field such that the first quasi-homogeneous component is polynomially integrable and irreducible, doing a change of variables and a reparametrization of the time (see [11, Proposition 2.12]), can be written as

$$\mathbf{F}(x) = \mathbf{F}_r(x) + \sum_{j>r} \mathbf{F}_j(x), \qquad \mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}, \tag{1.4}$$

where $\mathbf{F}_r$ is one of the following quasi-homogeneous vector fields

**i)** $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$ with $h = -\frac{1}{2}y^2 + \frac{1}{2n+1}x^{2n+1}$, $r = 2n-1$ and $\mathbf{t} = (2, 2n+1)$.

**ii)** $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$ with $h = -\frac{1}{2}\left(y^2 + \sigma x^{2(n+1)}\right)$, $\sigma = \pm 1$, $r = n$ and $\mathbf{t} = (1, n+1)$.

**iii)** $\mathbf{F}_r = \mathbf{X}_h + \mu_r \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ with $h = -\frac{1}{2}\left(y^2 - x^{2(n+1)}\right)$, $\mu_r = dx^n$, $d = \frac{m_1 - m_2}{m_1 + m_2} \neq 0$, $m_1, m_2 \in \mathbb{N}$ coprimes, $r = n$, $\mathbf{t} = (1, n+1)$ where a first integral of $\mathbf{F}_r$ is $I = (y - x^{n+1})^{m_1}(y + x^{n+1})^{m_2}$.

Next result gives a sufficient condition of analytic integrability of the nilpotent differential systems with the first quasi-homogeneous component irreducible and with non-zero divergence, by means of the existence of an algebraic inverse integrating factor.

**Theorem 1.5** *Let $\mathbf{F}$ be the vector field (1.4) with $\mathbf{F}_r$ of type* **iii)**. *If there exists $\frac{p}{q} \in \mathbb{Q}$ with $\frac{(n+1)(m_1+m_2)p}{q} \notin \mathbb{N}$ such that*

$$W = (y - x^{n+1} + \cdots)^{1 + \frac{p}{q} m_1} (y + x^{n+1} + \cdots)^{1 + \frac{p}{q} m_2}$$

*is an algebraic inverse integrating factor of $\mathbf{F}$ then we have that $\mathbf{F}$ is analytically integrable.*

The proof of Theorem 1.5 is given in Section 3.

**Remark.** The hypothesis $\frac{(n+1)(m_1+m_2)p}{q} \notin \mathbb{N}$ is necessary. For instance if we consider the vector field $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ with $\mathbf{F}_j \in \mathcal{Q}_i^{\mathbf{t}}$, $\mathbf{t} = (1,2)$, $\mathbf{F}_1 = (y - \frac{1}{3}x^2, 2x^3 - \frac{2}{3}xy)^T$, $\mathbf{F}_2 = x^2(x, 2y)^T$ we have that in this case $n = 1$, $m_1 = 1$, $m_2 = 2$, and $I_6 = (y - x^2)(y + x^2)^2 \in \mathcal{P}_6^{\mathbf{t}}$ is a first integral of $\mathbf{F}_1$, where $\mathbf{F}_1$ is irreducible (the origin is an isolated singularity). Moreover,

$$W = (y - x^2)^{1+\frac{p}{q}} (y + x^2)^{1+\frac{2p}{q}},$$

with $p = 1$ and $q = 6$, is an algebraic inverse integrating factor of $\mathbf{F}$. Nevertheless, $\mathbf{F}$ is not formally integrable, see Theorem 4.19 below. In this case the hypothesis that fails is $\frac{(n+1)(m_1+m_2)p}{q} = 1 \in \mathbb{N}$.

Hence, the existence of an algebraic inverse factor does not guarantee the analytic integrability. For instance the differential system $\dot{x} = y + 2x^2$, $\dot{y} = x^2 + 3xy$ is not formally integrable but it has an algebraic integrating factor of the form $W = h^{7/6}$, with $h = -\frac{1}{6}(3y^2 - 2x^3)$, see [6, Theorem 2].

From Theorem 1.5 we have the following corollary for the resonant saddle case.

**Corollary 1.6** *Consider the system $\dot{x} = -\lambda_1 x + \cdots$, $\dot{y} = \lambda_2 y + \cdots$ with $\lambda_1, \lambda_2 \in \mathbb{N}$, $(\lambda_1, \lambda_2) = 1$, $\lambda_1/\lambda_2 \neq 1$. If there exist $p, q \in \mathbb{N}$ such that $(\lambda_1 + \lambda_2)p/q \notin \mathbb{N}$ and*

$$W = (x + \cdots)^{1+\frac{p}{q}\lambda_2} (y + \cdots)^{1+\frac{p}{q}\lambda_1}$$

*is an algebraic inverse integrating factor of such system then it is analytically integrable.*

**Remark.** The condition $(\lambda_1 + \lambda_2)p/q \notin \mathbb{N}$ is necessary as the next example shows. The system $\dot{x} = -x + x^3 y$, $\dot{y} = 2y + x^2 y^2$ has the inverse integrating factor $W = x^3 y^2$ but it is not analytically integrable. In this case we have $p = 1$, $q = 1$ and $(\lambda_1 + \lambda_2)p/q = 3 \in \mathbb{N}$.

## 2 Proof of the Theorem 1.3

In order to prove Theorem 1.3, we need to recall the following theorem proved in [4] and there also called Theorem 1.3.

**Theorem 2.7** *Let $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ be an analytic vector field with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$. Then, $\mathbf{F}$ and $\mathbf{F}_r$ are formally orbital equivalent if and only if there exist a formal vector field $\mathbf{G} = \sum_{j \geq 0} \mathbf{G}_j$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$, with $\mathbf{G}_0 = \mathbf{D}_0$ and a formal scalar function $f$ with $f(\mathbf{0}) = r$ verifying $[\mathbf{F}, \mathbf{G}] = f\mathbf{F}$.*

The following result corresponds to Proposition 18 of [8].

**Proposition 2.8** *Let $\Phi$ be a diffeomorphism and $\eta$ a scalar function on $U \subset \mathbb{R}^2$ such that $\det(D\phi)$ has no zero on $U$ and $\eta(\mathbf{0}) \neq 0$. If $V \in \mathbb{C}[[x, y]]$ is an inverse integrating factor of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, then $\eta(\mathbf{y}) (\det(D\phi(\mathbf{y})))^{-1} V(\Phi(\mathbf{y}))$ is an inverse integrating factor of $\dot{\mathbf{y}} = \Phi_* (\eta \mathbf{F})(\mathbf{y}) := (D\Phi)^{-1} \eta(\mathbf{y}) \mathbf{F}(\Phi(\mathbf{y}))$.*

**Definition 2.9** *$f \in \mathbb{C}[[x, y]]$ is an invariant curve of (1.2) if there exists $K \in \mathbb{C}[[x, y]]$ such that $\nabla f \cdot \mathbf{F} = K\mathbf{F}$, where $K$ is called the cofactor of $f$.*

Now we provide some technical results and their proofs in order to prove the main result of the paper.

**Lemma 2.10** *Let $f \in \mathcal{P}_s^{\mathbf{t}}$ be an invariant curve of $X_g$, with $g \in \mathcal{P}_k^{\mathbf{t}}$, then there exists $\lambda \in \mathcal{P}_{k-s}^{\mathbf{t}}$ such that $g = \lambda f$.*

*Proof.* As $g$ is a first integral of $X_g$ then all the solutions of the system associated to $X_g$ are in the level curves of $g$. If $f$ is an invariant curve of $X_g$ then $f = 0$ is formed by solution curves of the system associated to $X_g$. Thus all the solutions curves that are in $f = 0$ are in the same level curve of $g$, that is, $g$ vanishes in all the points where $f$ vanishes. Therefore, as $g$ and $f$ are quasi-homogeneous polynomials, there exists $\lambda \in \mathcal{P}_{k-s}^{\mathbf{t}}$ such that $g = \lambda f$. ■

**Lemma 2.11** *Consider $f \in \mathcal{P}_s^{\mathbf{t}}$ and $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$. The following statements hold.*

- *i) If $f$ is an invariant curve of $\mathbf{F}$ then $f$ is an invariant curve of $\mathbf{F}_j$ for all $j \geq r$.*
- *ii) If $f$ is an inverse integrating factor of $\mathbf{F}$ then $f$ is an inverse integrating factor of $\mathbf{F}_j$ for all $j \geq r$.*

*Proof.* First, we prove statement i). If $f$ is an invariant curve of $\mathbf{F}$ then there exists $K = \sum_{j \geq r} K_j$, $K_j \in \mathcal{P}_j^{\mathbf{t}}$ (cofactor) such that $\nabla f \cdot \mathbf{F} = K f$ and consequently

$$0 = \sum_{j \geq r} \nabla f \cdot \mathbf{F}_j - \sum_{j \geq r} K_j f = \sum_{j \geq r} \left( \nabla f \cdot \mathbf{F}_j - K_j f \right).$$

Thus for all $j \geq r$, we have $\nabla f \cdot \mathbf{F}_j - K_j f = 0$ and $f$ is an invariant curve of $\mathbf{F}_j$.

Second, we prove statement ii). If $f$ is an inverse integrating factor of $\mathbf{F}$ then $\nabla f \cdot \mathbf{F} - f \operatorname{div}(\mathbf{F}) = 0$ and consequently

$$0 = \sum_{j \geq r} \nabla f \cdot \mathbf{F}_j - \sum_{j \geq r} f \operatorname{div}(\mathbf{F}_j) = \sum_{j \geq r} \left( \nabla f \cdot \mathbf{F}_j - f \operatorname{div}(\mathbf{F}_j) \right).$$

Thereby for all $j \geq r$ we have $\nabla f \cdot \mathbf{F}_j = f \operatorname{div}(\mathbf{F}_j)$, hence $f$ is an inverse integrating factor of $\mathbf{F}_j$. ■

The following result is proved in [11] there called Lemma 3.17. We adapt the statement to our purposes.

**Lemma 2.12** *Let $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$ be and $\mathbf{F}_r = \mathbf{X}_h + \mu_r \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$. Every $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ with $j > r$ can be expressed by $\mathbf{F}_j = \mathbf{X}_{g_{j+|\mathbf{t}|}} + \eta_j \mathbf{D}_0 + \lambda_{j-r} \mathbf{F}_r$, with $\lambda_{j-r} \in \mathcal{P}_{j-r}^{\mathbf{t}}$, $\eta_j \in \mathcal{P}_j^{\mathbf{t}}$ and $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$, being $\Delta_{j+|\mathbf{t}|}$ a complementary subspace of $h\mathcal{P}_{j-r}^{\mathbf{t}}$ in $\mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$, i.e. $\mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}} = \Delta_{j+|\mathbf{t}|} \bigoplus h\mathcal{P}_{j-r}^{\mathbf{t}}$.*

The following result describes the vector fields which have $h$ as an invariant curve. For similar results inside the inverse problem, see [14, 15].

**Proposition 2.13** *Consider $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$ with $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{F}_r = \mathbf{X}_h + \mu_r \mathbf{D}_0$, $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T$. If $h$ is an invariant curve of $\mathbf{F}$ then $\mathbf{F} = (1 + \lambda)\mathbf{F}_r + \mu \mathbf{D}_0$ with $\lambda = \sum_{j > r} \lambda_j$, $\lambda_j \in \mathcal{P}_j^{\mathbf{t}}$ and $\mu = \sum_{j \geq r} \mu_j$, $\mu_j \in \mathcal{P}_j^{\mathbf{t}}$*

*Proof.* If $h$ is an invariant curve of $\mathbf{F}$ and $h$ is a quasi-homogeneous polynomial then by Lemma 2.11 statement **i)** follows that $h$ is an invariant curve of $\mathbf{F}_j$ for each $j > r$ with cofactor $K_j$. On the other hand, applying Lemma 2.12 $\mathbf{F}_j = \mathbf{X}_{g_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0 + \lambda_{j-r} \mathbf{F}_r$ with $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$, $\mu_j \in \mathcal{P}_j^{\mathbf{t}}$ and $\lambda_j \in \mathcal{P}_{j-r}^{\mathbf{t}}$. Thus we have:

$$\begin{aligned}
\nabla h \cdot \mathbf{F}_j &= \nabla h \cdot \left( \mathbf{X}_{g_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0 + \lambda_{j-r} \mathbf{F}_r \right) \\
&= \nabla h \cdot \mathbf{X}_{g_{j+|\mathbf{t}|}} + (r + |\mathbf{t}|)\mu_j h + \lambda_{j-r} \nabla h \cdot (\mathbf{X}_h + \mu_r \mathbf{D}_0) \\
&= \nabla h \cdot \mathbf{X}_{g_{j+|\mathbf{t}|}} + (r + |\mathbf{t}|) \left( \mu_j + \lambda_{j-r} \mu_r \right) h = K_j h.
\end{aligned}$$

Consequently $\nabla h \cdot \mathbf{X}_{g_{j+|\mathbf{t}|}} = (K_j - (r+|\mathbf{t}|)\,(\mu_j + \lambda_{j-r}\mu_r))h$. Hence $h$ is an invariant curve of $\mathbf{X}_{g_{j+|\mathbf{t}|}}$ and applying Lemma 2.10 we deduce that $g_{j+|\mathbf{t}|}$ is multiple of $h$ and $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$ hence $g_{j+|\mathbf{t}|} = 0$ and this completes the proof. ■

Next result characterizes the vector fields which have $h$ as inverse integrating factor.

**Proposition 2.14** *Consider* $\mathbf{F} = \sum_{j\geq r} \mathbf{F}_j$ *with* $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{F}_r = \mathbf{X}_h + \mu_r \mathbf{D}_0$, $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T$. *If $h$ is an inverse integrating factor of* $\mathbf{F}$ *then* $\mathbf{F} = (1+\lambda)\mathbf{F}_r - (\nabla\tilde{\lambda}\cdot\mathbf{F}_r)\,\mathbf{D}_0$ *with* $\lambda = \sum_{j\geq 1}\lambda_j$, $\lambda_j \in \mathcal{P}_j^{\mathbf{t}}$ *and* $\tilde{\lambda} = \sum_{j\geq 1}\lambda_j/j$.

*Proof.* If $h$ is an inverse integrating factor of $\mathbf{F}$ as $h$ is a quasi-homogeneous polynomial then by Lemma 2.11 statement **ii)** $h$ is an inverse polynomial integrating factor of $\mathbf{F}_j$ for all $j \geq r$. For $j = r$ is fulfilled because

$$\begin{aligned}\nabla h \cdot \mathbf{F}_r - h\mathrm{div}\,(\mathbf{F}_r) \quad &= \nabla h \cdot (\mathbf{X}_h + \mu_r \mathbf{D}_0) - h\mathrm{div}\,(\mu_r \mathbf{D}_0)\\ &= (r+|\mathbf{t}|)h\mu_r - (r+|\mathbf{t}|)h\mu_r = 0\end{aligned}$$

If for $j > r$ $h$ is an inverse integrating factor of $\mathbf{F}_j$, then $h$ is an invariant curve of $\mathbf{F}_j$ and by Proposition 2.13 we have $\mathbf{F}_j = \lambda_{j-r}\mathbf{F}_r + \mu_j\mathbf{D}_0$ with $\mu_j \in \mathcal{P}_j^{\mathbf{t}}$ and $\lambda_{j-r} \in \mathcal{P}_{j-r}^{\mathbf{t}}$. Therefore, we get

$$\begin{aligned}0 \quad &= \quad \nabla h\cdot\mathbf{F}_j - h\mathrm{div}\,(\mathbf{F}_j) = \nabla h\cdot(\lambda_{j-r}\mathbf{F}_r + \mu_j\mathbf{D}_0) - h\mathrm{div}\,(\lambda_{j-r}\mathbf{F}_r + \mu_j\mathbf{D}_0)\\ &= \quad \lambda_{j-r}\nabla h\cdot\mathbf{F}_r + \mu_j\nabla h\cdot\mathbf{D}_0 - h\,(\nabla\lambda_{j-r}\mathbf{F}_r + \lambda_{j-r}\mathrm{div}\,(\mathbf{F}_r) + (j+|\mathbf{t}|)\mu_j)\\ &= \quad (r+|\mathbf{t}|)\lambda_{j-r}\mu_r h + (r+|\mathbf{t}|)\mu_j h - h\,(\nabla\lambda_{j-r}\mathbf{F}_r + (r+|\mathbf{t}|)\lambda_{j-r}\mu_r + (j+|\mathbf{t}|)\mu_j)\\ &= \quad (r-j)\mu_j h - h\nabla\lambda_{j-r}\cdot\mathbf{F}_r = (r-j)h\left(\mu_j + \frac{\nabla\lambda_{j-r}\cdot\mathbf{F}_r}{j-r}\right).\end{aligned}$$

Consequently, $\mu_j = -(\nabla\lambda_{j-r}\cdot\mathbf{F}_r)/(j-r)$ and this completes the proof. ■

The following result provides an orbital normal form for a vector field with an inverse integrating factor of the form $V = h + \cdots$.

**Proposition 2.15** *Consider* $\mathbf{F} = \sum_{j\geq r}\mathbf{F}_j$ *with* $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{F}_r = \mathbf{X}_h + \mu_r\mathbf{D}_0$, $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $\mu_r \in \mathcal{P}_r^{\mathbf{t}} \setminus \{0\}$, $\mathbf{D}_0 = (t_1x, t_2y)^T$. $\mathbf{F}$ *is formally orbitally equivalent to* $\mathbf{F}_r$ *if, and only if,* $V = h + \cdots$ *is a formal inverse integrating factor of* $\mathbf{F}$ *and there exists a change of variables* $\Phi$ *such that* $V \circ \Phi = h$.

*Proof.* If $V = h + \cdots$ is an inverse integrating factor of $\mathbf{F}$, taking $\eta = \det\,(D\Phi)$ and applying Proposition 2.8 we have that $h$ is an inverse integrating factor of $\tilde{\mathbf{F}} = \Phi_*\,(\eta\mathbf{F})$. Applying now Proposition 2.14 we obtain that $\tilde{\mathbf{F}} = (1+\lambda)\mathbf{F}_r - (\nabla\tilde{\lambda}\cdot\mathbf{F}_r)\,\mathbf{D}_0$ with $\lambda = \sum_{j\geq 1}\lambda_j$, $\lambda_j \in \mathcal{P}_j^{\mathbf{t}}$ and $\tilde{\lambda} = \sum_{j\geq 1}\lambda_j/j$.

Next we claim that $\frac{1}{1+\lambda}\mathbf{D}_0$ is a commutator of $\tilde{\mathbf{F}}$. In order to see this we use the property $[\mu\mathbf{F},\mathbf{G}] = (\nabla\mu\cdot\mathbf{G})\,\mathbf{F} + \mu[\mathbf{F},\mathbf{G}]$. Hence we have

$$\begin{aligned}\left[\tilde{\mathbf{F}}, \tfrac{1}{1+\lambda}\mathbf{D}_0\right] \quad &= \quad \left[(1+\lambda)\mathbf{F}_r, \tfrac{1}{1+\lambda}\mathbf{D}_0\right] + \left[-\left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\mathbf{D}_0, \tfrac{1}{1+\lambda}\mathbf{D}_0\right]\\ &= \quad \tfrac{\nabla(1+\lambda)\cdot\mathbf{D}_0}{1+\lambda}\mathbf{F}_r + (1+\lambda)\left[\mathbf{F}_r, \tfrac{1}{1+\lambda}\mathbf{D}_0\right]\\ &\qquad -\tfrac{\nabla\left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\cdot\mathbf{D}_0}{1+\lambda}\mathbf{D}_0 - \left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\left[\mathbf{D}_0, \tfrac{1}{1+\lambda}\mathbf{D}_0\right]\\ &= \quad \tfrac{\nabla\lambda\cdot\mathbf{D}_0}{1+\lambda}\mathbf{F}_r + \tfrac{\nabla\lambda\cdot\mathbf{F}_r}{1+\lambda}\mathbf{D}_0 + [\mathbf{F}_r,\mathbf{D}_0] - \tfrac{\nabla\left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\cdot\mathbf{D}_0}{1+\lambda}\mathbf{D}_0\\ &\qquad -\left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\tfrac{\nabla\lambda\cdot\mathbf{D}_0}{(1+\lambda)^2}\mathbf{D}_0\end{aligned}$$

Taking into account that

$$
\begin{aligned}
[\mathbf{F}_r, \mathbf{D}_0] &= r\mathbf{F}_r, \\
\nabla \mu_k \cdot \mathbf{D}_0 &= k\mu_k \quad \text{(by the Euler's theorem for } \mu_k \in \mathcal{P}_k^{\mathbf{t}}) \\
\nabla\left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\cdot\mathbf{D}_0 &= \nabla\left(\sum_{j\geq 1}\tfrac{\nabla\lambda_j\cdot\mathbf{F}_r}{j}\right)\cdot\mathbf{D}_0 = \sum_{j\geq 1}\tfrac{\nabla(\nabla\lambda_j\cdot\mathbf{F}_r)\cdot\mathbf{D}_0}{j} = \left[\nabla\lambda_j\cdot\mathbf{F}_r \in \mathcal{P}_{j+r}^{\mathbf{t}}\right] \\
&= \sum_{j\geq 1}\tfrac{j+r}{j}\nabla\lambda_j\cdot\mathbf{F}_r = \sum_{j\geq 1}\nabla\lambda_j\cdot\mathbf{F}_r + r\sum_{j\geq 1}\nabla\tfrac{\lambda_j}{j}\cdot\mathbf{F}_r \\
&= \nabla\lambda\cdot\mathbf{F}_r + r\nabla\tilde{\lambda}\cdot\mathbf{F}_r
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left[\tilde{\mathbf{F}}, \tfrac{1}{1+\lambda}\mathbf{D}_0\right] &= \tfrac{\nabla\lambda\cdot\mathbf{D}_0}{1+\lambda}\mathbf{F}_r + \tfrac{\nabla\lambda\cdot\mathbf{F}_r}{1+\lambda}\mathbf{D}_0 + r\mathbf{F}_r - \tfrac{\nabla\lambda\cdot\mathbf{F}_r}{1+\lambda}\mathbf{D}_0 - r\tfrac{\nabla\tilde{\lambda}\cdot\mathbf{F}_r}{1+\lambda}\mathbf{D}_0 \\
&\quad - \left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\tfrac{\nabla\lambda\cdot\mathbf{D}_0}{(1+\lambda)^2}\mathbf{D}_0 \\
&= \left(r + \tfrac{\nabla\lambda\cdot\mathbf{D}_0}{1+\lambda}\right)\mathbf{F}_r - \tfrac{\nabla\tilde{\lambda}\cdot\mathbf{F}_r}{1+\lambda}\left(r + \tfrac{\nabla\lambda\cdot\mathbf{D}_0}{1+\lambda}\right)\mathbf{D}_0 \\
&= \tfrac{r(1+\lambda)+\nabla\lambda\mathbf{D}_0}{(1+\lambda)^2}\left((1+\lambda)\mathbf{F}_r - \left(\nabla\tilde{\lambda}\cdot\mathbf{F}_r\right)\mathbf{D}_0\right) \\
&= \tfrac{r(1+\lambda)+\nabla\lambda\mathbf{D}_0}{(1+\lambda)^2}\tilde{\mathbf{F}}
\end{aligned}
$$

Applying Theorem 2.7 we obtain that $\tilde{\mathbf{F}}$ is formally orbitally equivalent to $\mathbf{F}_r$.

The sufficiency is trivial because $h$ is an inverse integrating factor of $\mathbf{F}_r$ and it is sufficient to apply the Proposition 2.8. ■

Now, we can give the proof of the main result.

**Proof of Theorem 1.3** The necessity is trivial because if $\mathbf{F}$ is analytically integrable by Theorem 1.2 we have that $\mathbf{F}$ is formally orbitally equivalent to $\mathbf{F}_r$ (the first quasi-homogeneous component). Taking into account that $\mathbf{F}_r$ has the inverse integrating factor $\tilde{V} = h$ then by Proposition 2.8 we deduce that $\mathbf{F}$ has the inverse integrating factor of the form $V = h + \cdots$.

Let us now the sufficiency. As $\mathbf{F}$ has non-null linear part and its first quasi-homogeneous component is irreducible and polynomially integrable, the unique possibilities are:

**a)** $\mathbf{F} = (-y, x)^T + \cdots$ (perturbation of a linear center type)

**b)** $\mathbf{F} = (-\lambda_1 x, \lambda_2 y)^T + \cdots$, $\lambda_1, \lambda_2 \in \mathbb{N}$ (perturbation of a resonant saddle)

**c)** $\mathbf{F}$ is of the form (1.4) with $\mathbf{F}_r$ of type **i)**, **ii)** or **iii)**.

The case **a)** is proved in [27]. For the case **b)** by means of a linear change of variables an a reparametrization of time we can transform the original system into the vector field $(y, x)^T + \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}(x, y)^T + \cdots$ which is a vector field of the form (1.4) with $\mathbf{F}_r$ of type **iii)** with $n = 0$. Therefore it is enough to show the sufficient condition for the case **c)**.

In all the cases **i)**, **ii)** or **iii)**, the inverse integrating factor of $\mathbf{F}$ is $V = h + \cdots$, where $h(x, y) = y^2 + Ax^m$, with $m \in \mathbb{N}$, and $A \neq 0$. It is proved in [12] that there exists a change of variables near to identity $\Phi$ such that $V \circ \Phi = h$. Applying now Proposition 2.15 we have that $\mathbf{F}$ is formally orbitally equivalent to $\mathbf{F}_r$ and $\mathbf{F}_r$ is polynomially integrable so we deduce that $\mathbf{F}$ is formally integrable and by [24, Theorem A] $\mathbf{F}$ is analytically integrable. ■

# 3 Proof of the Theorem 1.5

Now we give the following technical result and its proof in order to prove the theorem 1.5. For that we need the following linear operator (Lie derivate of $\mathbf{F}_r$)

$$\begin{array}{rcl} \ell_j & : & \mathcal{P}^{\mathbf{t}}_{j-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_{j} \\ & & \mu_{j-r} \longrightarrow \nabla\mu_{j-r}\cdot\mathbf{F}_r, \end{array}$$

**Lemma 3.16** *Let $\hat{\mathbf{F}} = \mathbf{F}_r + \sum_{j>r}\mu_j\mathbf{D}_0$, be the vector field (1.4) with $\mu_j \in \mathrm{Cor}\,(\ell_j)$ (where $\mathrm{Cor}\,(\ell_j)$ is a complementary subspace to $\mathrm{Range}\,(\ell_j)$). Let $V = \frac{V_{s_1}}{W_{s_2}}U$, $V_{s_1} \in \mathcal{P}^{\mathbf{t}}_{s_1}$, $W_{s_2} \in \mathcal{P}^{\mathbf{t}}_{s_2}$ and $U$ a formal unity series, i.e., $U(\mathbf{0}) = 1$, such that for some $m \in \mathbb{N}$ is verified the equation*

$$\nabla V\cdot\hat{\mathbf{F}} - mV\mathrm{div}\left(\hat{\mathbf{F}}\right) = 0. \tag{3.5}$$

*If $s_1 - s_2 \neq m(r+i+|\mathbf{t}|)$ for all $i \in \mathbb{N}$, then $\mu_j = 0$ for all $j > r$*

*Proof.* The equation (3.5) is

$$0 = \nabla\left(\frac{V_{s_1}}{W_{s_2}}U\right)\cdot\hat{\mathbf{F}} - m\frac{V_{s_1}}{W_{s_2}}U\mathrm{div}\left(\hat{\mathbf{F}}\right) = U\nabla\frac{V_{s_1}}{W_{s_2}}\cdot\hat{\mathbf{F}} + \frac{V_{s_1}}{W_{s_2}}\nabla U\cdot\hat{\mathbf{F}} - m\frac{V_{s_1}}{W_{s_2}}U\mathrm{div}(\hat{\mathbf{F}}).$$

On the other hand this equation (3.5) for degree $r+s_1-s_2$ is $0 = \nabla\frac{V_{s_1}}{W_{s_2}}\cdot\hat{\mathbf{F}}_r - m\frac{V_{s_1}}{W_{s_2}}\mathrm{div}\left(\hat{\mathbf{F}}_r\right)$. Moreover, as $V_{s_1} \in \mathcal{P}^{\mathbf{t}}_{s_1}$, $W_{s_2} \in \mathcal{P}^{\mathbf{t}}_{s_2}$ is fulfilled $\nabla\frac{V_{s_1}}{W_{s_2}}\cdot\mathbf{D}_0 = (s_1 - s_2)\frac{V_{s_1}}{W_{s_2}}$. Hence

$$\begin{aligned} 0 &= U\nabla\frac{V_{s_1}}{W_{s_2}}\cdot\hat{\mathbf{F}}_r + (s_1-s_2)U\frac{V_{s_1}}{W_{s_2}}\sum_{j>r}\mu_j + \frac{V_{s_1}}{W_{s_2}}\nabla U\cdot\hat{\mathbf{F}} - m\frac{V_{s_1}}{W_{s_2}}U\mathrm{div}(\hat{\mathbf{F}}_r) \\ &\quad -m\frac{V_{s_1}}{W_{s_2}}U\sum_{j>r}(j+|\mathbf{t}|)\mu_j = \frac{V_{s_1}}{W_{s_2}}\left[\nabla U\cdot\hat{\mathbf{F}} - U\sum_{j>r}A_j\mu_j\right], \end{aligned}$$

where $A_j = m(j+|\mathbf{t}|) - (s_1 - s_2)$.

In this way the equation (3.5) is $\nabla U\cdot\hat{\mathbf{F}} - U\sum_{j>r}A_j\mu_j = 0$. When developing it in quasi-homogeneous degrees results

$$\begin{aligned} 0 &= \sum_{i\geq1}\nabla U_i\cdot\left(\hat{\mathbf{F}}_r + \sum_{j>r}\mu_j\mathbf{D}_0\right) - \left(1+\sum_{i\geq1}U_i\right)\sum_{j>r}A_j\mu_j \\ &= \sum_{i\geq1}\left(\nabla U_i\cdot\hat{\mathbf{F}}_r + \sum_{k=1}^{i-1}\mu_{r+i-k}\nabla U_k\cdot\mathbf{D}_0 - \left(A_{i+r}\mu_{i+r} + \sum_{k=1}^{i-1}A_{r+i-k}\mu_{r+i-k}U_k\right)\right) \\ &= \sum_{i\geq1}\left(\nabla U_i\cdot\hat{\mathbf{F}}_r - A_{r+i}\mu_{r+i} + \sum_{k=1}^{i-1}(k-A_{r+i-k})\mu_{r+i-k}U_k\right). \end{aligned}$$

We claim that $\mu_{r+j} = 0$ for all $j \geq 1$ because otherwise we can consider $j_0 = \min\{j \in +\mathbb{N} : \mu_{r+j} \neq 0\}$ and equation (3.5) for degree $r+j_0$ is

$$\nabla U_{j_0}\cdot\hat{\mathbf{F}}_r - A_{r+j_0}\mu_{r+j_0} + \sum_{k=1}^{j_0-1}(k-A_{r+j_0-k})\mu_{r+j_0-k}U_k = 0$$

Taking into account that $\mu_{r+j_0-k} = 0$ for $1 \leq k \leq j_0 - 1$, we have that $\nabla U_{j_0}\cdot\hat{\mathbf{F}}_r = A_{r+j_0}\mu_{r+j_0}$. However $A_{r+j_0} \neq 0$ because $m(r+j_0+|\mathbf{t}|) \neq (s_1-s_2)$ therefore $A_{r+j_0}\mu_{r+j_0} \in \mathrm{Cor}\,(\ell_{r+j_0})\setminus\{0\}$ and $\nabla U_{j_0}\cdot\hat{\mathbf{F}}_r \in \mathrm{Range}\,(\ell_{r+j_0})$, consequently $\mu_{r+j_0} = 0$ which is a contradiction and the claim is proved. ■

**Proposition 3.17** *Let* $\hat{\mathbf{F}} = \mathbf{F}_r + \sum_{j>r} \mu_j \mathbf{D}_0$, *be the vector field (1.4) with* $\mathbf{F}_r$ *of type* **iii)** *and* $\mu_j \in \mathrm{Cor}\,(\ell_j)$ (*where* $\mathrm{Cor}\,(\ell_j)$ *is a complementary subspace to* $\mathrm{Range}\,(\ell_j)$). *The unique solution curves passing through the origin are* $y - x^{n+1} = 0$ *and* $y + x^{n+1} = 0$.

*Proof.* The curves $y - x^{n+1} = 0$ and $y + x^{n+1} = 0$ are invariant curves of $\mathbf{F}_r$ and applying Euler's theorem it is easy to prove that they are also invariant curves of $\mathbf{D}_0$. Consequently they are invariant curves of $\hat{\mathbf{F}}$.

We are going to see that they are the only ones. By applying the blow-up $x = u$, $y = u^{n+1}v$ to the differential system $\dot{\mathbf{x}} = \hat{\mathbf{F}}(\mathbf{x})$ and the scaling of time $dT = u^n dt$, we get the differential system

$$\begin{aligned} \tfrac{du}{dT} &= u\big[v + d + \sum_{j>n} \mu_j(1,v)u^{j-n}\big], \\ \tfrac{dv}{dT} &= (n+1)\left[1 - v^2\right]. \end{aligned}$$

The singular points in $u = 0$ are $v = \pm 1$ and the linearization matrix at these equilibrium points are

$$\begin{aligned} D(0,1) &= \mathrm{diag}\,(d+1, -2(n+1)) = \mathrm{diag}\left(\tfrac{2m_1}{m_1+m_2}, -2(n+1)\right), \\ D(0,-1) &= \mathrm{diag}\,(d-1, 2(n+1)) = \mathrm{diag}\left(-\tfrac{2m_2}{m_1+m_2}, 2(n+1)\right). \end{aligned}$$

So both singular points are hyperbolic saddles and therefore have a unique curve entering to the origin. From here we deduce that the solution curves are unique. ■

**Proof of Theorem 1.5** Let $\mathbf{F}$ be the vector field (1.4) with $\mathbf{F}_r$ of type **iii)**, by [11, Theorems 3.23 and 3.24], $\mathbf{F}$ is formally orbitally equivalent, by a near identity transformation, to $\hat{\mathbf{F}} = \mathbf{F}_r + \sum_{j>r} \mu_j \mathbf{D}_0$, with $\mu_j \in \mathrm{Cor}\,(\ell_j)$, where $\mathrm{Cor}\,(\ell_j)$ is a complementary subspace to $\mathrm{Range}\,(\ell_j)$.

If $W = (y - x^{n+1} + \cdots)^{1+\frac{p}{q}m_1}(y + x^{n+1} + \cdots)^{1+\frac{p}{q}m_2}$ with $\frac{(n+1)(m_1+m_2)p}{q} \notin \mathbb{N}$, $\frac{p}{q} \in \mathbb{Q}$, is an algebraic inverse integrating factor of $\mathbf{F}$ then by Proposicion 2.8 we can affirm that, $\tilde{W} = (y - x^{n+1} + \cdots)^{1+\frac{p}{q}m_1}(y + x^{n+1} + \cdots)^{1+\frac{p}{q}m_2}$ with $\frac{(n+1)(m_1+m_2)p}{q} \notin \mathbb{N}$, is an algebraic inverse integrating factor of $\hat{\mathbf{F}}$.

On the other hand the factors of $\tilde{W}$, i.e., $y - x^{n+1} + \cdots$, $y + x^{n+1} \cdots$ are invariant curves of $\hat{\mathbf{F}}$. Applying Proposition 3.17, we know that there exist unity series $U_1$, $U_2$ such that $y - x^{n+1} + \cdots = (y - x^{n+1})U_1$ and $y + x^{n+1} + \cdots = (y + x^{n+1})U_2$. Thus we have that $\tilde{W} = CU$ with $C = \left[(y - x^{n+1})^{q+pm_1}(y + x^{n+1})^{q+pm_2}\right]^{1/q}$ and where $U$ is a unity series.

Moreover, $C$ satisfies the equation (3.5) for $m = q$, $s_1 - s_2 = (n+1)(q + pm_1 + q + pm_2) = 2(n+1)q + (n+1)(m_1 + m_2)p \neq q(r + |\mathbf{t}| + i) = q(2(n+1) + i)$ with $i \in \mathbb{N}$ because $\frac{(n+1)(m_1+m_2)p}{q} \neq i$ for all $i \in \mathbb{N}$. By applying Lemma 3.16 $\hat{\mathbf{F}} = \mathbf{F}_r$ is polynomially integrable, therefore $\mathbf{F}$ is formally integrable. From [24, Theorem A], we deduce that $\mathbf{F}$ is analytically integrable. ■

## 4 Applications

We consider the integrability problem for the following nilpotent vector field with non-hamiltonian first quasi-homogeneous component

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} a_0x^3 + a_1xy \\ b_0x^4 + b_1x^2y + b_2y^2 \end{pmatrix}. \tag{4.6}$$

The first quasi-homogeneous component of the vector field respect to the type $(1,2)$ is

$$\mathbf{F}_1 = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} \in \mathcal{Q}_1^{(1,2)},$$

being $\mathbf{F}_1 = \mathbf{X}_h + \mu_1 \mathbf{D}_0$ where $h = -\frac{1}{2}\left(y^2 - x^4\right)$, $\mu = -\frac{1}{3}x$ and $\mathbf{D}_0 = (x, 2y)^T$. Therefore the origin of system (4.6) is not monodromic, see [1, 7]. In fact, the origin is a saddle with two invariant curves $C_1 = (y - x^2) + \cdots$ and $C_2 = (y + x^2) + \cdots$, see [11]. Moreover $\mathbf{F}_1$ has a primitive first integral given by $I_6 = (y - x^2)(y + x^2)^2$. The following result proved in [11] provides a normal form under orbital equivalence of a perturbation of $\mathbf{F}_1$.

**Lemma 4.18** *There exist a change of variables in the state variables and the time, $\mathbf{x} = \Phi(\mathbf{y})$, $dT = \mu(\mathbf{y})dt$, such that system (4.6) is transformed into*

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \alpha_2^{(0)} x^2 \mathbf{D}_0 + \alpha_4^{(0)} h \mathbf{D}_0 \\ &\quad + \sum_{l \geq 1} \alpha_0^{(l)} I_6^l \mathbf{D}_0 + \alpha_1^{(l)} x I_6^l \mathbf{D}_0 + \alpha_2^{(l)} x^2 I_6^l \mathbf{D}_0 + \alpha_4^{(l)} h I_6^l \mathbf{D}_0 \end{aligned} \tag{4.7}$$

The conservative-dissipative splitting, (see [2, Prop.2.7]), allows to write system (4.6) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{X}_{-\frac{1}{2}(y^2 - x^4)} - \tfrac{1}{3}x\mathbf{D}_0 + \mathbf{X}_{c_0x^5 + c_1x^3y + c_2xy^2} + \left(d_0x^2 + d_1y\right)\mathbf{D}_0, \tag{4.8}$$

where

$$c_0 = \frac{b_0}{5},\ c_1 = \frac{b_1 - 2a_0}{5},\ c_2 = \frac{b_2 - 2a_1}{5},\ d_0 = \frac{3a_0 + b_1}{5},\ d_1 = \frac{a_1 + 2b_2}{5}.$$

Next result solves the integrability problem for the previous system.

**Theorem 4.19** *System (4.8) is analytically integrable if, and only if, it satisfies one of the following conditions:*

**i)** $c_1 - c_0 + 7d_1 = c_2 + 7d_1 = 7d_0 + 3c_0 + 14d_1 = 0.$

**ii)** $c_1 + c_0 + 8d_1 = c_2 - 8d_1 = 8d_0 + 3c_0 - 16d_1 = 0.$

**iii)** $c_1 = c_0 + c_2 = 3d_0 + 2c_0 + d_1 = 0.$

**iv)** $12c_0 + 7d_1 = 3c_1 - 4d_1 = 4c_2 + 3d_1 = 3d_0 + d_1 = 0.$

**v)** $9c_0 - 16d_1 = 28d_1 - 9c_1 = 3c_2 - 4d_1 = 9d_0 + 7d_1 = 0.$

**vi)** $115d_1 - 189c_0 = 63c_1 + 38d_1 = 7c_2 - d_1 = 21d_0 + 11d_1 = 0.$

**vii)** $189c_0 - 19d_1 = 21c_1 + 2d_1 = 7c_2 - d_1 = 3d_0 + d_1 = 0.$

**viii)** $12c_0 + 5d_1 = 6c_1 + 7d_1 = 4c_2 + 3d_1 = 3d_0 + d_1 = 0.$

**ix)** $9c_0 - 20d_1 = 9c_1 + 32d_1 = 3c_2 - 4d_1 = 9d_0 + 7d_1 = 0.$

**x)** $21c_0 - 11d_1 = 3c_1 - 2d_1 = 7c_2 - d_1 = 21d_0 + 11d_1.$

**xi)** $27c_0 + 35d_1 = 3c_1 + 2d_1 = c_2 - 3d_1 = d_0 - d_1 = 0.$

*Proof.* The necessity is proved computing successively the coefficients of the dissipative part of the normal form (4.7) and imposing their vanishing because otherwise by Theorem 1.2 they prevent integrability. The first coefficient of the dissipative part of the normal form (4.7) is

$$\alpha_2^{(0)} := 336d_0 - 89c_2 + 9c_1 + 135c_0 + 112d_1.$$

The next coefficients of the dissipative part of the normal form (4.7) are big expressions that we do not present here. We define the ideal $J = \langle \alpha_2^{(0)}, \alpha_4^{(0)}, ...\rangle$ generated by these coefficients. By the Hilbert Basis theorem we know that is finitely generated which implies that there exist a set of generators $g_1, g_2, ..., g_k$ such that $J = \langle g_1, g_2, ..., g_k\rangle$. We call $\lambda \in \mathbb{R}^5$ the set of parameters of system (4.6). The affine variety $V(J) = \{\lambda \in \mathbb{R}^5 \mid g_i(\lambda) = 0\}$ is the integrable variety of system (4.6). This variety provides a finite set of necessary and sufficient conditions to have integrability around the origin for system (4.6). We compute a certain number of coefficients to find a set of generators. In particular, we have computed six coefficients. We decompose the algebraic set of the computed coefficients into its irreducible components using a computer algebra system Singular [21]. In fact, we use the routine `minAssGTZ` [16] based on the Gianni-Trager-Zacharias algorithm [18]. The decomposition of the ideal has been possible in the field of rational numbers, hence we know that the decomposition of the variety is complete. The obtained decomposition consists of 12 components defined by the following prime ideals:

(1) $\langle c_1 - c_0 + 7d_1, c_2 + 7d_1, 7d_0 + 3c_0 + 14d_1\rangle$,

(2) $\langle c_1 + c_0 + 8d_1, c_2 - 8d_1, 8d_0 + 3c_0 - 16d_1\rangle$,

(3) $\langle c_1, c_2 + c_0, 3d_0 + 2c_0 + d_1\rangle$,

(4) $\langle 12c_0 + 7d_1, 3c_1 - 4d_1, 4c_2 + 3d_1, 3d_0 + d_1\rangle$,

(5) $\langle 9c_0 - 16d_1, 9c_1 - 28d_1, 3c_2 - 4d_1, 9d_0 + 7d_1\rangle$,

(6) $\langle 189c_0 - 115d_1, 63c_1 + 38d_1, 7c_2 - d_1, 21d_0 + 11d_1\rangle$,

(7) $\langle 189c_0 - 19d_1, 21c_1 + 2d_1, 7c_2 - d_1, 3d_0 + d_1\rangle$,

(8) $\langle 12c_0 + 5d_1, 6c_1 + 7d_1, 4c_2 + 3d_1, 3d_0 + d_1\rangle$,

(9) $\langle 9c_0 - 20d_1, 9c_1 + 32d_1, 3c_2 - 4d_1, 9d_0 + 7d_1\rangle$,

(10) $\langle 21c_0 - 11d_1, 3c_1 - 2d_1, 7c_2 - d_1, 21d_0 + 11d_1\rangle$,

(11) $\langle 27c_0 + 35d_1, 3c_1 + 2d_1, c_2 - 3d_1, d_0 - d_1\rangle$.

(12) $\langle 21c_1 + 315c_0 - 179d_1, 7c_2 - d_1, 21d_0 + 11d_1, 173282571c_0^4 - 429580746c_0^3d_1 + 396959976c_0^2d_1^2 - 162133286c_0d_1^3 + 24706845d_1^4\rangle$.

However the component (12) is a fake component because does not vanish the next coefficients of the dissipative part of the normal form (4.7). Hence the first 11 components are the necessary conditions for system (4.6) be integrable and they are the conditions that appear in Theorem 4.19.

Now we give the sufficiency for each condition.

**i)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} \frac{5}{3}(7d_1 + 2d_0)x^3 + 15d_1xy \\ -\frac{35}{3}(2d_1 + d_0)x^4 - 5(d_0 + 7d_1)x^2y - 5d_1y^2 \end{pmatrix}$$

This vector field has two invariant curves $C_i$ with cofactors $K_i$, respectively, given by $C_1 = y - x^2 + 5(2d_1 + d_0)x^3 + 15d_1xy$, $K_1 = -8x/3 + 10d_0x^2 + 10d_1y$, $C_2 = y + x^2$, $K_2 = 4x/3 - 5d_0x^2 - 5d_1y$. As $\mathrm{div}(\mathbf{F}) = K_1 + K_2$, then $V = C_1C_2 = h + \cdots$ is a inverse integrating factor of system (4.6). Hence by Theorem 1.3 we have that it is analytically integrable. Moreover an analytic first integral is given by $I = C_1C_2^2$.

**ii)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} \frac{5}{3}(8d_1 - d_0)x^3 - 15d_1xy \\ \frac{40}{3}(2d_1 - d_0)x^4 + 10(d_0 - 4d_1)x^2y + 10d_1y^2 \end{pmatrix}$$

The vector field has also two invariant curves $C_i$ with cofactors $K_i$ given by $C_1 = y-x^2$, $K_1 = -8x/3+10d_0x^2+10d_1y$, $C_2 = y+x^2+5(2d_1-d_0)x^3-15d_1xy$, $K_2 = 4x/3-5d_0x^2-5d_1y$. As $\operatorname{div}(\mathbf{F}) = K_1+K_2$, then $V = C_1C_2 = h+\cdots$ is a inverse integrating factor of system (4.6). Applying Theorem 1.3 we have that it is analytically integrable. Moreover an analytic first integral is $I = C_1C_2^2$.

**iii)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y-\frac{1}{3}x^2 \\ 2x^3-\frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} d_0x^3-3d_0xy \\ -\frac{5}{2}(d_1+3d_0)x^4+2d_0x^2y+\frac{1}{2}(5d_1+3d_0)y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, $i=1,2,3$ given by $C_1 = y-x^2$, $K_1 = -8x/3+(19d_0+5d_1)x^2/2+(5d_1+3d_0)y/2$, $C_2 = y+x^2$, $K_2 = 4x/3-(11d_0+5d_1)x^2/2+(5d_1+3d_0)y/2$, $C_3 = 1-3d_0x$, $K_3 = d_0x^2-3d_0y$. As $\operatorname{div}(\mathbf{F}) = K_1+K_2+K_3$ then $V = C_1C_2C_3 = h+\cdots$ is a inverse integrating factor of system (4.6). Hence by Theorem 1.3 it is also analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^A$, where $A = (5d_1+3d_0)/(2d_0)$ if $d_0 \neq 0$ and $I = C_1C_2^2\exp(-\frac{15}{2}d_1x)$ if $d_0 = 0$.

**iv)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y-\frac{1}{3}x^2 \\ 2x^3-\frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} -\frac{5}{3}d_1x^3+\frac{5}{2}d_1xy \\ -\frac{35}{12}d_1x^4+\frac{10}{3}d_1x^2y+\frac{5}{4}d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, $i=1,2,3$ given by $C_1 = y-x^2$, $K_1 = -\frac{8}{3}x-\frac{5}{12}d_1x^2+\frac{5}{4}d_1y$, $C_2 = y+x^2-\frac{5}{4}d_1x^3+\frac{5}{4}d_1xy+\frac{15}{64}d_1^2y^2-\frac{15}{32}d_1^2x^2y+\frac{15}{64}d_1^2x^4$, $K_2 = \frac{4}{3}x-\frac{5}{6}d_1x^2+\frac{5}{2}d_1y$, $C_3 = 1+\frac{5}{4}d_1x-\frac{75}{128}d_1^2x^2+\frac{75}{128}d_1^2y$, $K_3 = -\frac{5}{12}d_1x^2+\frac{5}{4}d_1y$. As $\operatorname{div}(\mathbf{F}) = K_1+K_2+K_3$ then $V = C_1C_2C_3 = h+\cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^{-5}$.

**v)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y-\frac{1}{3}x^2 \\ 2x^3-\frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} -\frac{35}{9}d_1x^3-\frac{5}{3}d_1xy \\ \frac{80}{9}d_1x^4+\frac{70}{9}d_1x^2y+\frac{10}{3}d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, for $i=1,2,3$ given by $C_1 = y-x^2-\frac{20}{3}d_1x^3-\frac{1525}{72}d_1^2x^4-\frac{25}{4}d_1^2x^2y+\frac{75}{8}d_1^2y^2-\frac{125}{4}d_1^3x^5-\frac{125}{2}d_1^3x^3y-\frac{125}{4}d_1^3xy^2+\frac{625}{24}d_1^4x^6+\frac{625}{12}d_1^4x^4y+\frac{625}{24}d_1^4x^2y^2$, $K_1 = -\frac{8}{3}x-\frac{50}{9}d_1x^2+\frac{10}{3}d_1y$, $C_2 = y+x^2$, $K_2 = \frac{4}{3}x+\frac{10}{9}d_1x^2+\frac{10}{3}d_1y$, $C_3 = 1+\frac{10}{3}d_1x+\frac{100}{9}d_1^2x^2+\frac{25}{3}d_1^2y$, $K_3 = -\frac{10}{9}d_1x^2+\frac{10}{3}d_1y$. As $\operatorname{div}(\mathbf{F}) = K_1+K_2+K_3$ then $V = C_1C_2C_3 = h+\cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^{-3}$.

**vi)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y-\frac{1}{3}x^2 \\ 2x^3-\frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} \frac{5}{63}d_1x^3+\frac{5}{7}d_1xy \\ \frac{575}{189}d_1x^4-\frac{20}{7}d_1x^2y+\frac{15}{7}d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, for $i=1,2,3$ given by $C_1 = y-x^2-\frac{145}{63}d_1x^3+\frac{15}{7}d_1xy-\frac{225}{392}d_1^2y^2+\frac{375}{196}d_1^2x^2y-\frac{625}{392}d_1^2x^4$, $K_1 = -\frac{8}{3}x-\frac{10}{3}d_1x^2+\frac{20}{7}d_1y$, $C_2 = y+x^2+\frac{80}{63}d_1x^3$, $K_2 = \frac{4}{3}x+\frac{5}{21}d_1x^2+\frac{15}{7}d_1y$, $C_3 = 1+\frac{20}{7}d_1x+\frac{425}{147}d_1^2x^2-\frac{25}{49}d_1^2y+\frac{2875}{3087}d_1^3x^3-\frac{125}{343}d_1^3xy$, $K_3 = -\frac{20}{21}d_1x^2+\frac{20}{7}d_1y$. As $\operatorname{div}(\mathbf{F}) = K_1+K_2-\frac{1}{2}K_3$ then $V = C_1C_2C_3^{-1/2} = h+\cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^{-3}$.

**vii)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} -\frac{5}{21}d_1x^3 + \frac{5}{7}d_1xy \\ \frac{95}{189}d_1x^4 - \frac{20}{21}d_1x^2y + \frac{15}{7}d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, for $i = 1, 2, 3$ given by $C_1 = y - x^2 - \frac{10}{63}d_1x^3$, $K_1 = -\frac{8}{3}x - \frac{5}{7}d_1x^2 + \frac{15}{7}d_1y$, $C_2 = y + x^2 + \frac{20}{63}d_1x^3$, $K_2 = \frac{4}{3}x - \frac{5}{7}d_1x^2 + \frac{15}{7}d_1y$, $C_3 = 1 + \frac{5}{7}d_1x$, $K_3 = -\frac{5}{21}d_1x^2 + \frac{5}{7}d_1y$. As $\mathrm{div}(\mathbf{F}) = K_1 + K_2 - K_3$ then $V = C_1C_2C_3^{-1} = h + \cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^{-12}$.

**viii)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} \frac{5}{6}d_1x^3 + \frac{5}{2}d_1xy \\ -\frac{25}{12}d_1x^4 - \frac{25}{6}d_1x^2y + \frac{5}{4}d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, for $i = 1, 2, 3$ given by $C_1 = y - x^2 + \frac{5}{4}d_1x^3 + \frac{5}{4}d_1xy - \frac{75}{256}d_1^2y^2 - \frac{75}{128}d_1^2x^2y - \frac{75}{256}d_1^2x^4$, $K_1 = -\frac{8}{3}x - \frac{5}{6}d_1x^2 + \frac{5}{2}d_1y$, $C_2 = y + x^2$, $K_2 = \frac{4}{3}x - \frac{5}{12}d_1x^2 + \frac{5}{4}d_1y$, $C_3 = 1 + \frac{5}{4}d_1x - \frac{75}{64}d_1^2x^2 - \frac{75}{64}d_1^2y$, $K_3 = -\frac{5}{12}d_1x^2 + \frac{5}{4}d_1y$. As $\mathrm{div}(\mathbf{F}) = K_1 + K_2 + K_3$ then $V = C_1C_2C_3 = h + \cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^{-4}$.

**ix)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} \frac{25}{9}d_1x^3 - \frac{5}{3}d_1xy \\ \frac{100}{9}d_1x^4 - \frac{110}{9}d_1x^2y + \frac{10}{3}d_1y^2 \end{pmatrix}$$

For this vector field we have been able to find two invariant curves $C_i$ with cofactors $K_i$, for $i = 1, 2$ given by $C_1 = y - x^2$, $K_1 = -\frac{8}{3}x - \frac{50}{9}d_1x^2 + \frac{10}{3}d_1y$, $C_2 = 1 + \frac{10}{3}d_1x + \frac{125}{18}d_1^2x^2 - \frac{25}{6}d_1^2y$, $K_2 = -\frac{10}{9}d_1x^2 + \frac{10}{3}d_1y$. As $\mathrm{div}(\mathbf{F}) = \frac{1}{2}K_1 + K_2$ then $W = C_1^{1/2}C_2 = (y - x^2 + \cdots)^{1/2}$ is an algebraic inverse integrating factor of system (4.6), i.e., $W = (y - x^2 + \cdots)^{1+\frac{p}{q}}(y + x^2 + \cdots)^{1+\frac{p}{q}2}$ with $p = -1$, $q = 2$. In this case $\frac{(n+1)(m_1+m_2)p}{q} = -2 \notin \mathbb{N}$ for $m_1 = 1$, $m_2 = 2$, and by Theorem 1.5 we have that it is analytically integrable.

Anyway, in this case using this inverse integrating factor $W$ we can compute a first integral given by

$$H = \frac{4(5d_1x-3)\sqrt{y-x^2}}{25d_1^2} + \frac{12\sqrt{6}}{125d_1^3}\operatorname{arctanh}\left(\frac{5\sqrt{6}d_1\sqrt{y-x^2}}{2(5d_1x+3)}\right).$$

This first integral $H$ is not analytic at the origin, however taking into account that $\operatorname{arctanh}(z) = \sum_{j\geq 0}\frac{1}{2j+1}z^{2j+1} = z(1 + \sum_{j\geq 1}\frac{1}{2j+1}z^{2j})$ and defining $z = \frac{5\sqrt{6}d_1\sqrt{y-x^2}}{2(5d_1x+3)}$ we have that

$$\begin{aligned} H &= \sqrt{y-x^2}\left(\frac{4(5d_1x-3)}{25d_1^2} + \frac{12\sqrt{6}}{125d_1^3}\frac{5\sqrt{6}d_1}{2(5d_1x+3)}(1 + \sum_{j\geq 1}\frac{1}{2j+1}z^{2j})\right) \\ &= \sqrt{y-x^2}\left(\tfrac{2}{3}(y+x^2) + \cdots\right). \end{aligned}$$

Consequently $H^2$ is an analytic first integral at the origin. Moreover $V = WH = \frac{2}{3}h + \cdots$ is an analytic inverse integrating factor and we can also apply Theorem 1.3 for deducing that the above vector field is analytically integrable.

**x)** In this case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} -\frac{25}{21}d_1x^3 + \frac{5}{7}d_1xy \\ \frac{55}{21}d_1x^4 + \frac{20}{21}d_1x^2y + \frac{15}{7}d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, for $i = 1, 2, 3$ given by $C_1 = y - x^2 + \frac{5}{7}d_1x(-5x^2 + 3y) + \frac{25}{392}d_1^2(9y + 17x^2)(3y - 5x^2) - \frac{1000}{343}d_1^3x^3(x^2 + y)$, $K_1 = -\frac{8}{3}x - \frac{10}{3}d_1x^2 + \frac{30}{7}d_1y$, $C_2 = y + x^2$, $K_2 = \frac{4}{3}x + \frac{5}{21}d_1x^2 + \frac{15}{7}d_1y$, $C_3 = 1 + \frac{20}{7}d_1x + \frac{25}{7}d_1^2x^2 + \frac{75}{49}d_1^2y + \frac{375}{343}d_1^3x^3 + \frac{375}{343}d_1^3xy$, $K_3 = -\frac{20}{21}d_1x^2 + \frac{20}{7}d_1y$. As $\mathrm{div}(\mathbf{F}) = K_1 + K_2 - \frac{1}{2}K_3$ then $V = C_1C_2C_3^{-1/2} = h + \cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2C_3^{-3}$.

**xi)** In this last case, the vector field (4.6) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \frac{1}{3}x^2 \\ 2x^3 - \frac{2}{3}xy \end{pmatrix} + \begin{pmatrix} \frac{5}{3}d_1x^3 - 5d_1xy \\ -\frac{175}{27}d_1x^4 + 5d_1y^2 \end{pmatrix}$$

This vector field has three invariant curves $C_i$ with cofactors $K_i$, for $i = 1, 2, 3$ given by $C_1 = y - x^2 + \frac{35}{9}d_1x^3 - 5d_1xy$, $K_1 = -\frac{8}{3}x + \frac{40}{3}d_1x^2$, $C_2 = y + x^2 - \frac{25}{9}d_1x^3 - 5d_1xy$, $K_2 = \frac{4}{3}x - \frac{20}{3}d_1x^2$, $C_3 = 1 - 5d_1x$, $K_3 = \frac{5}{3}d_1x^2 - 5d_1y$. As $\mathrm{div}(\mathbf{F}) = K_1 + K_2 - K_3$ then $V = C_1C_2C_3^{-1} = h + \cdots$ is a inverse integrating factor of system (4.6). By Theorem 1.3 we have that it is analytically integrable. Moreover, it has an analytic first integral of the form $I = C_1C_2^2$.

■

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