

Solitons and cavitons in a nonlocal Whitham equation

N. Kulagin¹, L. Lerman^{2,3}, A. Malkin¹,

¹Institute of Physical Chemistry and Electrochemistry,
Russian Academy of Science, Moscow, Russia,

²Dept. of Fund. Math. of Higher School of Economics, and

³Lobachevsky State University of Nizhny Novgorod, Russia

Abstract

Solitons and cavitons (localized solutions with singularities) for the nonlocal Whitham equations are studied. The equation of a fourth order with a parameter in front of fourth derivative for traveling waves is reduced to a reversible Hamiltonian system defined on a two-sheeted four-dimensional space. When this parameter is small we get a slow-fast Hamiltonian system. Solutions of the system which stay on one sheet represent smooth solutions of the equation but those which perform transitions through the branching plane represent solutions with jumps. They correspond to solutions with singularities – breaks of the first and third derivatives but continuous even derivatives. The system has two types of equilibria on different sheets, they can be of saddle-center or saddle-foci. Using analytic and numerical methods we found many types of homoclinic (and periodic as well) orbits to these equilibria both with a monotone asymptotics and oscillating ones. They correspond to solitons and cavitons of the initial equation. When we deal with homoclinic orbits to a saddle-center the values of the second parameter (physical wave speed) is discrete but for the case of a saddle-center it is continuous. The presence of majority such solutions displays the very complicated dynamics of the system.

1 Introduction

Nonlinear nonlocal Whitham equation

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} dx' R(x - x') V(x', t) \quad (1)$$

represents a wide class of equations which are of great interest for nonlinear wave theory. It combines the typical hydrodynamic nonlinearity and an integral term descriptive of dispersion of the linear theory. The kernel of the integral term is conventionally defined by the dispersion relation $\omega = k\tilde{R}(k)$ with

$$R(x) = \int_{-\infty}^{\infty} \tilde{R}(k) e^{-ikx} dk. \quad (2)$$

Eq.(1) with $\tilde{R} = (1 + k^2)^{-1}$ was proposed by G. Whitham instead of Korteweg-de Vries equation in order to describe sharp crests of the water waves of a greatest height [1].

The usage of relatively simple Whitham type equations appeared to be very fruitful for various physical applications. A number of special cases of Eq.(1) were examined in detail. Among them are the Benjamin-Ono [2, 3] and Joseph [4] equations describing internal waves in stratified fluids of infinite and finite depth. These equations appeared to be integrable by the inverse scattering technique and the behavior of their solutions has been studied rather well. The Benjamin-Ono and Joseph equations are however the only representatives of the Whitham equations possessing this property [5]. Another widely known equations of that class were studied not so exhaustively, although the literature on the subject is quite extensive. A list of well-known Whitham equations involves the Leibovitz one for the waves in rotating fluid [6], the Klimontovich equation for magnetohydrodynamic waves in non-isothermal collision-less plasma [7], equations for shallow water waves [1], capillary [10] and hydroelastic [11] waves. The review on nonlinear nonlocal equations in theory of waves is presented in detailed monograph [8].

The characteristic feature of conservative Whitham equations is the existence of solitary wave solutions. For all just listed equations these solutions are smooth except some limiting cases of peaking for the waves of greatest amplitude. Besides, the amplitudes and velocity spectra of solitons can be bounded or not. But in any case the spectra are continuous. These properties are believed to be typical, but, as will be shown below, they are not essential for the solitons of Whitham equations.

Here we examine a particular case of the Whitham equation with a resonance dispersion relation

$$\tilde{R}(k) = \frac{1}{1 - k^2 + D^2 k^4}. \quad (3)$$

That equation has been proposed for nonlinear acoustic waves in simple peristaltic systems [45]. With small D^2 it is also applicable to the waves in a medium with internal oscillators [12]. A tentative analysis of some peculiarities of solitons to that equation has been performed in a short communication [13].

We study here specific features of solitary wave solutions to Eqs.(1)-(3). It is shown that this equation possesses both smooth and singularity involving solitons with exponential asymptotics, bound states of solitons and solitary waves with oscillating asymptotics. The velocity spectra of exponentially localized solitons turn out to be discrete ones.

2 Equation for traveling waves and its reduction

Hereafter we shall study an ordinary differential equation that obtained by the inversion of integral operator defined by Eqs (1)-(3) and transferring to the traveling wave solutions. The result takes the form of the fourth order differential equation

$$D^2 S^{(IV)} + S'' + S = V, \quad S = \lambda V + \frac{1}{2} V^2, \quad (4)$$

with traveling coordinate $y = x + \lambda t$, boundary conditions $\lim S(y) = 0$, as $|y| \rightarrow \infty$, and parameters $D^2 \ll 1$, $0 \leq \lambda \leq 1$. Thus, physically treated, the problem of searching for solutions of this type can be thought as a nonlinear boundary value problem for the parameter λ .

This equation for very small D^2 is of the singularly perturbed type similar to many such equations, see, for instance, [10, 30, 14, 10, 32, 37, 21, 16]. Another feature of this equation is its relation with the class of implicit differential equations [9]. To see this, let us introduce, similar to [38], new variables $u_1 = S$, $u_2 = S'$, $v_1 = -S' - D^2 S'''$, $v_2 = DS''$. Then the equation is reduced to the Hamiltonian system

$$u_1' = u_2, \quad v_1' = u_1 - V(u_1), \quad Du_2' = v_2, \quad Dv_2' = -u_2 - v_1, \quad (5)$$

where $V(u_1)$ is given by solving the equation $2u_1 = 2\lambda V + V^2$ w.r.t. V . This system is also reversible w.r.t. the involution $L : (u_1, v_1, u_2, v_2) \rightarrow (u_1, -v_1, -u_2, v_2)$, i.e. if $(u_1(y), v_1(y), u_2(y), v_2(y))$ is a solution to the system, then $(u_1(-y), -v_1(-y), -u_2(-y), v_2(-y))$ is as well [19]. The fixed point set $Fix(L)$ of L is 2-plane $v_1 = u_2 = 0$.

The quadratic equation for $V(u_1)$ has generally either two or no real solutions, so function V is two-valued. To keep this into account in more convenient way, let us consider the space \mathbb{R}^5 with coordinates (u_1, v_1, u_2, v_2, V) and its smooth 4-dimensional submanifold M given by real solutions of the equation $V^2 + 2\lambda V - 2u_1 = 0$. This is a two-sheeted submanifold with respect to the projection $\pi : (u_1, v_1, u_2, v_2, V) \rightarrow (u_1, v_1, u_2, v_2)$, its image is half-space $u_1 \geq -\lambda^2/2$. Both sheets are glued along 3-dimensional branching 3-plane $u_1 = -\lambda^2/2$, $V = -\lambda$. The shape of this submanifold is the direct product of a parabola and a 3-plane.

On each sheet (upper one $V(u_1) = -\lambda + \sqrt{\lambda^2 + 2u_1}$ and lower one $V = -\lambda - \sqrt{\lambda^2 + 2u_1}$) the system generates its own differential system. In the half-space $u_1 \geq -\lambda^2/2$ on every sheet the Peano theorem on the existence of solutions is valid [31], but in the open half-space $u_1 > -\lambda^2/2$ the usual theorem of existence and uniqueness of solutions works, so, despite of the non-smoothness of the function $\sqrt{\lambda^2 + 2u_1}$ on the boundary, through every point in the closed sheet of the open half-space an orbit passes and the only orbit through the point on a sheet over the open half-space. There are two questions here: 1) how does one need to adjoin orbits from different sheets in order to preserve continuity of $S(y)$, when the related orbits hit the boundary of a sheet, i.e. they satisfy the equality $u_1(y_0) = -\lambda^2/2$ and for this y_0 we have $u_2(y_0) \neq 0$, and 2) about possible non-uniqueness for orbits through boundary points where $u_1 = -\lambda^2/2$ (and $V = -\lambda$).

The inequality $u_2(y_0) \neq 0$ means that the related orbit must leave a sheet or enter to a sheet, in dependence on the sign $u_2(y_0)$. Indeed, the restrictions of both vector fields to the branching plane $u_1 = -\lambda^2/2$ coincide. Hence, if $u_2(y_0) > 0$ at the boundary point, then the orbit looks inward the region $u_1 > -\lambda^2/2$ on both sheets and should enter to both of them (we remind $u_1' = u_2$). But it is impossible, if $u_1(y)$ (i.e. $S(y)$) varies continuously in y near this value y_0 .

If, on the contrary, one gets $u_2(y_0) < 0$, then the orbits look outward in both sheets and should leave an either sheet. Thus, in order to preserve continuity of u_1 we need to use the

reversibility of vector fields and make jumps on the boundary of both sheets in accordance to the action of L . How does this save the situation, one can see as follows. Suppose $u_2(y_0) > 0$ at the boundary point $X_0 = (u_1(y_0), v_1(y_0), u_2(y_0), v_2(y_0))$, $u_1(y_0) = -\lambda^2/2$, the vector field looks inward. For $u_2 \neq 0$, the boundary point does not belong to the set $Fix(L)$ and its L -image is another boundary point $X_0^{(1)} = L(X_0)$ with coordinates $(u_1(y_0), -v_1(y_0), -u_2(y_0), v_2(y_0))$. Thus, the vector field at $X_0^{(1)}$ looks outward the region $u_1 > -\lambda^2/2$. So, suppose we move for $y < y_0$, as y increases to y_0 , in the lower sheet along the orbit which hits the branching plane at the point $X_0^{(1)}$. Next we jump to the point $L(X_0^{(1)}) = X_0$ and move further in the upper sheet along the orbit through the point X_0 . In this motion the value of $u_1 = S$ varies continuously, but $u_2 = S'$ undergoes a jump at $y=y_0$. There is a pairing composed orbit, where we move first for $y < y_0$ on the upper sheet till the point $X_0^{(1)}$, then jump to X_0 and go further on the lower sheet along the orbit through X_0 . Observe that if an orbit in a sheet does not cross the branching point its behavior is defined by the smooth (in fact – analytic) vector field and any tool working in this case can be used. Our main concern below will be on solutions to (5) which are homoclinic orbits to equilibria that exist in the system. As we shall see, these solutions are symmetric but they can be either smooth or with singularities (they cross the branching plane several times). Smooth homoclinic orbits we call sometimes solitons and those with singularities cavitons. Here we follow our terminology in [13].

In order to facilitate simulations, we can eliminate jumps as follows. The submanifold M is the graph of a smooth function $u_1 = \lambda V + V^2/2$ in variables (V, v_1, u_2, v_2) . Let us rewrite the system in variables (r, v_1, u_2, v_2) , $r = V + \lambda$

$$rr' = u_2, \quad v_1' = \lambda(1 - \lambda/2) - r + r^2/2, \quad Du_2' = v_2, \quad Dv_2' = -(u_2 + v_1).$$

The system obtained has singularities along 3-plane $r = 0$ (it is not defined). For upper sheet we get $r = V + \lambda = \sqrt{\lambda^2 + 2u_1} > 0$, but for lower sheet the sign is opposite $r = V + \lambda = -\sqrt{\lambda^2 + 2u_1} < 0$. In order to eliminate the singularity, we multiply equations 2-4 at r and change the “time” to s , $ds = dy/r$, obtaining a smooth differential system. The orbits and direction of moving along orbits of this smooth system coincide on the upper sheet $r > 0$ with those for orbits of the initial system (5), but on the lower sheet $r < 0$ the true direction of moving along orbits is opposite. In particular, this approach allows one to assert that through any point of the boundary $r = 0$, if this point is not singular, a unique orbit can pass on either sheet. This smooth system looks as follows

$$\frac{dr}{ds} = u_2, \quad \frac{dv_1}{ds} = \lambda(1 - \lambda/2)r - r^2 + r^3/2, \quad D\frac{du_2}{ds} = rv_2, \quad D\frac{dv_2}{ds} = -r(u_2 + v_1). \quad (6)$$

Additional equilibria of the system, appeared due to the change of “time”, fill the plane $r = 0$, $u_2 = 0$.

The system (6) is Hamiltonian and reversible w.r.t. the involution $L : (r, v_1, u_2, v_2) \rightarrow (r, -v_1, -u_2, v_2)$ with the smooth Hamiltonian

$$H = u_2v_1 + \frac{1}{2}(u_2^2 + v_2^2) - \lambda(1 - \lambda/2)r^2/2 + \frac{1}{3}r^3 - \frac{1}{8}r^4.$$

The symplectic structure here is not standard and defined by the matrix J

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & rD^{-1} \\ 0 & 0 & -rD^{-1} & 0 \end{pmatrix}, \quad \dot{X} = J\nabla H.$$

It is seen that this structure degenerates at $r = 0$.

The transition to the smooth system sets the question: what are interrelations between orbits of the smooth system (6) and orbits of the initial system on the upper and lower sheets? The answer is the same as was done above: for orbits not crossing the plane $r = 0$ the behavior is the same as for its counterpart. For $r > 0$ direction in y and direction in s are the same, for $r < 0$ orbits are the same but direction of motion is opposite. Situation for orbits crossing $r = 0$ appears as before: if an orbit from upper sheet ($r > 0$) hits the plane $r = 0$ at a point X_0 the true motion is described as above. We make jump by the action of L , $X_1 = L(X_0)$ and after that we continue by the orbit through X_1 in the lower sheet ($r < 0$). If we start from the lower sheet, the procedure is similar. At such transformation the function S varies continuously, as well as its S_{yy} , S_{yyyy} but S_y and S_{yyy} change their signs. This leads to the solutions with a singularities (sharpening). The shape of a dependence $S(y)$, its smoothen variant is plotted on Fig. 1a,b and the projection on (r, u_2) -plane is on Fig. 2. The explanations on the orbit behavior will be given below.

Remark 1 *It is worth remarking that when we deal with symmetric orbits (invariant w.r.t. the action of L), even though they cross the plane $r = 0$, the orbits (curves in the phase space) will be the same as for related orbits of the smooth system, since, due to symmetry, the passage to the symmetric point occurs on the same orbit. We use this under simulations.*

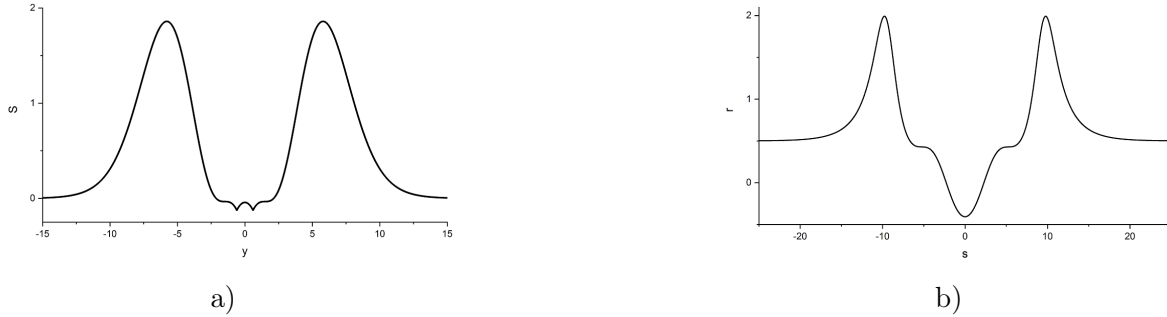


Figure 1: (a) Graph of a true 2-round soliton-caviton $S(y)$; (b) and its smoothing $r(s)$.

3 Slow system

Let us demonstrate this approach for the limiting system as $D = 0$ (slow system [9]). Then third and fourth equation give the representation for the so-called slow manifold $v_2 = 0, u_2 = -v_1$.

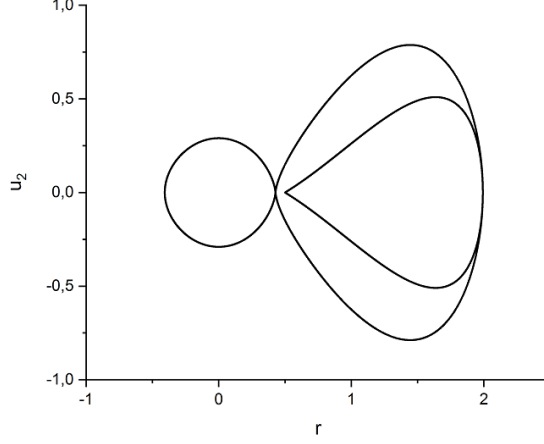


Figure 2: Projection of smoothen soliton-caviton on (u_2, r) -plane

Inserting them into the first and second equations we get the slow system in the half-plane $u_1 \geq -\lambda^2/2$

$$u'_1 = -v_1, \quad v'_1 = u_1 + \lambda \mp \sqrt{\lambda^2 + 2u_1}, \quad (7)$$

where upper sign corresponds to the upper system and lower sign does to the lower system. First we investigate the systems on the upper and lower sheets separately and compare their behavior with the smooth system obtained by the change of “time” y .

Both systems are Hamiltonian ones and reversible w.r.t. the symmetry $(u_1, v_1) \rightarrow (u_1, -v_1)$. The upper system has two equilibria $(0, 0)$ and $2(1 - \lambda)$, being a saddle and a center. The lower system has not equilibria in the half-plane $u_1 \geq -\lambda^2/2$ and all their orbits go from the points $u_1 = -\lambda^2/2, |v_1| < \infty$ from negative v_1 to hit this line at points with positive v_1 . An orbit on the upper sheet hitting, as y increases, the line $u_1 = -\lambda^2/2$ at the point $(-\lambda^2/2, v_1^*)$, has to be continued from the point $(-\lambda^2/2, -v_1^*)$ with the further increasing y . The similar is done for orbits on the lower sheet. Under this procedure we get either periodic orbits (smooth, if it belongs to the upper sheet, or with a sharpening, if this orbit intersects the line $u_1 = -\lambda^2/2$). The phase portraits are the following (see, Fig. 3) where solid lined represent orbits of the upper system and dashed line do those for the lower system.

Now let us go to the smooth system in variables (r, v_1) , here r can take any sign.

$$r' = -v_1, \quad v'_1 = \lambda(1 - \lambda/2)r - r^2 + r^3/2. \quad (8)$$

The Hamiltonian of the system is

$$h = \frac{v_1^2}{2} + \frac{\lambda(2 - \lambda)r^2}{4} - \frac{r^3}{3} + \frac{r^4}{8}.$$

The system is also reversible with respect to the involution $L : (r, v_1) \rightarrow (r, -v_1)$. This implies that if $(r(s), v_1(s))$ is a solution, then $(r(-s), -v_1(-s))$ as well.

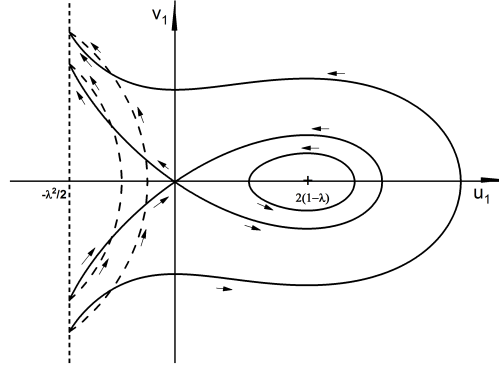


Figure 3: Slow manifold dynamics with jumps.

The equilibria of the system are $v_1 = 0, r = 0, \lambda, 2 - \lambda$, of them the first and third are centers, the second is a saddle with two separatrix loops. The left center corresponds to the point of the glued system on the line $u_1 = -\lambda^2/2$ where $v_1 = 0$ (the point of tangency for orbits of the line). Here all orbits, except for two loops and equilibria, are periodic ones. Thus we get the following plot (Fig. 4).

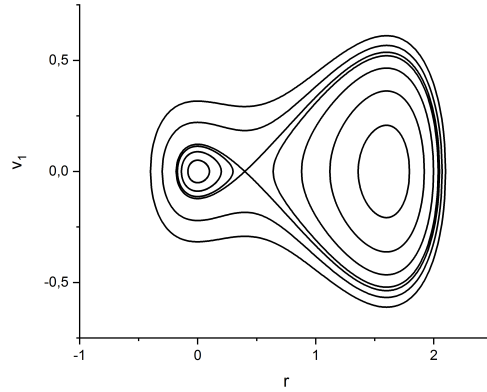


Figure 4: Phase portrait for smoothen system (8).

4 $\lambda = 0$

Here we consider first the degenerated limiting case $\lambda = 0$ (in fact, physically not relevant). This consideration is useful to compare with results for values $\lambda > 0$. Thus, we get two systems

defined in the half-space $u_1 \geq 0$, on the upper sheet

$$u'_1 = u_2, v'_1 = u_1 - \sqrt{2u_1}, Du'_2 = v_2, Dv'_2 = -u_2 - v_1, \quad (9)$$

with equilibria $(0, 0, 0, 0)$ and $(2, 0, 0, 0)$ and on the lower sheet

$$u'_1 = u_2, v'_1 = u_1 + \sqrt{2u_1}, Du'_2 = v_2, Dv'_2 = -u_2 - v_1 \quad (10)$$

with the only equilibrium $(0, 0, 0, 0)$.

Smoothing the system in variables (r, v_1, u_2, v_2) with changing time gives the system (r is arbitrary)

$$r' = u_2, v'_1 = -r^2 + r^3/2, Du'_2 = rv_2, Dv'_2 = -r(u_2 + v_1). \quad (11)$$

with integral $H_0 = u_2v_1 + (u_2^2 + v_2^2)/2 + r^3/3 - r^4/8$. Also the system is reversible w.r.t. the involution $L : (r, v_1, u_2, v_2) \rightarrow (r, -v_1, -u_2, v_2)$, its fixed points plane is given as $v_1 = u_2 = 0$.

The system has a plane P_0 of equilibria $r = u_2 = 0$. Only a curve from this plane belongs to the level $H = c$. In particular, in the level $H = 0$ where orbits asymptotic to equilibrium at the origin $r = v_1 = u_2 = v_2 = 0$ lie, the intersection of P_0 with this level is given in coordinates v_1, v_2 as straight line $(v_1^0, 0)$.

The system obtained has a complex equilibrium O at the origin: all four its eigenvalues are zeroth. Hence the study of its local orbit behavior is a rather complicated problem. In the first turn we are interested in its orbits that enter and leave the equilibrium O as $s \rightarrow \pm\infty$. Such orbits, if they exist, have power asymptotics in $s \rightarrow \infty$. Let us find these asymptotics using the following ansatz

$$r = As^{-\alpha}(1 + o(1)), v_1 = Bs^{-\beta}(1 + o(1)), u_2 = Cs^{-\gamma}(1 + o(1)), v_2 = Es^{-\delta}(1 + o(1)), \quad (12)$$

with unknown coefficients A, B, C, E and exponents $\alpha, \beta, \gamma, \delta$ to be found. Inserting these functions into differential equations (11) we get exponents $\alpha = 4/3, \beta = 5/3, \gamma = 7/3, \delta = 2 = 6/3$ and coefficients

$$A = \frac{2\sqrt[3]{35}}{3}D^{2/3}, B = \frac{4}{15}\sqrt[3]{35^2}D^{4/3}, C = -\frac{8\sqrt[3]{35}}{9}D^{2/3}, E = \frac{28}{9}D. \quad (13)$$

The same calculation with the change $s = -s_1$ to get orbits tending O as $s \rightarrow -\infty$ gives naturally the same exponents but reverses signs for B and C staying signs of A, E the same. This calculation shows that the system have to possess by one orbit entering the equilibrium O as $s \rightarrow \infty$ and one orbit entering O as $s \rightarrow -\infty$. Both these orbits belong to the half-space $r > 0$ since $A > 0$ in both cases. In particular, the dependence u_2 in r is of power type $u_2 \sim r^{7/4}$. This is seen in the Fig. 5a where a homoclinic orbit found is shown (soliton). It is clearly seen the tangency at the entrance to the equilibrium. The related homoclinic orbits for $0 < \lambda < 1$ enter and leave the equilibrium as $s \rightarrow \pm\infty$ with different asymptotics that can be seen on the Fig. 9a. This becomes clear below.

Sometimes our simulations performed with the initial equation (4) not with the system. Then, to find symmetric homoclinic and periodic solution we sought for orbits which intersect

either one or two times the fixed point set $Fix(L)$. For initial differential equation (4) this plane defines by relations $S' = 0, S''' = 0$. The 3-plane $S' = 0$ is a cross-section for the majority of its points, so zeroes of the graph $S'''(D^2)$ give values of D^2 for which homoclinic orbits exist. As an example, such plot is presented on Fig. 5b.

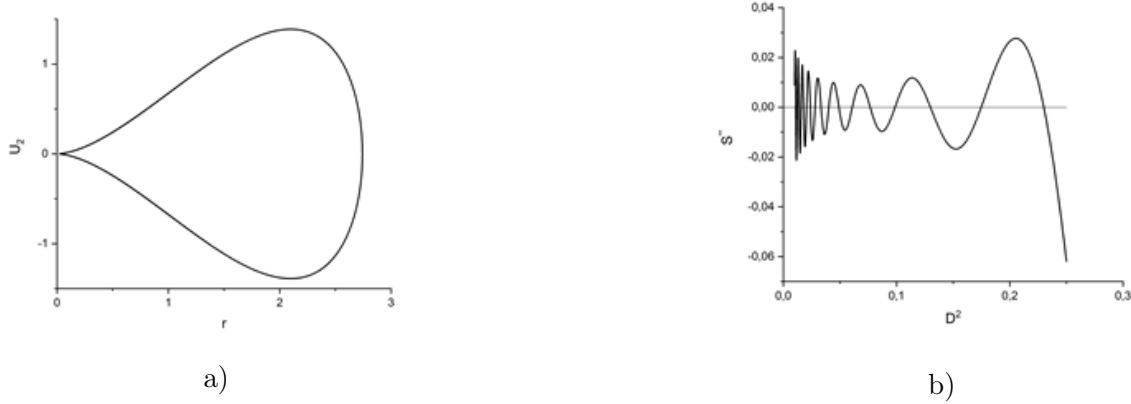


Figure 5: (a) The projection of the homoclinic orbit on the plane (r, u_2) ; (b) Graph $S'''(D^2)$.

As we shall see, the orbit behavior of the system reminds that for the case of a saddle-center equilibrium in a Hamiltonian system in two degrees of freedom [39, 35, 28, 47] (see details below). Again, for small positive D system (11) is slow-fast with slow variables (r, v_1) and fast variables (u_2, v_2) . In this case it is informative to investigate slow and fast systems [21] separately. Slow system is derived if we set $D = 0$ in the third and fourth equations, then we have either $r = 0$ or $v_2 = 0, u_2 = -v_1$. 3-plane $r = 0$ divides the phase space into two half-spaces $r > 0$ and $r < 0$. We shall work in the half-space $r > 0$, here we get slow 2-plane $v_2 = 0, u_2 = -v_1$. On this plane the slow system is given as

$$\dot{r} = -v_1, \quad \dot{v}_1 = -r^2 + r^3/2.$$

Its structure is plotted on Fig.2.

The fast systems is derived, if one changes $s \rightarrow s/D = \tau$, as a result small parameter D arises as a multiplier in the first two equations. Then one sets $D = 0$ that makes variables r, v_1 be parameters. The third and fourth equation with parameters r^0, v_1^0 have integral $h = (u_2 + v_1^0)^2 + v_2^2$. The periods $2\pi/r_0$ of these linear systems depend on r^0 . The motions in the full system, as $D > 0$ small, in some thin neighborhood of the slow manifold is a combination of two motions: the slow motion along the orbits of the slow system and fast rotation around the slow orbit. This follows from results of [21]. In particular, if one moves along the homoclinic orbit of the slow system, then the orbit behavior looks very similar to that which is observed near a homoclinic orbit to a saddle-center in a Hamiltonian system with two degrees of freedom [39, 35].

5 $\lambda \neq 0$: equilibria

Now we turn to the case $D > 0$ small enough. One of our main concern is to find soliton solutions to the equation (5). This corresponds to homoclinic orbits for equilibrium that exists on the upper sheet (see below). In fact, there are two equilibria on this sheet but only one of them has outgoing and ingoing orbits (separatrices). The situation under consideration depends heavily on the value of parameter λ and changes at the ends of the segment $\lambda \in [0, 1]$.

For positive λ in the half-space $u_1 > -\lambda^2/2$ both systems on 4-dimensional sheets $V > -\lambda$ and $V < -\lambda$ are analytic Hamiltonian ones

$$u'_1 = u_2, \quad v'_1 = u_1 + \lambda \mp \sqrt{\lambda^2 + 2u_1}, \quad Du'_2 = v_2, \quad Dv'_2 = -u_2 - v_1, \quad (14)$$

with related Hamiltonians

$$H = u_2 v_1 - \frac{u_1^2}{2} - \lambda u_1 + \frac{u_2^2 + v_2^2}{2} \pm \frac{1}{3}(\lambda^2 + 2u_1)^{3/2}.$$

So, all available methods can be applied to the study, in particular, it concerns existence of homoclinic orbits and nearby dynamics. The equilibrium at the origin $O(0, 0, 0, 0)$ on the upper sheet for $0 < \lambda < 1$ is a saddle-center, its eigenvalues are a pair of pure imaginary numbers and two reals. Indeed, linearizing at O gives a linear Hamiltonian system, its characteristic polynomial is

$$D^2 \sigma^4 + \sigma^2 - (1 - \lambda)/\lambda$$

with roots

$$\pm i D^{-1} \sqrt{(1 + \sqrt{1 + 4D^2(1 - \lambda)/\lambda})/2}, \quad \pm \sqrt{\frac{2(1 - \lambda)}{\lambda + \sqrt{\lambda^2 + 4D^2(1 - \lambda)\lambda}}}.$$

Coordinates of the second equilibrium on the upper sheet are $(2(1 - \lambda), 0, 0, 0)$, its characteristic polynomial is

$$D^2 \sigma^4 + \sigma^2 + (1 - \lambda)/(2 - \lambda)$$

with pure imaginary roots (the elliptic point)

$$\sigma_1^2 = \frac{-1 - \sqrt{1 - 4D^2(1 - \lambda)/(2 - \lambda)}}{2D^2}, \quad \sigma_2^2 = -\frac{2(1 - \lambda)/(2 - \lambda)}{1 + \sqrt{1 - 4D^2(1 - \lambda)/(2 - \lambda)}}.$$

As λ approach to $1 - 0$, both equilibria coalesce into one equilibrium with non semi-simple double zero and two pure imaginary eigenvalues. The lower sheet does not contain equilibria at all.

5.1 Positive $\lambda > 1$

For $\lambda > 1$ two equilibria exist on the upper sheet, O and another one $P_+ = (2(1 - \lambda), 0, 0, 0)$, the latter exists for $1 < \lambda < 2$. Their characteristic equations are

$$D^2 \sigma^4 + \sigma^2 + \frac{\lambda - 1}{\lambda} = 0, \quad D^2 \sigma^4 + \sigma^2 + \frac{1 - \lambda}{2 - \lambda} = 0.$$

On the lower sheet there is an only equilibrium $P_- = (2(1 - \lambda), 0, 0, 0)$ which exists for $\lambda > 2$ with the characteristic equation $D^2\sigma^4 + \sigma^2 + \frac{\lambda-1}{\lambda-2} = 0$.

The types of these equilibria depend on the value of D^2 . These are as follows. For the upper sheet we have

1. if $0 < D^2 < 1/4$, then O is an elliptic point;
2. if $D^2 > 1/4$, then O is an elliptic point for $1 < \lambda < \lambda_0$, $\lambda_0 = 4D^2/(4D^2 - 1)$, and O is a saddle-focus for $\lambda > \lambda_0$;
3. P_+ exists for $1 < \lambda < 2$, within this interval it is a saddle-center.

For the lower sheet at P_-

1. P_- exists for $\lambda > 2$, within this interval: if $0 < D^2 < 1/4$ it is a saddle-focus for $1 < \lambda < \lambda_0$, $\lambda_0 = 1 + 1/(1 - 4D^2)$, and is an elliptic point for $2 > \lambda > \lambda_0$;
2. if $D^2 > 1/4$, then P_- is a saddle-focus for $\lambda > 2$.

5.2 Negative λ

Negative values of the parameter λ also have physical sense. Let us first investigate the type of the equilibrium at O . Here we have another distribution of equilibria on sheets in comparison with $\lambda > 0$.

On the upper sheet we have the only equilibrium $S_+ = (2(1 - \lambda), 0, 0, 0)$ and on the lower sheet the unique equilibrium is $O = (0, 0, 0, 0)$. Their types are as follows. The characteristic equation for S_+ is $D^2\sigma^4 + \sigma^2 + (1 - \lambda)/(2 - \lambda) = 0$ and for O is $D^2\sigma^4 + \sigma^2 - (1 - \lambda)/\lambda = 0$. For S_+ we get the following.

1. If $0 < D^2 < 1/4$, then S_+ is an elliptic point for any $\lambda < 0$;
2. if $1/4 < D^2 < 1/2$, then S_+ is an elliptic point for $\lambda_+ < \lambda < 0$ and is a saddle-focus point for $-\infty < \lambda < \lambda_+$, $\lambda_+ = -2\frac{2D^2-1}{1-4D^2}$;
3. if $D^2 > 1/2$, then S_+ is a saddle-focus point for any $\lambda < 0$.

For O we have

1. If $0 < D^2 < 1/4$, then O is a saddle-focus for $\lambda_- < \lambda < 0$, $\lambda_- = 4D^2/(4D^2 - 1)$, and O is an elliptic point for $-\infty < \lambda < \lambda_-$;
2. if $D^2 > 1/4$, then O is a saddle-focus for any $\lambda < 0$.

Suppose some solution of the upper system hits the branching plane $u_1 = -\lambda^2/2$ at a finite value y_0 of “time” y and its tangent vector at this point is directed outward, i.e. to the half-space

$u_1 < -\lambda^2/2$, that is, $\dot{u}_1(y_0) = u_2(y_0) < 0$. To continue this solution by means of the lower system we apply to the point $m_+ = (-\lambda^2/2, v_1(y_0), u_2(y_0), v_2(y_0))$ the involution

$$m_- = L(-\lambda^2/2, v_1(y_0), u_2(y_0), v_2(y_0)) = (-\lambda^2/2, -v_1(y_0), -u_2(y_0), v_2(y_0)).$$

At this symmetric point coordinates of \dot{u}_1 and \dot{v}_2 of the vector field change signs but coordinates of \dot{v}_1 and \dot{u}_2 are the same. The lower vector field on the 3-plane $u_1 = -\lambda^2/2$ coincides with the upper vector field. Now we proceed the orbit from the point m_- using the lower vector field for $y > y_0$, this trajectory enters to the half-space $u_1 > -\lambda^2/2$. If its continuation reach again the branching plane, we do the same using upper vector field. As was said above, the value of S does not change at these switchings but S' and S''' do. In this way we can get cavitons being non-smooth homoclinic orbits which represent orbits joining stable and unstable separatrices of the saddle-center and crossing 3-plane $u_1 = -\lambda^2/2$ under their journey.

In what follows, we perform simulations with the smooth system (6). If some solution to this system stay all time in the half-space $r > 0$, then this solution corresponds to the upper sheet system. In particular, solitons correspond to homoclinic orbits of the equilibrium $O(\lambda, 0, 0, 0)$. Homoclinic orbits to O , which spend part time in the half-space $r < 0$, correspond to cavitons.

6 Solitons and cavitons: homoclinic loops of saddle-center

The existence of homoclinic loops to a saddle-center is a rather delicate problem, since one needs to find the merge of one-dimensional stable and unstable manifolds of the saddle-center within 3-dimensional singular level of the Hamiltonian. This level is singular (it is not a smooth manifold at any its point) because this level has a cone-type singularity at the equilibrium. Thus, such a problem should be studied in a two-parameter unfolding generally. The task becomes easier, if one considers reversible Hamiltonian systems and searches for symmetric homoclinic orbits. Then generally an unfolding has to be one-parametric (in fact, this depends on the type of an action of the reversible involution near a saddle-center, [39, 47]). If we investigate 2-parameter families of reversible Hamiltonian systems, then one expects a possibility to construct curves in the parameter plane along which the systems has homoclinic orbits to the related saddle-centers. Remind that existence of a saddle-center for Hamiltonian systems is a structurally stable phenomenon.

Since we are of interest with spectra on parameter λ , for which the system has homoclinic orbits to the saddle-center, we recall the result proved first in [47]. There a general one-parameter unfolding of reversible Hamiltonian systems was studied under an assumption that it unfolds a system with a homoclinic orbit to a saddle-center and for this orbit a genericity condition holds found first in [39]. Then it was proved that the set of parameter values which correspond to systems with symmetric homoclinic orbits to the saddle-center (not obligatory 1-round ones) is self-limiting and self-similar: each point is an accumulation point for this set. It is worth noting that our system can have both solitons and cavitons. On the mathematical language this corresponds to the case when both unstable separatrices of the saddle-center can merge with

stable ones forming one or even two homoclinic loops. In case of one loop this can be impossible, if one deals with the case B as was discovered in [39] and was indicated in [47].

The orbit structure of an analytic Hamiltonian system near a homoclinic orbit to a saddle-center was studied first in [39, 34] and then this was extended to different situations including reversible systems [35, 47, 28, 29, 53, 27]. The study is based on the reduction of instead of studying the flow to the investigation of Poincaré map and its orbit structure generated by the flow on some cross-section to a homoclinic orbit. This is heavily facilitated by the usage of a local normal form near a saddle-center due to Moser [49]: there exists an analytic symplectic local coordinates (x_1, y_1, x_2, y_2) such that the Hamiltonian H in these coordinates casts in the form $H = h(\xi, \eta)$, where h is an analytic function in variables $\xi = x_1 y_1$, $\eta = (x_2^2 + y_2^2)/2$, $h = \sigma \xi + \omega \eta + \dots$, $\sigma \omega \neq 0$. Such normal form is integrable, local functions ξ, η , as functions in (x_1, y_1, x_2, y_2) , are local integrals of the flow generated by Hamiltonian H . This easily allows one to construct local map from a cross-section to stable separatrix to a cross-section to unstable separatrix. This map has a singularity at the trace of the stable separatrix but can be redefined to get a continuous map everywhere and analytic at all points except for the trace of the separatrix. Orbits of the system correspond to orbits of Poincaré map. so its studying gives a complete information concerning orbit behavior of the flow. Principal elements of this picture were found in [39, 47, 28, 29]. In particular, suppose a homoclinic orbit to the saddle-center exist and some genericity condition holds for it, then each Lyapunov periodic orbit possesses within its level of H four transverse homoclinic Poincaré orbits [39] implying the existence of complicated (chaotic) dynamics nearby [52, 51].

Separatrices of the saddle-center are orbits (different from the equilibrium itself) on two invariant analytic curves through the equilibrium, in Moser coordinates they are $x_1 = x_2 = y_2 = 0$ and $y_1 = x_2 = y_2 = 0$ (strong stable W^s and strong unstable W^u local manifolds). A two-dimensional center manifold near the saddle-center is given as $x_1 = y_1 = 0$, it is filled with Lyapunov saddle periodic orbits lying each in its own level of the Hamiltonian. These periodic orbits are saddle ones in the related level of H .

The continuation of an unstable separatrix by the flow within the singular level can lead to its merge with one of two stable separatrices forming a homoclinic orbit to the saddle-center. The local orbit structure of the flow near such orbit is rather well known since [39] (see also [35, 28, 47, 28, 29, 53]). The orbit behavior depends essentially on the case which is realized of two possible ones here [39]. To explain this, let us remind the local structure of the Hamiltonian near a saddle-center (see, Fig. 6-8). We present here only related pictures (see details in [41]). On these pictures it is seen the local behavior of orbits, as well.

The cases mentioned depend on how the homoclinic orbit connects cutting disks of two solid cylinders: the orbit can connect disks from the same solid cylinder (case 1) or two different ones (case 2). As simulations show that we deal with the case 2 for the system under study. Thus, two separatrices going to the half-space $r > 0$ may form homoclinic orbits corresponding to solitons, and two remaining going to the half-space $r < 0$ may form homoclinic orbits corresponding to cavitons. Related orbits have been found, as an example, they are plotted in Fig.9a,b. They

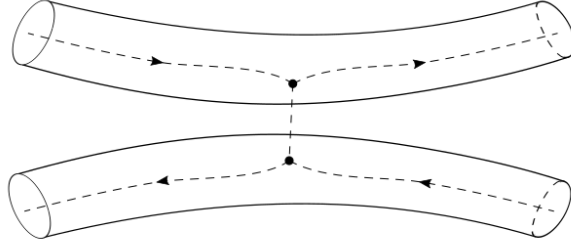


Figure 6: $c = 0$, observe two glued points – saddle-center

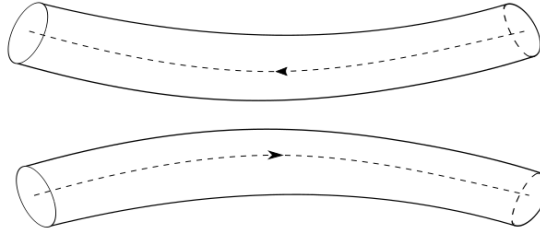


Figure 7: $c < 0$

represent both solitons and solitons.

If the genericity condition mentioned above holds for a homoclinic orbit of the saddle-center, then in the level $H = 0$ containing the equilibrium and the homoclinic orbit there exist also countably many saddle long periodic orbits accumulating at the homoclinic orbit to the saddle-center [39, 35]. As an example, such periodic orbit is shown on Fig. 17 but its fact there are many of them. For the system we study all this picture takes place at fixed values of parameters D_*, λ_* for which a homoclinic orbit to a saddle-center exists.

When varying parameters D, λ , the orbit structure of the flow varies. In particular, a homoclinic orbit to the saddle-center generically fails to exist (it is destroyed). Instead, multi-round homoclinic orbits to O can arise [36]. Because saddle periodic orbits accumulate to the former homoclinic loop, a situation may occur, when an orbit on an one-dimensional unstable manifold of the saddle-center (which persists under small changes D, λ) gets lie on the stable manifold of some saddle periodic orbit γ in the same level of H . Since the system under consideration is, in addition, reversible, and if saddle-center O and saddle periodic orbit γ are symmetric, then pairing orbit of the stable manifold of the saddle-center gets lie by symmetry on the unstable manifold of γ . Thus, in this case a heteroclinic connection is made up of two heteroclinic orbits, a symmetric saddle-center and a symmetric saddle periodic orbit γ .

Such heteroclinic connection can be of two types in dependence of how two heteroclinic orbits are displaced with respect to two local solid cylinders $H = H(p)$, described above. Namely, they can either intersect both the same cylinder (case 1) or one heteroclinic orbit intersects one

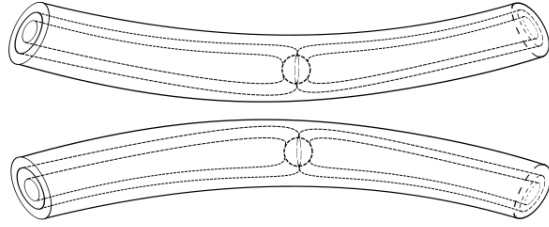
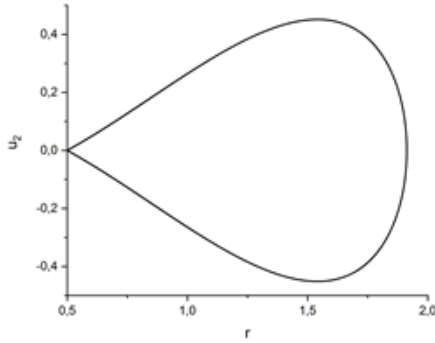
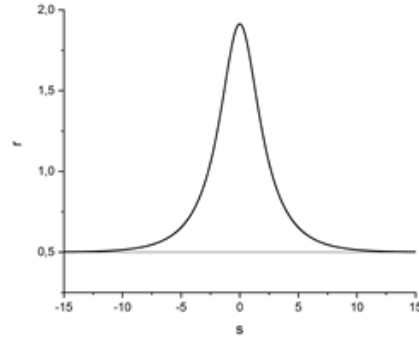


Figure 8: $c > 0$



a)



b)

Figure 9: (a) 1-round homoclinic, $\lambda = 0.5$, $D^2 = 0.23$; (b) Unfolding of this 1-round soliton.

cylinder, but another one does another cylinder (case 2). Our simulations show that we deal here with the case 2. This implies that orbits leaving the unstable manifold of γ can return to lie on its stable manifold only making at least one passage near two remaining stable and unstable manifolds of the saddle-center. For our case this means that such orbits have to intersect the plane $r = 0$ before they return to the stable manifold of γ . Existence of a heteroclinic connection of the type indicated is shown in the Fig.10. Studying an orbit behavior near this connection was performed recently [40].

Another feature of the system under varying parameters (D, λ) is the appearance of new homoclinic orbits to the saddle-center. Due to reversibility of the systems and the type of action of the reversor L locally (the intersection of the fixed point set of L with the singular level $H = 0$ is the curve through p) both homoclinic orbits (solitons) and cavitons are usually symmetric orbits (i.e. invariant w.r.t. L). So, at a fixed D^2, λ , $0 < \lambda < 1$, only one symmetric soliton can exist and only one symmetric caviton. Under varying parameters these homoclinic orbits usually are destroyed, but can exist multi-round homoclinic orbits which before closing make several excursions near the former 1-round homoclinic orbits. Moreover, for a reversible system in the plane of parameters (D^2, λ) there are usually countably many bifurcation curves accumulating to the curve of 1-round homoclinic orbits [47, 29]. Our calculations show just this behavior, see, Figs 12-16.

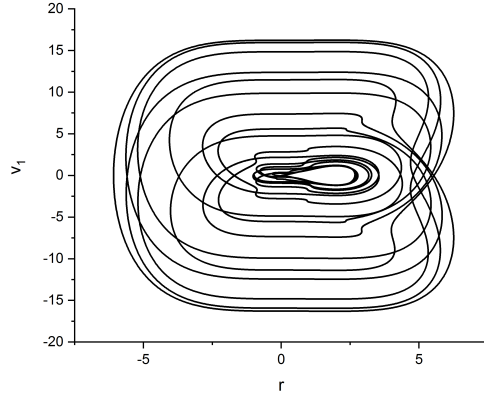


Figure 10: Heteroclinic connection in the singular level of H .

One more situation that can arise under varying parameters (D^2, λ) , is the existence of non-symmetric homoclinic orbits. By symmetry, if such an orbit exists, there is another homoclinic orbit being the symmetric counterpart of the former. The existence of such nonsymmetric homoclinic orbit requires the 2-parameter analysis, they exist at selected points (see Fig.11) a,b.

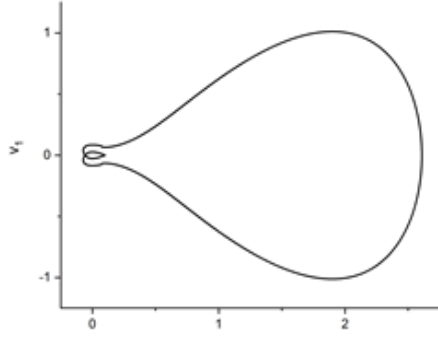


Figure 11: (a) Nonsymmetric soliton at $(\lambda = 0.0887472, D^2 = 0.247906)$; (b) Its unfolding.

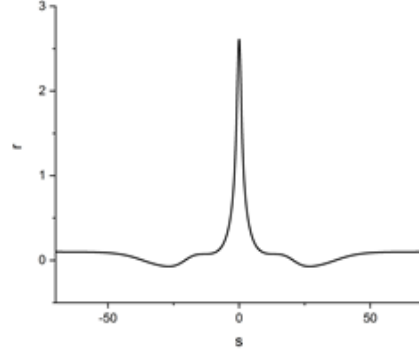
7 Small positive $1 - \lambda$

Let us now study the problem for small positive $1 - \lambda$ near the point $(0, 0, 0, 0)$ on the upper sheet. As was said above, this equilibrium is degenerate at $\lambda = 1$ with double zero eigenvalue and two imaginary eigenvalues $\pm i\omega$. Let us scale the initial equation (5):

$$\lambda = 1 - \varepsilon^2, \quad \tau = \varepsilon y, \quad \frac{d}{dy} = \varepsilon \frac{d}{d\tau}, \quad S = \varepsilon^2 X, \quad D = \frac{\kappa}{\varepsilon}.$$

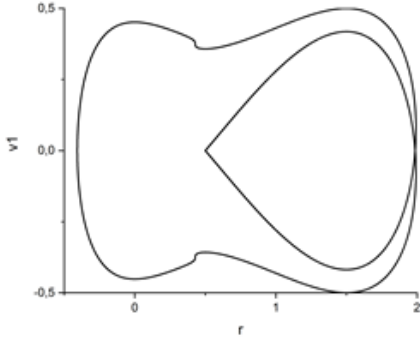


a)

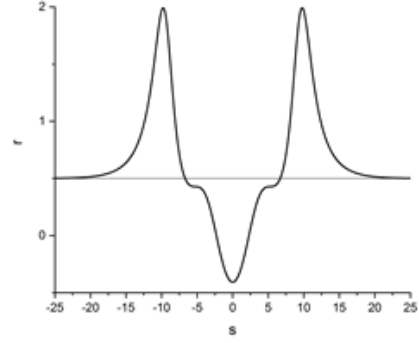


b)

Figure 12: (a) Simplest caviton; (b) Its unfolding.

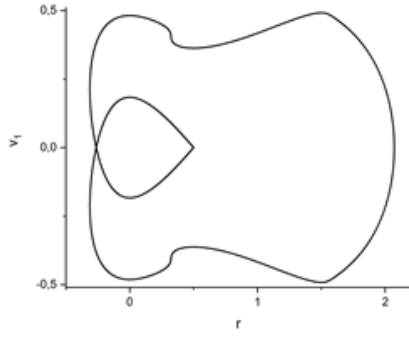


a)

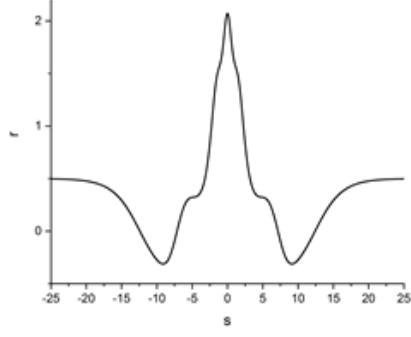


b)

Figure 13: (a) 2-round homoclinic orbit, $\lambda = 0.5, D^2 = 0.47$; (b) Its unfolding – 2-hump soliton.

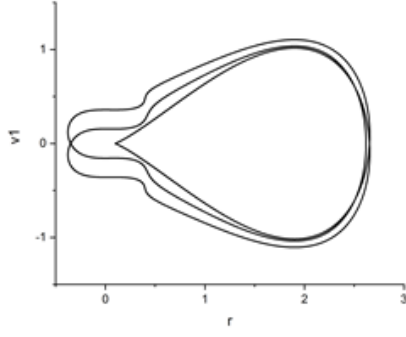


a)

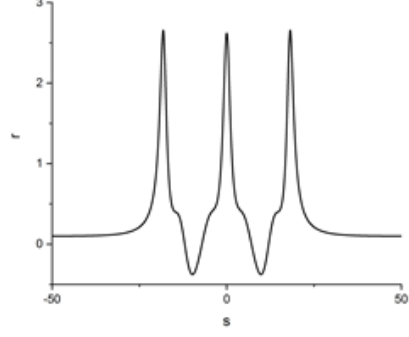


b)

Figure 14: (a) 2-round homoclinic orbit with sharpening, $\lambda = 0.5, D^2 = 0.188847$; (b) Its unfolding – 2-hump caviton.

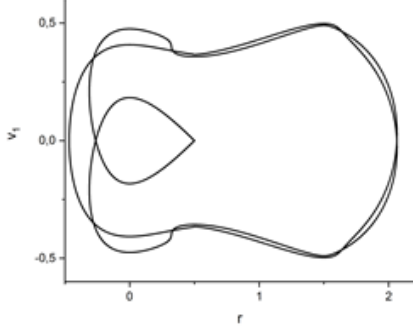


a)

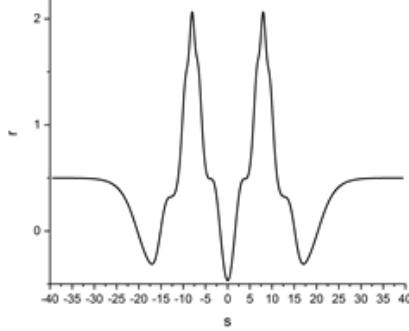


b)

Figure 15: (a) 3-round homoclinic orbit, $\lambda = 0.1, D^2 = 0.315485$; (b) Its unfolding – 3-hump soliton.



a)



b)

Figure 16: (a) 3-round homoclinic orbit with sharpening, $\lambda = 0.5, D^2 = 0.181595$; (b) Its unfolding – 3-hump caviton.

As a result of these transformations we come to the following equation

$$\kappa^2 \frac{d^4 X}{d\tau^4} + \frac{d^2 X}{d\tau^2} - X + \frac{1}{2}X^2 - \varepsilon^2 X(1 - \frac{3}{2}X + \frac{1}{2}X^2) + \varepsilon^4(-X + 3X^2 - \frac{5}{2}X^3 + \frac{5}{8}X^4) + \dots, \quad (15)$$

that defines the behavior of solutions as $\lambda \rightarrow 1 - 0$. As above, let us reduce the equation to the Hamiltonian system by means of the change of variables $u_1 = X$, $u_2 = X'$, $v_1 = -X' - \kappa^2 X'''$, $v_2 = \kappa X''$. The equation is reduced to the slow-fast Hamiltonian system with respect to the symplectic form $dv_1 \wedge du_1 + \kappa dv_2 \wedge du_2$

$$\begin{aligned} u_1' &= u_2, \quad v_1' = -u_1 + \frac{1}{2}u_1^2 - \varepsilon^2 u_1(1 - \frac{3}{2}u_1 + \frac{1}{2}u_1^2) + \varepsilon^4(-u_1 + 3u_1^2 - \frac{5}{2}u_1^3 + \frac{5}{8}u_1^4) + \dots, \\ \kappa u_2' &= v_2, \quad \kappa v_2' = -(u_2 + v_1). \end{aligned} \quad (16)$$

The Hamiltonian of the system is

$$H = u_2 v_1 + \frac{1}{2}(u_2^2 + v_2^2) + \frac{1 + O(\varepsilon^2)}{2}u_1^2 - \frac{1 + O(\varepsilon^2)}{6}u_1^3 + \frac{\varepsilon^2(1 + O(\varepsilon^2))}{8}u_1^4 - \frac{\varepsilon^4(1 + O(\varepsilon^2))}{8}u_1^5 + \dots$$

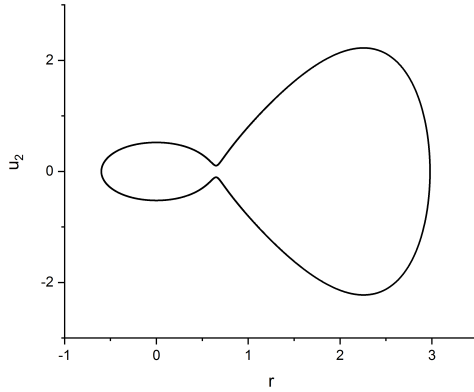


Figure 17: Saddle periodic orbit in the singular level of H .

In this form we get a problem about an orbit behavior near a ghost separatrix on an almost invariant elliptic manifold where a saddle equilibrium with a homoclinic orbit exists. Such problem was studied partially in [22].

The system (16) is also reversible with respect to the involution $L : (u_1, v_1, u_2, v_2) \rightarrow (u_1, -v_1, -u_2, v_2)$ with its fixed point set $Fix(L) = \{v_1 = 0, u_2 = 0.\}$

In order to find a homoclinic orbit to the saddle-center it is important to keep in mind that we have a reversible slow-fast Hamiltonian system whose fast system is a fast rotation. Indeed, to get the fast system, we do the scaling of the independent variable $\tau/\kappa = \xi$, then the small multiplier κ appears in the right hand sides of the first and second differential equations. Then, setting $\kappa = 0$ we get u_1, v_1 as parameters of the system of two remaining equations. They are linear and have an equilibrium – center – on any leaf $u_1 = u_1^0, v_1 = v_1^0$. Thus, all assumptions of the theorem 1 from [21] hold and therefore there is a neighborhood U of a compact region in the slow plane $u_2 = -v_1, v_2 = 0$, where analytic Hamiltonian by an analytic symplectic change of variables is transformed to the function $H = H_0(I, u, v, \kappa) + R(x, y, u, v, \kappa)$, $I = (x^2 + y^2)/2$, $|R| = O(\varepsilon[-c/\kappa])$. Thus, in U the Hamiltonian is exponentially close to an integrable Hamiltonian H_0 with I being an additional integral. In particular, this theorem works for a region U which contains on the slow plane the separatrix loop of the saddle. Also, in this case a theorem from [22] holds which asserts the validity of the Moser normal form [49] for H in some neighborhood of saddle-center of the size $O(C\varepsilon)$. These two theorems allows two prove the following theorem being an analog of the theorem 1 from [14]

Theorem 1 *For a small positive κ in the whole phase space a neighborhood of the order κ of the homoclinic orbit on the slow manifold exist such that two branches of stable and unstable separatrices of the saddle-center which cut the cross-section $x_2 = 0$ first time are displaced on the distance of the order $R \exp[-c/\kappa]$ with some positive constants R, c .*

For $\varepsilon = 0$ the equation above is the well studied, it also models the form of stationary water

waves on the surface of a liquid with the surface tension [10, 30, 14], if we eliminate terms of the order ε^2 and higher. It was proved for small κ [14] this equation to have not localized solutions, or, in other terms, no homoclinic solutions to the corresponding saddle-center exist for the related slow-fast Hamiltonian system. Nevertheless, our simulations have shown the existence of homoclinic orbits under varying ε, κ (see, as a hint, Fig.5b). More exactly, the following hypothesis seems to be valid

Hypothesis. There is a neighborhood of the point $(0, 0)$ on the parameter plane (κ, ε) such that a countable set of bifurcation curve exists which correspond to the existence of homoclinic orbits of any roundness.

8 Homoclinic loops to saddle-focus

The calculations of equilibria and their types show, in particular, that if $D^2 > 1/4$, then for positive $\lambda > \lambda_0$ the equilibrium O on the upper sheet is the saddle-focus. The simulations discovered the abundance of symmetric homoclinic orbits to this equilibrium (see Fig.18-21). These homoclinic orbits are usually transverse in the following sense. The related singular 3-dimensional level of the Hamiltonian (containing the saddle-focus) includes both smooth 2-dimensional stable and unstable manifolds of the saddle-focus and their intersection along the homoclinic orbit is transverse within this level. The theory of the complicated orbit behavior near a saddle-focus loop was developed by Shilnikov [50] for general systems and later adapted [18] to cover the case of Hamiltonian systems (see also [19, 24, 17] where some elements of complex dynamics were proved for reversible systems. An overview of these results can be found in [25]). It says that near a transverse homoclinic orbit there exists multi-pulse homoclinic orbits and a complicated behavior of nearby orbits (hyperbolic subsets) [18]. Moreover, varying levels of the Hamiltonian leads to many bifurcations of hyperbolic sets, creations of elliptic periodic orbits, etc. [42, 43]. Also, for the system under study there are those symmetric homoclinic orbits which intersect during their travel the branching plane $u_1 = -\lambda^2/2$. Such homoclinic orbits can also be named *cavitons with oscillating asymptotics* at infinity. Near them multi-pulse cavitons also exist as well as a complicated orbit structure.

Let us prove the existence of two symmetric homoclinic orbits to the equilibrium O on the upper sheet for small enough $\lambda - \lambda_0$. To do this, we use results of studying the Hamiltonian Hopf bifurcation [46] and their realization for the Swift-Hohenberg equation [23, 26]. Recall that the Hamiltonian Hopf bifurcation is the bifurcation in an one-parameter family of Hamiltonian systems in two degrees of freedom having equilibria for all values of a parameter and at some critical value of a parameter the related equilibrium has two double pure imaginary eigenvalues each with the 2-dimensional Jordan box (non-semisimple case). The type of bifurcation that occurs under transition through this critical value of the parameter depends on the sign of some coefficient in the normal form of the Hamiltonian near this equilibrium. In particular, if this coefficient is positive, then for those values of the parameter, when the equilibrium is a saddle-focus, the system, if it is, in addition, reversible, the saddle-focus gives the birth of two small

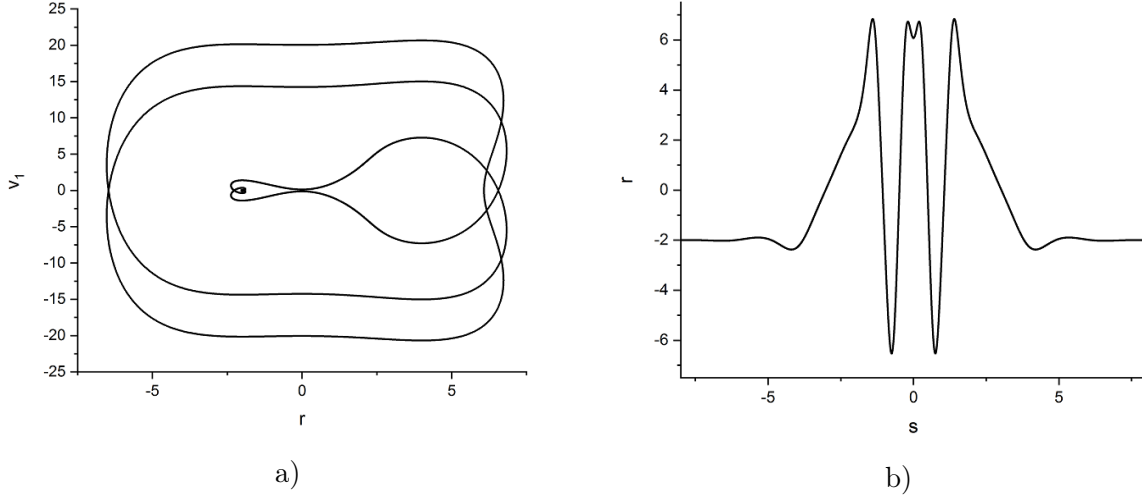


Figure 18: (a) Saddle-focus homoclinics, $D^2 = 0.36$, $\lambda = -2$; (b) and its unfolding.

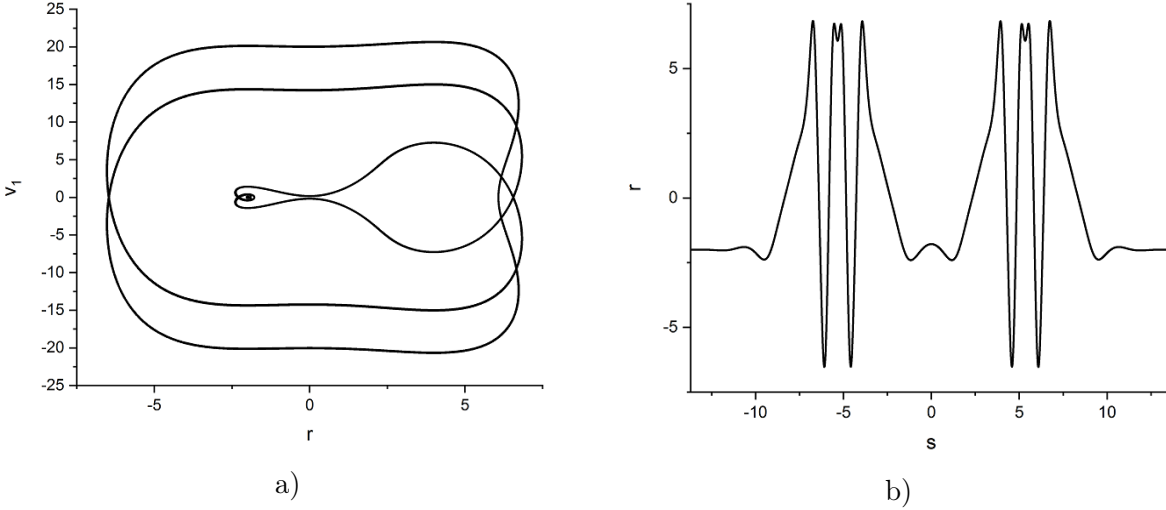


Figure 19: (a) Saddle-focus homoclinics, $D^2 = 0.36$, $\lambda = -2$, 2-rounded; (b) and its unfolding.

symmetric homoclinic orbits. The reversibility here guarantees their existence, otherwise, to find such orbits is a very delicate problem related (for analytic systems) with exponentially small splitting of stable and unstable manifolds of the saddle-focus [20].

We prove the result reducing our problem to that being similar to the problem as for the Swift-Hohenberg equation. To that end, let us scale the traveling coordinate $y = \gamma\xi$, $\gamma = \sqrt{2D}$, in the initial equation (4). After scaling and dividing at D^2 we get the equation

$$u^{(IV)} + 2u'' + u = (1 - 4D^2 + \frac{4D^2}{\lambda})u - \frac{2D^2}{\lambda^3}u^2 + \frac{2D^2}{\lambda^5}u^3 + \dots$$

In notations of [23] we get $\alpha = 1 - 4D^2 + \frac{4D^2}{\lambda}$, $\beta = -\frac{2D^2}{\lambda^3}$. The change $u \rightarrow -u$ allows one to make β positive as in [23]. Thus we get the criterion of the birth of homoclinic orbit when crossing

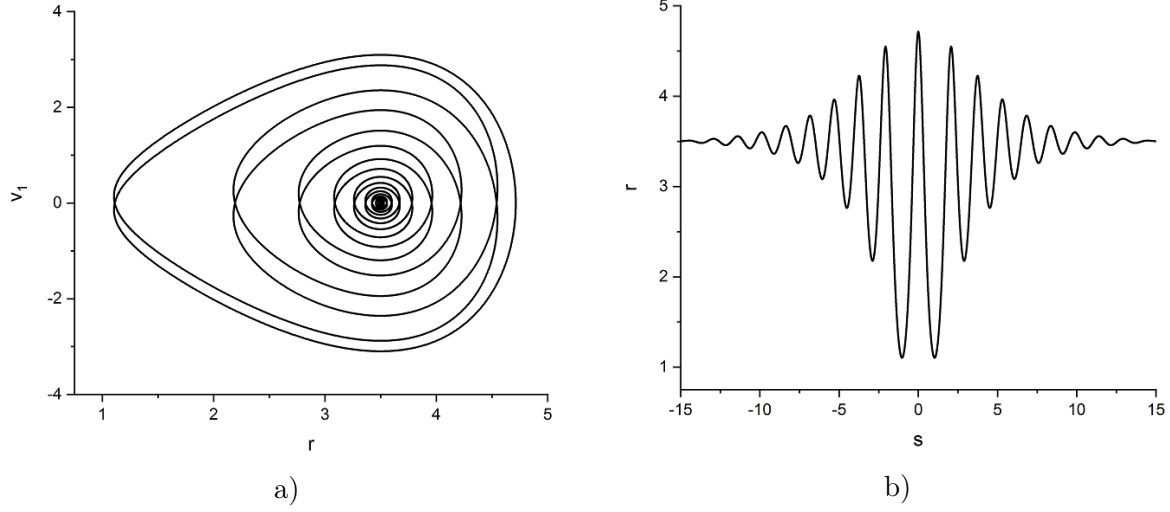


Figure 20: (a) Saddle-focus homoclinics, $D^2 = 0.36$, $\lambda = 3.5$, and (b) its unfolding.

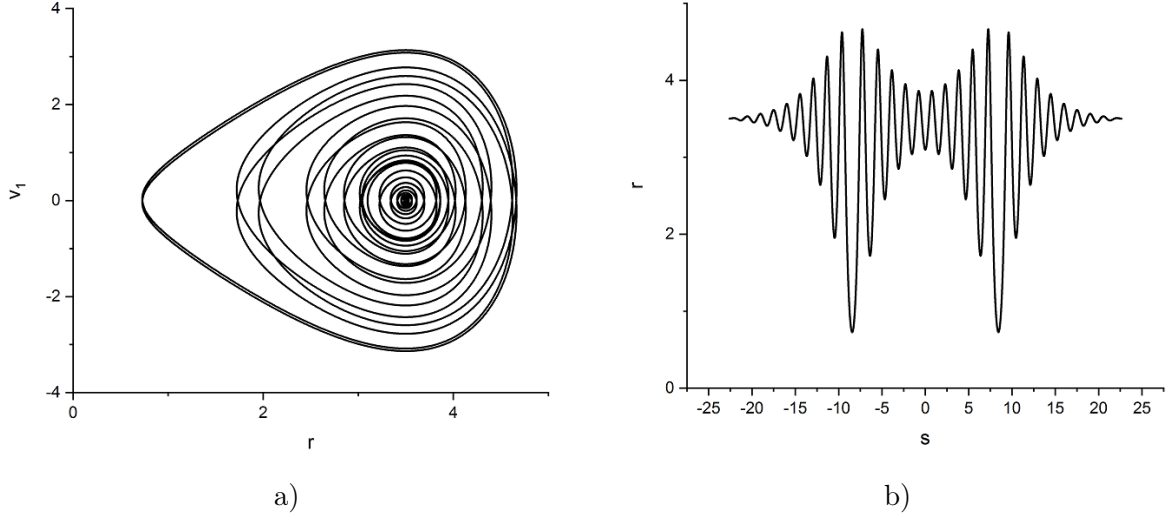


Figure 21: (a) Saddle-focus homoclinics, $D^2 = 0.36$, $\lambda = 3.5$, 2-rounded and (b) its unfolding.

$\alpha = 0$, i.e. we just get $\lambda = \lambda_0$. Nevertheless, the equation differs from [23], since coefficient at the term with u^3 is positive. So, we need to calculate the needed coefficient in the normal form directly. We perform this calculation using the averaging. This was done long ago [44] but unpublished.

We calculate the coefficient, we remark that saddle-foci appear as $\lambda > \lambda_0$ as $D^2 > 1/4$. Denote $-\nu = 1 - 4D^2 + \frac{4D^2}{\lambda}$ and consider ν as small positive parameter. After scaling $u = -\kappa u$ with $\kappa = \sqrt{2}D/\lambda^{5/2}$ we come to the equation of the form (we preserve old notations)

$$u^{(IV)} + 2u'' + u = -\nu u + \beta u^2 + u^3 + \dots. \quad (17)$$

Let us write the equation in the form of two second order equations

$$u'' + u = v, \quad v'' + v = -\nu u + \beta u^2 + u^3 + \dots$$

After scaling $u \rightarrow \sqrt{\nu}$, $v \rightarrow \nu v$ and denoting $\mu = \sqrt{\nu}$, we get the system

$$u'' + u = \mu v, \quad v'' + v = \beta u^2 - \mu(u - u^3) + O(\mu^2).$$

At $\mu = 0$ we have the system whose solutions are of the form

$$u = A \exp[i\xi] + \bar{A} \exp[-i\xi], \quad v = B \exp[i\xi] + \bar{B} \exp[-i\xi] - \frac{\beta}{3}(A^2 \exp[2i\xi] + \bar{A}^2 \exp[-2i\xi]) + 2\beta|A|^2.$$

We add here new variables $u' = p$, $v' = q$ and differentiation of above equalities under an assumption that A, B are constant gives the relations for p, q through A, \bar{A}, B, \bar{B} . We consider these relations as the change of variables $(u, v, p, q) \rightarrow (A, \bar{A}, B, \bar{B})$. Observe that this change of variables depend 2π -periodically in ξ .

Performing this change of variables, we come to the system of four first order differential equations in variables (A, \bar{A}, B, \bar{B}) which is the 2π -periodic system in the so-called standard form of the averaging method (see, [15]) $X' = \mu F(X, \xi)$. Averaging this system in ξ gives the average system $Y' = \mu F_0(Y)$,

$$F_0(Y) = \frac{1}{2\pi} \int_0^{2\pi} F(Y, \xi) d\xi.$$

For our case we have

$$A' = -i\mu \frac{B}{2}, \quad B' = i\mu \frac{A}{2} \left[1 - \frac{27 + 2\beta^2}{9}|A|^2\right], \quad \bar{A}' = c.c., \quad \bar{B}' = c.c. \quad (18)$$

The coefficient we sought for is $\frac{27+2\beta^2}{9}$. It is positive that means the existence of the homoclinic skirt in the system (18) which is integrable and the existence of two symmetric homoclinic orbits in the initial system due to its reversibility [33]. The structure of the averaged system is easily restored if introduce real variables (a, b, c, d) , $A = a + ib$, $B = c + id$. In these variables we have a Hamiltonian system

$$a' = c, \quad c' = \frac{a}{4}(1 - L(a^2 + b^2)), \quad b' = d, \quad d' = \frac{b}{4}(1 - L(a^2 + b^2)), \quad L = \frac{27 + 2\beta^2}{9}$$

with Hamiltonian

$$H = \frac{c^2 + d^2}{2} - \frac{a^2 + b^2}{8} + \frac{L}{16}(a^2 + b^2)^2$$

and an additional integral $K = ad - bc$. The common level $H = K = 0$ gives the homoclinic skirt, i.e. one-parameter family of homoclinic orbits to the equilibrium O of a saddle type with merged 2-dimensional stable and unstable manifolds.

In the similar way one can check that zero equilibrium O on the lower sheet for negative λ also gives the birth of homoclinic orbits to a saddle-focus as $0 < D^2 < 1/4$ at $\lambda = \lambda_-$.

As we have seen above, saddle-foci in the system exist both for negative λ and for positive λ . Results of our simulations in these cases are plotted in Fig. 18,19 for negative λ . For positive λ we get the following plots, Fig. 20,21.

9 Conclusion

In this work we have studied localized traveling wave solutions of the nonlocal Whitham equation by means of the reduction to a Hamiltonian system. This initial equation is of the fourth order with a nonlinearity being double-valued. The reduction allows to derive a two degrees of freedom Hamiltonian system but it defined on the two-sheeted space due to the type of nonlinearity. In addition, the system is reversible with respect to some involution. This permitted to obtain a clear geometric description of both smooth solutions and solutions with singularities and apply to the problem of developed methods of the theory of Hamiltonian dynamics, in particular, theory of homo- and heteroclinic orbits. The search for homoclinic and heteroclinic orbits in dynamical systems is a very nontrivial problem, being global in its own nature. Therefore, numerical methods with the sharp set up allow to solve this problem for the concrete equation like that under study. But the numerical search can be made much more rigorous if we have some points in the parameter space (our (D^2, λ)) at which the system has degenerate equilibria. Then bifurcation methods allows one to find homoclinic orbits through the bifurcation. We do this using Hamiltonian Hopf bifurcation and calculation the needed coefficients in the local normal form to determine the type of the bifurcation. All this together allowed one to investigate the system with many details.

10 Acknowledgement

The work by L.Lerman was partially supported by the Laboratory of Topological Methods in Dynamics NRU HSE, of the Ministry of Science and Higher Education of RF, grant #075-15-2019-1931 and by Ministry of Science and Higher Education of Russian Federation (Project #0729-2020-0036). Numerical simulations were performed under a support of the Russian Foundation of Basic Research (grants 18-29-10081, 19-01-00607).

References

- [1] G.B. Whitham. Linear and Nonlinear Waves. Pure and Appl. Math. , Wiley, New York, 1974.
- [2] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, J. Fluid Mech., v.29 (1967), 559-592.
- [3] K.M. Case, The Benjamin-Ono equation: a remarkable dynamical system, Ann. Nuclear Energy, v.7 (1980), 264-266.
- [4] R.J. Joseph, Solitary waves in a finite depth fluid, J. Phys. A, v.10 (1977), 225-287.
- [5] R. Benguria, M. Depassier, Equations of Korteweg - de Vries type with conserved quantities, J. Phys. A 22 (1989), 4135-4142.
- [6] S.D. Leibovitz, Weakly nonlinear waves in rotating fluid, J. Fluid Mech., v.42 (1970), 803-822.
- [7] Yu.L. Klimontovich. Statistic Theory of Non-Equilibrium Processes in Plasma, MIT Press, Cambridge, MA, 1967.
- [8] P.I. Naumkin, I.A. Shishmarev. Nonlinear nonlocal equations in the theory of waves, Amer. Math. Soc., Providence, R.I., v.133 (1998).
- [9] V.I. Arnold. Geometrical Methods in the Theory of Ordinary Differential Equations, Second Edition, Springer, 1988.
- [10] C.J. Amick, K. Kirschgässner, A theory of solitary water-waves in the presence of surface tension, Arch. Rational Mech. Anal. 105 (1988), 1-50.
- [11] E. Dinvari, H. Kalish, D. Moldabayev, E.I. Parah, The Whitham equation for hydroelastic waves, Appl. Ocean Res., v.89 (2019), 202-210.
- [12] N.N. Myagkov. On hydrodynamic instabilities of chemical oscillations, JETP, v.81(6)(1985), 1108-1116.
- [13] A. Gevorgyan, N. Kulagin, L. Lerman, A. Malkin, Solitons of Whitham equation with resonance dispersion, submitted.
- [14] Amick C.J. and McLeod J.B. A singular perturbation problem in water waves, Stab. Appl. Anal. Cont. Media, v.1 (1991), 127-148.
- [15] N.N. Bogoliubov, Yu.A. Mitropolsky. Asymptotic Methods in the Theory of Non-linear Oscillations (Hindustan Publishing Company, Delhi, 1961).

- [16] A.R. Champneys, Codimension-one persistence beyond all orders of homoclinic orbits to singular saddle centres in reversible systems, *Nonlinearity*, v.14 (2001), 87–112.
- [17] A.R. Champneys, J. Härterich, Cascades of homoclinic orbits to a saddle-centre for reversible and perturbed Hamiltonian systems, *Dyn. Stab. Syst.*, v.15 (2000), No.3, 231-252.
- [18] R. Devaney. Homoclinic orbits in Hamiltonian systems. *J. Diff. Equat.*, V.21 (1976), 431–439.
- [19] R. Devaney, Blue sky catastrophes in reversible and Hamiltonian systems, *Indiana Univ. Math. J.*, v.26 (1977), 247–263.
- [20] V. Gelfreich, J.P. Gaivão, Splitting of separatrices for the Hamiltonian-Hopf bifurcation with the Swift–Hohenberg equation as an example, *Nonlinearity*, v.24 (2011), 677–698.
- [21] V. Gelfreich, L. Lerman, Almost invariant elliptic manifold in a singularly perturbed Hamiltonian system, *Nonlinearity*, v. 15 (2002), 447-457.
- [22] V. Gelfreich, L. Lerman, Slow-fast Hamiltonian Dynamics Near a Ghost Separatrix Loop, *J. Math. Sci.*, Vol. 126 (2005), No.5, 1445-1466.
- [23] L.Yu. Glebsky, L.M. Lerman, On small stationary localized solutions for the generalized 1D Swift-Hohenberg equation, *Intern. J. Nonlin. Sci.: Chaos*, v.5 (1995), No.2, 424-431.
- [24] J.R. Härterich, Cascades of reversible homoclinic orbits to a saddle-focus equilibrium, *Physica D* 112, No.1-2 (1998), 187-200.
- [25] A.J. Homburg, B. Sandstede, Homoclinic and Heteroclinic Bifurcations in Vector Fields, in “Handbook of Dynamical Systems”, v.3, Eds. B.Hasselblatt, H.W.Broer, F.Takens, North Holland, Amsterdam, 2010, pp.379-524.
- [26] J. Burke, E. Knobloch, Normal form for spatial dynamics in the Swift-Hohenberg equation, *Discr. and Cont. Dyn. Syst., Suppl.*, 2007, 170-180.
- [27] W. Giles, J.S.W. Lamb, D. Turaev, On homoclinic orbits to center manifolds of elliptic-hyperbolic equilibria in Hamiltonian systems, *Nonlinearity*, v.29, No.10, 3148-3173.
- [28] C. Grotta Ragazzo, Nonintegrability of some Hamiltonian systems, scattering and analytic continuation, *Comm. Math. Phys.* Vol.166 (1994), 255–277.
- [29] C. Grotta Ragazzo, Irregular dynamics and homoclinic orbits to Hamiltonian saddle-centers. *Commun. Pure Appl. Math.*, v.50 (1997), No.2, 105-147.
- [30] J.M. Hammersley, G. Mazzarino, Computational aspects of some autonomous differential equations, *Proc. R. Soc. Lond. A*, v.424 (1989), 19–37.
- [31] P. Hartman. *Ordinary Differential Equations*, Academic Press, 1970.

- [32] Iooss G and Kirchgassner K 1992 Water waves for small surface tension: an approach via normal forms Proc. R. Soc. Edin. A 112 267–300.
- [33] G. Iooss, M.C. Peroéme, Perturbed Homoclinic Solutions in Reversible 1:1 Resonance Vector Fields, J. Diff. Equat., v.101 (1993), 62-88.
- [34] Koltsova O. Yu., Lerman L. M., Periodic and homoclinic orbits in a two-parameter unfolding of a Hamiltonian system with a homoclinic orbit to a saddle-center, Int. J. Bifurcation & Chaos. 1995. Vol. 5, No. 2. P. 397–408.
- [35] Koltsova O. Yu., Lerman L. M., Families of Transverse Poincaré Homoclinic Orbits in 2N-Dimensional Hamiltonian Systems close to The System with a Loop to a Saddle-center, Int. J. Bifurcation & Chaos, v.6 (1996), No.6, 991-1006.
- [36] O.Yu. Koltsova, Families of multi-round homoclinic and periodic orbits near a saddle-center equilibrium, Reg. Chaot. Dynam., v.8 (2003), No.2, 1-10.
- [37] M.D. Kruskal, H. Segur, Asymptotics beyond all orders in a model of crystal growth, Stud. Appl. Math. v.85 (1991), 129–181.
- [38] N.E. Kulagin, L.M. Lerman, T.G. Shmakova, Fronts, Traveling Fronts, and Their Stability in the Generalized Swift–Hohenberg Equation, Comput. Math. & Math. Phys., Vol.48 (2008), No. 4, 659–676.
- [39] L.M. Lerman, Hamiltonian systems with a separatrix loop of a saddle-center, in “Methods of Qualitative Theory of Diff. Equations” (Ed. E.A. Leontovich-Andronova), Gorky State Univ., 1987. 89–103 (in Russian) (Engl. transl. Selecta Math. Sov., 1991, vol. 10, No. 3, p. 297–306).
- [40] L.M. Lerman, K.N. Trifonov, Dynamics of a reversible Hamiltonian systems in a neighborhood of symmetric heteroclinic contour (submitted).
- [41] L. M. Lerman, Ya. L. Umanskii, Classification of four-dimensional Hamiltonian systems and Poisson actions of \mathbb{R}^2 in extended neighborhood of simple singular points, I, Matem. Sbornik, (in Russian), 1992, vol. 183, 141-176.
- [42] L.M. Lerman. Complex dynamics and bifurcations in Hamiltonian systems having the transversal homoclinic orbit to a saddle-focus, Chaos: Interdisc. J. Nonlin. Sci. 1991. V.1. No.2. P. 174–180.
- [43] L. Lerman. Dynamical phenomena near a saddle-focus homoclinic connection in a Hamiltonian system, J. Stat. Physics, v. 101, No. 1-2 (2000), 357-372.
- [44] L.M. Lerman, Normalization and averaging, a manuscript, 2001 (unpublished).

- [45] Malkin A.I., Acoustic solitons in elastic pipes filled with a liquid, Doklady RAS, Mechanics, v.342 (1995), No.5, 621-625 (in Russian).
- [46] J.-C. van der Meer. The Hamiltonian Hopf Bifurcation, Lect. Notes in Math., v.1160, Springer-Verlag, Berlin, 1985.
- [47] Mielke A., Holmes P., O'Reilly O., Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle-center, J. Dyn. Diff. Equat., v.4 (1992), No.1. 95–126.
- [48] J. Milnor. Morse theory, Princeton University Press, Princeton, NJ, 1963.
- [49] Moser J., On the generalization of a theorem of A. Liapunoff, Comm. Pure Appl. Math., v.11 (1958), 257–271.
- [50] L.P. Shilnikov, A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type, Math. USSR-Sb., 10:1 (1970), 91–102.
- [51] L.P. Shilnikov, On a Poincaré-Birkhoff problem, Math. USSR-Sb., 3:3 (1967), 353–371.
- [52] S. Smale, Diffeomorphisms with many periodic points, in “Differential and Combinatorial Topology”, S.S. Carnes, Ed., Princeton Univ. Press, Princeton, NJ, 1965, 63-80.
- [53] K. Yagasaki, Horseshoes in Two-Degree-of-Freedom Hamiltonian Systems with Saddle-Centers, Arch. Rat. Mech. Anal., v.154 (2000), 275–296.