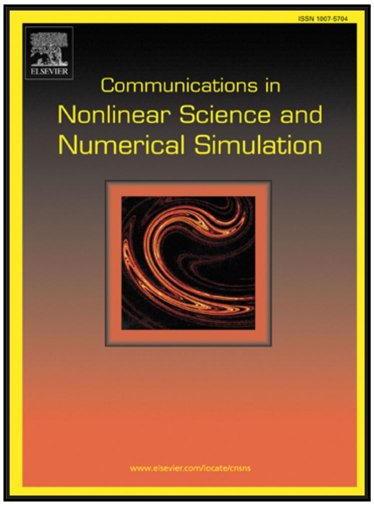
Journal Pre-proof



Stability Analysis by Contraction Principle for Impulsive Systems with Infinite Delays

Xinzhi Liu, Cesar Ramirez

PII: S1007-5704(19)30340-5
DOI: https://doi.org/10.1016/j.cnsns.2019.105021
Reference: CNSNS 105021

To appear in: *Communications in Nonlinear Science and Numerical Simulation*

Highlights

- This paper studies a class of quasi-linear impulsive systems of functional differential equations with infinite time delay.
- It uses the Banach contraction principle
- It has established criteria on uniform stability and asymptotic stability
- The proposed approach utilizes the idea of averaging instead of the point-wise estimate in the Lyapunov method.
- It shows that the Banach contraction principle can be used as a possible alternative to Lyapunov methods for stability analysis when the conditions of Lyapunov method fails to hold.

Stability Analysis by Contraction Principle for Impulsive Systems with Infinite Delays

Xinzhi Liu and Cesar Ramirez
Department of Applied Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
E-mail: xzliu@uwaterloo.ca; c3ramire@uwaterloo.ca.

Abstract This paper studies a class of quasi-linear impulsive systems of functional differential equations with infinite time delay. By employing the contraction principle, several criteria on uniform stability and asymptotic stability are established. The proposed approach utilizes the idea of averaging instead of the point-wise estimate in the Lyapunov method. Our results show that the Banach contraction principle can be used as a possible alternative to Lyapunov methods for stability analysis when the conditions of Lyapunov method fails to hold. Several examples are discussed to illustrate the ideas of our results.

Keywords: Contraction principle, stability, impulsive system, time delay.

1 Introduction

Impulsive systems have attracted an increased interest of the world research community due to their important applications in many areas[1, 15, 22, 28, 30, 31, 36, 37]. Some typical real world processes which exhibit impulsive behavior include impacts in constrained mechanics, sudden population decrease of a species due to natural disasters such as earthquakes or fire, burst activity in the propagation of action potentials of neurons. In other situations, they arise as a from of control that is designed to enable the dynamics in a desired manner. Examples include orbital transfer of satellite, vibration suppression of flexible structure in a space craft, interest adjustment in financial and economics management, maintains of a species through periodic stocking or harvesting, and consensus via impulsive protocol in multi-agent systems. An impulsive system normally contains three main elements: a continuous system of differential equations, which governs the evolution of the system between impulsive events; a discrete system of difference equations, which govern the way the system states are instantaneously mapped when a setting event occurs; and a criterion for determining when the states of the system are to be reset. The theory of impulsive systems of ordinary differential equations (ODEs) has been well-developed, see [15, 17, 18, 21, 22, 28] and references therein. However, the corresponding theory for impulsive systems of functional differential equations (FDEs) is yet to be fully developed. There are a few challenges one must face in studying impulsive FDEs. For example, in the theory of FDEs, it is shown that the continuity of a function $x(t)$ in $\Re^n$ implies the continuity of the functional x_t in C^n. This fact is crucial in proving the existence of solutions of FDEs [13]. However, a solution $x(t)$ of an impulsive FDEs is piecewise continuous, but the functional x_t is in general not piecewise continuous. In fact it could be discontinuous in an entire interval. Thus even if $f(t, \psi)$ is well-behaved and very smooth in its two variables one cannot, in general, confirm anything about the composition function $f(t, x_t)$ whenever $x(t)$ is piecewise continuous. This problem was solved in [3, 24], where existence and uniqueness results for impulsive FDEs are established.

The method of Lyapunov has been a very effective tool in the study of stability problems for impulsive FDEs. The manifest advantage of this method is that it does not require the knowledge of solutions and therefore has great power in applications. The stability analysis may be divided into two approaches. In one approach, Lyapunov functions are used, where the derivative of the Lyapunov function is estimated with respect to an appropriate minimal class of functionals. This approach is known as the Lyapunov-Ruzumikhin technique[4, 19, 23, 25, 33, 36]. The other approach employs Lyapunov functionals, where the corresponding derivative can be estimated without demanding minimal classes of functionals. Numerous interesting stability results have been established for impulsive systems of FDEs, see [2, 4, 5, 12, 19, 20, 23, 29, 30, 33, 35, 36, 38] and references therein. However, there have been some known difficulties in employing this method. It is needless to say that there is difficulty constructing Lyapunov function and Lyapunov functionals. Even if a suitable Lyapunov function or functional is found there still remains significant problem to determine a set where its derivative along solutions of the concerning system is definite. These difficulties motivate us to consider other methods in conjunction with the Lyapunov method. In relatively recent studies, the fixed point method for stability analysis has been successfully applied to FDEs, see [6, 7, 9, 10, 39], and references therein. The advantage of the fixed point method is the idea of averaging while the Lyapunov method requires pointwise estimate. However, this method is yet to become popular, especially for impulsive system of FDEs.

In this paper, We shall consider a class of impulsive FDEs and establish some stability criteria utilizing the ideas given in [9, 39]. We thus generalize the previous results to a theoretical framework of impulsive FDEs, and in doing so we will obtain some insight into how difficult it may be to fit in these "harmless" perturbations of the previous result. Harmless in the sense that they do not break the contraction requirement of the previous result by [9, 39]. We will notice that we also obtain global existence of solutions as a by-product, just like Lyapunov stability methods can do. The fixed point method here gives a global existence and uniqueness result, whereas existence results such as those of [3, 24] give local existence and uniqueness. Determining global existence is not a trivial matter for impulsive differential equations, as is illustrated in [32, 34]. Although perhaps, it is not a surprise that in both of these aforementioned results, fixed point methods are used in order to prove the existence of global solutions. However, instead of the Banach contraction principle, which we shall use, the aforesaid papers use the fixed point theorem by Schaefer.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. Then in Section 3, we obtain some criteria on uniform stability and asymptotic stability for impulsive systems of FDEs. Section 4 discusses the issues about uniqueness of solutions which is a required condition in our approach. We consider some special cases in Section 4 and give a conclusion in Section 5. Several remarks and examples are given to illustrate the ideas of our results.

2 Preliminaries

Let $a < b$ with $a, b \in \mathbb{R}$ and $D \subset \mathbb{R}^n$ be open. We define the following classes of functions.

$$\begin{aligned} PC([a,b],D) = \{x : [a,b] \longrightarrow D \mid x(t) = x(t^+),\ \forall t \in [a,b);\ x(t^-) \text{ exists},\ \forall t \in (a,b]; \\ x(t^-) = x(t) \text{ for all but at most a \textit{finite} number of points } t \in (a,b]\}. \end{aligned}$$

$$\begin{aligned} PC([a,b),D) = \{x : [a,b) \longrightarrow D \mid x(t) = x(t^+),\ \forall t \in [a,b);\ x(t^-) \text{ exists},\ \forall t \in (a,b); \\ x(t^-) = x(t) \text{ for all but at most a \textit{finite} number of points } t \in (a,b)\}. \end{aligned}$$

These classes describe spaces that are right-continuous with left limits everywhere, and they are left continuous except possibly on a finite number of points where they are defined.

Notice the previous intervals of definition are finite in length. For infinite intervals we have

$$PC([a,\infty),D)=\{x:[a,\infty)\longrightarrow D\mid \forall c>a, x|_{[a,c]}\in PC([a,c],D)\}.$$

$$\begin{aligned}PC\big((-\infty,b],D\big)=\{x:(-\infty,b]\longrightarrow D\mid\ & x(t)=x(t^+),\ \forall t\in(-\infty,b);\\ & x(t^-)\text{ exists in }D,\ \forall t\in(-\infty,b];\ x(t^-)=x(t)\\ & \text{for all but a \textit{countable} number of points } t\in(-\infty,b],\\ & \text{and discontinuities do not have finite accumulation points}\}.\end{aligned}$$

$$PC(\mathbb{R},D)=\{x:\mathbb{R}\longrightarrow D\mid \forall b\in\mathbb{R}, x|_{(-\infty,b]}\in PC\big((-\infty,b],D\big)\}.$$

$PCB([a,b],D)=PC([a,b],D)$.
$PCB([a,b),D)=\{x\in PC([a,b),D)\mid x \text{ is bounded on } [a,b)\}$.
$PCB([a,\infty),D)=\{x:[a,\infty)\longrightarrow D\mid \forall c>a, x|_{[a,c]}\in PC([a,c],D),\ x\text{ is bounded on }[a,\infty)\}$.
$PCB\big((-\infty,b],D\big)=\{x\in PC\big((-\infty,b],D\big)\mid x\text{ is bounded on }(-\infty,b]\}$.
$PCB(\mathbb{R},D)=\{x\in PC(\mathbb{R},D)\mid\ x\text{ is bounded on }\mathbb{R}\}$.

Remark 2.1. We will be interested in the space $PCB([-r,0],D)$, with $0\le r\le\infty$. We will sometimes omit the open set $D\subset\mathbb{R}^n$ when this set is implicitly understood to be fixed. In the case when $r=\infty$, we still denote the space $PCB(-\infty,0]$ by the notation $PCB[-r,0]$, by considering for this special case $[-r,0]$ to mean the *infinite* interval $(-\infty,0]$, and using the piecewise continuous bounded functions on $(-\infty,0]$. Of course, $PCB[-r,0]=PC[-r,0]$ when $r<\infty$. If $r=\infty$, the state x_t always contains part of the initial functions. As a result of this, each different phase space in general requires a new and separate development for the theory [13].

Consider the impulsive delayed differential equation

$$\begin{aligned} x'(t)&=A(t)x(t)+g(t,x_t), && t\neq\tau_k, t\ge 0\\ \Delta x(t)&=I(t,x_{t^-}), && t=\tau_k, t\ge 0\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^n$, $g,I:J\times PCB([-r,0],D)\longrightarrow\mathbb{R}^n$ with $J\subset\mathbb{R}^+$ an interval, $D\subset\mathbb{R}^n$ an open set and $\Delta x(t)=x(t)-x(t^-)$. The impulse times τ_k satisfy $0=\tau_0<\tau_1<\cdots$ and $\lim_{k\to\infty}\tau_k=\infty$. $A(t)$ is an $n\times n$ matrix of continuous function, in the sense that each entry of $A(t)$ is a continuous function of t in the interval of definition of the functional differential equation (1).
The initial condition for equation (1) will be given for $t_0\ge 0$ as

$$x_{t_0}=\phi\tag{2}$$

for $t_0\in J$, and $\phi\in PCB([-r,0],D)$.

The norm that we use on $PCB([-r,0],D)$ will be

$$\|\psi\|_r:=\sup_{s\in[-r,0]}|\psi(s)|,$$

where of course for $r=\infty$ this norm is $\|\psi\|_r=\sup_{s\in(-\infty,0]}|\psi(s)|$. Wherever the norm symbol $\|\cdot\|$ is used, we refer to the norm on $PCB([-r,0],D)$. We will denote the Euclidean norm by $|x|$ whenever no confusion should arise. Let us define

$$\|\psi\|^{[s,t]}:=\sup_{u\in[s,t]}|\psi(u)|.$$

If for some $\sigma \in \mathbb{R}$, $a > 0$ we have a continuous function $x : [\sigma - r, a] \longrightarrow \mathbb{R}^n$, then for each $t \in [\sigma, a]$ we denote by x_t the element in $PCB[-r, 0]$ defined explicitly as

$$x_t(\theta) := x(t + \theta) \qquad \text{for } \theta \in [-r, 0] \tag{3}$$

Note that if $x : [\sigma - r, a] \longrightarrow \mathbb{R}^n$, then for each $t \in [\sigma, a]$, x_t simply denotes the restriction of $s \mapsto x(s)$ to the interval $s \in [t - r, t]$.

By x_{t^-} in (1) we refer to the function defined by a given $x \in PCB([t_0 - r, b], D)$ through the assignment

$$\begin{aligned} x_{t^-}(s) &= x_t(s) \quad \text{ for } s \in [-r, 0) \\ x_{t^-}(0) &= \lim_{u \to t^-} x(u) = x(t^-). \end{aligned} \tag{4}$$

This is a way of getting a well defined function in $PCB[-r, 0]$, that takes into account only the information available right until before the jump occurs right at $t = \tau_k$. In this way, the mapping I induces a jump from $x(t^-)$ to a value $x(t)$, using the information available until just before the impulse occurs at time t.

As in the convention used in [3], we do not ask for the jump condition in (1) to be satisfied at t_0, since this imposes an unnecessary restriction on the initial condition.

We now give the definitions of stability for impulsive FDEs. Let us have

$$\begin{aligned} x'(t) &= f(t, x_t), \qquad t \neq \tau_k, t \geq t_0 \\ \Delta x(t) &= I(t, x_{t^-}), \qquad t = \tau_k, t > t_0. \end{aligned} \tag{5}$$

We have $f : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J = [0, \infty)$. Let each $t_0 \in J, \phi \in PCB([-r, 0], D)$ induce an initial value problem by appending to (5) the initial condition

$$x_{t_0} = \phi. \tag{6}$$

Remark 2.2. For stability analysis, we assume that $0 \in D$, which implies that $0 \in PCB([-r, 0], D)$ and that $f(t, 0) \equiv 0$ for all $t \in J$, and $I(\tau_k, 0) \equiv 0$ for all k. Thus 0 is an equilibrium solution.

It should be noted that the Euclidean norm is denoted by $|\cdot|$.

Definition 2.1. The zero solution of (5) is said to be

[S1] *stable* if for each $\epsilon > 0$ and $t_0 \in J$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that for $\phi \in PCB([-r, 0], D)$ with $\|\phi\|_r \leq \delta$, and any solution $x(t) = x(t; t_0, \phi)$ of the induced IVP (5)-(6) satisfies

$$|x(t; t_0, \phi)| \leq \epsilon, \qquad \forall t \geq t_0; \tag{7}$$

[S2] *uniformly stable* if the δ in [S1] is independent of t_0;

[S3] *asymptotically stable* if [S1] holds and for every $t_0 \in J$ there is a constant $c = c(t_0) > 0$ such that for $\phi \in PCB([-r, 0], D)$ with $\|\phi\|_r \leq c$, then $x(t) = x(t; t_0, \phi) \to 0$ as $t \to \infty$;

[S4] *unstable* if [S1] fails to hold.

In order to extend results to time-dependent impulsive systems, will use the theoretical framework of impulsive systems as done in [23, 26]. In order for the necessary integrals to exist (namely those of nonlinear part g), we will assume that g is composite-PCB.

Definition 2.2. A mapping $g : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$, where $0 \le r \le \infty$, is said to be *composite-PCB* if for each $t_0 \in J$ and $\beta > 0$ where $[t_0, t_0 + \beta] \subset J$, if $x \in PCB([t_0 - r, t_0 + \beta], D)$, and x is continuous at each $t \neq \tau_k$ in $(t_0, t_0 + \beta]$ then the composite function $t \mapsto g(t, x_t)$ is an element of the function class $PCB([t_0, t_0 + \beta], \mathbb{R}^n)$.

Remark 2.3. We denote by $B(L) \subset PCB[-r, 0]$ the closed ball of radius L in $PCB[-r, 0]$:

$$B(L) = \{\psi \in PCB[-r, 0] : \|\psi\|_r \le L\}.$$

3 Main Results

We shall state and prove our main results in this section. For this purpose, we shall use the fundamental solution matrix $\Phi(t, t_0)$ of the system of linear ODEs

$$\begin{aligned} y'(t) &= A(t)y(t) \\ y(t_0) &= y_0 \end{aligned} \tag{8}$$

such that the solution of IVP (8) is

$$y(t) = \Phi(t, t_0)y_0.$$

For a matrix M we use the standard linear operator norm induced by the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$:

$$\|M\| := \|M\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{|y|=1} |My|.$$

We will use the inequality $|My| \le \|M\||y|$ for $y \in \mathbb{R}^n$.
Notice that the previous results in [9, 39] can be generalized to the n-dimensional case by noticing that we have that for t_1, t_2 in the scalar case:

$$\Phi(t_2, t_1) = e^{-\int_{t_1}^{t_2} a(u)du}.$$

Therefore it follows that a way of determining that $\|\Phi(t, 0)\| \longrightarrow 0$ as $t \to \infty$ for the 1-dimensional case, is by observing that $\displaystyle\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ is a sufficient condition.

In the main result of [39], asymptotic stability is concluded for the continuous (non-impulsive) scalar version of (1), namely

$$x'(t) = -a(t)x(t) + g(t, x_t), \tag{9}$$

with respective continuity assumptions on $g(t, x_t)$. Notice that in the proof of Theorem 2.1 in [39], a constant K is defined as

$$K = \sup_{t \ge t_0}\{e^{-\int_{t_0}^{t} a(s)ds}\} \tag{10}$$

order to prove asymptotic stability. Said constant K depends on the particular t_0 of interest, so that $K = K(t_0)$. Given this scalar case, we notice that $K \ge 1$, and $K > 1$ is possible when $a(t)$ takes on negative values $a(t) < 0$. We quickly illustrate this dependence of K on t_0 for the scalar case, through the following example.

Example 3.1. Suppose

$$a(t) = \begin{cases} 5\sin(t) & \text{if } t \in [0, \pi] \\ \sin(2t - \pi) & \text{if } t \in [\pi, 2\pi] \\ t - 2\pi & \text{if } t \ge 2\pi. \end{cases}$$

We have that $a(t) < 0$ if $t \in (\pi, \frac{3}{2}\pi)$. As mentioned previously, negative values of $a(t)$ might make $K = K(t_0) > 1$. Nonetheless, the most negative contribution of $\int_\pi^t a(s)ds$ for $t \in (\pi, \frac{3}{2}\pi)$, does not affect if $t_0 = 0$, when defining $K(0)$, since

$$\int_\pi^{\frac{3\pi}{2}} a(t)dt = \int_\pi^{\frac{3\pi}{2}} \sin(2t - \pi)dt = -\frac{1}{2}\cos(2t - \pi)\Big|_{t=\pi}^{t=\frac{3\pi}{2}} = -1 \tag{11}$$

is cancelled out by the positive contribution from the interval $[0, \pi]$ of $\int_0^\pi a(s)ds$

$$\int_0^\pi a(t)dt = 5\int_0^\pi \sin(t)dt = 10. \tag{12}$$

This makes, if $t_0 = 0$, $K := K(0) = \sup_{t\geq 0}\left(e^{-\int_0^t a(u)du}\right) = 1$, since afterwards, on the interval $[\frac{3}{2}\pi, \infty)$, we only have positive contributions to the integral.

However, the case is different if we now take $t_0 = \pi$. This is because of (11), so that we have

$$K = K(\pi) = \sup_{t\geq\pi}\left(e^{-\int_\pi^t a(u)du}\right) = e > 1$$

with the maximum value achieved at $t = \frac{3\pi}{2}$, since the integral $\int_\pi^t a(u)du$ is decreasing on $(\pi, \frac{3\pi}{2})$, and afterwards, positive contributions come to the integral after this time, making $t \mapsto \int_\pi^t a(s)ds$ increasing on $(\frac{3\pi}{2}, \infty)$. On $(2\pi, \infty)$ it is positive and increasing such that the overall dominant behavior of the positiveness causes $\int_\pi^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ so that $e^{-\int_\pi^t a(u)du} \longrightarrow 0$. Thus, this is how K depends on the initial time t_0 taken into account.

Remark 3.1. The previous example gives insight into how to calculate, for the scalar case, a K that is independent of the initial time t_0, by focusing on the longest interval where $a(t)$ in (9) is negative. The condition $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ makes it clear that *overall* $a(t)$ is positive and that negative values of $a(t)$ are transient behavior, or not as dominant as the positive values. The constant K can be taken as a measure of how bad things get before the positive values of $a(t)$ overtake.

Through a modest modification, we can slightly improve to uniform stability plus asymptotic stability, by making K independent of t_0, as we do in the following result.

Theorem 3.1. *Suppose that there exists positive constants α, L and continuous functions $b, c : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the following conditions hold:*

(i) For all $s_2 \geq s_1 \in [0, \infty)$, let us have the uniform bound $\|\Phi(s_2, s_1)\| \leq K < \infty$, in other words let $\sup_{s_2\geq s_1\geq 0}(\|\Phi(s_2, s_1)\|) \leq K < \infty$.[1]

(ii) $|g(t, \phi) - g(t, \psi)| \leq b(t)\|\phi - \psi\|$ for all $\phi, \psi \in B(L)$, and $g(t, 0) = 0$.

(iii) $|I(t, \phi) - I(t, \psi)| \leq c(t)\|\phi - \psi\|$ for all $\phi, \psi \in B(L)$ and $I(t, 0) = 0$.

(iv) For all $t \geq 0$

$$\int_0^t b(s)\|\Phi(t, s)\|ds + \sum_{0<\tau_k\leq t} c(\tau_k)\|\Phi(t, \tau_k)\| \leq \alpha < 1. \tag{13}$$

(v) For every $\epsilon > 0$ and $T_1 \geq 0$, there exists a $T_2 > T_1$ such that $t \geq T_2$ and $x_t \in B(L)$ implies

$$\begin{aligned} |g(t, x_t)| &\leq b(t)\left(\epsilon + \|x\|^{[T_1,t]}\right) \\ |I(t, x_{t^-})| &\leq c(t)\left(\epsilon + \|x\|^{[T_1,t]}\right). \end{aligned} \tag{14}$$

[1] Notice that $\Phi(t, t) = I_d$ for every t implies $K \geq 1$.

(vi) $\|\Phi(t,0)\| \longrightarrow 0$ *as* $t \to \infty$.

Then zero solution of (1) *is uniformly stable and asymptotically stable.*

Proof. We show that if

$$\delta_0 < \frac{(1-\alpha)}{K} L, \tag{15}$$

then for an initial condition[2] $\|\phi\| \le \delta_0$, the zero solution of (1) is uniformly stable and asymptotically stable.

For an initial condition $\|\phi\| \le \delta_0$, let us define the space

$$\begin{aligned} \mathcal{S} = \{ & x \in PCB([t_0 - r, \infty), D) : x_{t_0} = \phi,\, x_t \in B(L) \text{ for } t \ge t_0, \\ & x \text{ is discontinuous only at impulsive moments } t = \tau_k, \text{ and } x(t) \to 0 \text{ as } t \to \infty \}. \end{aligned}$$

$\mathcal{S}$ is a nonempty complete metric space under the metric

$$d_{\mathcal{S}}(x,y) = \sup_{s \in [t_0 - r, \infty)} |x(s) - y(s)| = \sup_{s \in [t_0, \infty)} |x(s) - y(s)| \text{ for } x, y \in \mathcal{S},$$

where we note that we can disregard the contribution on the subinterval $[t_0 - r, t_0]$ because of the definition of $\mathcal{S}$, and we remind the reader that $[t_0 - r, t_0] = (-\infty, t_0]$ when $r = \infty$.

To obtain a mapping suitable for the Banach fixed point theorem, we make the following observation. For $s \in [\tau_{k-1}, \tau_k)$, we have that, using the fundamental matrix and the functional differential equation (1):

$$\begin{aligned} x(t) &= \Phi(t, \tau_{k-1}) x(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \Phi(t,s) g(s, x_s) ds \\ &= \Phi(t, \tau_{k-1}) \left[x(\tau_{k-1}^-) + I\left(\tau_{k-1}, x_{\tau_{k-1}^-}\right) \right] + \int_{\tau_{k-1}}^{t} \Phi(t,s) g(s, x_s) ds \end{aligned}$$

Note that the necessary integrals will exist because $g(t, x_t)$ is composite-PCB as defined above.

The first line follows from variation of parameters for ODEs, as follows. Assume that a solution in the interval $s \in [\tau_{k-1}, \tau_k)$ is given by $x(t) = \Phi(t, \tau_{k-1}) m(t)$, where $m(t)$ is a differentiable vector valued function to be determined in the following fashion. By the product rule for differentiation we have that

$$\begin{aligned} x'(t) =& \Phi'(t, \tau_{k-1}) m(t) + \Phi(t, \tau_{k-1}) m'(t) \\ =& A(t) \Phi(t, \tau_{k-1}) m(t) + \Phi(t, \tau_{k-1}) m'(t) \end{aligned}$$

By the differential equation that $x(t)$ satisfies on $[\tau_{k-1}, \tau_k)$, this implies

$$A(t)\Phi(t, \tau_{k-1}) m(t) + \Phi(t, \tau_{k-1}) m'(t) = A(t) \Phi(t, \tau_{k-1}) m(t) + g(t, x_t).$$

Thus

$$m'(t) = [\Phi(t, \tau_{k-1})]^{-1} g(t, x_t) = \Phi(\tau_{k-1}, t) g(t, x_t)$$

The previous expression implies, after integrating from τ_{k-1} to t and using $m(\tau_{k-1}) = x(\tau_{k-1})$ that

$$m(t) = x(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \Phi(\tau_{k-1}, s) g(s, x_s) ds$$

[2]Notice that $\delta_0 < L$ since $K \ge 1$ and $1 - \alpha < 1$.

so that

$$x(t) = \Phi(t, \tau_{k-1})x(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \Phi(t,s)g(s,x_s)ds.$$

Thus, for $t \in [\tau_{k-1}, \tau_k)$, we obtain the formula

$$x(t) = \Phi(t, \tau_{k-1})x(\tau_{k-1}^-) + \int_{\tau_{k-1}}^{t} \Phi(t,s)g(s,x_s)ds + \Phi(t,\tau_{k-1})I\left(\tau_{k-1}, x_{\tau_{k-1}^-}\right). \tag{16}$$

We stress that this formula holds for $t \in [\tau_{k-1}, \tau_k)$ only, but by backstepping we can express $x(t_{k-1}^-)$ using the analogous formula to (16) but for $t \in [\tau_{k-2}, \tau_{k-1})$, since $x(\tau_{k-1}^-)$ uses the expression for $x(t)$ for $t \in [\tau_{k-2}, \tau_{k-1})$, as $t \to \tau_{k-1}^-$. Backstepping in this way we get:

$$\begin{aligned}
x(\tau_{k-1}^-) &= \Phi(\tau_{k-1}, \tau_{k-2})x(\tau_{k-2}^-) + \int_{\tau_{k-2}}^{\tau_{k-1}} \Phi(\tau_{k-1},s)g(s,x_s)ds + \Phi(\tau_{k-1},\tau_{k-2})I\left(\tau_{k-2}, x_{\tau_{k-2}^-}\right)\\
&\vdots\\
x(\tau_2^-) &= \Phi(\tau_2,\tau_1)x(\tau_1^-) + \int_{\tau_2}^{\tau_1} \Phi(\tau_2,s)g(s,x_s)ds + \Phi(\tau_2,\tau_1)I\left(\tau_1, x_{\tau_1^-}\right)\\
x(\tau_1^-) &= \Phi(\tau_1,t_0)\phi(0) + \int_{t_0}^{\tau_1} \Phi(\tau_1,s)g(s,x_s)ds,
\end{aligned}$$

where we remind ourselves that $x(t_0) = \phi(0)$ and $t_0 > 0 = \tau_0$. By recursive substitution into (16) we get that in general, the solution $x(t)$ must satisfy:

$$x(t) = \Phi(t,t_0)\phi(0) + \int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds + \sum_{t_0<\tau_k\le t} \Phi(t,\tau_k)I\left(\tau_k, x_{\tau_k^-}\right)$$

This makes us define the mapping P by

$$(Px)_{t_0} = \phi,$$

and for $t \ge t_0$:

$$(Px)(t) = \Phi(t,t_0)\phi(0) + \int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds + \sum_{t_0<\tau_k\le t} \Phi(t,\tau_k)I\left(\tau_k, x_{\tau_k^-}\right). \tag{17}$$

To prove that P defines a contraction mapping on $\mathcal{S}$, we must prove first that P maps $\mathcal{S}$ to itself.

Clearly, Px has left limits well defined, since $\sum_{t_0<\tau_k\le t} \Phi(t,\tau_k)I\left(\tau_k, x_{\tau_k^-}\right)$ has limit from the left, since $\Phi(t,\tau_k)$ is continuous and each $I\left(\tau_k, x_{\tau_k^-}\right)$ is well defined thanks to x having limit from the left at each τ_k. Clearly the term

$$\Phi(t,t_0)\phi(0) + \int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds$$

has well defined limits at impulse times, since this part is even continuous at impulse moment τ_l, by continuity of the Riemann integral. Right continuity at each impulse time τ_l is reduced to verifying right continuity of

$$Q(t) := \sum_{t_0<\tau_k\le t} \Phi(t,\tau_k)I\left(\tau_k, x_{\tau_k^-}\right)$$

at τ_l. Choose $\eta > 0$ small enough such that $\tau_l + \eta < \tau_{l+1}$. Then

$$Q(\tau_l + \eta) - Q(\tau_l) =$$

$$\sum_{t_0<\tau_k\le\tau_l+\eta}\Phi(\tau_l+\eta,\tau_k)I\left(\tau_k,x_{\tau_k^-}\right)-\sum_{t_0<\tau_k\le\tau_l}\Phi(\tau_l,\tau_k)I\left(\tau_k,x_{\tau_k^-}\right)$$
$$=\sum_{t_0<\tau_k\le\tau_l}\left[\Phi(\tau_l+\eta,\tau_k)-\Phi(\tau_l,\tau_k)\right]I\left(\tau_k,x_{\tau_k^-}\right)\overset{\eta\to 0}{\longrightarrow}0$$

where we note that both sums have the same number of elements, due to $\tau_l+\eta<\tau_{l+1}$. Therefore for each $x\in\mathcal{S}$, we have that Px is right continuous and has left limits at impulse times, clearly it is continuous at nonimpulsive moments.

By definition of $\mathcal{S}$, we must show that $|(Px)(t)|\le L$ for every $t\ge 0$.
We remind the reader that $\|\phi\|\le\delta_0$, with δ_0 as defined in (15). We claim that $|(Px)(t)|\le L$ for all $t\ge t_0$. We have that, noticing that $|x(s)|\le L$ by definition of $\mathcal{S}$, so that the Lipschitz properties (ii) and (iii) hold, so that

$$\begin{aligned}|(Px)(t)|&\le\|\Phi(t,t_0)\||\phi(0)|+\int_{t_0}^{t}\|\Phi(t,s)\||g(s,x_s)|ds+\sum_{t_0<\tau_k\le t}\|\Phi(t,\tau_k)\||I\left(\tau_k,x_{\tau_k^-}\right)|\\&\le\delta_0\|\Phi(t,t_0)\|+\int_{t_0}^{t}b(s)\|\Phi(t,s)\|\|x_s\|ds+\sum_{t_0<\tau_k\le t}c(\tau_k)\|\Phi(t,\tau_k)\|\|x_{\tau_k^-}\|\\&\le\delta_0K+\sup_{\theta\in[t_0-r,t]}|x(\theta)|\left(\int_{t_0}^{t}b(s)\|\Phi(t,s)\|ds+\sum_{t_0<\tau_k\le t}c(\tau_k)\|\Phi(t,\tau_k)\|\right)\\&\le\delta_0K+\alpha L<L.\end{aligned}$$

Thus $|(Px)(t)|\le L$ for every $t\ge 0$.

By definition of $\mathcal{S}$, we have that $(Px)_{t_0}=\phi$. Now we show what $(Px)(t)\longrightarrow 0$ as $t\to\infty$.

For this, note that we can divide Px into

$$(Px)(t)=(P_1x)(t)+(P_2x)(t)$$

with

$$(P_1x)(t)=\Phi(t,t_0)\phi(0)+\sum_{t_0<\tau_k\le t}\Phi(t,\tau_k)I\left(\tau_k,x_{\tau_k^-}\right)$$

and

$$(P_2x)(t)=\int_{t_0}^{t}\Phi(t,s)g(s,x_s)ds.$$

By definition of $\mathcal{S}$, $x(t)\longrightarrow 0$ as $t\to\infty$. Thus we have that for any $\epsilon>0$ there exists $T_1>t_0$ such that

$$|x(t)|<\epsilon\qquad\text{for all }\ t\ge T_1.\tag{18}$$

By hypothesis (iv), given this ϵ and T_1, there exists $T_2>T_1$ such that $t\ge T_2$ implies

$$\begin{aligned}|g(t,x_t)|&\le b(t)\left(\epsilon+\|x\|^{[T_1,t]}\right)\\|I(t,x_{t^-})|&\le c(t)\left(\epsilon+\|x\|^{[T_1,t]}\right)\end{aligned}\tag{19}$$

Let us first analyze the term $(P_2x)(t)$. If $s\ge T_2>T_1$, by (18) we get

$$\|x\|^{[T_1,s]}<\epsilon.\tag{20}$$

By definition of $\mathcal{S}$, $\|x_t\|\le L$ for all $t\ge t_0$, $x\in\mathcal{S}$, and using the first inequality in (19) and inequality (20), we obtain that for $t>T_2$:

$$
\begin{aligned}
|(P_2x)(t)| &= \left|\int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds\right| \\
&\le \int_{t_0}^{T_2} |\Phi(t,s)g(s,x_s)|ds + \int_{T_2}^{t} |\Phi(t,s)g(s,x_s)|ds \\
&\le \int_{t_0}^{T_2} \|\Phi(t,s)\||g(s,x_s)|ds + \int_{T_2}^{t} \|\Phi(t,s)\||g(s,x_s)|ds \\
&\le \int_{t_0}^{T_2} b(s)\|\Phi(t,s)\|\|x_s\|ds + \int_{T_2}^{t} b(s)\|\Phi(t,s)\| \left(\epsilon + \|x\|^{[T_1,t]}\right) ds \qquad (21) \\
&\le L\int_{t_0}^{T_2} b(s)\|\Phi(t,s)\|ds + \int_{T_2}^{t} b(s)\|\Phi(t,s)\|(2\epsilon)ds \\
&= L\|\Phi(t,T_2)\| \int_{t_0}^{T_2} b(s)\|\Phi(T_2,s)\|ds + 2\epsilon \int_{T_2}^{t} b(s)\|\Phi(t,s)\|ds \\
&\le \alpha L\|\Phi(t,T_2)\| + 2\alpha\epsilon
\end{aligned}
$$

Since we have assumed that $\|\Phi(t,0)\| \longrightarrow \infty$ as $t \to \infty$, we see that given ϵ we can find $T > T_2$ such that

$$\alpha L\|\Phi(t,T_2)\| < \epsilon \quad \text{for} \quad t \ge T.$$

Substituting this last inequality into (21), we get that for $t > T$

$$|(P_2x)(t)| \le \epsilon + 2\alpha\epsilon = \epsilon(1+2\alpha)$$

This proves that $(P_2x)(t) \longrightarrow 0$ as $t \to \infty$. We now prove $(P_1x)(t) \longrightarrow 0$ as $t \to \infty$. It is similar to the way we proved this for P_2. Notice that using (19), (20) and (iv) we have that for $t > T_2$:

$$
\begin{aligned}
&\left|\sum_{t_0<\tau_k\le t} \Phi(t,\tau_k)I\left(\tau_k, x_{\tau_k^-}\right)\right| \\
\le& \sum_{t_0<\tau_k\le T_2} \|\Phi(t,\tau_k)\||I\left(\tau_k, x_{\tau_k^-}\right)| + \sum_{T_2<\tau_k\le t} \|\Phi(t,\tau_k)\||I\left(\tau_k, x_{\tau_k^-}\right)| \\
\le& \sum_{t_0<\tau_k\le T_2} c(\tau_k)\|\Phi(t,\tau_k)\|\|x_{\tau_k^-}\| + \sum_{T_2<\tau_k\le t} c(\tau_k)\|\Phi(t,\tau_k)\| \left(\epsilon + \|x\|^{[T_1,\tau_k]}\right) \\
=&\|\Phi(t,T_2)\| \sum_{t_0<\tau_k\le T_2} c(\tau_k)\|\Phi(T_2,\tau_k)\|\|x_{\tau_k^-}\| + \sum_{T_2<\tau_k\le t} c(\tau_k)\|\Phi(t,\tau_k)\| \left(\epsilon + \|x\|^{[T_1,\tau_k]}\right) \\
\le& L\|\Phi(t,T_2)\| \sum_{t_0<\tau_k\le T_2} c(\tau_k)\|\Phi(T_2,\tau_k)\| + 2\epsilon \sum_{T_2<\tau_k\le t} c(\tau_k)\|\Phi(t,\tau_k)\| \\
\le& \alpha L\|\Phi(t,T_2)\| + 2\alpha\epsilon
\end{aligned}
$$

In a similar way as we did for (P_2x), we can find some $T^* > T_2$, such that $t > T^*$ implies, adding the $\Phi(t,t_0)\phi(0)$ term, that

$$\|\Phi(t,t_0)\||\phi(0)| + \alpha L\|\Phi(t,T_2)\| < \epsilon.$$

This proves $(P_1x)(t) \longrightarrow 0$ as $t \to \infty$. Therefore choosing $\max\{T,T^*\}$ we have $(Px)(t) \longrightarrow 0$ as $t \to \infty$.

Thus $P : \mathcal{S} \longrightarrow \mathcal{S}$. We now prove that P is a contraction. For this, let $x, y \in \mathcal{S}$. By definition of $\mathcal{S}$ we have that $(Px)(t) - (Py)(t) = 0$ for $t \in [t_0 - r, t_0]$. For $t \ge t_0$ we get:

$$|(Px)(t) - (Py)(t)| =$$

$$\left| \int_{t_0}^{t} \Phi(t,s)[g(s,x_s) - g(s,y_s)]ds + \sum_{t_0<\tau_k\le t} \Phi(t,\tau_k) \left[I\left(\tau_k, x_{\tau_k^-}\right) - I\left(\tau_k, y_{\tau_k^-}\right) \right] \right|$$
$$\le \int_{t_0}^{t} b(s)\|\Phi(t,s)\|\|x_s - y_s\|ds + \sum_{t_0<\tau_k\le t} c(\tau_k)\|\Phi(t,\tau_k)\|\|x_{\tau_k^-} - y_{\tau_k^-}\|$$
$$\le d_{\mathcal{S}}(x,y)\left(\int_{t_0}^{t} b(s)\|\Phi(t,s)\|ds + \sum_{t_0<\tau_k\le t} c(\tau_k)\|\Phi(t,\tau_k)\| \right) \le \alpha d_{\mathcal{S}}(x,y)$$

where recall that $d_{\mathcal{S}}(x,y) = \sup_{s\in[t_0,\infty)} |x(s) - y(s)|$.

Thus P is a contraction on $\mathcal{S}$. Therefore, by the Banach fixed point theorem, we have found a solution to the impulsive delayed system. Moreover, the solution found through the Banach contraction principle is the unique solution to (1) with initial condition (2).[3]

To prove uniform stability, assume that we are given an $\epsilon > 0$. Choose $\delta < \epsilon$ such that $\delta K + \alpha\epsilon < \epsilon$, in other words, $\delta < \min\{\epsilon, (1-\alpha)\epsilon/K\}$. Notice that K as defined in (i) is independent of t_0, thus so is δ. This will give us uniform stability.

For $\|\phi\| \le \delta$, we claim that $|x(t)| \le \epsilon$ for all $t \ge t_0$. Note that if x is the unique solution corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < \epsilon$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > \epsilon$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > \epsilon\}.$$

By right continuity, either $|x(t^*)| = \epsilon$ if there is no impulsive moment at t^*, or $|x(t^*)| \ge \epsilon$ as a consequence of a jump at t^*. Whatever the case, we have $|x(s)| \le \epsilon$ for $s \in [t_0 - r, t^*)$, where $|x(t^*)| = \epsilon$ if this occurs at a non-impulsive moment. By the integral representation of $x(t)$, we have that

$$|x(t^*)| \le \|\Phi(t^*,t_0)\||\phi(0)| + \int_{t_0}^{t^*} \|\Phi(t^*,s)\||g(s,x_s)|ds + \sum_{t_0<\tau_k\le t^*} \|\Phi(t^*,\tau_k)\||I\left(\tau_k, x_{\tau_k^-}\right)|$$
$$\le \delta\|\Phi(t^*,t_0)\| + \int_{t_0}^{t^*} b(s)\|\Phi(t^*,s)\|\|x_s\|ds + \sum_{t_0<\tau_k\le t^*} c(\tau_k)\|\Phi(t^*,\tau_k)\|\|x_{\tau_k^-}\|$$
$$\le \delta K + \sup_{\theta\in[t_0-r,t^*)} |x(\theta)| \left(\int_{t_0}^{t^*} b(s)\|\Phi(t^*,s)\|ds + \sum_{t_0<\tau_k\le t^*} c(\tau_k)\|\Phi(t^*,\tau_k)\| \right)$$
$$\le \delta K + \alpha\epsilon < \epsilon$$

and this gives us the desired contradiction, by the definition of t^*, where $|x(t^*)| \ge \epsilon$. Therefore the solution is uniformly stable, and since $x(t)$ converges to zero as $t \to \infty$, we get uniform stability and asymptotic stability of trajectories. □

Remark 3.2. Perhaps condition (14) in Theorem 3.1 may seem somewhat contrived and difficult to identify in a system. Nonetheless, this aforementioned condition is what R. D. Driver in [11] comments about infinitely delayed FDEs requiring *fading memory conditions* in order to achieve asymptotic stability. Also, B. Zhang [39] on page 5 denotes this type of requirement as a fading memory condition. In an earlier work by Seifert [33] it is pointed out that some sort of decaying condition is required for the asymptotic stability of a general delay equation. For a physical system this can be interpreted as a system remembering its past (through the delay), but the influence of the past as time increases should diminish, which can be interpreted as "the memory fades with time". Intuitively, for *finitely delayed* systems, or bounded delays, a fading memory condition such as (14) should be satisfied, since after a finite time length, in this case, the maximum bound on the delay, the information from the past is left out. We quickly prove this in the following lemma.

[3]See the comments on uniqueness in Section 4.

Lemma 3.1. *Under the conditions of Theorem 3.1, if the delay $r < \infty$ is finite, then conditions* (ii) *and* (iii) *of Theorem 3.1 imply condition (v), namely, the fading memory requirement* (14) *is satisfied for finite delays.*

Proof. Let $\epsilon > 0$ and T_1 be given. Then if $T_2 = T_1 + r$ (which is finite, so well defined) then for any $t \geq T_2$, condition (ii) along with $g(t,0) = 0$ implies

$$\begin{aligned}|g(t,x_t)| \leq b(t)\|x_t\| &= b(t)\Big(\sup_{s\in[-r,0]}|x_t(s)|\Big)\\ &= b(t)\Big(\sup_{s\in[t-r,t]}|x(s)|\Big)\\ &\leq b(t)\Big(\sup_{s\in[T_2-r,t]}|x(s)|\Big) = b(t)\|x\|^{[T_1,t]}\end{aligned}$$

□

Thus fading memory conditions on systems are not so rare, and in fact we can easily find large groups of systems that satisfy this requirement. In this manner, finite delays are included in the following main result, with conditions in (v) being redundant for finitely delayed systems.

Remark 3.3. Notice that the fact that the solutions of the impulsive FDE remain bounded by L, is independent of the contraction mapping being restricted to $\mathcal{S}$. It is a property that depends solely on the variation or parameters formula, which necessarily any solution satisfies. This can be seen similar to the way we proved stability. When proving that $|(Px)(t)| \leq L$ above, we did assume that $|x(t)| \leq L$ for all t and $x \in \mathcal{S}$ so that we could apply the Lipschitz conditions (ii) and (iii), but we can still modify this.

Lemma 3.2. *Under the hypotheses stated in Theorem 3.1, we have that if* $\sup_{s_2\geq s_1}(\|\Phi(s_2,s_1)\|) \leq K < \infty$ *then the solutions of* (1) *with initial condition* $\|\phi\| \leq \delta_0 < \dfrac{(1-\alpha)}{K}L$ *remain bounded*[4] *by L, i.e., $|x(t)| < L$ for every t where x is defined .*

Proof. The proof is completely similar to the way in which we prove stability of the solution in Theorem 3.1, with the role of ϵ played by L this time.

For $\|\phi\| \leq \delta_0$, we claim that the solution $x(t)$ satisfies $|x(t)| < L$ for all $t \geq t_0$. Note that if x solves the impulsive FDE corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < L$ (remember that there is no impulsive moment at t_0). For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| \geq L$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| \geq L\}.$$

We have $|x(s)| < L$ for $s \in [t_0 - r, t^*)$. By the integral representation of $x(t)$, which all solutions to (1) satisfy with initial condition ϕ, we have that, since before t^* the paths are bounded by L, we can apply the Lipschitz conditions (ii) and (iii), so that

$$\begin{aligned}|x(t^*)| &\leq \|\Phi(t^*,t_0)\||\phi(0)| + \int_{t_0}^{t^*}\|\Phi(t^*,s)\||g(s,x_s)|ds + \sum_{t_0<\tau_k\leq t^*}\|\Phi(t^*,\tau_k)\||I\left(\tau_k, x_{\tau_k^-}\right)|\\ &\leq \delta_0\|\Phi(t^*,t_0)\| + \int_{t_0}^{t^*} b(s)\|\Phi(t^*,s)\|\|x_s\|ds + \sum_{t_0<\tau_k\leq t^*} c(\tau_k)\|\Phi(t^*,\tau_k)\|\|x_{\tau_k^-}\|\\ &\leq \delta_0 K + \sup_{\theta\in[t_0-r,t^*)}|x(\theta)|\left(\int_{t_0}^{t^*} b(s)\|\Phi(t^*,s)\|ds + \sum_{t_0<\tau_k\leq t^*} c(\tau_k)\|\Phi(t^*,\tau_k)\|\right)\\ &\leq \delta_0 K + \alpha L < L\end{aligned}$$

[4]Note that $\dfrac{(1-\alpha)}{K}L < L$, so that $\delta_0 < L$.

and this gives us the desired contradiction, since we have proved $|x(t^*)| < L$, and we assumed $|x(t^*)| = L$ if t^* is a continuity point, or $|x(t^*)| \geq L$ if t^* is a discontinuity point. □

Remark 3.4. We could have replaced L in the previous proof, by an $\epsilon < L$, so that even stability is independent of the Banach contraction principle. What we obtain through the fixed point method in the proof of Theorem 3.1 above is, the asymptotic convergence to zero of trajectories. We address this further in the next section.

4 An Observation on Uniqueness

The Banach Contraction Principle, when applied to a mapping $P : \mathcal{S} \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is a complete metric space, gives a unique solution *only within* $\mathcal{S}$, the space *where the mapping is restricted to.* The space $\mathcal{S}$ used in the proof of Theorem 3.1 is such that $\mathcal{S} \subset PCB\big([t_0-r,\infty), D\big)$ (*strict inclusion of sets*), so one might argue that there might be a solution to (1) with initial condition ϕ, for instance, $x_2(t;t_0,\phi) \in PCB\big([t_0-r,\infty), D\big)\setminus\mathcal{S}$, say, that does not converge to zero. Now, by definition, when speaking of "uniqueness", one must take note of *where is this uniqueness statement being held.* For impulsive FDEs, the definition of what a solution to an impulsive delayed system is, uses PCB-spaces, this can be seen in the theoretical background in of [23, 26], where solutions must be unique within the respective PCB-space, the space from which solution trajectories are taken to be. Further examination of definitions of what uniqueness means, point out that we must be specific about uniqueness. We do not ask for uniqueness in an L^p-space, for instance, as in Carátheodory solutions, since this space is too big. And uniqueness within $\mathcal{S} \subset PCB\big([t_0-r,\infty), D\big)$ (*strict inclusion*) is obviously not satisfactory, because this space is too small to be useful to guarantee uniqueness in a PCB-space. Thus we see here a caveat about what uniqueness by the Banach fixed point theorem really means, when applied to a subset of a PCB-space, such as $\mathcal{S}$ defined above in the proof. One must be careful in this sense.

Remark 4.1. The same argument applies to previous results in [9, 39], since they similarly define a complete metric space strictly contained in a space of bounded continuous mappings. In classical theory of delayed functional differential equations, uniqueness is defined within the space of bounded continuous mappings, not within the subset of functions that also converge to zero. The Banach fixed point theorem, as applied in the aforementioned papers, guarantees uniqueness within the latter space, not within the former, where uniqueness is required. Nevertheless, said vector fields in these papers mentioned satisfy classical existence-uniqueness theory such as that of [11]. Therefore, the Banach fixed point method for stability is useful specifically for obtaining the asymptotic convergence to zero of solution trajectories, through the use of the complete metric space implicit in this method. The solution obtained through the Banach contraction principle is unique by agreeing with the unique solution obtained through general existence-uniqueness theory.

We can argue that the hypotheses supposed on the vector field are sufficient to establish uniqueness by other uniqueness results, such as that in a known result given in [26]. For the purpose of this, let us introduce terminology from the existence-uniqueness theory developed in [26].

Definition 4.1. (Locally Lipschitz) A functional $f : J \times PCB([-r,0], D) \longrightarrow \mathbb{R}^n$ is said to be locally Lipschitz in its second variable if for each $t_0 \in J$ and $\beta > 0$ such that $[t_0, t_0+\beta] \subset J$, and for each compact set $F \subset D$ there exists some constant $L = L(t_0,\beta,F) > 0$ such that $|f(t,\psi_1) - f(t,\psi_2)| \leq L\|\psi_1 - \psi_2\|_r$ for all $t \in [t_0, t_0+\beta]$ and $\psi_1, \psi_2 \in PCB([-r,0], F)$.

Now we can apply standard existence-uniqueness theory, as follows:

Proposition 4.1. *Under the hypotheses of Theorem 3.1, then the solution to the IVP induced by* (1) *with initial condition* ϕ *is unique, if* $\|\phi\| \leq \delta_0 < \dfrac{(1-\alpha)}{K}L =: \delta_{L,K}$, *and solutions remain bounded by* L.

Proof. If $L < \infty$, by hypothesis we have a local Lipschitz condition according to Definition 4.1, with $B_L(0) =: D$, the Euclidean open ball of size L. This is because any closed subset F of $B_L(0)$ would give us a compact subset as required in the definition of a local Lipschitz condition. If t is in a compact set, then $b(t)$ is bounded and gives us the necessary Lipschitz constants for the use of Definition 4.1. Since we assumed $g(s, x_s)$ is composite-PCB, we are actually satisfying the hypotheses required in the *local* existence-uniqueness result of [26]. This guarantees uniqueness of solutions, even for infinite delay. As shown in Lemma 3.2, using the variation of parameter formula with the bounds stated in Theorem 3.1, solutions with initial conditions $\|\phi\| \leq \delta_0$, cannot leave a ball of size L, on their maximal interval of existence. If $L = \infty$, we have a global Lipschitz condition, and the previous analysis holds, taking compact subsets $F \subset \mathbb{R}^n$, to satisfy Definition 4.1 of a local Lipschitz condition. □

We thus have local existence and uniqueness by the previous result. The result in Lemma 3.2 guarantees us that, given the differential equation (1), the solution $x(t)$ with initial condition ϕ satisfying

$$\|\phi\| \leq \delta_0 < \frac{(1-\alpha)}{K}L =: \delta_{L,K},$$

will remain in a ball of size L. The Banach fixed point theorem guarantees that the solutions are asymptotically stable. The solution found by the contraction mapping principle is unique in a satisfactory way, and whatever we achieve through the contraction method, must hold for each unique solution.

Thus $\delta_{L,K}$ clearly gives an upper threshold on the initial conditions for an initial value problem. Below the upper threshold, we can guarantee the conclusions of Theorem 3.1. The additional information that we are obtaining from using the contraction mapping is the asymptotic stability of the unique solutions to each initial value problem.

5 Particular Cases of Main Result

Notice that the condition

$$\int_0^t b(s)\|\Phi(t,s)\|ds + \sum_{0<\tau_k \leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \leq \alpha < 1 \tag{22}$$

is not easy to evaluate, unless we know some bounds. For the scalar case, let us concentrate on guaranteeing

$$\sum_{0<\tau_k \leq t} c(\tau_k) e^{-\int_{\tau_k}^t a(u)du} \leq \frac{\alpha}{2} \tag{23}$$

for a given $\alpha < 1$. We already know, from examples in [39], how to make the first contribution in (22) from the integral less than $\alpha/2$, by a simple rescaling by the $1/2$ factor. Notice that if $t \in [\tau_{n-1}, \tau_n)$, for $n \geq 2$ (since for $n = 1$, $t \in [\tau_0, \tau_1)$, so no jumps have occurred, we do not even need to worry about this contribution at $n = 1$) we have that[5]

$$\sum_{0<\tau_k\le t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du}$$
$$= c(\tau_1)e^{-\int_{\tau_1}^t a(u)du} + c(\tau_2)e^{-\int_{\tau_2}^t a(u)du} + \cdots + c(\tau_{n-2})e^{-\int_{\tau_{n-2}}^t a(u)du} + c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du}$$
$$= e^{-\int_{\tau_{n-1}}^t a(u)du} \sum_{m=1}^{n-2} c(\tau_{n-1-m}) \left(\prod_{j=1}^{m} e^{-\int_{\tau_{n-1-j}}^{\tau_{n-j}} a(u)du} \right) + c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du}, \tag{24}$$

where we have used that for each m:

$$e^{-\int_{\tau_{n-1}}^t a(u)du} \left(\prod_{j=1}^{m} e^{-\int_{\tau_{n-1-j}}^{\tau_{n-j}} a(u)du} \right) = e^{-\int_{\tau_{n-1-m}}^t a(u)du}.$$

Suppose that we allow sufficient time between jumps so that the "good" behavior of $a(t)$ dominates on each continuous subinterval so that $e^{-\int_{\tau_j}^{\tau_{j+1}} a(u)du} \le \beta < \frac{1}{2}$. Now, notice how we always obtain a left over term

$$+c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du}, \tag{25}$$

and that $e^{-\int_{\tau_{n-1}}^t a(u)du}$ might be relatively large, at least not smaller than β, for example, if $a(u)$ is negative at the beginning of the impulse at τ_{n-1}. Maybe there still has not been enough time for the good behavior of $a(u)$ to have the good effects that allow for asymptotic stability. Suppose that the worst that can happen is captured as $e^{-\int_{s_1}^{s_2} a(u)du} \le K$ for every $s_1 \le s_2 \in [0, \infty)$.

Remark 5.1. Notice that $K \ge 1$, since $e^{-\int_{s_2}^{s_2} a(u)du} = 1$. In case that $a(u) \ge 0$ always, then $K = 1$ automatically.

Thus we have that if, say, $c(\tau_m) \le \frac{\alpha}{4K}$, and $\beta < \frac{1}{2}$, then

$$\begin{aligned} &e^{-\int_{\tau_{n-1}}^t a(u)du} \sum_{m=1}^{n-2} c(\tau_{n-1-m}) \left(\prod_{j=1}^{m} e^{-\int_{\tau_{n-1-j}}^{\tau_{n-j}} a(u)du} \right) + c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du} \\ &\le K \sum_{m=1}^{n-2} c(\tau_{n-1-m})\beta^m + Kc(\tau_{n-1}) \\ &\le \frac{\alpha}{4} \sum_{m=1}^{\infty} \beta^m + \frac{\alpha}{4} \\ &\le \frac{\alpha}{4} \frac{\beta}{1-\beta} + \frac{\alpha}{4} < \frac{\alpha}{2} \end{aligned} \tag{26}$$

where we have used that $\frac{\beta}{1-\beta} < 1$ because $\beta < \frac{1}{2}$. So we have shown that

$$\sum_{0<\tau_k\le t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du} \le \frac{\alpha}{2},$$

as long as the intervals $[\tau_{k-1}, \tau_k)$ between jumps allow sufficient time for $e^{-\int_{\tau_{k-1}}^{\tau_k} a(u)du} \le \beta < \frac{1}{2}$, the condition $e^{-\int_{s_1}^{s_2} a(u)du} \le K$ holds for every $s_1 \le s_2 \in [0, \infty)$ and the Lipschitz weighting function of the jumps satisfies $c(\tau_m) \le \frac{\alpha}{4K}$ for all $m \ge 1$.

[5]For $n = 2$, we use as notational convention $\sum_{m=1}^{0} (\cdot) = 0$, so that only the term $c(\tau_1)e^{-\int_{\tau_1}^t a(u)du}$ is left for this special case.

Remark 5.2. Through a similar analysis to the one we did in the previous chapter for continuous delayed functions, we can have an idea of how to calculate the maximum bound K. As can be remembered from the previous chapter, such as in Example 3.1 and ensuing remarks there, a good candidate to finding K to obtain a uniform bound in t_0 is to look for the longest interval where $a(t)$ is negative.

The previous example motivates the following corollary, which could serve as a criterion to determine if the hypotheses of Theorem 3.1 hold. Of course, different criteria can be obtained, this is just one of many possible that give sufficient conditions for the application of Theorem 3.1.

Corollary 5.1. *Suppose that the conditions of Theorem 3.1 hold, except that instead of condition* (13), *we have that there exists an* $\alpha \in (0,1)$ *such that*

$$\sup_{t\geq 0}\left(\int_0^t b(s)\|\Phi(t,s)\|ds\right) \leq \frac{\alpha}{2} \tag{27}$$

and the following conditions hold. The intervals $[\tau_{k-1},\tau_k)$ *between impulses satisfy that for every* $k \geq 1$

$$\|\Phi(\tau_k,\tau_{k-1})\| \leq \beta < \frac{1}{2}, \tag{28}$$

$\|\Phi(s_2,s_1)\| \leq K$ *holds for every* $s_1 \leq s_2 \in [0,\infty)$, *and the Lipschitz weighting function of the impulses satisfies* $c(\tau_m) \leq \frac{\alpha}{4K}$ *for all* $m \geq 1$. *Then the trivial solution of* (1) *is uniformly stable and asymptotically stable.*

Proof. We just need to prove that the hypotheses of this proposition imply that

$$\sum_{0<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \leq \frac{\alpha}{2},$$

so that, along with (27) we have that condition (13) of Theorem 3.1 holds. If $t \in [\tau_{n-1},\tau_n)$, then

$$\begin{aligned}
&\sum_{0<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \\
&\quad= \sum_{m=1}^{n-2} c(\tau_{n-1-m})\left\|\Phi(t,\tau_{n-1})\left(\prod_{j=1}^{m}\Phi(\tau_{n-j},\tau_{n-1-j})\right)\right\| + c(\tau_{n-1})\|\Phi(t,\tau_{n-1})\| \\
&\quad\leq \|\Phi(t,\tau_{n-1}\|\sum_{m=1}^{n-2} c(\tau_{n-1-m})\left(\prod_{j=1}^{m}\|\Phi(\tau_{n-j},\tau_{n-1-j})\|\right) + c(\tau_{n-1})\|\Phi(t,\tau_{n-1})\| \\
&\quad\leq K\sum_{m=1}^{n-2} c(\tau_{n-1-m})\beta^m + Kc(\tau_{n-1}) \\
&\quad\leq \frac{\alpha}{4}\sum_{m=1}^{\infty}\beta^m + \frac{\alpha}{4} \leq \frac{\alpha}{4}\frac{\beta}{1-\beta} + \frac{\alpha}{4} < \frac{\alpha}{2}.
\end{aligned}$$

The rest follows from the main result, Theorem 3.1.

□

Example 5.1. Let us have

$$\begin{aligned}
x'(t) &= -a(t)x(t) + q(t)x^2(t-r_1(t)) && t \neq \tau_k, t \geq 0 \\
\Delta x(t) &= \frac{1}{6}x_{t^-}(-r_2(t)), && t = \tau_k
\end{aligned} \tag{29}$$

with $a(t), q(t)$ continuous real-valued functions on $\mathbb{R}^+$, $t - r_i(t) \longrightarrow \infty$ as $t \to \infty$, for $i = 1, 2$, $0 \le r_i(t) \le r \le \infty$. Remember x_{t^-} as defined in (4) above. Suppose we are interested in a ball of size L around the origin, where we will specify certain conditions on L. We will see that the larger L is, the more difficult it is to obtain stability conclusions. The point of this example is to understand the role of L and how it can be chosen for certain cases. Let the following conditions hold:

(i) $a(t) \ge c > 0$ for c constant, for all $t \ge 0$.

(ii) The impulsive moments satisfy $\min_{k\ge1}\{\tau_k - \tau_{k-1}\} > \frac{\ln(2)}{c}$.

(iii) There is a $2 < J \le 3$ such that $a(t) \ge 2LJ|q(t)|$ for all $t \ge 0$.[6]

Then the origin is uniformly stable and asymptotically stable.

Proof. We verify that the hypotheses of the scalar version of Corollary 5.1 are satisfied. We have $g(t, x_t) := q(t)x^2(t - r_1(t))$, and so, if $\|\phi\|_r, \|\psi\|_r \le L$, we have that

$$\begin{aligned}|g(t,\phi) - g(t,\psi)| &= |q(t)||\phi^2(-r_1(t)) - \psi^2(-r_1(t))|\\ &\le 2L|q(t)||\phi(-r_1(t)) - \psi(-r_1(t))|\\ &\le 2L|q(t)|\|\phi - \psi\|_r,\end{aligned}$$

so that if $b(t) = 2L|q(t)|$, we have the necessary weighted Lipschitz-type condition. Clearly the impulsive operator is linear in x_t, so we have a Lipschitz condition as well, with $c(t) = \frac{1}{6}$. The fading memory conditions (14) are satisfied, since divergence to infinity of $t - r_1(t)$ implies that given $\epsilon > 0$ and $T_1 \ge 0$, there exists $T_2^* > T_1$ such that $t - r_1(t) \ge T_1$ for all $t \ge T_2^*$. Given that $r_1(t) \ge 0$, this implies that for T_2^* as defined, it is true that $t - r_1(t) \in [T_1, t]$ for every $t \ge T_2^*$. Putting together the information we have so far, we have that given $\epsilon > 0$ and $T_1 \ge 0$, it is true that there exists a $T_2^* > T_1$ such that using $\|x\|^{[T_1,t]} = \sup_{\theta\in[T_1,t]} |x(\theta)|$:

$$|x_t(-r_1(t))| = |x(t - r_1(t))| \le \|x\|^{[T_1,t]} \le \epsilon + \|x\|^{[T_1,t]} \quad \text{for } t \ge T_2^*,$$

because $t - r_1(t) \in [T_1, t]$ for every $t \ge T_2^*$. Thus, if $t \ge T_2^*$

$$|g(t, x_t)| \le b(t)|x_t(-r_1(t)) \le b(t)\Big(\epsilon + \|x\|^{[T_1,t]}\Big).$$

Similarly, for the impulsive operator, there exists, given the same $\epsilon > 0$, T_1, a $\hat{T}_2 > T_1$, such that using $r_2(t)$ this time,

$$|x_{t^-}(-r_2(t))| \le \|x\|^{[T_1,t]} \le \epsilon + \|x\|^{[T_1,t]} \quad \text{for } t \ge \hat{T}_2,$$

because $t - r_2(t) \in [T_1, t]$ for every $t \ge \hat{T}_2$. And this gives similarly the fading memory condition for $I(t, x_{t^-}) = \frac{1}{6}x_{t^-}(-r_2(t))$. Let $T_2 = \max\{T_2^*, \hat{T}_2\}$, to obtain condition (14). Clearly $e^{-\int_0^t a(s)ds} \longrightarrow 0$ as $t \to \infty$, because of (i). We must only verify condition (13), using Corollary 5.1. By (iii), we have

$$\begin{aligned}\int_0^t e^{-\int_s^t a(u)du} b(s)ds = \int_0^t e^{-\int_s^t a(u)du} 2L|q(s)|ds &\le \frac{1}{J}\int_0^t e^{-\int_s^t a(u)du} a(s)ds\\ &= \frac{1}{J}e^{-\int_s^t a(u)du}\Big|_{s=0}^{s=t} = \frac{1}{J}\left(1 - e^{-\int_0^t a(u)du}\right).\end{aligned}$$

Thus $\sup_{t\ge0}\left\{\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds\right\} \le \frac{1}{J} = \frac{\alpha}{2}$, with $\alpha = \frac{2}{J} < 1$, and condition (27) of Corollary 5.1 is satisfied. We have $K = 1$, since $a(t) \ge 0$, and $c(t) = \frac{1}{6} \le \frac{1}{2J} = \frac{\alpha}{4K}$, because $J \le 3$ For condition (28), we have that using (i), (ii), we get that for every k:

$$e^{-\int_{\tau_{k-1}}^{\tau_k} a(s)ds} \le e^{-c(\tau_k - \tau_{k-1})} \le \sup_k\{e^{-c(\tau_k - \tau_{k-1})}\} = \beta < \frac{1}{2}.$$

[6] This is how we realize that the larger L is, the harder it is to satisfy this condition.

Thus all of the conditions of Corollary 5.1 are satisfied, so that we have uniform stability, and asymptotic stability of the trivial solution. Notice that by (15), the initial conditions must be less than $\delta_0 < L\frac{1-\alpha}{K} = L\frac{J-2}{J}$.

□

Example 5.2. Let us have in the previous Example 5.1, $a(t) = t + \frac{1}{2}$, $q(t) = \frac{t}{16}$, $r_1(t) = \frac{t}{2}, r_2(t) = \frac{t}{3}$. Thus we have the impulsive delayed differential equation

$$\begin{aligned} x'(t) &= -\left(t+\frac{1}{2}\right)x(t) + \frac{t}{16}x^2\left(\frac{t}{2}\right) \qquad && t \neq \tau_k, t \geq 0 \\ \Delta x(t) &= \frac{1}{6}x\left(\frac{2t}{3}\right), && t = \tau_k = 2k, \quad k \in \mathbb{N} = \{1,2,3,..\} \end{aligned} \tag{30}$$

We take that we are interested in a ball of radius $L = 3$ around the origin. Here, we see from the definition of $a(t)$, $c = 1/2$. We can directly verify the hypotheses of the main result, or use the previous nonlinear general model of (29), with $q(t) = \frac{t}{16}$, with $J = \frac{5}{2}$. Notice that $\alpha = \frac{2}{J} = \frac{4}{5}$, with this value of J. We need initial functions ϕ less than $\delta_0 < \frac{L(1-\alpha)}{K} = \frac{3}{5}$ in norm. Also, notice that we have $\min_k\{\tau_k - \tau_{k-1}\} = 2 \geq \frac{\ln(2)}{c} = 2\ln(2)$. Numerical simulation is illustrated in Figure 1.

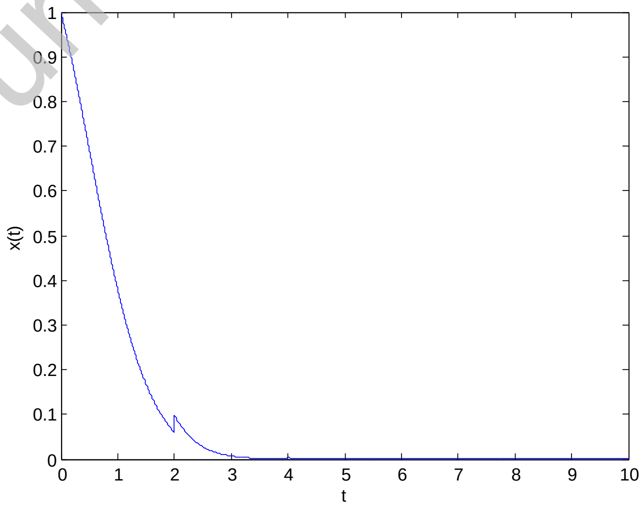


Figure 1: Simulation result for Example 5.2. with $a(t) = t+\frac{1}{2}$, $q(t) = \frac{t}{16}$, $r_1(t) = \frac{t}{2}, r_2(t) = \frac{t}{3}$.

A Particular Linear Impulsive System

Suppose that we have the following particular linear case of (1):

$$\begin{aligned} x'(t) &= A(t)x(t) + M(t)x(t-r(t)), \qquad && t \neq \tau_k, t \geq 0 \\ \Delta x(t) &= I(t)x(t^-), && t = \tau_k, t \geq 0 \end{aligned} \tag{31}$$

with $M(t)$ a continuous time-varying matrix of dimension $n \times n$.

Corollary 5.2. *Suppose that in the linear FDE* (31)*, $t - r(t) \longrightarrow \infty$ as $t \to \infty$, that there exists a positive constant α, and a continuous function $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the following conditions hold:*

(i) For all $s_2 \geq s_1 \in [0, \infty)$, let us have on the fundamental matrix Φ of the time-varying system (8) *induced by $A(t)$, the uniform bound $\|\Phi(s_2, s_1)\| \leq K < \infty$, in other words let $\sup_{s_2 \geq s_1 \geq 0} (\|\Phi(s_2, s_1)\|) \leq K < \infty$*

(ii) $M(t)$ has its operator norm bounded $\|M(t)\| \leq b(t)$, for all $t \geq 0$, and similarly $\|I(t)\| \leq c(t)$, for all $t \geq 0$,[7] such that for all $t \geq 0$:

$$\int_0^t b(s)\|\Phi(t,s)\|ds + \sum_{0<\tau_k \leq t} c(\tau_k)\|\Phi(t, \tau_k)\| \leq \alpha < 1. \tag{32}$$

(iii) $\|\Phi(t, 0)\| \longrightarrow 0$ as $t \to \infty$.

Then zero solution of (31) *is uniformly stable and asymptotically stable, with initial condition arbitrarily large, so that the stability conclusions are global.*

Proof. Notice that the δ_0 in (15) depends on L proportionally, and L is where the Lipschitz conditions $(ii), (iii)$ hold. But in this case, we do not have a nonlinearity that forces a local Lipschitz condition, so L can be arbitrarily large. Thus asymptotic convergence holds, no matter how large the initial condition is.

We now just need to prove that the fading memory condition holds in case of infinite delay. For finite delays, if $0 \leq r(t) \leq r$ then $t - r(t) \longrightarrow \infty$. The proof that condition (14) holds is similar to how we did in Lemma 3.1, as we illustrate: By hypothesis, we have that $t - r(t) \longrightarrow \infty$ as $t \to \infty$. This divergence to infinity implies that given $\epsilon > 0$ and $T_1 \geq 0$, there exists $T_2 > T_1$ such that $t - r(t) \geq T_1$ for all $t \geq T_2$. Given that $r(t) \geq 0$, this implies that for T_2 as defined, it is true that $t - r(t) \in [T_1, t]$ for every $t \geq T_2$. Putting together the information we have so far, we have that given $\epsilon > 0$ and $T_1 \geq 0$, it is true that there exists a $T_2 > T_1$ such that using $\|x\|^{[T_1,t]} = \sup_{\theta \in [T_1,t]} |x(\theta)|$:

$$|x_t(-r(t))| = |x(t - r(t))| \leq \|x\|^{[T_1,t]} \leq \epsilon + \|x\|^{[T_1,t]} \quad \text{for } t \geq T_2,$$

because $t - r(t) \in [T_1, t]$ for every $t \geq T_2$. Thus

$$\begin{aligned} |g(t, x_t)| \leq \|M(t)\||x_t(-r(t))| &\leq |b(t)||x_t(-r(t))| \\ &\leq |b(t)|\Big(\epsilon + \|x\|^{[T_1,t]}\Big). \end{aligned}$$

□

Example 5.3. Suppose we have

[7] Notice that $b(t) := \|M(t)\|$, $c(t) := \|I(t)\|$ also work, but perhaps knowing this exactly is too difficult, so using matrix bounds one can settle for an upper estimate.

$$
\begin{aligned}
x'(t) &= -a(t)x(t) + b(t)x(t/6), && t \geq 0, \quad t \neq \tau_k = 3 + \frac{k-1}{2} \\
\Delta x(t) &= \frac{1}{20}x(t^-), && t = \tau_k = 3 + \frac{k-1}{2}, k \in \mathbb{N},
\end{aligned} \tag{33}
$$

as well as $\tau_0 = 0$, with

$$
a(t) = \begin{cases} -1 & \text{if } t \in [0,1] \\ -1 + 2(t-1) & \text{if } t \geq 1. \end{cases}
$$

$$
b(t) = \begin{cases} \frac{1}{25} & \text{if } t \in \left[0, \frac{39}{25}\right] \\ \frac{1}{25} + \frac{1}{2}\left(t - \frac{39}{25}\right) & \text{if } t \geq \frac{39}{25}. \end{cases}
$$

Let us define $t_1 = \frac{39}{25}$, to shorten notation. Notice that for $t \in [0, t_1]$, since $a(t) \geq -1$:

$$
\int_0^t e^{-\int_s^t a(u)du} b(s)ds \leq \int_0^t e^{\int_s^t du} b(s)ds \leq \frac{1}{25}(e^{t_1} - 1) \quad t \in [0, t_1].
$$

Now, for $t \geq t_1$, notice that at t_1, $\frac{3}{25} = a(t) = 3b(t)$. Afterwards, it is true that for $t \geq t_1$, $a(t) \geq 3b(t)$. Therefore, the following inequality holds, for $t \geq t_1$:

$$
\begin{aligned}
\int_0^t e^{-\int_s^t a(u)du} b(s)ds &\leq \frac{1}{25}(e^{t_1} - 1) + \int_{t_1}^t e^{-\int_s^t a(u)du} b(s)ds \\
&\leq \frac{1}{25}(e^{t_1} - 1) + \int_{t_1}^t e^{-\int_s^t a(u)du} \frac{a(s)}{3} ds \\
&= \frac{1}{25}(e^{t_1} - 1) + \frac{1}{3}\left(1 - e^{-\int_{t_1}^t a(u)du}\right) \qquad (t \geq t_1) \\
&\leq \frac{1}{25}(e^{t_1} - 1) + \frac{1}{3} =: \frac{\alpha}{2},
\end{aligned}
$$

with $\alpha = 2\left(\frac{1}{25}(e^{t_1} - 1) + \frac{1}{3}\right) \approx 0.96737 < 1$, since $t_1 = \frac{39}{25}$. We will use Corollary 5.1 to verify the hypothesis

$$
\int_0^t b(s)\|\Phi(t,s)\|ds + \sum_{0 < \tau_k \leq t} c(\tau_k)\|\Phi(t, \tau_k)\| \leq \alpha < 1
$$

in order to apply Corollary 5.2. We already have, from the previous calculation,

$$
\sup_{t \geq 0}\left(\int_0^t b(s)\|\Phi(t,s)\|ds\right) = \sup_{t \geq 0}\left(\int_0^t e^{-\int_s^t a(u)du} b(s)ds\right) \leq \frac{\alpha}{2}.
$$

Notice that the negative contribution of $a(t)$ on the interval $\left[0, \frac{3}{2}\right]$ makes the value of the constant K bigger, where K such that $K \geq \sup_{s_2 \geq s_1 \geq 0}\{e^{-\int_{s_1}^{s_2} a(s)ds}\}$. This is because, if $a(t) \geq 0$ always, then $e^{-\int_{s_1}^{s_2} a(s)ds} \leq 1$ for all $s_2 \geq s_1$, since the integral $\int_{s_1}^{s_2} a(s)ds$ is nonnegative. Therefore $K = 1$ is good enough for nonnegative functions $a(t)$. In case $a(t) < 0$, as happens here, a good candidate to determine K is to search for the largest interval where $a(t)$ is negative. This largest interval is $\left[0, \frac{3}{2}\right]$. We have that

$$
\int_0^{3/2} a(s)ds = \int_0^1 (-1)ds + \int_1^{3/2} [-1 + 2(s-1)]ds = -\frac{5}{4}.
$$

so that

$$
\sup_{s_2 \geq s_1 \geq 0}\{e^{-\int_{s_1}^{s_2} a(s)ds}\} = e^{-\int_0^{3/2} a(s)ds} = e^{5/4} \approx 3.49.
$$

Thus $K = \frac{7}{2}$ is good enough. Notice that we have for the impulsive operator that $c(t) = \frac{1}{20} \leq \frac{\alpha}{4K}$. Thus, we must just verify that the impulsive moments $\{\tau_k\}_{k\geq 0}$ satisfy that

$$e^{-\int_{\tau_{k-1}}^{\tau_k} a(u)du} \leq \beta < \frac{1}{2},$$

for every k. Notice that for $k \geq 2$, we have that $\tau_k - \tau_{k-1} \geq \frac{1}{2}$. Also, for $t \geq \tau_1 = 3$, we have that $a(t) \geq 3$. Therefore, for every $k \geq 2$

$$e^{-\int_{\tau_{k-1}}^{\tau_k} a(u)du} \leq e^{-3(\tau_k - \tau_{k-1})} \leq e^{-\frac{3}{2}} < \frac{1}{2}.$$

On the interval $[\tau_0, \tau_1] = [0, 3]$, we have that $\int_0^3 a(s)ds = 1$, so that $e^{-\int_{\tau_0}^{\tau_1} a(u)du} = e^{-\int_0^3 a(s)ds} = \frac{1}{e} < \frac{1}{2}$. Letting $\beta := \frac{1}{e} < \frac{1}{2}$, we have that $e^{-\int_{\tau_{k-1}}^{\tau_k} a(u)du} \leq \beta$ for every $k \geq 1$. Thus, we have verified, after applying Corollary 5.1 and Corollary 5.2, the uniform stability and asymptotic stability of the trivial solution, for arbitrarily large initial conditions ϕ. Numerical simulation is illustrated in Figure 2.

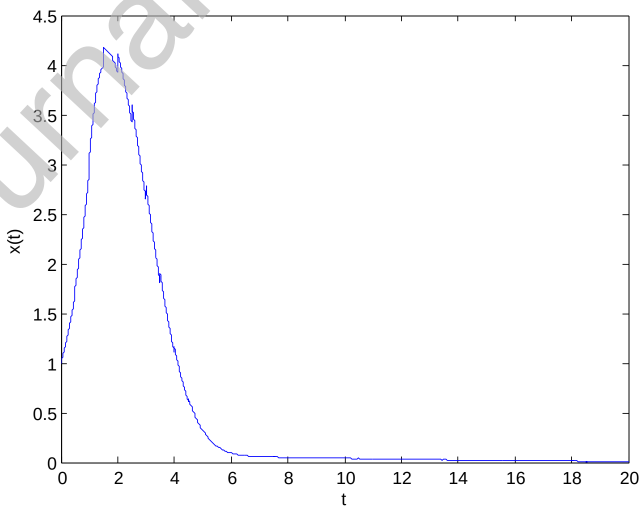


Figure 2: Simulation result for Example 5.3. with $a(t), b(t)$ piece-wisely defined.

Remark 5.3. In the previous example, notice the following. Suppose that we have the non-impulsive version

$$x'(t) = -a(t)x(t) + b(t)x(t - r(t)), \qquad t \geq 0, \tag{34}$$

with

$$a(t) = \begin{cases} -1 & \text{if } t \in [0,1] \\ -1 + 2(t-1) & \text{if } t \geq 1. \end{cases}$$

$$b(t) = \begin{cases} \frac{1}{25} & \text{if } t \in \left[0, \frac{39}{25}\right] \\ \frac{1}{25} + \frac{1}{2}\left(t - \frac{39}{25}\right) & \text{if } t \geq \frac{39}{25}. \end{cases}$$

Suppose $t - r(t) \longrightarrow \infty$ as $t \to \infty$. Then, by a similar calculation to the one done in Example 5.3, we have that $\int_0^t e^{-\int_s^t a(u)du} b(s)ds \leq \alpha < 1$, and the rest of the hypotheses of Theorem 2.1 in [39] are satisfied, so that we have asymptotic stability of the trivial solution. Notice that this example has that for $t \in [0, 1)$, $a(t) < 0$. Therefore, this example does not satisfy the sufficient conditions for stability given through the Lyapunov stability result given in [11]. In said result, using a Lyapunov function, it is determined that a sufficient condition for asymptotic stability requires that $a(t) \geq c > 0$, $a(t) \geq J|b(t)|$ for every $t \geq 0$, for some $J > 1$ and $c > 0$. The example illustrated here shows how the Banach contraction method for stability can be used as a possible alternative to Lyapunov methods, which can improve classical Lyapunov stability analysis. In Example 5.3, of course, we have added impulses, and verified that these discontinuities still give asymptotic stability.

6 Conclusion

We have studied, in this paper, a class of quasi-linear impulsive systems of functional differential equations with infinite time delays. We have adopted a new approach to overcome the usual difficulties of constructing the Lyapunov function and Lyapunov functionals as well as the estimate of their derivatives along the solutions. By employing the contraction principle, we have established some criteria on uniform stability and asymptotic stability for impulsive system of FDEs. The proposed approach utilizes the idea of averaging instead of the point-wise estimate in the Lyapunov method. Our results show that the Banach contraction principle can be used as a possible alternative to Lyapunov methods for stability analysis when the conditions of Lyapunov method fails to hold. Similar approaches by using other fixed point theorems may be employed for stability analysis of impulsive systems of FDEs in the futre.

7 Acknowledgements

The Authors would like to thank the reviewers for their constructive comments which have improved the quality of the paper. This research was supported by NESRC Canada which is gratefully acknowledged.

References

[1] Alwan M, Liu X. Theory of Hybrid Systems, Deterministic and Stochastic, High Education Press and Springer, 2018.

[2] Anokhin A, Berezansky L, Braverman E. Exponential stability of linear delay impulsive differential equations, J. Math. Anal. Appl. 193(1995), 923-941.

[3] Ballinger G, Liu X. Existence and uniqueness results for impulsive delay differential equations. Dynam. Contin. Discrete Impuls. Systems, 5:579-591, 1999.

[4] Ballinger G, Liu X. Practical stability of impulsive delay differential equations and applications to control problems, Optimization methods and applications, Kluwer Acad. Publ., Dordrecht, 2001.

[5] Berezansky L, Idels L. Exponential stability of some scalar impulsive delay differential equation, Commun. Appl. Math. Anal. 2(1998), 301-309.

[6] Burton TA. Stability by fixed point theory or Liapunov's theory: a comparison. fixed point theory, 4:15-32, 2003.

[7] Burton TA. Stability by fixed point methods for highly nonlinear delay equations. Fixed Point Theory, 5(1):3-20, 2004.

[8] Burton TA. Stability by Fixed Point Theory for Functional Differential Equations, Dover Publications, New York 2006.

[9] Burton TA, Furumochi T. Fixed points and problems in stability theory for ordinary and functional differential equations. Dynam. Systems Appl., 10:89-116, 2001.

[10] Burton TA, Furumochi T. A note on stability by Schauder's theorem. Funkcial. Ekvac., 44:73-82, 2001.

[11] Driver RD. Existence and stability of solutions of a delay-differential system. Arch. Rational Mech. Anal., 10:401-426, 1962.

[12] Gopalsamy K, Zhang BG. On delay differential equations with impulses, J. Math. Anal. Appl., 139(1989), 110-122.

[13] Hale JK, Lunel SMV. Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[14] Kolmanovskii VB, Nosov VR. Stability of Functional Differential Equations, Academic Press Inc., London, 1986.

[15] Lakshmikantham V, Bainov DD, Simeonov PS. Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[16] Lakshmikantham V, Leela S, Martynyuk AA. Stability Analysis of Nonlinear Systems, Marcel Dekker, New York, 1989.

[17] Lakshmikantham V, Liu X. Stability criteria for impulsive differential equations in terms of two measures, J. Math. Anal. Appl. 137(1989), 591-604.

[18] Lakshmikantham V, Liu X. Stability Analysis in Terms of Two Measures, World Scientific, Singapore, 1993.

[19] Li X, Ding Y. Razumikhin-type theorems for time-delay systems with persistent impulses. Systems Control Lett. 107 (2017), 22–27.

[20] Li X, Yang X, Huang T, Persistence of delayed cooperative models: Impulsive control method, Applied Math. Comput., 342 (2019) 130-146.

[21] Liu X. Impulsive stabilization of nonlinear systems, IMA J. Math. Control Inform. 10(1993), 11-19.

[22] Liu X. Stability results for impulsive differential systems with applications to population growth models, Dynamics and Stability of Systems 9(1994), 163-174.

[23] Liu X, Ballinger G. Uniform asymptotic stability of impulsive delay differential equations. Computers Math. Appl., 41:903-915, 2001.

[24] Liu X, Ballinger G. Existence and continuability of solutions for differential equations with delays and state-dependent impulses, Nonlinear Analysis, 51(2002), 633-647.

[25] Liu X, Ballinger G. Boundedness for impulsive delay differential equations and applications to population growth models, Nonlinear Analysis, Theory, Methods and Applications, 53(2003), 1041-1062.

[26] Liu X, Stechlinski P. Existence results for a class of hybrid systems with infinite delay. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms, 20:591-623, 2013.

[27] Liu X, Wang Q. The method of Lyapunov functionals and exponential stability of impulsive systems with time delay. Nonlinear Anal.66 (2007), no. 7, 1465-1484.

[28] Liu X, Stechlinski P. Infectious Disease Modeling, A Hybrid System Approach, Springer, New York, 2017.

[29] Liu X, Zhang K. Stabilization of nonlinear time-delay systems: distributed-delay dependent impulsive control, Systems & Control Letters, vol. 120, pp. 17-23, 2018.

[30] Liu X, Zhang K. Impulsive Systems on Hybrid Time Domains, Springer,Switzerland, 2019.

[31] Liu X, Zhang K, Xie W. Stabilization of time-delay neural networks via delayed pinning impulses. Chaos Solitons Fractals 93 (2016), 223-234.

[32] Luo Z, Shen J. Global existence results for impulsive functional differential equations. J. Math. Anal. Appl., 323:644-653, 2006.

[33] Seifert G. Liapunov-Razumikhn conditions for stability and boundedness of functional differential equations of Volterra type. J. Differential Equations, 14:424-430, 1973.

[34] Shen J, Liu X. Global existence results for impulsive differential equations. J. Math. Anal. Appl., 314:546-557, 2006.

[35] Shen J, Luo Z, Liu X. Impulsive stabilization of functional differential equations via Liapunov functionals, J. Math. Anal. Appl. 240(1999), 1-5.

[36] Stamova IM, Stamov GT. Lyapunov-Razumikhin method for impulsive functional equations and applications to the population dynamics, J. Compt. Appl. Math. 130(2001), 163-171.

[37] Wang X, Liu X, Shi K, Zhong S. Pinning impulsive synchronization of complex dynamical networks with various time-varying delay sizes. Nonlinear Anal. Hybrid Syst. 26 (2017), 307-318.

[38] Yang X, Li X, Xi Q, Duan P. Review of stability and stabilization for impulsive delayed systems, Math. Biosciences Engineering, 15 (2018) 1495-1515.

[39] Zhang B. Contraction mapping and stability in a delay-differential equation. Proceedings of Dynamic Systems and Applications, 4:189-190, 2004.