

# A minimax approach for inverse variational inequalities

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## Abstract

In this work, we characterize the existence of a solution for a certain variational inequality by means of a classical minimax theorem. In addition, we propose a numerical algorithm for the solution of an inverse problem associated with a variational inequality. To this end we state a collage-type result in this variational framework.

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## 1 Introduction

The well-known *collage theorem* [2], a direct consequence of the Banach fixed point theorem, states that for the unique fixed point  $x_0$  of a  $c$ -contractive self-mapping  $\Psi$  on a complete metric space  $(X, d)$ , there holds

$$x \in X \Rightarrow d(x, x_0) \leq \frac{1}{1-c} d(x, \Psi(x)).$$

This result has become a fundamental tool for a class of numerical methods providing the solution of some inverse problems: see, for instance, [15, 18, 19]. Furthermore, if one replaces the Banach

fixed point theorem by the Lax–Milgram theorem or some of its generalizations ([13, 31]), it is possible to establish results along the lines of the collage theorem: these are the so-called *generalized collage theorems* ([5, 16, 17, 18]). Our main aim in this paper is to state a collage-type result starting from the Stampacchia theorem to deal with inverse problems related to a variational inequality.

First of all, we prove that the existence of a solution for a variational inequality in a reflexive Banach space is equivalent to the existence of a constant satisfying an adequate convexity condition. To this end, we make use of the classical minimax theorem of J. von Neumann and K. Fan. In addition, we show how our result clearly implies the Stampacchia theorem. Once the existence of a solution is studied, we deal with the inverse problem associated with a variational inequality. The above-mentioned collage theorem –a stability result derived from the Stampacchia theorem– and the use of an adequate Schauder basis in the involved reflexive Banach space, allow us to design a numerical method for the solution of the variational inverse problem. We illustrate our results with a numerical test.

Let us recall some standard notations. For a real normed space  $E$ ,  $B_E$  denotes its *closed unit ball* and  $E^*$  its *topological dual*, that is, the Banach space of those continuous and linear functionals on  $E$ . If  $m \geq 1$ ,  $\Delta_m$  stands for the *unit simplex* of  $\mathbb{R}^m$ , that is,

$$\Delta_m := \{t = (t_1, \dots, t_m) \in \mathbb{R}^m : t_1, \dots, t_m \geq 0, t_1 + \dots + t_m = 1\}.$$

## 2 Existence of a solution for variational inequalities

We adopt a minimax approach for dealing with variational inequalities, as in [12, 30], unlike that of K. Fan [7] (see also [1]), where an equilibrium result is the main tool. In this way, we not only derive sufficient but also necessary conditions for the existence of a solution for a certain variational inequality.

We make use of the following classical minimax theorem ([8, 14, 23]). More general or different versions can be found, for instance, in [25, 26, 27, 33, 32, 35, 36].

**Theorem 2.1** *Assume that  $X$  is a nonempty, convex and compact subset of a real topological vector space,  $Y$  is a nonempty set and  $g : X \times Y \longrightarrow \mathbb{R}$  is continuous and concave on  $X$ . Then,*

$$\text{there exists } x_0 \in X : \inf_{y \in Y} \max_{x \in X} g(x, y) \leq \inf_{y \in Y} g(x_0, y)$$

if (and only if)

$$\left. \begin{array}{l} m \geq 1, \quad t \in \Delta_m \\ y_1, \dots, y_m \in Y \end{array} \right\} \Rightarrow \inf_{y \in Y} \max_{x \in X} g(x, y) \leq \max_{x \in X} \sum_{j=1}^m t_j g(x, y_j).$$

This result has been used, in an equivalent form of theorem of the alternative, or that of Hahn–Banach type result, to characterize the existence of a solution for nonlinear infinite programs: see [20, 21, 22, 28]. Let us also mention that the convexity condition in Theorem 2.1 is the so-called *infsup*-convexity (see [14, 29]).

Now we state a general theorem of existence for certain variational inequalities that implies the classical Stampacchia theorem ([34]).

**Theorem 2.2** *Let  $E$  be a real reflexive Banach space,  $x_0^* \in E^*$ ,  $a : E \times E \longrightarrow \mathbb{R}$  be a continuous bilinear form, and  $Y$  be a nonempty weak closed subset of  $E$ . Then*

$$\text{there exists } x_0 \in Y : y \in Y \Rightarrow x_0^*(y - x_0) \leq a(y, y - x_0) \quad (2.1)$$

if, and only if, for some  $\alpha \geq 0$ ,  $Y \cap \alpha B_E \neq \emptyset$  and

$$\left. \begin{array}{l} m \geq 1, \quad t \in \Delta_m \\ y_1, \dots, y_m \in Y \end{array} \right\} \Rightarrow \sum_{j=1}^m t_j (x_0^*(y_j) - a(y_j, y_j)) \leq \max_{x \in Y \cap \alpha B_E} \left( x_0^*(x) - a\left(\sum_{j=1}^m t_j y_j, x\right) \right). \quad (2.2)$$

PROOF. The fact that (2.1) implies (2.2) is very easy to check: it suffices to consider  $\alpha := \|x_0\|$  and use the linearity of  $a$  at its first variable.

And conversely. Let  $X := Y \cap \alpha B_E$  and observe that, thanks to (2.2) with  $m = 1$ , it holds that

$$\begin{aligned} 0 &\leq \inf_{y \in Y} \left( a(y, y) - x_0^*(y) - \max_{x \in X} (x_0^*(x) - a(y, x)) \right) \\ &= \inf_{y \in Y} \max_{x \in X} (a(y, y - x) - x_0^*(y - x)). \end{aligned}$$

Then, we write

$$\mu := \inf_{y \in Y} \max_{x \in X} (a(y, y - x) - x_0^*(y - x))$$

and so,  $\mu \in \mathbb{R}$ . If we define the bifunction  $f : X \times Y \longrightarrow \mathbb{R}$  by

$$f(x, y) := a(y, y - x) - x_0^*(y - x) - \mu, \quad (x \in X, y \in Y),$$

then

$$0 = \inf_{y \in Y} \max_{x \in X} f(x, y),$$

and the minimax theorem, Theorem 2.1, applies when we endow  $E$  with its weak topology, since  $X$  is a nonempty, weak compact and convex subset of  $E$  ( $Y$  is weak closed and  $B_E$  is weak compact) and  $f$  is concave and weak continuous on  $X$ . Therefore, there exists  $x_0 \in X$  such that

$$\inf_{y \in Y} \max_{x \in X} f(x, y) \leq \inf_{y \in Y} f(x_0, y),$$

i.e., there holds (2.1), if

$$\left. \begin{array}{l} m \geq 1, \ t \in \Delta_m \\ y_1, \dots, y_m \in Y \end{array} \right\} \Rightarrow 0 \leq \max_{x \in X} \sum_{j=1}^m t_j f(x, y_j),$$

which is clearly equivalent to the condition (2.2).  $\square$

When, in addition,  $Y$  is convex –in this case  $Y$  is closed for the norm topology, since it is weakly closed and convex– and  $a$  is non-negative at the diagonal, the condition (2.2) is simpler and (2.1) can be equivalently reformulated:

**Corollary 2.3** *Under the same assumptions that in Theorem 2.2, let us also suppose that  $Y$  is convex and that, for all  $x \in E$ ,  $a(x, x) \geq 0$ . Then*

$$\text{there exists } x_0 \in Y : y \in Y \Rightarrow x_0^*(y - x_0) \leq a(x_0, y - x_0) \quad (2.3)$$

*if, and only if, there exists  $\alpha \geq 0$  such that  $Y \cap \alpha B_E \neq \emptyset$  and*

$$y \in Y \Rightarrow x_0^*(y) - a(y, y) \leq \sup_{x \in Y \cap \alpha B_E} (x_0^*(x) - a(y, x)). \quad (2.4)$$

PROOF According to [6, Lemma 4.1] and the positivity condition on the diagonal of  $a$ , the variational problem (2.3) is equivalent to (2.1). Therefore, in view of Theorem 2.2, we will state the equivalence of (2.4) and (2.2).

It is clear that (2.2)  $\Rightarrow$  (2.4), hence we focus on proving the converse. So, let  $m \geq 1$ ,  $t \in \Delta_m$  and  $y_1, \dots, y_m \in Y$ . Then, the convexity of  $Y$ , the linearity of  $f$  and the convexity of the quadratic form  $x \in E \mapsto a(x, x)$  (taking into account that  $a$  is non-negative on its diagonal) yield

$$\begin{aligned} \sum_{j=1}^m t_j (x_0^*(y_j) - a(y_j, y_j)) &= \sum_{j=1}^m t_j x_0^*(y_j) - \sum_{j=1}^m t_j a(y_j, y_j) \\ &\leq x_0^* \left( \sum_{j=1}^m t_j y_j \right) - a \left( \sum_{j=1}^m t_j y_j, \sum_{j=1}^m t_j y_j \right) \\ &\leq \sup_{x \in Y \cap \alpha B_E} \left( x_0^*(x) - a \left( \sum_{j=1}^m t_j y_j, x \right) \right). \end{aligned}$$

□

The existence of solution for (systems of) variational equations or different variational inequalities has been previously established by means of minimax inequalities or the Hahn–Banach theorem [12, 13, 27, 30, 31].

Let us note that Stampacchia’s theorem is a direct consequence of the previous result: assume that  $E$  is a Hilbert space,  $Y$  is a nonempty, closed and convex subset of  $E$ ,  $a : E \times E \rightarrow \mathbb{R}$  is  $\rho$ -coercive ( $\rho > 0$ ), continuous and bilinear and that  $x_0^* \in E^*$  –here we do not use the Riesz identification of  $E$  with its topological dual  $E^*$ –. These hypotheses are sufficient for the existence of  $x_0 \in E$  with

$$y \in Y \Rightarrow x_0^*(y - x_0) \leq a(x_0, y - x_0).$$

To prove this, let  $\beta > 0$  such that  $Y \cap \beta B_E \neq \emptyset$  and note that

$$\begin{aligned} \frac{x_0^*(y) - a(y, y)}{\|y\|} - \frac{\sup_{x \in Y \cap \beta B_E} (x_0^*(x) - a(y, x))}{\|y\|} &\leq \|x_0^*\| - \rho\|y\| + \beta \frac{\|x_0^* - a(y, \cdot)\|}{\|y\|} \\ &\leq \|x_0^*\| \left( 1 + \frac{\beta}{\|y\|} \right) + \beta \|a\| - \rho\|y\|, \end{aligned}$$

hence in particular, for some  $\alpha > \beta$  we have that  $(Y \cap \alpha B_E \neq \emptyset$  and)

$$y \in Y, \|y\| > \alpha \Rightarrow x_0^*(y) - a(y, y) \leq \sup_{x \in Y \cap \alpha B_E} (x_0^*(x) - a(y, x)),$$

while, trivially,

$$y \in Y \cap \alpha B_E \Rightarrow x_0^*(y) - a(y, y) \leq \sup_{x \in Y \cap B_E} (x_0^*(x) - a(y, x)),$$

so condition (2.4) is valid and then the variational problem (2.3) admits a solution.

### 3 The inverse problem

The Stampacchia theorem straightforwardly implies the following collage-type result, which will allow us to deal with an inverse problem associated with a certain variational inequality. The role of this result is the same as that of the Banach fixed point in the collage treatment of some inverse problems (see, for instance, [15]). In fact, such an approach is motivated by this stability idea, previously considered for differential and variational equations in terms of the Lax–Milgram theorem or some of its generalizations, which is known as the *generalized collage theorem*: see [5, 16, 17, 18, 19].

**Theorem 3.1** *Let  $J$  be a nonempty set,  $E$  be a real Hilbert space,  $Y$  be a nonempty, closed and convex subset of  $E$ , and for all  $j \in J$  let  $x_j^* \in E^*$ ,  $a_j : E \times E \rightarrow \mathbb{R}$  be continuous, bilinear and such that, for some  $\rho_j > 0$ ,*

$$y \in E \Rightarrow \rho_j \|y\|^2 \leq a_j(y, y).$$

*If in addition  $x_j \in Y$  is the solution of the variational inequality*

$$y \in Y \Rightarrow x_j^*(y - x_j) \leq a(x_j, y - x_j),$$

*then,*

$$y \in Y, j \in J \Rightarrow \|y - x_j\| \leq \frac{\|a_j(y, \cdot) - x_j^*\|}{\rho_j}.$$

PROOF Given  $y \in Y$  and  $j \in J$ , the announced inequality follows from this chain of inequalities:

$$\begin{aligned}
\rho \|y - x_j\|^2 &\leq a_j(y - x_j, y - x_j) \\
&= a_j(y, y - x_j) - a_j(x_j, y - x_j) \\
&\leq a_j(y, y - x_j) - x_j^*(y - x_j) \\
&= (a_j(y, \cdot) - x_j^*)(y - x_j) \\
&\leq \|a_j(y, \cdot) - x_j^*\| \|y - x_j\|.
\end{aligned}$$

□

**Remark 3.2** We point out that, if  $Y$  is a closed affine subspace of  $E$ , then we can replace in the theorem  $\|a_j(y, \cdot) - x_j^*\|$  by  $\|(a_j(y, \cdot) - x_j^*)|_{Y_0}\|$ , where  $Y_0$  is the closed linear subspace  $Y - Y$  of  $E$ .

As mentioned above, our work is motivated by the collage treatment in [17], here for the following inverse problem: let us assume the hypotheses in Theorem 3.1 hold. If  $y \in Y$  is a given target element, then we want to determine, if possible, that  $j_0 \in J$  minimizing the distance  $\|y - x_j\|$ , that is,

$$\|y - x_{j_0}\| = \inf_{j \in J} \|y - x_j\|.$$

However, such an optimization problem is very difficult to solve, since we should not only minimize that function but also solve all the variational problems: find  $x_j \in Y$  with

$$y \in Y \Rightarrow x_j^*(y - x_j) \leq a(x_j, y - x_j). \quad (3.5)$$

If, in addition, the family of continuous and bilinear forms is uniformly coercive, in the sense that

$$\rho := \inf_{j \in J} \rho_j > 0,$$

then, in view of this assumption, Theorem 3.1 and Remark 3.2, if  $Y$  is a closed affine subspace of  $E$  and  $y \in Y$  is a given target element, we can replace that optimization problem by this other one:

$$\inf_{j \in J} \|(a_j(y, \cdot) - x_j^*)|_{Y_0}\|. \quad (3.6)$$

Let us note that the data of this nonlinear program only depend on the data of the variational inequalities (3.5). Moreover, this new optimization problem can be approximately solved by means of an orthogonal basis in the Hilbert space  $E$  ([17]), or even by a Schauder basis in  $E$  ([5, 16, 18, 19]). Let us notice that Schauder bases are tools for the numerical solution of a wide variety of differential, integral and integro-differential problems [3, 4, 10, 11, 24]. It is worth mentioning that, unlike in our motivating works [5, 16, 17], the choice of the target function is obtained by an approximation of the exact solution provided by a Galerkin scheme for the direct problem. Therefore, such a numerical method will become an auxiliary tool for dealing with the inverse problem.

Now, we illustrate this collage-based numerical method for the solution of the inverse problem associated with the family of variational inequalities (3.5). Before this, we introduce the above-mentioned numerical method for approximating the direct problem.

**Example 3.3** Consider the boundary value problem

$$\begin{cases} -u''(x) + ju(x) = f(x) & \text{on } (0, 1) \\ u(0) = \alpha, u(1) = \beta, \end{cases}$$

where  $\alpha, \beta, j \in \mathbb{R}$  with  $j > 0$ , and  $f \in L^\infty(0, 1)$ . Standard reasoning leads to the following variational formulation of this nonhomogeneous problem: let  $v$  be a test function in the closed and convex subset  $Y$  of  $H^1(0, 1)$

$$Y := \{v \in H^1(0, 1) : v(0) = \alpha, v(1) = \beta\},$$

multiply the second order differential equation by  $v - u$  and integrate by parts to arrive at

$$v \in Y \Rightarrow \int_0^1 u'(v - u)' + j \int_0^1 u(v - u) \geq \int_0^1 f(v - u). \quad (3.7)$$

Then, the Stampacchia theorem, when applied to the continuous, bilinear and 1-coercitive form  $a_j : H^1(0, 1) \times H^1(0, 1) \rightarrow \mathbb{R}$

$$a_j(u, v) := \int_0^1 u'v' + j \int_0^1 uv, \quad (u, v \in H^1(0, 1))$$

and the continuous and lineal functional  $x_0^* : H^1(0, 1) \rightarrow \mathbb{R}$

$$x_0^*(v) := \int_0^1 fv, \quad (v \in H^1(0, 1))$$



guarantees the existence of a unique solution  $u \in Y$  of the variational inequality (3.7).

Now we introduce a Galerkin method based upon the properties of a certain Schauder basis in  $H_0^1(0, 1)$ . To this end, if we write (3.7) with  $\omega \in H_0^1(0, 1)$ ,  $\omega := v - u$  we obtain

$$\omega \in H_0^1(0, 1) \Rightarrow \int_0^1 u' \omega' + j \int_0^1 u \omega = \int_0^1 f \omega.$$

In order to generate an increasing sequence of finite dimensional linear subspaces of  $H_0^1(0, 1)$  whose union is dense in this space, let us consider the Haar  $\{h_k\}_{k \geq 1}$  in  $L^2(0, 1)$  and define the sequence in  $H^1(0, 1)$

$$g_1(x) := 1, \quad (x \in [0, 1])$$

and for any  $k \geq 2$ ,

$$g_k(x) := \int_0^x h_{k-1}(t) dt, \quad (x \in [0, 1]).$$

This sequence is a Schauder basis for  $H^1(0, 1)$  and the sequence  $\{g_{k+2}\}_{k \geq 1}$  is also a Schauder basis for  $H_0^1(0, 1)$  (see [9, Propositions 4.7 and 4.8]). Let us note that we can express any element  $v \in Y$  as

$$v = \alpha + (\beta - \alpha)x + \sum_{k=1}^{\infty} \alpha_k g_{k+2},$$

for some scalars  $\alpha_k$ . Then, we define the aforementioned  $m$ -dimensional subspaces

$$H_m := \text{span} \{g_3, \dots, g_{m+2}\}, \quad (m \geq 1).$$

In the following table we collect the errors generated when approximating the exact solution  $u$  of the variational inequality (3.7) by the solution  $u_m \in Y$  of the  $m$ -dimensional problem, with the data  $\alpha = -3$ ,  $\beta = -4$ ,  $j = \sqrt{2}$ ,

$$f(x) := -2 + \sqrt{2}(x^2 - 2x - 3), \quad (x \in [0, 1])$$

and  $m = 3, 7, 15, 31, 63$ .

$m$	$\ u - u_m\ _{L_2}$	$\ u' - u'_m\ _{L_2}$	$\ u - u_m\ _{H_1}$
3	0.0105048	0.144383	0.144765
7	0.00261572	0.0721747	0.0722221
15	0.000653279	0.0360851	0.0360911
31	0.000163279	0.0180423	0.018043
63	0.0000408172	0.00902111	0.0090212

Table 1: Errors of the direct method.

Now we can address the inverse problem:

**Example 3.4** Let us now introduce the following inverse problem associated with the variational equation (3.7):

$$\begin{cases} -u''(x) + ju(x) = f(x) & \text{on } (0, 1) \\ u(0) = -3, \quad u(1) = -4, \end{cases}$$

where  $1 \leq j \leq 4$  and

$$f(x) := -2 + \sqrt{2}(x^2 - 2x - 3), \quad (x \in [0, 1]).$$

We analyze the performance of the collage-based method. In order to illustrate it, we take  $j = \sqrt{2}$  and obtain a target function  $u_m \in Y$  by the Galerkin method described in Example 3.3. Then, in order to solve the inverse problem given in (3.6), we observe that in our problem

$$\inf_{j \in [1, 4]} \|a_j(y, \cdot) - x^*\| = \inf_{j \in [1, 4]} \sup_{\substack{\omega \in H_0^1(0, 1) \\ \|\omega\| = 1}} |a_j(y, \omega) - x^*(\omega)|.$$

Now, we note that  $\omega \in H_0^1(0, 1)$  can be written as

$$\omega = \sum_{k=1}^{\infty} \alpha_k g_{k+2},$$

for some scalars  $\alpha_k$  verifying that  $|\alpha_k| \leq 1$ . Then

$$\inf_{j \in [1, 4]} \sup_{\substack{\omega \in H_0^1(0, 1) \\ \|\omega\| = 1}} |a_j(y, \omega) - x^*(\omega)| \leq \inf_{j \in [1, 4]} \left| \sum_{k=1}^{\infty} (a_j(y, g_{k+2}) - x^*(g_{k+2})) \right|.$$

The next table shows the approximations obtained for  $j$  solving the minimizing problem

$$\inf_{j \in [1,4]} \left| \sum_{k=1}^n (a_j(y_m, g_{k+2}) - x^*(g_{k+2})) \right|.$$

for the value  $n = 31$ , with different target elements  $u_m$  of the previous example.

$n \backslash m$	3	7	15	31
31	1.53389	1.46679	1.43170	1.41421

Table 2: Numerical results for the inverse problem.

## References

- [1] J.P. Aubin, *Optima and equilibria. An introduction to nonlinear analysis*, Second edition, Graduate Texts in Mathematics **140**, Springer-Verlag, Berlin, 1998.
- [2] M.F. Barnsley, V. Ervin, D. Hardin, J. Lancaster, *Solution of an inverse problem for fractals and other sets*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), 1975–1977.
- [3] M.I. Berenguer, M.V. Fernández Muñoz, A.I. Garralda-Guillem, M. Ruiz Galán, *Numerical treatment of fixed point applied to the nonlinear Fredholm integral equation*, Fixed Point Theory Appl. 2009 (2009), Art. ID 735638.
- [4] M.I. Berenguer, M.A. Fortes, A.I. Garralda-Guillem, M. Ruiz Galán, *Linear Volterra integro-differential equation and Schauder bases*, Appl. Math. Comput. 159 (2004), 495–507.
- [5] M.I. Berenguer, H. Kunze, D. La Torre, M. Ruiz Galán, *Galerkin method for constrained variational equations and a collage-based approach to related inverse problems*, J. Comput. Appl. Math. 292 (2016), 67–75.
- [6] A. Capatina, *Variational inequalities and frictional contact problems*, Adv. Mech. Math. 31, Springer, Cham, 2014.
- [7] K. Fan, *A minimax inequality and applications*, Inequalities, III (Proc. Third Sympos. Univ. California, Los Angeles, CA, 1969; dedicated to the memory of Theodore S. Motzkin), 103–113, Academic Press, New York, 1972.
- [8] K. Fan, *Minimax theorems*, Proceedings of the National Academy of Sciences of the United States of America **39** (1953), 42–47.

- [9] S. Fučík, *Fredholm alternative for nonlinear operators in Banach Spaces and its applications to differential and integral equations*, časopis Pěst. Mat. 96 (1971), 371–390.
- [10] D. Gámez, A.I. Garralda-Guillem, M. Ruiz Galán, *Nonlinear initial-value problems and Schauder bases*, Nonlinear Anal. 63 (2005), 97–105.
- [11] D. Gámez, A.I. Garralda-Guillem, M. Ruiz Galán, *High-order nonlinear initial-value problems countably determined*, J. Comput. Appl. Math. 228 (2009), 77–82.
- [12] A.I. Garralda-Guillem, M. Ruiz Galán, *A minimax approach for the study of systems of variational equations and related Galerkin schemes*, J. Comput. Appl. Math. 354 (2019), 103–111.
- [13] A.I. Garralda-Guillem, M. Ruiz Galán, *Mixed variational formulations in locally convex spaces*, J. Math. Anal. Appl. 414 (2014), 825–849.
- [14] G. Kassay, J. Kolumbán, *On a generalized sup-inf problem*, J. Optim. Theory Appl. 91 (1996), 651–670.
- [15] H. Kunze, G. Gomes, *Solving an inverse problem for Urison-type integral equations using Banach’s fixed point theorem*, Inverse Problems 19 (2003), 411–418.
- [16] H. Kunze, D. La Torre, K. Levere, M. Ruiz Galán, *Inverse problems via the generalized collage theorem for vector-valued Lax-Milgram-based variational problems*, Math. Probl. Eng. 2015 (2015), Art. ID 764643.
- [17] H. Kunze, D. La Torre, E.R. Vrscay, *A generalized collage method based upon the Lax-Milgram functional for solving boundary value inverse problems*, Nonlinear Anal. 71 (2009), 1337–1343.
- [18] H. Kunze, F. Menvivil, D. La Torre, E.R. Vrscay, *Fractal-Based Methods in Analysis*, Springer, New York, 2012.
- [19] H. Kunze, E.R. Vrscay, *Solving inverse problems for ordinary differential equations using the Picard contraction mapping*, Inverse Problems 15 (1999), 745–770.
- [20] P. Montiel López, M. Ruiz Galán, *Infinite programming and theorems of the alternative*, Math. Meth. Appl. Sci. 42(2019), 5769–5778.
- [21] P. Montiel López, M. Ruiz Galán, *Revisiting the Hahn-Banach theorem and nonlinear infinite programming*, J. Math. Anal. Appl. 455 (2017), 1037–1050.
- [22] P. Montiel López, M. Ruiz Galán, *Nonlinear programming via Königs maximum theorem*, J. Optim. Theory Appl. 170 (2016), 838–852.

- [23] J. von Neumann, *Zur theorie der gesellschaftsspiele*, Mathematische Annalen **100** (1928), 295–320.
- [24] A. Palomares, M. Ruiz Galán, *Isomorphisms, Schauder bases in Banach spaces, and numerical solution of integral and differential equations*, Numer. Funct. Anal. Optim. 26 (2005), 129–137.
- [25] M. Patriche, *Applications of the KKM property to coincidence theorems, equilibrium problems, minimax inequalities and variational relation problems*, Numer. Funct. Anal. Optim. 38 (2017), 523–548.
- [26] B. Ricceri, *On a minimax theorem: an improvement, a new proof and an overview of its applications*, Minimax Theory Appl. 2 (2017), 99–152.
- [27] M. Ruiz Galán, *Elementary convex techniques for equilibrium, minimax and variational problems*, Optim. Lett. 12 (2018), 137–154.
- [28] M. Ruiz Galán, *A sharp Lagrange multiplier theorem for nonlinear programs*, J. Global Optim. 65 (2016), 513–530.
- [29] M. Ruiz Galán, *The Gordan theorem and its implications for minimax theory*, J. Nonlinear Convex Anal. 17 (2016), 2385–2405.
- [30] M. Ruiz Galán, *An intrinsic notion of convexity for minimax*, J. Convex Anal. 21 (2014), 1105–1139.
- [31] M. Ruiz Galán, *A version of the Lax-Milgram theorem for locally convex spaces*, J. Convex Anal. 16 (2009), 993–1002.
- [32] M. Ruiz Galán, S. Simons, *A new minimax theorem and a perturbed James’s theorem*, Bull. Austral. Math. Soc. 66 (2002), 43–56.
- [33] J. Saint Raymond, *A new minimax theorem for linear operators*, Minimax Theory Appl. 3 (2018), 131–160.
- [34] G. Stampacchia, *Formes bilinaires coercives sur les ensembles convexes*, C. R. Acad. Sci. Paris 258 (1964), 4413–4416.
- [35] M. Syga, *Minimax theorems for extended real-valued abstract convex-concave functions*, J. Optim. Theory Appl. 176 (2018), 306–318.
- [36] G. Tian, *Full characterizations of minimax inequality, fixed point theorem, saddle point theorem, and KKM principle in arbitrary topological spaces*, J. Fixed Point Theory Appl. 19 (2017), 1679–1693.