On Babuška's model for asymmetric hysteresis

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Abstract

Hysteresis is a complex phenomenon that occurs in many areas of science and engineering. To model hysteresis, scientists combine first laws of physics with fictitious equations to obtain phenomenological models that aim to match the macroscopic behavior of hysteresis processes. In this paper we focus on a model of hysteresis that was proposed by the physicist P. Duhem starting from thermodynamical considerations. On the one hand, we strengthen the results obtained by the mathematician I. Babuška in relation with the existence and uniqueness of a global solutions and of a periodic solution when the input is periodic. On the other hand, we specialize into a specific form of the Duhem model proposed by Babuška. For Babuška's model we determine the explicit analytic expression of the hysteresis loop. We apply our findings to dry friction modeling and to asymmetric hysteresis loop matching.

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Duhem model, Babuška's model, Asymmetric hysteresis loop.
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1. Introduction and problem statement

The term *hysteresis* refers to a special type of behaviour found in many processes like magnetism [1], geophysics [2, Chapter 6], economics [3], biology [4], and several other fields. A survey of the issue "*what is hysteresis*" is provided in [5, Section 2]. In short, when excited by a slow periodic input, a hysteresis system produces a loop in the steady state part of the graph output-versus-input [6].

The first dynamical model of hysteresis is–to the best of our knowledge–due to French physicist P. Duhem [7]. For an input u and a state x, Duhem's model has the form

$$\begin{cases} \dot{x} = f_1(x,u)\dot{u}, & \dot{u} \geq 0, \\ \dot{x} = f_2(x,u)\dot{u}, & \dot{u} \leq 0, \end{cases} \tag{1}$$

where $\dot{u}$ refers to the derivative of u with respect to time, and f_1, f_2 are functions.

In a series of articles published between 1896 and 1902, Duhem analyzes model (1) using geometric methods and relates the properties of the model to the observed behaviour of some hysteresis processes [7]–[13].

Whilst the methods used by Duhem are intuitive, they lack in general the standards of today's mathematical rigor. For example, no proof is provided for the existence and uniqueness of the solutions of the differential equation (1). Also, Duhem assumes the existence of a periodic solution x for a periodic input u.

To the best of our knowledge, the first mathematically rigorous work on Duhem's model has been done by I. Babuška in 1959 [14]. This paper has gone largely unnoticed in the current English-dominated literature as [14] is written in Russian. Babuška proposes sufficient conditions on functions f_1 and f_2 that guarantee the existence and uniqueness of the solutions of the differential equation (1). Moreover, these conditions guarantee also the existence and uniqueness of a periodic solution of (1) when u is periodic. Examples of functions f_1, f_2 that satisfy Babuška's conditions are provided in [14, Examples 1 and 2] as

$$f_1(a,b) = h_1(a)g_1(b), \tag{2a}$$
$$f_2(a,b) = h_2(a)g_2(b), \tag{2b}$$

where the functions h_1, h_2, g_1, g_2 are taken to be specific polynomials on some intervals. The plots in [14, Section 3] show the shape of the hysteresis loop that corresponds to [14, Example 1].

Both in Duhem's works [7]–[13] and Babuška's paper [14] no explicit analytic expression of the hysteresis loop is provided; and this is the main motivation for the present research work.

Why an explicit analytic expression of the hysteresis loop is important? To illustrate this point consider for instance the Bouc-Wen model which is a particular case of the model (1)–(2) for which

$$h_1(x) = \rho\big(1-\sigma|x|^{n-1}x+(\sigma-1)|x|^n\big), \tag{3a}$$
$$h_2(x) = \rho\big(1+\sigma|x|^{n-1}x+(\sigma-1)|x|^n\big), \tag{3b}$$
$$g_1(u) = 1, \tag{3c}$$
$$g_2(u) = 1, \tag{3d}$$

where $\rho > 0$, $\sigma \geq 0.5$, and $n \geq 1$ are the model parameters [15, pp. 41-42]. The analytic expression of the hysteresis loop is derived in [15, Theorem 3, p. 47] as an explicit function of the model parameters. For materials described by the Bouc-Wen model, the corresponding hysteresis loop is shown to be divided into two regions of transition, a linear region, and a plastic region, see [15, Figure 4.3, p. 68]. The points that define these regions are explicitly given, and the variation of the hysteresis loop with the model parameters is studied analytically thanks to the analytic expression of the hysteresis loop [15, Chapter 4].

We classify the contributions of the present paper into three categories:

(i) Improvement of the results obtained by Babuška in relation with the existence and uniqueness of global solutions and of periodic solutions.

(ii) Obtention of new results regarding the explicit analytic description of the hysteresis loop.

(iii) Application of these results to dry friction and hysteresis loop matching.

Regarding point (i) we have relaxed the assumptions used in [14] as follows:

- We have removed the assumption labeled Vc in [14] by using techniques that were made popular by Filippov in the 1980' [16], that is more than 20 years after the publication of Babuška's paper [14]. The result of this relaxation is Theorem 4.1 on the existence and uniqueness of a global solution of the Duhem model (1).

- We have relaxed the assumption on the input from continuously differentiable to absolutely continuous by using a normalized set of variables that we introduced in our work [17]. This is of practical importance in the context of hysteresis systems since–in general–differentiability cannot be assumed everywhere in applications, see for instance the displacement input of [18, Figure 6] which is continuous but not differentiable everywhere. The result of this relaxation is Theorem 7.1 on the existence and uniqueness of a periodic solution of the Duhem model (1) when the input u is periodic.

Regarding point (ii) we have considered Babuška's hysteresis model formed by the Duhem model (1) where the field is given by (2). We have identified four cases κ_1, κ_2, κ_3, κ_4 for the hysteresis loop and given for each case necessary and sufficient explicit conditions for the case κ_i to occur. Also, for each case κ_i we provided the explicit analytic expression of the hysteresis loop. The results of our findings are summarized in Theorem 9.1.

To stress the importance of Theorem 9.1 we remark the following:

- Although Babuška proves that the Duhem model (1) has a periodic solution for a periodic input, this periodic solution is not obtained as an explicit function of the field.

- Take for instance the Bouc-Wen model (3). A general form of the model has been proposed first by Bouc in 1971 [19] and later on specified by Wen in 1976 [20]. This model attracted a lot of attention as mentioned in the survey paper [21]. However it was not until 2005 that the hysteresis loop of the model was described analytically [22]. The main difficulty in studying this model is that the state appears nonlinearly in the state equation. To the best of our knowledge, the Bouc-Wen model is–up to date–the only particular case of the Duhem model with the state appearing nonlinearly, and for which an explicit analytic expression of the hysteresis loop is available.

- For a range of initial conditions, Babuška's model is a generalization of the Bouc-Wen model in which instead of the expressions (3) the functions h_1, h_2, g_1, g_2 are much more general: h_1 is an arbitrary decreasing function, h_2 an arbitrary increasing function, and g_1, g_2 arbitrary positive functions. Note that the state appears nonlinearly when the functions h_1 and h_2 are nonlinear.

Finally, regarding point (iii) we have obtained the following results:

- We proposed in Section 10 explicit sufficient conditions on Babuška's model to be compatible with the Coulomb model for dry friction, and we have compared the hysteresis loops of the LuGre and Dahl models to that of Babuška's model.

- In Section 11 we have shown that, for a wide range of experimental hysteresis loops that may be symmetric or asymmetric, Babuška's model can be chosen in such a way to make its hysteresis loop mach exactly the experimental one. The fact that this model can generate a wide range of hysteresis loops–both symmetric and asymmetric–is of particular interest in practice, see for instance [23, Figure 2.2, p. 19]. As a matter of fact, all hysteresis loops that appear in that figure can be generated by Babuška's model.

The paper is organized as follows. Section 2 presents the mathematical notation used in the text. Section 3 introduces the conditions that were proposed by Babuška to guarantee the existence and uniqueness of a global solution of the Duhem model, and a periodic solution when the input is periodic. Section 4 presents the theorem of existence and uniqueness of a solution for Duhem's model under Babuška's conditions. The version that we present in Theorem 4.1 is stronger than [14, Theorem 1] since the assumptions that we use are weaker whilst the obtained existence and uniqueness result is the same. Section 5 presents those results from Refs. [17] and [5] that are needed in this paper. This is the case in particular for the normalized variables which allow to write the Duhem model (1) in an equivalent simpler form; this is done in Section 6. The normalized form of the Duhem model allows to get a stronger theorem for the existence and uniqueness of a periodic solution x when the input is periodic. Indeed, instead of assuming that the input is continuously differentiable as in [14, Theorem 4], Theorem 7.1 considers that the input u is absolutely continuous, which means that u is allowed to lose differentiability on a set of measure zero. Theorem 7.1 uses weaker assumptions to get the same results as [14, Theorem 4]; this is done in Section 7. Using a result of the mathematician Filippov, Section 8 derives the analytic expression of the hysteresis loop of the Duhem model. However, this expression is not explicit since the initial condition of the periodic solution is given as a limit of a sequence of internal states, and also since the state x appears in the expression of the hysteresis loop. The main reason for this lack of explicit expression is that the state equations (1) cannot in general be integrated to obtain x. This is why Section 9 specializes into Babuška's model (1)–(2) since, in this case, the state equations can be integrated by separation of the variables. However, special care has to be taken in this integration since we have to ensure that the left-hand side function–that contains x–is well defined. The conditions for this well-definiteness is the object of Propositions 9.2–9.4. The conditions obtained directly from Propositions 9.2–9.4 are not explicit since they depend on information of the internal state x. This is why the object of whole analysis of Section 9 revolves around the idea of getting explicit conditions on the fields h_1, h_2, g_1, g_2, along with the input u to ensure the well-definiteness that allows the integration of (1)–(2) by separation of the variables. The main results of Section 9 are summarized in Theorem 9.1: we have identified four cases κ_1, κ_2, κ_3, κ_4 for the hysteresis loop and given for each case necessary and sufficient explicit conditions for the case κ_i to occur, along with the explicit analytic expression of the hysteresis loop in each case κ_i. We have also provided numerical simulations to illustrate these cases. As applications to the obtained results we considered dry friction modeling in Section 10, and hysteresis loop matching in Section 11. In particular we have derived the conditions under which Babuška's model is compatible with the Coulomb model, and compared its hysteresis loop to that of the LuGre and the Dahl models. We have also shown that the hysteresis loop of Babuška's model can match a wide range of experimental hysteresis loops that may be symmetric or asymmetric.

2. Mathematical notation

An ordered pair a, b is denoted (a, b) whilst the open interval $\{t \in \mathbb{R} \mid a < t < b\}$ is denoted $]a, b[$. The set of nonnegative integers is denoted $\mathbb{N} = \{0, 1, \ldots\}$ and the set of nonnegative real numbers is denoted $\mathbb{R}_+ = [0, \infty[$.

The Lebesgue measure on $\mathbb{R}$ is denoted μ. We say that a subset of $\mathbb{R}$ is measurable when it is Lebesgue measurable. Consider a function $g : I \subset \mathbb{R}_+ \to \mathbb{R}$ where I is an interval. We say that g is measurable when $\{x \in I : g(x) > a\}$ is measurable for all $a \in \mathbb{R}$. For a measurable function $g : I \to \mathbb{R}$, $\|g\|$ denotes the essential supremum of the function $|g|$.

$C^0(J_1, J_2)$ denotes the space of continuous functions $f : J_1 \to J_2$.

$\mathcal{S}(I, \mathbb{R})$ denotes the space of absolutely continuous functions $\phi : I \to \mathbb{R}$ such that $\|\phi\| < \infty$ and $\|\dot{\phi}\| < \infty$ where $\dot{\phi}$ is the derivative of ϕ.

For any $\gamma \in \,]0, \infty[$ define the linear change in time scale $s_\gamma : \mathbb{R} \to \mathbb{R}$ by $s_\gamma(t) = t/\gamma, \forall t \in \mathbb{R}$.

For any $a \in \mathbb{R}$ define the translation $\tau_a : \mathbb{R} \to \mathbb{R}$ by $\tau_a(t) = t + a, \forall t \in \mathbb{R}$.

Let $T \in \,]0, \infty[$. A function $u : \mathbb{R}_+ \to \mathbb{R}$ is said to be T–periodic, or periodic of period T, if $u(t + T) = u(t), \forall t \in \mathbb{R}_+$.

The symbol $\wedge$ stands for the logical AND, and the symbol $\neg$ sets for the logical NOT.

3. Babuška's Conditions

Consider the Duhem model:

$$\begin{aligned} \dot{x}(t) &= f_1\big(x(t), u(t)\big)\dot{u}(t), \text{ for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \geq 0, && \text{(4a)} \\ \dot{x}(t) &= f_2\big(x(t), u(t)\big)\dot{u}(t), \text{ for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \leq 0, && \text{(4b)} \\ x(0) &= x_0, && \text{(4c)} \end{aligned}$$

where $x_0 \in \mathbb{R}$ the initial condition, $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ the input, and $x : \mathbb{R}_+ \to \mathbb{R}$ the state. We refer to the differential equation (4) as the *scalar* Duhem model as $x(t) \in \mathbb{R}$.

We consider that f_1 and f_2 satisfy Conditions (i)–(v).

(i) $f_1, f_2 \in C^0(\mathbb{R}^2, \mathbb{R})$.

(ii) $f_1(a, b) \geq 0$ and $f_2(a, b) \geq 0$ for all $a, b \in \mathbb{R}$.

(iii) For all $a_1, a_2, b \in \mathbb{R}$ such $a_1 > a_2$ the following holds:

- (iii-1) $f_1(a_1, b) \leq f_1(a_2, b)$.
- (iii-2) $f_2(a_1, b) \geq f_2(a_2, b)$.
- (iii-3) $f_1(a_1, b) = f_1(a_2, b) \Leftrightarrow f_1(a_1, b) = f_1(a_2, b) = 0$.
- (iii-4) $f_2(a_1, b) = f_2(a_2, b) \Leftrightarrow f_2(a_1, b) = f_2(a_2, b) = 0$.

(iv) There exist constants $D_1 > 0$ and $D_2 > 0$ such that the following holds:

(iv-1) $f_1(a,b) = 0$ for all $(a,b) \in [D_1, \infty[\times \mathbb{R}$.

(iv-2) $f_2(a,b) = 0$ for all $(a,b) \in]-\infty, -D_2] \times \mathbb{R}$.

(v) Define $M_1 = \{(a,b) \in \mathbb{R}^2 \mid f_1(a,b) = 0\}$ and $M_2 = \{(a,b) \in \mathbb{R}^2 \mid f_2(a,b) = 0\}$. Then $M_1 \cap M_2 = \emptyset$.

We call Conditions (i)–(v) Babuška's Conditions as they first appear in [14], and refer to them as BC. As a matter of fact, [14] contains an additional assumption labeled Vc[1] which is used in the proof of the existence of a solution of (4). By using an alternative proof in Theorem 4.1, we show that this assumption is not needed.

An example of a scalar Duhem model that satisfies BC is provided in Section 9.

4. Existence and uniqueness of a global solution

The proof of the existence and uniqueness of a global solution for the differential equation (4) under BC is given in [14]. Since the text of that reference is in Russian we provide here that proof for the English speaking audience with improvements of ours which include removing the assumption labeled Vc in [14].

Theorem 4.1. *The differential equation (4) has a unique Carathéodory solution x on $\mathbb{R}_+$. Moreover $x \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ and $\|x\| \le \max\big(|x_0|, D_1, D_2\big)$.*

Proof. The differential equation (4) satisfies the Carathéodory conditions thus an absolutely continuous solution exists on some interval $[0,d]$ with $d > 0$ [16, Theorem 1, p. 4]. Consider the Lyapunov candidate function $z : [0,d] \to \mathbb{R}_+$ such that

$$z(t) = \frac{1}{2}x^2(t),$$

where x is a Carathéodory solution of (4). Then z is absolutely continuous [24, Exercise 3.6(ii)] and $\dot{z}(t) = x(t)\dot{x}(t)$ almost everywhere in $]0,d[$. Thus by BC(iv) $\dot{z}(t) = 0$ whenever $x(t) \ge D_1$ or $x(t) \le -D_2$. Then it follows from [25, Lemma 17] that $|x(t)| \le \max\big(|x_0|, D_1, D_2\big)$ for all $t \in [0,d]$. This fact implies by [16, Theorem 4, p. 7] that the solution x can be continued on $\mathbb{R}_+$.

To prove the uniqueness of the solution x we suppose that there exist two different solutions x_1 and x_2 to (4) on $\mathbb{R}_+$. Let $t_1 > 0$ be such that $x_1(t_1) \neq x_2(t_1)$. Without loss of generality suppose that $x_1(t_1) > x_2(t_1)$. Define $\xi = x_1 - x_2$, then we have $\xi(0) = 0$ and $\xi(t_1) > 0$. Define the set $A = \{t \in [0,t_1] \mid \xi(t) = 0\}$. Then $A \neq \emptyset$ as it contains 0 so that A has a least upper bound $t_2 \in [0,t_1]$. Owing to the continuity of ξ we have $\xi(t_2) = 0$ so that $t_2 < t_1$. Again the continuity of ξ implies that $\xi(t) > 0$ for all $t \in]t_2, t_1]$.

[1]Assumption Vc is stated as follows in [14]. For all $a, b_1, b_2 \in \mathbb{R}$ such $b_1 > b_2$ the following holds: $f_1(a,b_1) \ge f_1(a,b_2)$, $f_2(a,b_1) \le f_2(a,b_2)$, $f_1(a,b_1) = f_1(a,b_2) \Leftrightarrow f_1(a,b_1) = f_1(a,b_2) = 0$, and $f_2(a,b_1) = f_2(a,b_2) \Leftrightarrow f_2(a,b_1) = f_2(a,b_2) = 0$. Assumption Vc is not used in our paper.

On the other hand, owing to (4a)–(4b) the function ξ satisfies:

$$\dot{\xi}(t) = \Big(f_1\big(\xi(t) + x_2(t), u(t)\big) - f_1\big(x_2(t), u(t)\big) \Big) \dot{u}(t), \quad \text{for almost all } t \in [t_2, t_1] \text{ such that } \dot{u}(t) \geq 0, \tag{5}$$

$$\dot{\xi}(t) = \Big(f_2\big(\xi(t) + x_2(t), u(t)\big) - f_2\big(x_2(t), u(t)\big) \Big) \dot{u}(t), \quad \text{for almost all } t \in [t_2, t_1] \text{ such that } \dot{u}(t) \leq 0, \tag{6}$$

Since $\xi(t) > 0$ for all $t \in]t_2, t_1]$ it comes from BC(iii-1) that $f_1\big(\xi(t) + x_2(t), u(t)\big) - f_1\big(x_2(t), u(t)\big) \leq 0$ and from BC(iii-2) that $f_2\big(\xi(t) + x_2(t), u(t)\big) - f_2\big(x_2(t), u(t)\big) \geq 0$ for all $t \in]t_2, t_1]$. This fact leads from Equations (5)–(6) to $\dot{\xi}(t) \leq 0$ for almost all $t \in [t_2, t_1]$ which contradicts the fact that $\xi(t_2) = 0$ and $\xi(t_1) > 0$. □

Theorem 4.1 implies that we can define an operator $\mathcal{H}_s : \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \to \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ such that $\mathcal{H}_s(u, x_0) = x$.

Proposition 4.1. *Consider the differential equation (4). If for some $t_0 \in \mathbb{R}_+$ we have $x(t_0) \leq D_1$ then for all $t \geq t_0$ we have $x(t) \leq D_1$. Also if $x(t_0) \geq -D_2$ then for all $t \geq t_0$ we have $x(t) \geq -D_2$*

Proof. This is a direct consequence of [25, Lemma 17]. □

5. Background results

One of the main tools for the study of the Duhem model is the use of a different set of variables we call *normalized*. The theory behind the process of normalization has been introduced in [17] and expanded in [5]. We present in this section a brief overview of these normalized variables.

5.1. The normalized input

For $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0, t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)| \,\mathrm{d}\tau \in \mathbb{R}_+$, $\forall t \in \mathbb{R}_+$. The function ρ_u is well defined, nondecreasing and absolutely continuous. Observe that ρ_u may not be invertible (this happens when the input u is constant on some interval or intervals). Denote $\rho_{u,\max} = \lim_{t \to \infty} \rho_u(t)$ and let

- $I_u = [0, \rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t \in \mathbb{R}_+$ (in this case the interval I_u is finite),
- $I_u = [0, \rho_{u,\max}[$ if $\rho_{u,\max} > \rho_u(t)$ for all $t \in \mathbb{R}_+$ (in this case the interval I_u may be finite or infinite).

Lemma 5.1. *[17] Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be non constant so that the interval I_u is not reduced to a single point. Then there exists a unique function $\psi_u \in \mathcal{S}(I_u, \mathbb{R})$ that satisfies $\psi_u \circ \rho_u = u$. Moreover, the function ψ_u satisfies $\|\psi_u\| = \|u\|$, $\|\dot{\psi}_u\| = 1$ and*

$$\mu\left(\left\{\varrho \in I_u \mid \dot{\psi}_u(\varrho) \textit{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right) = 0.$$

The function ψ_u is constructed as follows. Let $\varrho \in I_u$, then there exists $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ (note that t_ϱ is not necessarily unique as ρ_u is not necessarily invertible). Then $u(t_\varrho)$ is independent of the particular choice of t_ϱ, and $\psi_u(\varrho)$ is defined by the relation $\psi_u(\varrho) = u(t_\varrho)$ [17].

Lemma 5.1 shows that the input u has been 'normalized' so that the resulting function ψ_u is such that $\dot{\psi}_u$ has norm 1 with respect to the new time variable ϱ. For this reason, we call function ψ_u the *normalized input*.

Lemma 5.2. *[17]* $\forall \gamma \in]0, \infty[$, $I_{u \circ s_\gamma} = I_u$ *and* $\psi_{u \circ s_\gamma} = \psi_u$.

Lemma 5.3. *[17] Let* $T \in]0, \infty[$. *If* $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ *is nonconstant and* T*–periodic, then* $I_u = \mathbb{R}_+$ *and* $\psi_u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ *is* $\rho_u(T)$*–periodic.*

5.2. An illustrative example of the normalized input

Let $u_{\min}, u_{\max}, T_1, T \in \mathbb{R}$ be such that $u_{\min} < u_{\max}$ and $0 < T_1 < T$. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be a T-periodic input which is strictly increasing on the interval $[0, T_1]$ and strictly decreasing on the interval $[T_1, T]$, with $u(0) = u(T) = u_{\min}$ and $u(T_1) = u_{\max}$, see Figure 1.

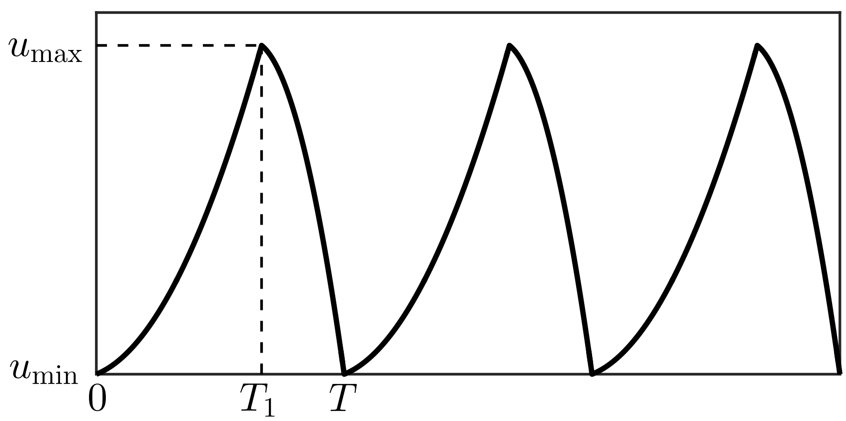

Figure 1: $u(t)$ versus t.

To find its corresponding normalized function ψ_u we proceed as follows. Note that ρ_u is strictly increasing so that it is invertible, and ρ_u^{-1} is also strictly increasing. From Lemma 5.1 it comes that $\psi_u = u \circ \rho_u^{-1}$ so that ψ_u is strictly increasing on the interval $[0, \varrho_1]$, where $\varrho_1 = \rho_u(T_1)$. Thus $\dot{\psi}_u(\varrho) \geq 0$ when $\varrho \in]0, \varrho_1[$ and $\dot{\psi}_u(\varrho)$ exists. On the other hand, by Lemma 5.1 the set on which $\dot{\psi}_u$ is not defined or is different from ± 1 has measure zero. Thus $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in]0, \varrho_1[$. Using the fact that ψ_u is absolutely continuous we obtain from the Fundamental Theorem of Calculus that

$$\psi_u(\varrho) - \psi_u(0) = \int_0^\varrho \dot{\psi}_u(\tau)\, \mathrm{d}\tau = \int_0^\varrho \mathrm{d}\tau = \varrho, \forall \varrho \in [0, \varrho_1].$$

Taking into account that $\psi_u\big(\rho_u(0)\big) = u(0)$ it comes that $\psi_u(0) = u_{\min}$ so that

$$\psi_u(\varrho) = \varrho + u_{\min}, \forall \varrho \in [0, \varrho_1]. \tag{7}$$

The value of ϱ_1 is determined as follows:

$$\varrho_1 = \rho_u(T_1) = \int_0^{T_1} |\dot{u}(t)|\mathrm{d}t = \int_0^{T_1} \dot{u}(t)\mathrm{d}t = u(T_1) - u(0) = u_{\max} - u_{\min}. \tag{8}$$

Also, $\psi_u = u \circ \rho_u^{-1}$ so that ψ_u is strictly decreasing on the interval $[\varrho_1, \varrho_2]$, where $\varrho_2 = \rho_u(T)$. Thus $\dot{\psi}_u(\varrho) \le 0$ when $\varrho \in]\varrho_1, \varrho_2[$ and $\dot{\psi}_u(\varrho)$ exists. On the other hand, by Lemma 5.1 the set on which $\dot{\psi}_u$ is not defined or is different from ± 1 has measure zero. Thus $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in]\varrho_1, \varrho_2[$. Using the fact that ψ_u is absolutely continuous we obtain from the Fundamental Theorem of Calculus that

$$\psi_u(\varrho) - \psi_u(\varrho_1) = \int_{\varrho_1}^{\varrho} \dot{\psi}_u(\tau)\,\mathrm{d}\tau = \int_{\varrho_1}^{\varrho} -1\,\mathrm{d}\tau = \varrho_1 - \varrho, \text{ for all } \varrho \in [\varrho_1, \varrho_2],$$

which leads to

$$\psi_u(\varrho) = \psi_u(\varrho_1) + \varrho_1 - \varrho, \forall \varrho \in [\varrho_1, \varrho_2]. \tag{9}$$

The value of $\psi_u(\varrho_1)$ is determined from Equation (7) as $\psi_u(\varrho_1) = u_{\max}$, and the value of ϱ_2 is determined as follows:

$$\varrho_2 = \rho_u(T) = \int_0^{T_1} |\dot{u}(t)|\mathrm{d}t + \int_{T_1}^{T} |\dot{u}(t)|\mathrm{d}t = \varrho_1 - \int_{T_1}^{T} \dot{u}(t)\mathrm{d}t = \varrho_1 + u(T_1) - u(T) = 2\varrho_1. \tag{10}$$

As a conclusion, we have

$$\begin{aligned} \psi_u(\varrho) &= \varrho + u_{\min}, \text{ for } \varrho \in [0, \varrho_1], \\ \psi_u(\varrho) &= -\varrho + 2u_{\max} - u_{\min}, \text{ for } \varrho \in [\varrho_1, \varrho_2]. \end{aligned} \tag{11}$$

Finally, note that by Lemma 5.3 the function ψ_u is ϱ_2–periodic, see Figure 2.

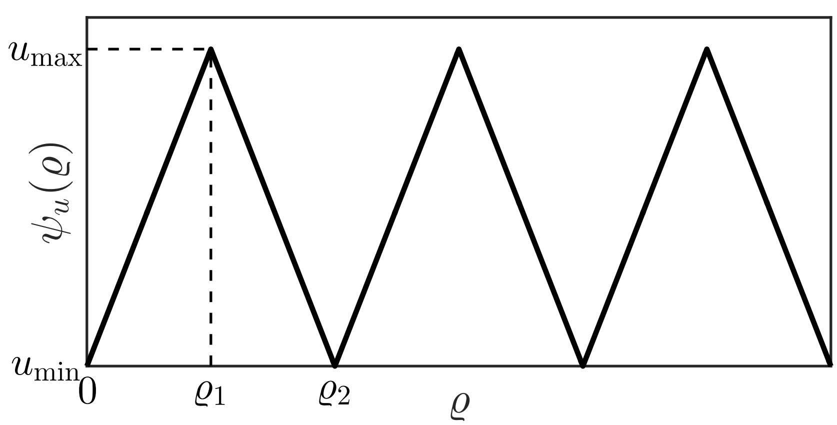

Figure 2: $\psi_u(\varrho)$ versus ϱ.

Comment. Note that ψ_u is independent of T_1 and T, and is also independent of the particular shape of u. The normalized function ψ_u depends solely on $u_{\max}$, $u_{\min}$, and on the fact that u is strictly increasing from $u_{\min}$ to $u_{\max}$ and strictly decreasing from $u_{\max}$ to $u_{\min}$.

5.3. The normalized input-derivative

Lemma 5.4. *[5, Lemma 13, p. 994]. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be nonconstant. There exists a measurable function $v_u : I_u \to \mathbb{R}$ defined by $v_u \circ \rho_u = \dot{u}$. The function v_u is unique up to a set of measure zero. Moreover, $\|v_u\| \leq \|\dot{u}\|$ and v_u is nonzero almost everywhere on I_u.*

The function v_u is called the *normalized input-derivative* and is constructed as follows. Let $\varrho \in I_u$, then there exists $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ (t_ϱ is not necessarily unique as ρ_u is not necessarily invertible). Then, whenever $\dot{u}(t_\varrho)$ exists for all t_ϱ such that $\rho_u(t_\varrho) = \varrho$, the quantity $\dot{u}(t_\varrho)$ is independent of the particular choice of t_ϱ, and $v_u(\varrho)$ is defined almost everywhere by the relation $v_u(\varrho) = \dot{u}(t_\varrho)$ [5, p. 982].

5.4. The normalized state

Lemma 5.5. *[17] Let $(u, x_0) \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$ with u nonconstant. Define $x = \mathcal{H}_s(u, x_0)$. Then, there exists a unique function $x_u \in \mathcal{S}(I_u, \mathbb{R})$ that satisfies $x_u \circ \rho_u = x$. Moreover we have $\|x_u\| = \|x\|$.*

The function x_u is called the *normalized state* and is constructed as follows. Let $\varrho \in I_u$, then there exists $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ (t_ϱ is not necessarily unique as ρ_u is not necessarily invertible). Then $x(t_\varrho)$ is independent of the particular choice of t_ϱ, and $x_u(\varrho)$ is defined by the relation $x_u(\varrho) = x(t_\varrho)$ [17].

5.5. Definition of strong consistency and hysteresis loop

Definition 5.1. *[17] Let $x_0 \in \mathbb{R}$. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be such that the input u is nonconstant and T–periodic where $T \in]0, \infty[$. Note that in this case $I_u = \mathbb{R}_+$. For any nonnegative integer k, define the function $x_{u,k} \in \mathcal{S}\big([0, \rho_u(T)], \mathbb{R}\big)$ by $x_{u,k}(\varrho) = x_u\big(\rho_u(T)k + \varrho\big), \forall \varrho \in [0, \rho_u(T)]$. The operator $\mathcal{H}_s$ is said to be strongly consistent with respect to (u, x_0) if there exists $x_u^\circ \in C^0\big([0, \rho_u(T)], \mathbb{R}\big)$ such that $\lim_{k\to\infty} \|x_{u,k} - x_u^\circ\| = 0$.*

Definition 5.2. *[17, 5] Let $x_0 \in \mathbb{R}$ and let $T > 0$. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be non constant and T–periodic. Assume that the operator $\mathcal{H}_s$ is strongly consistent with respect to (u, x_0). We call hysteresis loop of the operator $\mathcal{H}_s$ with respect to (u, x_0) the set*

$$\mathcal{G}_{u,x_0} = \left\{ \big(\psi_u(\varrho), x_u^\circ(\varrho)\big), \varrho \in [0, \rho_u(T)] \right\}. \tag{12}$$

6. The normalized scalar Duhem model

In this section we express the Duhem model in terms of the normalized variables. Recall that the normalized state is defined in Lemma 5.4 uniquely by the relation $x_u \circ \rho_u = x$ where x is the unique solution of (4) under BC. Then we get from Equation (4a) that

$$\begin{aligned}\dot{\overbrace{x_u \circ \rho_u}}(t) = f_1\big(x_u \circ \rho_u(t), \psi_u \circ \rho_u(t)\big) \cdot v_u \circ \rho_u(t), \\ \text{for almost all } t \in \mathbb{R}_+ \text{ such that } v_u \circ \rho_u(t) \geq 0.\end{aligned} \tag{13}$$

The chain rule can be used as x_u, ρ_u and x are absolutely continuous so that

$$\dot{\overbrace{x_u \circ \rho_u}}(t) = \dot{x}_u\big(\rho_u(t)\big)\dot{\rho}_u(t), \text{ for almost all } t \in \mathbb{R}_+, \tag{14}$$

where $\dot{x}_u\big(\rho_u(t)\big)\dot{\rho}_u(t)$ is interpreted to be zero whenever $\dot{\rho}_u(t) = 0$ even if x_u is not differentiable at $\rho_u(t)$ [24, Theorem 3.44]. On the other hand it comes from [24, Lemma 3.31] that

$$\dot{\rho}_u(t) = |\dot{u}(t)| = |v_u \circ \rho_u(t)|, \text{ for almost all } t \in \mathbb{R}_+. \tag{15}$$

From $u = \psi_u \circ \rho_u$, $\dot{u} = v_u \circ \rho_u$, and Equation (15) we get

$$v_u \circ \rho_u(t) = \dot{u}(t) = \dot{\overbrace{\psi_u \circ \rho_u}}(t) = \dot{\psi}_u\big(\rho_u(t)\big)|v_u \circ \rho_u(t)|, \text{ for almost all } t \in \mathbb{R}_+, \tag{16}$$

where $\dot{\psi}_u\big(\rho_u(t)\big)\dot{\rho}_u(t)$ is interpreted to be zero whenever $\dot{\rho}_u(t) = 0$ even if ψ_u is not differentiable at $\rho_u(t)$.

Combining Equations (13)–(16) it comes that

$$\begin{aligned} |v_u \circ \rho_u(t)| \cdot \dot{x}_u\big(\rho_u(t)\big) = f_1\big(x_u \circ \rho_u(t), \psi_u \circ \rho_u(t)\big) \cdot \dot{\psi}_u\big(\rho_u(t)\big)|v_u \circ \rho_u(t)|,\\ \text{for almost all } t \in \mathbb{R}_+ \text{ such that } v_u \circ \rho_u(t) \geq 0. \end{aligned} \tag{17}$$

At this point we express Equation (17) in terms of the normalized time $\varrho = \rho_u(t)$ as follows. Define

$$F = \{t \in \mathbb{R}_+ \mid v_u \circ \rho_u(t) \geq 0\},$$

then Equation (17) holds on $F \setminus E$ where $E \subseteq \mathbb{R}_+$ is such that $\mu(E) = 0$. The function ρ_u is absolutely continuous, thus $\mu\big(\rho_u(E)\big) = 0$ [24, Theorem 3.12] so that we get from Equation (17) that

$$\begin{aligned} |v_u(\varrho)| \cdot \dot{x}_u(\varrho) = f_1\big(x_u(\varrho), \psi_u(\varrho)\big) \cdot \dot{\psi}_u(\varrho)|v_u(\varrho)|,\\ \text{for almost all } \varrho \in I_u \text{ such that } v_u(\varrho) \geq 0. \end{aligned} \tag{18}$$

Using Lemma 5.4 we obtain from Equation (18) that

$$\dot{x}_u(\varrho) = f_1\big(x_u(\varrho), \psi_u(\varrho)\big) \cdot \dot{\psi}_u\big(\varrho\big), \text{ for almost all } \varrho \in I_u \text{ such that } v_u(\varrho) > 0. \tag{19}$$

On the other hand, Equation (16) can be rewritten in terms of the normalized time $\varrho = \rho_u(t)$ as

$$v_u(\varrho) = \dot{\psi}_u(\varrho)|v_u(\varrho)|, \text{ for almost all } \varrho \in I_u. \tag{20}$$

Combining Equations (19) and (20) it comes that

$$\dot{x}_u(\varrho) = f_1\big(x_u(\varrho), \psi_u(\varrho)\big), \text{ for almost all } \varrho \in I_u \text{ such that } \dot{\psi}_u(\varrho) = 1. \tag{21}$$

Following the same argument we come to

$$\dot{x}_u(\varrho) = -f_2\big(x_u(\varrho), \psi_u(\varrho)\big), \text{ for almost all } \varrho \in I_u \text{ such that } \dot{\psi}_u(\varrho) = -1. \tag{22}$$

The initial condition of x_u is obtained from Equation (4c) as

$$x_u(0) = x_0, \tag{23}$$

since $\rho_u(0) = 0$.

The normalized Duhem model (21)–(23) is equivalent to (4) by Lemma 5.4. Also by Theorem 4.1 the differential equation (21)–(23) has a unique Carathéodory solution x_u on I_u. Moreover $x_u \in \mathcal{S}(I_u, \mathbb{R})$, $x_u \circ \rho_u = x$, and $\|x_u\| = \|x\| \leq \max\big(|x_0|, D_1, D_2\big)$.

7. Existence and uniqueness of a periodic solution

In this section we consider that the input is periodic. The proof of the existence and uniqueness of a periodic solution for the differential equation (4a)–(4b) is given in [14]. Since the text of that reference is in Russian we provide here that proof for the English speaking audience with an improvement of ours which consists in relaxing the assumption on the input from continuously differentiable in [14] to absolutely continuous. The main ingredient for getting this relaxation is the use of the normalized variables.

Proposition 7.1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $S_1, S_2, E \subseteq [a, b]$ be such that $S_1 \cap S_2 = \emptyset$, $\mu(S_1) \neq 0$, $\mu(S_2) \neq 0$, $\mu(E) = 0$, and $S_1 \cup S_2 \cup E = [a, b]$. Let $\bar{S}$ be the closure of the set S. Then $\bar{S}_1 \cap \bar{S}_2 \neq \emptyset$.*

Proof. Define $S_3 = S_1 \cup S_2$. We show first that $\bar{S}_3 = [a, b]$. Let $t \in E$, then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \in S_3$ such that $\lim_{n\to\infty} t_n = t$. Otherwise there would exist an $\epsilon > 0$ such that $]t - \epsilon, t] \subseteq E$ or $[t, t + \epsilon[\subseteq E$. Thus $\mu(E) \geq \epsilon$ which contradicts $\mu(E) = 0$. Then we conclude that $\bar{S}_3 = [a, b] = \bar{S}_1 \cup \bar{S}_2$.

On the other hand $\mu(S_i) \neq 0$ thus $S_i \neq \emptyset$ and $S_i \subseteq \bar{S}_i \neq \emptyset$, $i = 1, 2$. Now, $\bar{S}_1$ and $\bar{S}_2$ are nonempty closed subsets of $[a, b]$ with $\bar{S}_1 \cup \bar{S}_2 = [a, b]$. Thus $\bar{S}_1 \cap \bar{S}_2 \neq \emptyset$ as the interval $[a, b]$ is connected [26, p. 14]. □

Theorem 7.1. *Let $T > 0$ and $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be a nonconstant and T–periodic input. Then there exists a unique initial condition $\zeta \in \mathbb{R}$ such that $\mathcal{H}_s(u, \zeta)$ is also periodic. Moreover $\zeta \in [-D_2, D_1]$, $\mathcal{H}_s(u, \zeta)$ is T–periodic, and $-D_2 \leq [\mathcal{H}_s(u, \zeta)](t) \leq D_1$ for all $t \in [0, T]$.*

Proof. **Uniqueness.** Suppose that we can find $\zeta_1, \zeta_2 \in \mathbb{R}$ with $\zeta_1 > \zeta_2$ such that $\mathcal{H}_s(u, \zeta_i) = x_i$ is T–periodic, $i = 1, 2$. Define $\xi = x_1 - x_2$, then $\xi(t_0) = \zeta_1 - \zeta_2 > 0$.

Claim 1. *$\xi(t) > 0$ for all $t \geq 0$.*

Proof. Suppose that there exists some $t' > 0$ with $\xi(t') = 0$. Due to the uniqueness of solutions of (4a)–(4b) we have $x_1(t) = x_2(t)$ for all $t \geq t'$. In particular, for $t = nT$ with n large enough to have $t \geq t'$ we have $x_1(nT) = x_2(nT)$. Owing to the T–periodicity of x_1 and x_2 we get $x_1(0) = \zeta_1 = x_2(0) = \zeta_2$ which is a contradiction. □

Claim 2. *$\xi(t) = \zeta_1 - \zeta_2$ for all $t \geq 0$.*

Proof. Using Equations (5)–(6) and the fact that $\xi(t) > 0$ for all $t \geq 0$ it comes that $\dot{\xi}(t) \leq 0$ for almost all $t \geq 0$ following the same argument as in the proof of Theorem 4.1. Since ξ is periodic and absolutely continuous we have

$$\xi(T) - \xi(0) = 0 = \int_0^T \dot{\xi}(t)\,\mathrm{d}t$$

which leads to $\dot{\xi}(t) = 0$ for almost all $t \in [0, T]$ since $\dot{\xi}(t) \leq 0$ almost everywhere in $[0, T]$. □

Substituting $\dot{\xi}(t) = 0$ in Equations (5)–(6) we get

$$f_1\big(\zeta_1 - \zeta_2 + x_2(t), u(t)\big) = f_1\big(x_2(t), u(t)\big) \text{ for almost all } t > 0 \text{ such that } \dot{u}(t) > 0, \quad (24)$$
$$f_2\big(\zeta_1 - \zeta_2 + x_2(t), u(t)\big) = f_2\big(x_2(t), u(t)\big) \text{ for almost all } t > 0 \text{ such that } \dot{u}(t) < 0. \quad (25)$$

Using BC(iii-3) and BC(iii-4) it follows from Equations (24)–(25) that

$$\begin{aligned} f_1\big(x_2(t), u(t)\big) &= 0 \text{ for almost all } t > 0 \text{ such that } \dot{u}(t) > 0, && (26)\\ f_2\big(x_2(t), u(t)\big) &= 0 \text{ for almost all } t > 0 \text{ such that } \dot{u}(t) < 0. && (27)\end{aligned}$$

Equations (26)–(27) combined with Equations (4a)–(4b) lead to x_2 is constant, that is $x_2(t) = \zeta_2$ for all $t \geq t_0$. A similarly argument holds for x_1. We thus conclude that

$$\begin{aligned} f_1\big(\zeta_2, u(t)\big) &= 0 \text{ for almost all } t > 0 \text{ such that } \dot{u}(t) > 0, && (28)\\ f_2\big(\zeta_1, u(t)\big) &= 0 \text{ for almost all } t > 0 \text{ such that } \dot{u}(t) < 0. && (29)\end{aligned}$$

We now use the normalized time $\varrho = \rho_u(t)$. Observe first that since u is nonconstant and periodic we have $I_u = \mathbb{R}_+$. Using an argument similar to that of Section 6 we obtain from Equations (28)–(29) that

$$\begin{aligned} f_1\big(\zeta_2, \psi_u(\varrho)\big) &= 0 \text{ for almost all } \varrho > 0 \text{ such that } \dot{\psi}_u(\varrho) = 1, && (30)\\ f_2\big(\zeta_1, \psi_u(\varrho)\big) &= 0 \text{ for almost all } \varrho > 0 \text{ such that } \dot{\psi}_u(\varrho) = -1. && (31)\end{aligned}$$

Define the sets

$$\begin{aligned} S_1 &= \{\varrho \in [0, \rho_u(T)] \mid \dot{\psi}_u(\varrho) = 1\},\\ S_2 &= \{\varrho \in [0, \rho_u(T)] \mid \dot{\psi}_u(\varrho) = -1\}.\end{aligned}$$

From Lemma 5.1 there exists a zero-measure set $E \subseteq [0, \rho_u(T)]$ such that

$$S_1 \cup S_2 \cup E = [0, \rho_u(T)]. \quad (32)$$

Claim 3. *$\mu(S_1) \neq 0$ and $\mu(S_2) \neq 0$.*

Proof. Suppose that $\mu(S_1) = 0$ and $\mu(S_2) = 0$. By Lemma 5.1 the function ψ_u is absolutely continuous. Thus for any $\varrho \in [0, \rho_u(T)]$ we have

$$\begin{aligned} \psi_u(\varrho) &= \psi_u(0) + \int_0^{\varrho} \dot{\psi}_u(\nu)\,\mathrm{d}\nu\\ &= \psi_u(0) + \int_{S_1\cap[0,\varrho]} \dot{\psi}_u(\nu)\,\mathrm{d}\nu + \int_{S_2\cap[0,\varrho]} \dot{\psi}_u(\nu)\,\mathrm{d}\nu + \int_{E\cap[0,\varrho]} \dot{\psi}_u(\nu)\,\mathrm{d}\nu\\ &= \psi_u(0),\end{aligned}$$

which contradicts the fact that u is nonconstant. Now suppose that $\mu(S_1) = 0$ then we must have that $\mu(S_2) \neq 0$. Then

$$\begin{aligned} \psi_u\big(\rho_u(T)\big) &= \psi_u(0) + \int_0^{\rho_u(T)} \dot{\psi}_u(\nu)\,\mathrm{d}\nu\\ &= \psi_u(0) + \int_{S_1} \dot{\psi}_u(\nu)\,\mathrm{d}\nu + \int_{S_2} \dot{\psi}_u(\nu)\,\mathrm{d}\nu + \int_{E} \dot{\psi}_u(\nu)\,\mathrm{d}\nu\\ &= \psi_u(0) + \int_{S_2} \dot{\psi}_u(\nu)\,\mathrm{d}\nu < \psi_u(0)\end{aligned}$$

which contradicts $\psi_u\big(\rho_u(T)\big) = u(T) = u(0) = \psi_u(0)$.

The same argument holds if we assume that $\mu(S_2) = 0$. $\square$

Combining Claim 3, Proposition 7.1, and Equation (32) it comes that $\bar{S}_1 \cap \bar{S}_2 \neq \emptyset$. Let

$$\varrho_0 \in \bar{S}_1 \cap \bar{S}_2 \subseteq [0, \rho_u(T)]. \tag{33}$$

Then using the continuity of f and u it comes from Equations (30)–(31) that

$$f_1\big(\zeta_2, \psi_u(\varrho_0)\big) = 0, \tag{34}$$
$$f_2\big(\zeta_1, \psi_u(\varrho_0)\big) = 0. \tag{35}$$

Define

$$\zeta_2 < \zeta_3 = \frac{\zeta_1 + \zeta_2}{2} < \zeta_1.$$

From Equation (34) and BC(iii-1) we get $0 = f_1\big(\zeta_2, \psi_u(\varrho_0)\big) \geq f_1\big(\zeta_3, \psi_u(\varrho_0)\big)$. This fact along with BC(ii) gives $f_1\big(\zeta_3, \psi_u(\varrho_0)\big) = 0$. A similar argument leads to $f_2\big(\zeta_3, \psi_u(\varrho_0)\big) = 0$. This is a contradiction with BC(v) which proves the uniqueness part of Theorem 7.1.

Existence. Define $x = \mathcal{H}_s(u, x_0)$ and

$$\xi(t) = x(t+T) - x(t), \ \forall t \geq 0.$$

Note that $x \circ \tau_T = \mathcal{H}_s\big(u, x(T)\big)$. If $\xi(0) = 0$ then $x(T) = x(0) = x_0$ so that $x \circ \tau_T = \mathcal{H}_s(u, x_0) = x$, that is $\xi(t) = 0$ for all $t \geq 0$. This means that x is the T–periodic solution. Thus we suppose in the following that $\xi(0) \neq 0$. Without loss of generality we consider that $\xi(0) > 0$.

Claim 4. $\xi(t) > 0$ *for all* $t > 0$.

Proof. Suppose that there exists $t_1 > 0$ such that $\xi(t_1) = 0$. Then as above it comes that $\xi(t) = 0$ for all $t \geq t_1$. In particular, taking an integer n such that $nT \geq t_1$ we get $\xi(nT) = 0$. Then as above it comes that x is T–periodic on the time interval $[nT, \infty[$. Due to the uniqueness of the solution of (4a)–(4b) it comes that x is also periodic on $[0, nT]$ which implies that $\xi(0) = 0$ which contradicts $\xi(0) > 0$. $\square$

Owing to (4a)–(4b) the function ξ satisfies:

$$\dot{\xi}(t) = \Big(f_1\big(\xi(t) + x_2(t), u(t)\big) - f_1\big(x_2(t), u(t)\big)\Big)\dot{u}(t), \quad \text{for almost all } t \geq 0 \text{ such that } \dot{u}(t) \geq 0, \tag{36}$$

$$\dot{\xi}(t) = \Big(f_2\big(\xi(t) + x_2(t), u(t)\big) - f_2\big(x_2(t), u(t)\big)\Big)\dot{u}(t), \quad \text{for almost all } t \geq 0 \text{ such that } \dot{u}(t) \leq 0, \tag{37}$$

From Claim 4 and BC(iii) it comes that $f_1\big(\xi(t) + x_2(t), u(t)\big) - f_1\big(x_2(t), u(t)\big) \leq 0$ and $f_2\big(\xi(t) + x_2(t), u(t)\big) - f_2\big(x_2(t), u(t)\big) \geq 0$ for all $t \geq 0$. This fact leads from Equations (36)–(37) to $\dot{\xi}(t) \leq 0$ for almost all $t \geq 0$. This fact along with Claim 4 shows that there exists $b \geq 0$ such that

$$\lim_{t \to \infty} \xi(t) = b. \tag{38}$$

Claim 5. *$b = 0$.*

Proof. Assume that $b > 0$. We have $\xi(nT) \geq b$ for all $n \in \mathbb{N}$, that is

$$x\big((n+1)T\big) - x(nT) \geq b, \tag{39}$$

so that $x\big((n+1)T\big) \geq x(0) + (n+1)b$. Taking n sufficiently large we get a contradiction with $\|x\| \leq \max\big(|x_0|, D_1, D_2\big)$, see Theorem 4.1. □

Combining Equation (38) and Claim 5 it follows that

$$\lim_{t\to\infty} \xi(t) = 0. \tag{40}$$

Define $\alpha_n = x(nT)$. From Inequality (39) and Claim 5 it comes that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is nondecreasing. Since $|\alpha_n| \leq \|x\| \leq \max\big(|x_0|, D_1, D_2\big)$ it follows that there exists $\bar{\alpha} \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \alpha_n = \bar{\alpha}.$$

Claim 6. *$\mathcal{H}_s(u, \bar{\alpha})$ is T–periodic.*

Proof. To prove Claim 6 it is enough to prove that $\big[\mathcal{H}_s(u,\bar{\alpha})\big](T) = \big[\mathcal{H}_s(u,\bar{\alpha})\big](0)$. From [16, Theorem 6, p. 11] it comes that the sequence $\mathcal{H}_s(u,\alpha_n)|_{[0,T]}$ converges uniformly to $\mathcal{H}_s(u,\bar{\alpha})|_{[0,T]}$ on the interval $[0,T]$. In particular we have

$$\begin{aligned} \lim_{n\to\infty} \big[\mathcal{H}_s(u,\alpha_n)\big](0) &= \big[\mathcal{H}_s(u,\bar{\alpha})\big](0), && (41)\\ \lim_{n\to\infty} \big[\mathcal{H}_s(u,\alpha_n)\big](T) &= \big[\mathcal{H}_s(u,\bar{\alpha})\big](T). && (42) \end{aligned}$$

On the other hand, we have

$$\mathcal{H}_s(u,\alpha_n)|_{[0,T]} = \mathcal{H}_s\big(u, x(nT)\big)|_{[0,T]} = \mathcal{H}_s(u,x_0)|_{[nT,(n+1)T]}$$

so that

$$\begin{aligned} \big[\mathcal{H}_s(u,\alpha_n)\big](0) &= x(nT) = \alpha_n, && (43)\\ \big[\mathcal{H}_s(u,\alpha_n)\big](T) &= x\big((n+1)T\big) = \alpha_{n+1}. && (44) \end{aligned}$$

Combining Equations (41)–(42) and (43)–(44) it comes that

$$\begin{aligned} \lim_{n\to\infty} \alpha_n &= \big[\mathcal{H}_s(u,\bar{\alpha})\big](0), && (45)\\ \lim_{n\to\infty} \alpha_{n+1} &= \big[\mathcal{H}_s(u,\bar{\alpha})\big](T), && (46) \end{aligned}$$

so that $\big[\mathcal{H}_s(u,\bar{\alpha})\big](0) = \big[\mathcal{H}_s(u,\bar{\alpha})\big](T) = \bar{\alpha}$. □

The existence part of Theorem 7.1 is thus established by taking $\zeta = \bar{\alpha}$. Remains to prove that $\zeta \in [-D_2, D_1]$.

Claim 7. *$\bar{x} = \mathcal{H}_s(u,\zeta)$ is not constant.*

Proof. Suppose that $\bar{x}$ is constant. Then we must have $\bar{x}(t) = \bar{x}(0) = \zeta$ for all $t \geq 0$ so that $\bar{x}_u(\varrho) = \zeta$ for all $\varrho \geq 0$. Then from Equations (21) and (22) we obtain

$$\begin{aligned} f_1\big(\zeta, \psi_u(\varrho)\big) &= 0, \text{ for almost all } \varrho > 0 \text{ such that } \dot{\psi}_u(\varrho) = 1, &(47)\\ f_2\big(\zeta, \psi_u(\varrho)\big) &= 0, \text{ for almost all } \varrho > 0 \text{ such that } \dot{\psi}_u(\varrho) = -1. &(48)\end{aligned}$$

Recall the value ϱ_0 defined in Equation (33). Owing to the continuity of f_1, f_2 and ψ_u it comes from Equations (47)–(48) that $f_1\big(\zeta, \psi_u(\varrho_0)\big) = 0$ and $f_2\big(\zeta, \psi_u(\varrho_0)\big) = 0$ which contradicts BC(v). □

By Claim 7 it comes that there exists $t_1 \in \,]0, T[$ such that $\bar{x}(t_1) \neq \bar{x}(0) = \zeta$. Suppose that $\bar{x}(t_1) > \zeta$ (the case $\bar{x}(t_1) < \zeta$ is treated likewise).

Claim 8. *There exists $t_2 \in [0, t_1]$ such that $\bar{x}(t_2) \leq D_1$*

Proof. Suppose that for all $t \in [0, t_1]$ we have $\bar{x}(t) > D_1$. Then for all $\varrho \in [0, \rho_u(t_1)]$ we have $\bar{x}_u(\varrho) > D_1$. From Equations (21)–(22), BC(ii) and BC(iv-1) it comes that

$$\dot{\bar{x}}_u(\varrho) \leq 0, \text{ for almost all } \varrho \in [0, \rho_u(t_1)].$$

Since $\bar{x}$ is absolutely continuous we can write

$$\bar{x}(t_1) - \zeta = \bar{x}\big(\rho_u(t_1)\big) - \bar{x}(0) = \int_0^{\rho_u(t_1)} \dot{\bar{x}}_u(\varrho) \, \mathrm{d}\varrho \leq 0,$$

which contradicts $\bar{x}(t_1) > \zeta$. □

Claim 9. *There exists $t_3 \in [t_1, T]$ such that $\bar{x}(t_3) \geq -D_2$*

Proof. Suppose that for all $t \in [t_1, T]$ we have $\bar{x}(t) < -D_2$. Then for all $\varrho \in [\rho_u(t_1), \rho_u(T)]$ we have $\bar{x}_u(\varrho) < -D_2$. From Equations (21)–(22), BC(ii) and BC(iv-2) it comes that

$$\dot{\bar{x}}_u(\varrho) \geq 0, \text{ for almost all } \varrho \in [\rho_u(t_1), \rho_u(T)].$$

Since $\bar{x}$ is absolutely continuous we can write

$$\zeta - \bar{x}(t_1) = \bar{x}\big(\rho_u(T)\big) - \bar{x}\big(\rho_u(t_1)\big) = \int_{\rho_u(t_1)}^{\rho_u(T)} \dot{\bar{x}}_u(\varrho) \, \mathrm{d}\varrho \geq 0,$$

which contradicts $\bar{x}(t_1) > \zeta$. □

Claims 8–9 and Proposition 4.1 show that for all $t \geq T$ we have $\bar{x}(t) \in [-D_2, D_1]$. Taking $t = T$ leads to $\zeta \in [-D_2, D_1]$. The fact that $-D_2 \leq \bar{x}(t) \leq D_1$ for all $t \in [0, T]$ derives from Proposition 4.1 and the fact that $\bar{x}$ is T–periodic.

□

Comment. We stress that Theorem 7.1 is valid for *any periodic input*, not only periodic inputs whose shape is that of Figure 1.

8. Hysteresis loop of the scalar Duhem model

Proposition 8.1. *Let $x_0 \in \mathbb{R}$ and $T > 0$. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be a nonconstant and T–periodic input. Then $\mathcal{H}_s$ is strongly consistent with respect to (u, x_0).*

Proof. Denote $x = \mathcal{H}_s(u, x_0)$. From [16, Theorem 6, p. 11] it comes that the sequence $\mathcal{H}_s(u, \alpha_n)$ converges uniformly to $\mathcal{H}_s(u, \bar{\alpha})$ on the interval $[0, T]$, where $\alpha_n = x(nT)$ and $\bar{\alpha} = \lim_{n\to\infty} \alpha_n$. Denote $\bar{x} = \mathcal{H}_s(u, \bar{\alpha})$. Observe that

$$\mathcal{H}_s(u, \alpha_n)|_{[0,T]} = \mathcal{H}_s\big(u, x(nT)\big)|_{[0,T]} = \mathcal{H}_s(u, x_0)|_{[nT,(n+1)T]} = x \circ \tau_{nT}|_{[0,T]},$$

so that $x \circ \tau_{nT}$ converges uniformly to $\bar{x}$ on $[0, T]$. This means that

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \mid \forall n \geq N_\epsilon, \big\| x \circ \tau_{nT}|_{[0,T]} - \bar{x}|_{[0,T]} \big\| < \epsilon. \tag{49}$$

On the other hand, we have

$$\rho_u(t + nT) = \rho_u(t) + n\rho_u(T)$$

so that we get from Definition 5.1 that

$$x \circ \tau_{nT} = x_u \circ \rho_u \circ \tau_{nT} = x_{u,n} \circ \rho_u.$$

Since $\bar{x} = \mathcal{H}_s(u, \bar{\alpha})$ it comes from Lemma 5.5 that there exists a unique function $\bar{x}_u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ that satisfies $\bar{x}_u \circ \rho_u = \mathcal{H}_s(u, x_0)$. These facts along with (49) give

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \mid \forall n \geq N_\epsilon, \big\| x_{u,n} - \bar{x}_u|_{[0,\rho_u(T)]} \big\| < \epsilon,$$

so that $\mathcal{H}_s$ is strongly consistent with respect to (u, x_0) and

$$x_u^\circ = \bar{x}_u|_{[0,\rho_u(T)]}. \tag{50}$$

□

Comment. By Proposition 8.1 the hysteresis loop $\mathcal{G}_{u,x_0}$ of the operator $\mathcal{H}_s$ with respect to (u, x_0) is given by Equation (12) where x_u° is given by Equation (50). This means that $\mathcal{G}_{u,x_0}$ depends on the normalized function ψ_u but not directly on the input u. In other words, $\mathcal{G}_{u,x_0}$ does not depend on T_1, T, or on the particular shape of the input u. The hysteresis loop $\mathcal{G}_{u,x_0}$ depends solely on $u_{\max}$, $u_{\min}$, and on the fact that u is strictly increasing from $u_{\min}$ to $u_{\max}$ and strictly decreasing from $u_{\max}$ to $u_{\min}$.

9. Case study: Babuška's hysteresis model

In Equation (50) x_u° is given as a function of $\mathcal{H}_s(u, \bar{\alpha})$ where $\bar{\alpha}$ is defined as the limit of a sequence. It is however desirable that the hysteresis loop be given as an explicit function of the data of the model, that is u, f_1 and f_2. This explicit expression of the hysteresis loop means an explicit integration of the possibly nonlinear differential equation (21)–(23) and the explicit determination of $\bar{\alpha}$.

We are not aware of any research work that provides an explicit integration of the differential equation (21)–(23). However, for some particular cases of the scalar Duhem model the corresponding hysteresis loop has been determined explicitly. This is the case when the state appears linearly as for the semilinear Duhem model [5, 27], the LuGre model [28, 29], and the Dahl model [30, 31]. The Bouc-Wen model is the only special case of the Duhem model we are aware of where the state appears nonlinearly and for which an explicit expression of the hysteresis loop is available [15].

In this section we consider a special case for the functions f_1 and f_2 proposed–but not studied–in [14]: $f_1(a,b) = h_1(a)g_1(b)$ and $f_2(a,b) = h_2(a)g_2(b)$, where the functions h_1, h_2, g_1, and g_2 may be nonlinear. Our aim is to provide the analytic explicit expression of the hysteresis loop in this case.

9.1. Babuška's model of hysteresis

Consider the following special case of the scalar Duhem model:

$$\begin{aligned}
\dot{x}(t) &= h_1\big(x(t)\big)\, g_1\big(u(t)\big)\, \dot{u}(t), \text{ for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \geq 0, && \text{(51a)}\\
\dot{x}(t) &= h_2\big(x(t)\big)\, g_2\big(u(t)\big)\, \dot{u}(t), \text{ for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \leq 0, && \text{(51b)}\\
x(0) &= x_0, && \text{(51c)}
\end{aligned}$$

where $x_0 \in \mathbb{R}$ is the initial condition, $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ the input, and $x : \mathbb{R}_+ \to \mathbb{R}$ the state. We call Equations (51) Babuška's model of hysteresis.

Babuška's Conditions translate as follows: the functions g_1, g_2, h_1, h_2 satisfy Conditions (A)–(E).

(A) $g_1, g_2, h_1, h_2 \in C^0(\mathbb{R}, \mathbb{R})$.

(B) There exist constants $D_1 > 0$ and $D_2 > 0$ such that the following holds:

  (B-1) $h_1(a) = 0$ for all $a \geq D_1$ and $h_1(a) \neq 0$ for all $a < D_1$.

  (B-2) $h_2(a) = 0$ for all $a \leq -D_2$ and $h_2(a) \neq 0$ for all $a > -D_2$.

(C) For all $a_1, a_2 \in \,]-\infty, D_1]$ such that $a_1 > a_2$ we have $h_1(a_1) < h_1(a_2)$.

(D) For all $a_1, a_2 \in [-D_2, \infty[$ such that $a_1 > a_2$ we have $h_2(a_1) > h_2(a_2)$.

(E) $g_1(b) > 0$ and $g_2(b) > 0$ for all $b \in \mathbb{R}$.

Observe that Conditions (B), (C), and (D) imply that $h_1(a) \geq 0$ and $h_2(a) \geq 0$ for all $a \in \mathbb{R}$.

Define the functions $\ell_1 : [-D_2, D_1] \to \mathbb{R} \cup \{\infty\}$ and $\ell_2 : [-D_2, D_1] \to \mathbb{R} \cup \{-\infty\}$ by

$$\ell_1(z) = \int_0^z \frac{1}{h_1(\nu)} \mathrm{d}\nu, \quad \ell_2(z) = \int_0^z \frac{1}{h_2(\nu)} \mathrm{d}\nu$$

where

$$\ell_1(D_1) = \lim_{z \uparrow D_1} \ell_1(z), \quad \ell_2(-D_2) = \lim_{z \downarrow -D_2} \ell_2(z)$$

may be finite or infinite. Then ℓ_1 and ℓ_2 are strictly increasing and of class C^1 on the interval $]-D_2, D_1[$. This fact implies that ℓ_1 and ℓ_2 are invertible and that their inverses ℓ_1^{-1} and ℓ_2^{-1} are strictly increasing and of class C^1 on the intervals $]\ell_1(-D_2), \ell_1(D_1)[$ and $]\ell_2(-D_2), \ell_2(D_1)[$ respectively.

Also define the functions $G_1, G_2 : \mathbb{R} \to \mathbb{R}$ by

$$G_1(z) = \int_0^z g_1(\nu)\mathrm{d}\nu, \quad G_2(z) = \int_0^z g_2(\nu)\mathrm{d}\nu.$$

Then G_1 and G_2 are strictly increasing and of class C^1 on $\mathbb{R}$.

By Theorem 4.1 the differential equation (51) has a unique Carathéodory solution x on $\mathbb{R}_+$. Moreover $x \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ and $\|x\| \leq \max\big(|x_0|, D_1, D_2\big)$.

Recall that the normalized state is defined uniquely by the relation $x_u \circ \rho_u = x$. Then from Equations (21)–(23) the normalized version of Babuška's model is

$$\dot{x}_u(\varrho) = h_1\big(x_u(\varrho)\big)\, g_1\big(\psi_u(\varrho)\big), \text{ for almost all } \varrho \in I_u \text{ such that } \dot{\psi}_u(\varrho) = 1, \tag{52a}$$
$$\dot{x}_u(\varrho) = -h_2\big(x_u(\varrho)\big)\, g_2\big(\psi_u(\varrho)\big), \text{ for almost all } \varrho \in I_u \text{ such that } \dot{\psi}_u(\varrho) = -1, \tag{52b}$$
$$x_u(0) = x_0. \tag{52c}$$

By Theorem 4.1, the differential equation (52) has a unique Carathéodory solution x_u on I_u. Moreover $x_u \in \mathcal{S}(I_u, \mathbb{R})$ and $\|x_u\| = \|x\| \leq \max\big(|x_0|, D_1, D_2\big)$.

Remark 9.1. *When the initial condition x_0 of the Bouc-Wen model (1)–(3) satisfies $|x_0| \leq 1$ we have $\|x\| \leq 1$ where x is the solution of (1)–(3) [22, Table 2]. Taking $D_1 = D_2 = 1$ makes the Bouc-Wen model a particular case of Babuška's model.*

9.2. Hysteresis loop of Babuška's model

In this section we consider a T–periodic input $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ such that u is strictly increasing on $[0, T_1]$ and strictly decreasing on $[T_1, T]$ for some $T_1 \in\,]0, T[$. The main result of this section is Theorem 9.1 which provides the explicit analytic expression of the hysteresis loop of Babuška's model under Conditions (A)–(E).

Define

$$u_{\min} = u(0),\ u_{\max} = u(T_1),\ \varrho_1 = \rho_u(T_1) = u_{\max} - u_{\min},\ \varrho_2 = \rho_u(T) = 2\varrho_1.$$

Then the corresponding normalized input is periodic of period $\rho_u(T) = \varrho_2$ and is given by Equation (11).

By Theorem 7.1 there exists a unique $\zeta \in [-D_2, D_1]$ such that the solution of (52a) and (52b) along with

$$x_u(0) = \zeta \tag{53}$$

is periodic of period ϱ_2. Denote this solution by $\bar{x}_u$.

Finally, observe that $I_u = \mathbb{R}_+$ and ρ_u is strictly increasing.

Proposition 9.1. $\zeta < D_1$.

Proof. Suppose that $\zeta = D_1$. By Proposition 4.1 we have $\bar{x}_u(\varrho) \leq D_1$ for all $\varrho \in [0, \varrho_1]$. On the other hand we get from Equation (52a) that $\bar{x}_u$ is nondecreasing so that $\bar{x}_u(\varrho) \geq \bar{x}_u(0) = \zeta = D_1$ for all $\varrho \in [0, \varrho_1]$. We thus get $\bar{x}_u(\varrho) = D_1, \forall \varrho \in [0, \varrho_1]$.

On the other hand, we get from Equation (52b) that $\dot{\bar{x}}_u(\varrho) \leq 0$ for almost all $\varrho \in]\varrho_1, 2\varrho_1[$. By Claim 7 in the proof of Theorem 7.1 we know that $\bar{x}_u$ cannot be constant on $\mathbb{R}_+$. This means that there exists $\varrho' \in]\varrho_1, \varrho_2]$ such that $\bar{x}_u(\varrho') \neq D_1$. Since $\bar{x}_u$ is absolutely continuous we have

$$\bar{x}_u(\varrho') - \bar{x}_u(\varrho_1) = \int_{\varrho_1}^{\varrho'} \dot{\bar{x}}_u(\varrho)\, \mathrm{d}\varrho \leq 0,$$

which gives $\bar{x}_u(\varrho') \leq \bar{x}_u(\varrho_1) = D_1$. This fact combined with $\bar{x}_u(\varrho') \neq D_1$ leads to $\bar{x}_u(\varrho') < D_1$.

On the other hand we have

$$\bar{x}_u(\varrho_2) - \bar{x}_u(\varrho') = \int_{\varrho'}^{\varrho_2} \dot{\bar{x}}_u(\varrho)\, \mathrm{d}\varrho \leq 0$$

which provides $\bar{x}_u(\varrho_2) \leq \bar{x}_u(\varrho') < D_1$. This inequality leads to a contradiction since $\bar{x}_u(\varrho_2) = \bar{x}_u(0) = \zeta = D_1$ owing to the periodicity of $\bar{x}_u$. □

Define the set

$$A = \big\{\varrho \in [0, \varrho_1] \mid G_1(\varrho + u_{\min}) - G_1(u_{\min}) + \ell_1(\zeta) \leq \ell_1(D_1)\big\}.$$

Observe that $0 \in A$ so that $A \neq \emptyset$. Let $\sup(A)$ be the supremum of A. Then $\sup(A) \in A$ owing to the continuity of the function G_1, and $\sup(A) > 0$ owing to Proposition 9.1.

Proposition 9.2. $\sup(A) < \varrho_1$ *if and only if*

$$G_1(u_{\max}) - G_1(u_{\min}) + \ell_1(\zeta) > \ell_1(D_1). \tag{54}$$

In this case we have

$$\sup(A) = G_1^{-1}\bigg(G_1(u_{\min}) - \ell_1(\zeta) + \ell_1(D_1)\bigg) - u_{\min}. \tag{55}$$

$\sup(A) = \varrho_1$ *if and only if*

$$G_1(u_{\max}) - G_1(u_{\min}) + \ell_1(\zeta) \leq \ell_1(D_1). \tag{56}$$

Proof. If $\ell_1(D_1) = \infty$ we have $\sup(A) = \varrho_1$ and Inequality (54) is false.

If $\ell_1(D_1) < \infty$ consider the function $q_A : [0, \varrho_1] \to \mathbb{R}$ defined by

$$q_A(\varrho) = G_1(\varrho + u_{\min}) - G_1(u_{\min}) + \ell_1(\zeta) - \ell_1(D_1).$$

Then $q_A(0) < 0$ owing to Proposition 9.1, q_A is continuous and strictly increasing. If Inequality (54) holds we get $q_A(\varrho_1) > 0$ so that q_A has a unique zero in the interval $]0, \varrho_1[$ which is $\sup(A)$. That is

$$G_1\big(\sup(A) + u_{\min}\big) - G_1(u_{\min}) + \ell_1(\zeta) - \ell_1(D_1) = 0 \tag{57}$$

which gives (55).

On the other hand, if Inequality (54) does not hold we have $q(\varrho_1) \leq 0$ so that $\varrho_1 \in A$ which means that $\sup(A) = \varrho_1$. □

Define the function $w_1 : [0, \sup(A)] \to \mathbb{R}$ by

$$w_1(\varrho) = \ell_1^{-1}\left(\ell_1(\zeta) + G_1(\varrho + u_{\min}) - G_1(u_{\min})\right).$$

Proposition 9.3. *$\bar{x}_u = w_1$ on $[0, \sup(A)]$. If $\sup(A) < \varrho_1$ then $\bar{x}_u(\varrho) = D_1$, for all $\varrho \in [\sup(A), \varrho_1]$.*

Proof. The function w_1 is well defined and strictly increasing on the interval $[0, \sup(A)]$, and is C^1 on the interval $]0, \sup(A)[$. For $\varrho \in]0, \sup(A)[$ we have

$$\begin{aligned} \dot{w}_1(\varrho) &= \frac{\dot{G}_1(\varrho + u_{\min})}{\dot{\ell}_1\left[\ell_1^{-1}\left(\ell_1(\zeta) + G_1(\varrho + u_{\min}) - G_1(u_{\min})\right)\right]} \\ &= \frac{\dot{G}_1(\varrho + u_{\min})}{\dot{\ell}_1\big[w_1(\varrho)\big]} = \frac{g_1(\varrho + u_{\min})}{1/h_1\big(w_1(\varrho)\big)} = h_1\big(w_1(\varrho)\big)g_1\big(\psi_u(\varrho)\big). \end{aligned}$$

Since $\bar{x}_u(0) = \zeta = w_1(0)$ it follows that $w_1 = \bar{x}_u$ on $[0, \sup(A)]$ owing to the uniqueness of the solutions of the differential equation (52a) $\wedge$ (52b) $\wedge$ (53).

There are two cases:

(i) $\sup(A) < \varrho_1$.
(ii) $\sup(A) = \varrho_1$

In Case (i) we have $\bar{x}_u\big(\sup(A)\big) = w_1\big(\sup(A)\big) = D_1$ owing to Equation (55). By Proposition 4.1 we have $\bar{x}_u(\varrho) \le D_1$ for all $\varrho \in [\sup(A), \varrho_1]$. On the other hand we get from Equation (52a) that $\bar{x}_u$ is nondecreasing on $[0, \varrho_1]$ so that $\bar{x}_u(\varrho) \ge \bar{x}_u\big(\sup(A)\big) = D_1$ for all $\varrho \in [\sup(A), \varrho_1]$. We thus get

$$\bar{x}_u(\varrho) = D_1, \forall \varrho \in [\sup(A), \varrho_1]. \tag{58}$$

In Case (ii) we have $w_1 = \bar{x}_u$ on $[0, \varrho_1]$. $\square$

We now analyze what happens on the interval $[\varrho_1, \varrho_2]$.

Define the set

$$B = \left\{\varrho \in [\varrho_1, \varrho_2] \mid \ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max}) \ge \ell_2(\zeta)\right\}.$$

Observe that $\bar{x}_u(\varrho_1) \ge \bar{x}_u\big(\sup(A)\big) = w_1\big(\sup(A)\big) > w_1(0) = \zeta$ so that $\ell_2\big(\bar{x}_u(\varrho_1)\big) > \ell_2(\zeta) \ge \ell_2(-D_2)$. Thus $\ell_2\big(\bar{x}_u(\varrho_1)\big)$ is finite and $\varrho_1 \in B$ so that $B \neq \emptyset$. Let $\sup(B)$ be the supremum of B. Then $\sup(B) \in B$ owing to the continuity of the function G_2, and $\sup(B) > \varrho_1$ since $\ell_2\big(\bar{x}_u(\varrho_1)\big) > \ell_2(\zeta)$.

Define the function $w_2 : [\varrho_1, \sup(B)] \to \mathbb{R}$ by

$$w_2(\varrho) = \ell_2^{-1}\left(\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max})\right).$$

Proposition 9.4. *$\bar{x}_u = w_2$ on $[\varrho_1, \sup(B)]$. Moreover,*

(i) $\sup(B) < \varrho_2$ *if and only if*

$$\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\min}) - G_2(u_{\max}) < \ell_2(-D_2). \tag{59}$$

In this case we have

$$\zeta = -D_2, \tag{60a}$$

$$\sup(B) = -G_2^{-1}\Bigg(\ell_2(-D_2) - \ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\max})\Bigg) + 2u_{\max} - u_{\min}, \tag{60b}$$

$$\bar{x}_u(\varrho) = -D_2, \forall \varrho \in [\sup(B), \varrho_2]. \tag{60c}$$

(ii) $\sup(B) = \varrho_2$ *if and only if*

$$\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\min}) - G_2(u_{\max}) \geq \ell_2(-D_2). \tag{61}$$

In this case we have $\bar{x}_u = w_2$ *for all* $\varrho \in [\varrho_1, \varrho_2]$ *and*

$$\zeta = \ell_2^{-1}\Big(\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\min}) - G_2(u_{\max})\Big). \tag{62}$$

Proof. The function w_2 is well defined and strictly decreasing on the interval $[\varrho_1, \sup(B)]$, and is C^1 on the interval $]\varrho_1, \sup(B)[$. For $\varrho \in]\varrho_1, \sup(B)[$ we have

$$\begin{aligned}
\dot{w}_2(\varrho) &= \frac{-\dot{G}_2(-\varrho + 2u_{\max} - u_{\min})}{\dot{\ell}_2\left[\ell_2^{-1}\Big(\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max})\Big)\right]} \\
&= -\frac{\dot{G}_2(-\varrho + 2u_{\max} - u_{\min})}{\dot{\ell}_2\big[w_2(\varrho)\big]} = -\frac{g_2(-\varrho + 2u_{\max} - u_{\min})}{1/h_2\big(w_2(\varrho)\big)} = -h_2\big(w_2(\varrho)\big)g_2\big(\psi_u(\varrho)\big).
\end{aligned}$$

Since $w_2(\varrho_1) = \bar{x}_u(\varrho_1)$ it follows that $w_2 = \bar{x}_u$ on $[\varrho_1, \sup(B)]$ owing to the uniqueness of the solutions of the differential equation (52a), (52b), (53).

We first consider the case $\ell_2(-D_2) \neq -\infty$. Then $\ell_2(\zeta) \neq -\infty$. Consider the function $q_B : [\varrho_1, \varrho_2] \to \mathbb{R}$ defined by

$$q_B(\varrho) = \ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max}) - \ell_2(\zeta).$$

Then q_B is continuous and strictly decreasing, $q_B(\varrho_1) > 0$ since $\bar{x}_u(\varrho_1) = \bar{x}_u\big(\sup(A)\big) = w_1\big(\sup(A)\big) > w_1(0) = \zeta$.

Proof of the if part of (i). If Inequality (59) holds then

$$\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\min}) - G_2(u_{\max}) < \ell_2(\zeta) \tag{63}$$

since $-D_2 \leq \zeta$. Thus $q_B(\varrho_2) < 0$ so that q_B has a unique zero in the interval $]\varrho_1, \varrho_2[$. It is immediate that this zero is $\sup(B)$ which gives

$$\sup(B) = -G_2^{-1}\Bigg(\ell_2(\zeta) - \ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\max})\Bigg) + 2u_{\max} - u_{\min}. \tag{64}$$

We have $\bar{x}_u\big(\sup(B)\big) = w_2\big(\sup(B)\big) = \zeta$ owing to Equation (64). On the other hand we get from Equation (52b) that $\bar{x}_u$ is nonincreasing on $[\sup(B), \varrho_2]$ so that $\bar{x}_u(\varrho) \leq \bar{x}_u(\sup(B)) = \zeta$ for all $\varrho \in [\sup(B), \varrho_2]$. Since $\bar{x}_u$ is ϱ_2–periodic we have $\bar{x}_u(\varrho_2) = \bar{x}_u(0) = \zeta$ so that

$$\bar{x}_u(\varrho) = \zeta, \forall \varrho \in [\sup(B), \varrho_2]. \tag{65}$$

Equation (65) implies that $\dot{\bar{x}}_u(\varrho) = 0$ for all $\varrho \in \,]\sup(B), \varrho_2[\, \neq \emptyset$. Substituting in (52b) gives $h_2\big(\bar{x}_u(\varrho)\big) = 0$ for all $\varrho \in [\sup(B), \varrho_2]$ so that $\bar{x}_u(\varrho) = -D_2$ for all $\varrho \in [\sup(B), \varrho_2]$. In particular $\bar{x}_u(\varrho_2) = -D_2 = \bar{x}_u(0) = \zeta$. Equations (60) are thus established.

Proof of the only if part of (i). If $\sup(B) < \varrho_2$ then $q_B\big(\sup(B)\big) = 0$ and $q_B(\varrho_2) < 0$. The argument follows as in the proof of the if part leading to Equations (60). Writing that $\sup(B) < \varrho_2$ where $\sup(B)$ is given by (60b) gives Inequality (59).

Proof of the only if part of (ii). If $\sup(B) = \varrho_2$ then $\varrho_2 \in B$ and the following inequality holds

$$\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\min}) - G_2(u_{\max}) \geq \ell_2(\zeta).$$

Since $\zeta \geq -D_2$ we obtain Inequality (61).

Proof of the if part of (ii). Suppose that Inequality (61) holds. If $\sup(B) < \varrho_2$ then following the same argument as in the proof of the if part of (i) we get Inequality (59) which contradicts (61).

Equation (62) is obtained by writing that $\zeta = \bar{x}_u(0) = \bar{x}_u(\varrho_2) = w_2(\varrho_2)$.

Finally, consider the case $\ell_2(-D_2) = -\infty$; then Inequality (59) cannot hold. If $\zeta = -D_2$ then the inequality that appears in the definition of the set B always holds which means that $\sup(B) = \varrho_2$. If $\zeta \neq -D_2$ then $\ell_2(\zeta) \neq -\infty$, so defining the function q_B as above and supposing that $\sup(B) < \varrho_2$ we get to a contradiction following the same argument as in the proof of the if part of (i). This means that $\sup(B) = \varrho_2$. $\square$

Proposition 9.5. *The hysteresis loop of the model (51) with respect to* (u, x_0) *is given by*

$$\mathcal{G}_{u,x_0} = \big\{\big(\psi_u(\varrho), \bar{x}_u(\varrho)\big), \varrho \in [0, \varrho_2]\big\}, \tag{66}$$

where the function $\bar{x}_u$ *is calculated as follows.*

Case κ_1*:* $(\sup(A) < \varrho_1) \wedge (\sup(B) < \varrho_2)$.

The quantities

$$\varrho_A = G_1^{-1}\bigg(G_1(u_{\min}) - \ell_1(-D_2) + \ell_1(D_1)\bigg) - u_{\min}, \tag{67}$$

$$\varrho_B = -G_2^{-1}\bigg(\ell_2(-D_2) - \ell_2\big(D_1\big) + G_2(u_{\max})\bigg) + 2u_{\max} - u_{\min}, \tag{68}$$

are well defined and we have

$$0 < \varrho_A < \varrho_1 \text{ and } \varrho_1 < \varrho_B < \varrho_2. \tag{69}$$

We have

$$\zeta = -D_2, \tag{70a}$$
$$\bar{x}_u(\varrho) = \ell_1^{-1}\Big(\ell_1(-D_2) + G_1(\varrho + u_{\min}) - G_1(u_{\min})\Big), \forall \varrho \in [0, \varrho_A], \tag{70b}$$
$$\bar{x}_u(\varrho) = D_1, \forall \varrho \in [\varrho_A, \varrho_1], \tag{70c}$$
$$\bar{x}_u(\varrho) = \ell_2^{-1}\Big(\ell_2(D_1) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max})\Big), \forall \varrho \in [\varrho_1, \varrho_B], \tag{70d}$$
$$\bar{x}_u(\varrho) = -D_2, \forall \varrho \in [\varrho_B, \varrho_2], \tag{70e}$$
$$\sup(A) = \varrho_A, \tag{70f}$$
$$\sup(B) = \varrho_B. \tag{70g}$$

Case κ_2*:* $(\sup(A) < \varrho_1) \wedge (\sup(B) = \varrho_2)$*.*
The quantities

$$\zeta = \ell_2^{-1}\Big(\ell_2(D_1) + G_2(u_{\min}) - G_2(u_{\max})\Big), \tag{71}$$
$$\varrho_A = G_1^{-1}\Big(G_1(u_{\min}) - \ell_1(\zeta) + \ell_1(D_1)\Big) - u_{\min}, \tag{72}$$

are well defined and we have $0 < \varrho_A < \varrho_1$*. We have*

$$\bar{x}_u(\varrho) = \ell_1^{-1}\Big(\ell_1(\zeta) + G_1(\varrho + u_{\min}) - G_1(u_{\min})\Big), \forall \varrho \in [0, \varrho_A], \tag{73a}$$
$$\bar{x}_u(\varrho) = D_1, \forall \varrho \in [\varrho_A, \varrho_1], \tag{73b}$$
$$\bar{x}_u(\varrho) = \ell_2^{-1}\Big(\ell_2(D_1) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max})\Big), \forall \varrho \in [\varrho_1, \varrho_2], \tag{73c}$$
$$\sup(A) = \varrho_A. \tag{73d}$$

Case κ_3*:* $(\sup(A) = \varrho_1) \wedge (\sup(B) < \varrho_2)$*.*
The quantities

$$\bar{x}_u(\varrho_1) = \ell_1^{-1}\Big(\ell_1(-D_2) + G_1(u_{\max}) - G_1(u_{\min})\Big), \tag{74}$$
$$\varrho_B = -G_2^{-1}\Big(\ell_2(-D_2) - \ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\max})\Big) + 2u_{\max} - u_{\min}, \tag{75}$$

are well defined and we have $\varrho_1 < \varrho_B < \varrho_2$*. We get*

$$\zeta = -D_2, \tag{76a}$$
$$\bar{x}_u(\varrho) = \ell_1^{-1}\Big(\ell_1(-D_2) + G_1(\varrho + u_{\min}) - G_1(u_{\min})\Big), \forall \varrho \in [0, \varrho_1], \tag{76b}$$
$$\bar{x}_u(\varrho) = \ell_2^{-1}\Big(\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max})\Big), \forall \varrho \in [\varrho_1, \varrho_B], \tag{76c}$$
$$\bar{x}_u(\varrho) = -D_2, \forall \varrho \in [\varrho_B, \varrho_2], \tag{76d}$$
$$\sup(B) = \varrho_B. \tag{76e}$$

Case *κ_4: $(\sup(A) = \varrho_1) \wedge (\sup(B) = \varrho_2)$.*
We have

$$\bar{x}_u(\varrho) = \ell_1^{-1}\Big(\ell_1(\zeta) + G_1(\varrho + u_{\min}) - G_1(u_{\min})\Big), \forall \varrho \in [0, \varrho_1], \tag{77a}$$

$$\bar{x}_u(\varrho) = \ell_2^{-1}\Big(\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max})\Big), \forall \varrho \in [\varrho_1, \varrho_2], \tag{77b}$$

where

$$\bar{x}_u(\varrho_1) = \ell_1^{-1}\Big(\ell_1(\zeta) + G_1(u_{\max}) - G_1(u_{\min})\Big), \tag{78}$$

and ζ satisfies

$$\ell_2^{-1}\Big(\ell_2(\zeta) + G_2(u_{\max}) - G_2(u_{\min})\Big) - \ell_1^{-1}\Big(\ell_1(\zeta) + G_1(u_{\max}) - G_1(u_{\min})\Big) = 0. \tag{79}$$

Proof. **Analysis in the case κ_1.**

By Proposition 9.3 we have $\bar{x}_u = w_1$ on $[0, \sup(A)]$, and $\bar{x}_u(\varrho) = D_1$, for all $\varrho \in [\sup(A), \varrho_1]$.

By Proposition 9.4 we have $\bar{x}_u = w_2$ on $[\varrho_1, \sup(B)]$, and Equations (60) hold.

Analysis in the case κ_2. By Proposition 9.3 we have $\bar{x}_u = w_1$ on $[0, \sup(A)]$, and $\bar{x}_u(\varrho) = D_1$, for all $\varrho \in [\sup(A), \varrho_1]$.

Since $\sup(B) = \varrho_2$ we have $\bar{x}_u = w_2$ on $[\varrho_1, \varrho_2]$ by Proposition 9.4. In particular $\zeta = \bar{x}_u(\varrho_2) = w_2(\varrho_2)$ which gives (71). Now that ζ has been determined we find $\sup(A)$ from Equation (55) which shows that $\sup(A) = \varrho_A$ where ϱ_A is given by Equation (72).

Analysis in the case κ_3.

We have that $\bar{x}_u = w_1$ on $[0, \varrho_1]$ by Proposition 9.3. Thus

$$\bar{x}_u(\varrho_1) = \ell_1^{-1}\Big(\ell_1(\zeta) + G_1(u_{\max}) - G_1(u_{\min})\Big). \tag{80}$$

By Proposition 9.4 we have $\bar{x}_u = w_2$ on $[\varrho_1, \sup(B)]$, and Equations (60) hold. Then $\bar{x}_u(\varrho_1)$ is obtained from Equation (80) as

$$\bar{x}_u(\varrho_1) = \ell_1^{-1}\Bigg(\ell_1(-D_2) + G_1(u_{\max}) - G_1(u_{\min})\Bigg). \tag{81}$$

Analysis in the case κ_4.

By Proposition 9.3 we have $\bar{x}_u = w_1$ on the interval $[0, \varrho_1]$. In particular

$$\bar{x}_u(\varrho_1) = \ell_1^{-1}\Bigg(\ell_1(\zeta) + G_1(u_{\max}) - G_1(u_{\min})\Bigg). \tag{82}$$

Since $\sup(B) = \varrho_2$ we have $\bar{x}_u = w_2$ on $[\varrho_1, \varrho_2]$ by Proposition 9.4. In particular $\bar{x}_u(\varrho_2) = w_2(\varrho_2) = \bar{x}_u(0) = \zeta$ which leads to

$$\zeta = \ell_2^{-1}\Bigg(\ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(u_{\min}) - G_2(u_{\max})\Bigg). \tag{83}$$

We get (79) combining (82) and (83).

$\square$

Comment. Proposition 9.5 reveals four mutually exclusive and collectively exhaustive cases that describe completely the hysteresis loop of Babuška's model. These cases are

- κ_1: $(\sup(A) < \varrho_1) \wedge (\sup(B) < \varrho_2)$.
- κ_2: $(\sup(A) < \varrho_1) \wedge (\sup(B) = \varrho_2)$.
- κ_3: $(\sup(A) = \varrho_1) \wedge (\sup(B) < \varrho_2)$.
- κ_4: $(\sup(A) = \varrho_1) \wedge (\sup(B) = \varrho_2)$.

In each case the analytic expression of the hysteresis loop is provided.

However, to know which case applies we need to know $\sup(A)$ and $\sup(B)$ which depend on ζ, and we have an explicit expression of ζ only when we know which case applies.

This means that we need explicit conditions that ensure which case applies. This is the aim of Propositions 9.6–9.10.

Proposition 9.6. *Suppose that $\ell_1(D_1) \neq \infty$ and $\ell_2(-D_2) \neq -\infty$. Then Case κ_1 applies if and only if we have*

$$G_1(u_{\max}) - G_1(u_{\min}) > \ell_1(D_1) - \ell_1(-D_2), \tag{84a}$$
$$G_2(u_{\max}) - G_2(u_{\min}) > \ell_2\big(D_1\big) - \ell_2(-D_2). \tag{84b}$$

Proof. **Proof of the if part.**

The fact that $\zeta \geq -D_2$ along with Inequality (84a) gives

$$\ell_1(\zeta) + G_1(u_{\max}) - G_1(u_{\min}) \geq \ell_1(-D_2) + G_1(u_{\max}) - G_1(u_{\min}) > \ell_1(D_1)$$

that is Inequality (54). Thus $\sup(A) < \varrho_1$ by Proposition 9.2. Consider the function $q_B : [\varrho_1, \varrho_2] \to \mathbb{R}$ defined by

$$q_B(\varrho) = \ell_2(D_1) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max}) - \ell_2(\zeta).$$

Then q_B is continuous and strictly decreasing, $q_B(\varrho_1) > 0$ by Proposition 9.1, $q_B(\varrho_2) < 0$ owing to Inequality (84b) and the fact that $\zeta \geq -D_2$. Thus q_B has a unique zero in the interval $]\varrho_1, \varrho_2[$ which is precisely $\sup(B)$. We thus have $\sup(B) < \varrho_2$.

Proof of the only if part.

By Proposition 9.4 Inequality (59) holds, and by Proposition 9.3 we have $\bar{x}_u(\varrho_1) = D_1$ which gives (84b).

By Proposition 9.2 Inequality (54) holds, and by Proposition 9.4 we have $\zeta = -D_2$ which gives (84a).

□

Proposition 9.7. *Suppose that $\ell_1(D_1) \neq \infty$. If*

$$G_1(u_{\max}) - G_1(u_{\min}) > \ell_1(D_1) - \ell_1(-D_2), \tag{85a}$$
$$G_2(u_{\max}) - G_2(u_{\min}) \leq \ell_2\big(D_1\big) - \ell_2(-D_2), \tag{85b}$$

then Case κ_2 applies. Conversely if Case κ_2 applies then Inequality (85b) holds.

Proof. The fact that $\zeta \geq -D_2$ along with Inequality (85a) give

$$\ell_1(\zeta) + G_1(u_{\max}) - G_1(u_{\min}) \geq \ell_1(-D_2) + G_1(u_{\max}) - G_1(u_{\min}) > \ell_1(D_1)$$

that is Inequality (54). Thus $\sup(A) < \varrho_1$ by Proposition 9.2 so that $\bar{x}_u(\varrho_1) = D_1$ by Proposition 9.3. Then (85b) is the same as (61) so that $\sup(B) = \varrho_2$.

Now, if Case κ_2 applies then Inequality (61) holds by Proposition 9.4 where $\bar{x}_u(\varrho_1) = D_1$ by Proposition 9.3 which gives (85b).

□

Proposition 9.8. *Suppose that $\ell_2(-D_2) \neq -\infty$. If*

$$G_1(u_{\max}) - G_1(u_{\min}) \leq \ell_1(D_1) - \ell_1(-D_2), \tag{86a}$$

$$G_2(u_{\max}) - G_2(u_{\min}) > \ell_2\big(D_1\big) - \ell_2(-D_2), \tag{86b}$$

then Case κ_3 applies. Conversely if Case κ_3 applies then Inequality (86a) holds.

Proof.

Claim 10. $\sup(A) = \varrho_1$.

Proof. Suppose that $\sup(A) < \varrho_1$. Then by Proposition 9.3 we get $\bar{x}_u(\varrho_1) = D_1$. Consider the function $q_B : [\varrho_1, \varrho_2] \to \mathbb{R}$ defined by

$$q_B(\varrho) = \ell_2(D_1) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max}) - \ell_2(\zeta).$$

Then q_B is continuous and strictly decreasing, $q_B(\varrho_1) > 0$ by Proposition 9.1, $q_B(\varrho_2) < 0$ owing to Inequality (86b) and the fact that $\zeta \geq -D_2$. Thus q_B has a unique zero in the interval $]\varrho_1, \varrho_2[$ which is precisely $\sup(B)$ so that $\sup(B) < \varrho_2$. By Proposition 9.4 we have $\zeta = -D_2$ obtaining that Inequality (54) contradicts Inequality (86a). □

Consider the function $q_B : [\varrho_1, \varrho_2] \to \mathbb{R}$ defined by

$$q_B(\varrho) = \ell_2\big(\bar{x}_u(\varrho_1)\big) + G_2(-\varrho + 2u_{\max} - u_{\min}) - G_2(u_{\max}) - \ell_2(\zeta).$$

Then q_B is continuous and strictly decreasing, $q_B(\varrho_1) > 0$ since $\bar{x}_u(\varrho_1) = w_1(\varrho_1) > w_1(0) = \zeta$, $q_B(\varrho_2) < \Big(\ell_2\big(\bar{x}_u(\varrho_1)\big) - \ell_2(D_1)\Big) + \Big(\ell_2(-D_2) - \ell_2(\zeta)\Big)$ owing to Inequality (86b). Since $\bar{x}_u(\varrho_1) \leq D_1$ and $\zeta \geq -D_2$ by Theorem 7.1 we get $q_B(\varrho_2) < 0$. Thus q_B has a unique zero in the interval $]\varrho_1, \varrho_2[$ which is precisely $\sup(B)$. We thus have $\sup(B) < \varrho_2$.

Now suppose that Case κ_3 applies. Then by Proposition 9.4 we have $\zeta = -D_2$. Also, by Proposition 9.2 Inequality (56) holds which gives (86a).

□

Proposition 9.9. *If*

$$G_1(u_{\max}) - G_1(u_{\min}) \leq \ell_1(D_1) - \ell_1(-D_2), \tag{87a}$$

$$G_2(u_{\max}) - G_2(u_{\min}) \leq \ell_2\big(D_1\big) - \ell_2(-D_2), \tag{87b}$$

then (i)–(vi) *hold.*

(i) *Case κ_1 cannot occur.*
(ii) *Case κ_2 holds if and only if*

$$\ell_2^{-1}\Big(\ell_2(D_1)+G_2(u_{\min})-G_2(u_{\max})\Big) > \ell_1^{-1}\Big(G_1(u_{\min})-G_1(u_{\max})+\ell_1(D_1)\Big). \quad (88)$$

(iii) *Case κ_3 holds if and only if*

$$\ell_1^{-1}\Big(\ell_1(-D_2)+G_1(u_{\max})-G_1(u_{\min})\Big) < \ell_2^{-1}\Big(G_2(u_{\max})-G_2(u_{\min})+\ell_2(-D_2)\Big). \quad (89)$$

(iv) *Inequalities (88) and (89) cannot hold simultaneously. That is (88) $\wedge$ (89) is false.*
(v) *Case κ_4 applies if and only if Inequalities (88) and (89) are both false. If Case κ_4 applies then Inequalities (87) hold.*
(vi) *Suppose that Case κ_4 applies. Denote $\nu_1 = G_1(u_{\max}) - G_1(u_{\min})$, $\nu_2 = G_2(u_{\max}) - G_2(u_{\min})$ and write Equation (98) as*

$$\nu_2 = \ell_2\Big[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\Big] - \ell_2(\zeta) = H_{\nu_1}(\zeta), \quad (90)$$

where

$$H_{\nu_1} : \Big[-D_2, \ell_1^{-1}\Big(\ell_1(D_1)-\nu_1\Big)\Big] \to \mathbb{R}.$$

Then H_{ν_1} is invertible and

$$\zeta = H_{\nu_1}^{-1}(\nu_2). \quad (91)$$

Proof. (i) Suppose that $\sup(A) < \varrho_1$ and $\sup(B) < \varrho_2$. By Proposition 9.3 we have $\bar{x}_u(\varrho_1) = D_1$ so that Inequality (59) contradicts (87b).

(ii) **Proof of the only if part of (ii).** Suppose that $\sup(A) < \varrho_1$ and $\sup(B) = \varrho_2$. By Proposition 9.3 we have $\bar{x}_u(\varrho_1) = D_1$ so that Equation (62) becomes

$$\zeta = \ell_2^{-1}\Big(\ell_2(D_1)+G_2(u_{\min})-G_2(u_{\max})\Big).$$

Then Inequality (54) gives (88). Note that both members of Inequality (88) are well defined owing to (87).

(iii) **Proof of the only if part of (iii).** Suppose that $\sup(A) = \varrho_1$ and $\sup(B) < \varrho_2$. By Proposition 9.4 we have $\zeta = -D_2$ and by Proposition 9.3 we have

$$\bar{x}_u(\varrho_1) = \ell_1^{-1}\Big(\ell_1(-D_2)+G_1(u_{\max})-G_1(u_{\min})\Big). \quad (92)$$

Then Inequality (59) leads to (89). Note that both members of Inequality (89) are well defined owing to (87).

(iv) Denote $\nu_1 = G_1(u_{\max})-G_1(u_{\min})$ and $\nu_2 = G_2(u_{\max})-G_2(u_{\min})$. Since $u_{\max} > u_{\min}$ it follows that $\nu_1 > 0$ and $\nu_2 > 0$. Also, owing to (87) we have $0 < \nu_1 \leq \nu_{1\,\max}$ and $0 < \nu_2 \leq \nu_{2\,\max}$ where $\nu_{1\,\max} = \ell_1(D_1) - \ell_1(-D_2)$ and $\nu_{2\,\max} = \ell_2(D_1) - \ell_2(-D_2)$. Then (88) is equivalent to

$$F_1(\nu_1) = \ell_2(D_1) - \ell_2\Big[\ell_1^{-1}\Big(-\nu_1+\ell_1(D_1)\Big)\Big] > \nu_2. \quad (93)$$

Observe that $F_1(0) = 0$, $F_1(\nu_{1\,\max}) = \nu_{2\,\max}$, F_1 is strictly increasing and continuous so that $F_1 : [0, \nu_{1\,\max}] \to [0, \nu_{2\,\max}]$, and $F_1 \in C^1\big(]0, \nu_{1\,\max}[, \mathbb{R}\big)$. The derivative of F_1 is

$$\dot{F}_1(\nu_1) = \frac{\dot{\ell}_2\left[\ell_1^{-1}\Big(-\nu_1 + \ell_1(D_1)\Big)\right]}{\dot{\ell}_1\left[\ell_1^{-1}\Big(-\nu_1 + \ell_1(D_1)\Big)\right]} = \frac{h_1\left[\ell_1^{-1}\Big(-\nu_1 + \ell_1(D_1)\Big)\right]}{h_2\left[\ell_1^{-1}\Big(-\nu_1 + \ell_1(D_1)\Big)\right]}$$

so that

$$\lim_{\nu_1 \downarrow 0} \dot{F}_1(\nu_1) = 0, \text{ and } \lim_{\nu_1 \uparrow \nu_{1\,\max}} \dot{F}_1(\nu_1) = \infty.$$

Note that numerator of $\dot{F}_1(\nu_1)$ is strictly increasing from 0 at $\nu_1 = 0$ to $h_1(-D_2) > 0$ at $\nu_1 = \nu_{1\,\max}$, and the denominator of $\dot{F}_1(\nu_1)$ is strictly decreasing from $h_2(D_1) > 0$ at $\nu_1 = 0$ to 0 at $\nu_1 = \nu_{1\,\max}$. Thus $\dot{F}_1(\nu_1)$ is strictly increasing from 0 at $\nu_1 = 0$ to ∞ at $\nu_1 = \nu_{1\,\max}$ so that F_1 is strictly convex on the interval $[0, \nu_{1\,\max}]$. Thus

$$F_1(\nu_1) < \frac{\nu_{2\,\max}}{\nu_{1\,\max}} \nu_1, \forall \nu_1 \in]0, \nu_{1\,\max}[. \tag{94}$$

On the other hand, (89) is equivalent to

$$F_2(\nu_1) = \ell_2\left[\ell_1^{-1}\Big(\ell_1(-D_2) + \nu_1\Big)\right] - \ell_2(-D_2) < \nu_2, \tag{95}$$

where $F_2 : [0, \nu_{1\,\max}] \to [0, \nu_{2\,\max}]$. Note that $F_2(0) = 0$, $F_2(\nu_{1\,\max}) = \nu_{2\,\max}$, F_2 is strictly increasing and continuous, and $F_2 \in C^1\big(]0, \nu_{1\,\max}[, \mathbb{R}\big)$. The derivative of F_2 is

$$\dot{F}_2(\nu_1) = \frac{\dot{\ell}_2\left[\ell_1^{-1}\Big(\ell_1(-D_2) + \nu_1\Big)\right]}{\dot{\ell}_1\left[\ell_1^{-1}\Big(\ell_1(-D_2) + \nu_1\Big)\right]} = \frac{h_1\left[\ell_1^{-1}\Big(\ell_1(-D_2) + \nu_1\Big)\right]}{h_2\left[\ell_1^{-1}\Big(\ell_1(-D_2) + \nu_1\Big)\right]}$$

so that

$$\lim_{\nu_1 \downarrow 0} \dot{F}_2(\nu_1) = \infty, \text{ and } \lim_{\nu_1 \uparrow \nu_{1\,\max}} \dot{F}_2(\nu_1) = 0.$$

Note that numerator of $\dot{F}_2(\nu_1)$ is strictly decreasing from $h_1(-D_2) > 0$ at $\nu_1 = 0$ to 0 at $\nu_1 = \nu_{1\,\max}$, and the denominator of $\dot{F}_2(\nu_1)$ is strictly increasing from 0 at $\nu_1 = 0$ to $h_2(D_1) > 0$ at $\nu_1 = \nu_{1\,\max}$. Thus $\dot{F}_2(\nu_1)$ is strictly decreasing from ∞ at $\nu_1 = 0$ to 0 at $\nu_1 = \nu_{1\,\max}$ so that F_2 is strictly concave on the interval $]0, \nu_{1\,\max}[$. Thus

$$F_2(\nu_1) > \frac{\nu_{2\,\max}}{\nu_{1\,\max}} \nu_1, \forall \nu_1 \in]0, \nu_{1\,\max}[. \tag{96}$$

Combining Inequalities (94) and (96) it comes that

$$F_2(\nu_1) > F_1(\nu_1), \forall \nu_1 \in]0, \nu_{1\,\max}[. \tag{97}$$

Take $(\nu_1, \nu_2) \in]0, \nu_{1\,\max}] \times]0, \nu_{2\,\max}]$ such that (88) holds. Then $F_1(\nu_1) > \nu_2$ by (93). If $\nu_1 \neq \nu_{1\,\max}$ then $F_2(\nu_1) > \nu_2$ by (97) so that (95) does not hold implying that (89) does not hold. If $\nu_1 = \nu_{1\max} = \ell_1(D_1) - \ell_1(-D_2)$ then substituting in (88) we get $\nu_2 < \nu_{2\max} = F_1(\nu_{1\max}) = F_2(\nu_{1\max})$ which again shows that (89) does not hold.

Finally, take $(\nu_1, \nu_2) \in \,]0, \nu_{1\max}] \times \,]0, \nu_{2\max}]$ such that (89) holds. Then $F_2(\nu_1) < \nu_2$ by (95). If $\nu_1 \neq \nu_{1\max}$ then $F_1(\nu_1) < \nu_2$ by (97) so that (93) does not hold implying that (88) does not hold. If $\nu_1 = \nu_{1\max} = \ell_1(D_1) - \ell_1(-D_2)$ then substituting in (89) we get to a contradiction with (87b). This means that the case $\nu_1 = \nu_{1\max}$ cannot occur when (89) holds.

(v) **Proof of the only if part of (v).** If Inequality (88) does not hold then from the only if part of (ii) Case κ_2 does not apply. If Inequality (89) does not hold then from the only if part of (iii) Case κ_3 does not apply. Since Case κ_1 does not hold by (i), Case κ_4 necessarily holds. Again, to assert that Inequalities (88) and (89) are both false we need the members of both inequalities to be well defined which is the case owing to Inequalities (87).

Proof of the if part of (v). If Case κ_4 applies then (61) leads to (87b) since $\bar{x}_u(\varrho_1) \leq D_1$ and (56) leads to (87a) since $\zeta \geq -D_2$. On the other hand Equation (79) is equivalent to

$$\nu_2 = \ell_2\Big[\ell_1^{-1}\Big(\ell_1(\zeta) + \nu_1\Big)\Big] - \ell_2(\zeta) = L_\zeta(\nu_1), \tag{98}$$

where $L_\zeta : [0, \nu_{1\zeta\max}] \to [0, \nu_{2\zeta\max}]$ where $0 < \nu_{1\zeta\max} = \ell_1(D_1) - \ell_1(\zeta) \leq \nu_{1\max}$ by (56), and $0 < \nu_{2\zeta\max} = \ell_2(D_1) - \ell_2(\zeta) \leq \nu_{2\max}$. Note that $L_\zeta(0) = 0$, L_ζ is strictly increasing and continuous, and $L_\zeta \in C^1\big(]0, \nu_{1\zeta\max}[, \mathbb{R}\big)$. The derivative of L_ζ is

$$\dot{L}_\zeta(\nu_1) = \frac{\dot{\ell}_2\Big[\ell_1^{-1}\Big(\ell_1(\zeta) + \nu_1\Big)\Big]}{\dot{\ell}_1\Big[\ell_1^{-1}\Big(\ell_1(\zeta) + \nu_1\Big)\Big]} = \frac{h_1\Big[\ell_1^{-1}\Big(\ell_1(\zeta) + \nu_1\Big)\Big]}{h_2\Big[\ell_1^{-1}\Big(\ell_1(\zeta) + \nu_1\Big)\Big]}$$

so that

$$\lim_{\nu_1 \downarrow 0} \dot{L}_\zeta(\nu_1) = \frac{h_1(\zeta)}{h_2(\zeta)} > 0 \text{ if } \zeta \neq -D_2, \lim_{\nu_1 \downarrow 0} \dot{L}_\zeta(\nu_1) = \infty \text{ if } \zeta = -D_2, \lim_{\nu_1 \uparrow \nu_{1\zeta\max}} \dot{L}_\zeta(\nu_1) = 0.$$

Note that numerator of $\dot{F}_2(\nu_1)$ is strictly decreasing from $h_1(\zeta) > 0$ at $\nu_1 = 0$ to 0 at $\nu_1 = \nu_{1\zeta\max}$, and the denominator of $\dot{L}_\zeta(\nu_1)$ is strictly increasing from $h_2(\zeta)$ at $\nu_1 = 0$ to $h_2(D_1) > h_2(\zeta)$ at $\nu_1 = \nu_{1\max}$. Thus $\dot{L}_\zeta(\nu_1)$ is strictly decreasing from ∞ at $\nu_1 = 0$ to 0 at $\nu_1 = \nu_{1\zeta\max}$ so that L_ζ is strictly concave on the interval $]0, \nu_{1\zeta\max}[$. Thus

$$L_\zeta(\nu_1) > \frac{\nu_{2\zeta\max}}{\nu_{1\zeta\max}}\nu_1, \forall \nu_1 \in \,]0, \nu_{1\zeta\max}[. \tag{99}$$

If $\zeta = -D_2$ then Equation (79) shows that (89) cannot hold. Also if $\zeta = -D_2$ then Equations (99) and (94) show that $\nu_2 = L_\zeta(\nu_1) > F_1(\nu_1), \forall \nu_1 \in \,]0, \nu_{1\max}[$ since $\nu_{1\zeta\max} = \nu_{1\max}$ and $\nu_{2\zeta\max} = \nu_{2\max}$. Thus (88) does not hold for $\nu_1 \in \,]0, \nu_{1\max}[$. For $\nu_1 = \nu_{1\max}$ we get from (79) that $\nu_2 = \nu_{2\max}$ so that (88) cannot hold.

If $\zeta \neq -D_2$ then

$$h_1\Big[\ell_1^{-1}\Big(\ell_1(\zeta) + \nu_1\Big)\Big] < h_1\Big[\ell_1^{-1}\Big(\ell_1(-D_2) + \nu_1\Big)\Big]$$

and

$$h_2\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right] > h_2\left[\ell_1^{-1}\Big(\ell_1(-D_2)+\nu_1\Big)\right]$$

so that

$$\dot{L}_\zeta(\nu_1) < \dot{F}_2(\nu_1), \forall \nu_1 \in \,]0, \nu_{1\zeta\,\max}[.$$

Thus

$$\nu_2 = L_\zeta(\nu_1) = \int_0^{\nu_1} \dot{L}_\zeta(\nu)\mathrm{d}\nu < \int_0^{\nu_1} \dot{F}_2(\nu)\mathrm{d}\nu = F_2(\nu_1), \forall \nu_1 \in \,]0, \nu_{1\zeta\,\max}],$$

meaning that (89) does not hold.

On the other hand, $L_\zeta(0) = 0 = F_1(0)$ and $L_\zeta(\nu_{1\zeta\,\max}) = \ell_2(D_1) - \ell_2(\zeta) = F_1(\nu_{1\zeta\,\max})$. Since L_ζ is strictly concave and F_1 is strictly convex we have $\nu_2 = L_\zeta(\nu_1) > F_1(\nu_1)$ for all $\nu_1 \in \,]0, \nu_{1\zeta\,\max}[$. Thus (88) does not hold for any $\nu_1 \in \,]0, \nu_{1\zeta\,\max}[$. For $\nu_1 = \nu_{1\zeta\,\max}$, substituting in (79) we get $\nu_2 = \nu_{2\zeta\,\max}$ so that (88) cannot hold.

(ii) **Proof of the if part of (ii).** Suppose (88). Then by (iv) Inequality (89) is false so that from the only if part of (iii) Case κ_3 does not apply. Also since (88) holds then by (v) Case κ_4 does not apply. Since Case κ_1 does not hold by (i), it comes that Case κ_2 applies.

(iii) **Proof of the if part of (iii).** Suppose (89). Then by (iv) Inequality (88) is false so that from the only if part of (ii) Case κ_2 does not apply. Also since (89) holds then by (v) Case κ_4 does not apply. Since Case κ_1 does not hold by (i), it comes that Case κ_3 applies.

(vi) We have $H_{\nu_1} \in C^1\left(\Big] - D_2, \ell_1^{-1}\Big(\ell_1(D_1)-\nu_1\Big)\Big[, \mathbb{R}\right)$.

Take $\zeta \in \Big] - D_2, \ell_1^{-1}\Big(\ell_1(D_1)-\nu_1\Big)\Big[$, then

$$\dot{H}_{\nu_1}(\zeta) = \frac{\dot{\ell}_2\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right]}{\dot{\ell}_1\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right]}\dot{\ell}_1(\zeta) - \dot{\ell}_2(\zeta) = \frac{h_1\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right]}{h_1(\zeta)h_2\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right]} - \frac{1}{h_2(\zeta)}.$$

We have $\nu_1 > 0$ so that $\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big) > \zeta$. Then $h_1\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right] < h_1(\zeta)$ and $h_2\left[\ell_1^{-1}\Big(\ell_1(\zeta)+\nu_1\Big)\right] > h_2(\zeta)$ so that

$$\dot{H}_{\nu_1}(\zeta) < \frac{h_1(\zeta)}{h_1(\zeta)h_2(\zeta)} - \frac{1}{h_2(\zeta)} = 0.$$

The latter operation is valid since $\zeta \neq D_1$ so that $h_1(\zeta) \neq 0$, and $\zeta \neq -D_2$ so that $h_2(\zeta) \neq 0$. Since $\dot{H}_{\nu_1}(\zeta) < 0$ it comes that $H_{\nu_1}(\zeta)$ is strictly decreasing, thus invertible. Then from (90) we obtain (91).

$\square$

Proposition 9.10. *We have*

(i) *If* $\ell_1(D_1) = \infty$ *and* $\ell_2(-D_2) = -\infty$ *then Case* κ_4 *applies.*
(ii) *If* $\ell_1(D_1) = \infty$ *and* $\ell_2(-D_2) \neq -\infty$ *then Cases* κ_1 *and* κ_2 *cannot occur.*
(iii) *If* $\ell_1(D_1) \neq \infty$ *and* $\ell_2(-D_2) = -\infty$ *then Cases* κ_1 *and* κ_3 *cannot occur.*

Proof. When $\ell_1(D_1) = \infty$ Inequality (54) does not hold which proves (ii) and eliminates Cases κ_1 and κ_2 in (i). When $\ell_2(-D_2) = -\infty$ Inequality (59) does not hold which proves (iii) and eliminates Case κ_3 in (i) completing thus the proof of (i). □

Theorem 9.1. *Consider the model (51a)–(51c) under Conditions* (A)-(E). *Consider that the input* $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ *is periodic with period* $T > 0$, *and is such that* u *is strictly increasing on* $[0, T_1]$ *and strictly decreasing on* $[T_1, T]$ *for some* $T_1 \in]0, T[$. *Define* $u_{\min} = u(0)$, $u_{\max} = u(T_1)$, $\varrho_1 = u_{\max} - u_{\min}$, *and* $\varrho_2 = 2\varrho_1$. *Then* $I_u = \mathbb{R}_+$ *and the normalized function* ψ_u *is given by Equation (11). The hysteresis loop of the model (51a)–(51c) with respect to* (u, x_0) *is independent of the initial state* x_0 *so that it is denoted* $\mathcal{G}_u$. *We have*

$$\mathcal{G}_u = \left\{ \left(\psi_u\left(\varrho\right), \bar{x}_u(\varrho) \right), \varrho \in [0, \varrho_2] \right\}, \tag{100}$$

where $\bar{x}_u$ *is given hereafter.*

(i) *If Inequalities (84) hold then Case* κ_1 *of Proposition 9.5 applies.*
(ii) *If Inequalities (85) hold then Case* κ_2 *of Proposition 9.5 applies.*
(iii) *If Inequalities (86) hold then Case* κ_3 *of Proposition 9.5 applies.*
(iv) *If Inequalities (87) hold and Inequality (88) also holds then Case* κ_2 *of Proposition 9.5 applies.*
(v) *If Inequalities (87) hold and Inequality (89) also holds then Case* κ_3 *of Proposition 9.5 applies.*
(vi) *If Inequalities (87) hold, and Inequality (88) does not hold, and Inequality (89) does not hold, then Case* κ_4 *of Proposition 9.5 applies, and* ζ *is given by Equation (91).*
(vii) *The cases* (i)–(vi) *are the only ones that can occur.*

Proof. (i) Combine Propositions 9.6 and 9.10.
(ii) Combine Propositions 9.7 and 9.10.
(iii) Combine Propositions 9.8 and 9.10.
(iv) Follows from Proposition 9.9 (ii).
(v) Follows from Proposition 9.9 (iii).
(vi) Follows from Proposition 9.9 (v) and (vi).
(vii) Inequalities (84), (85), (86), and (87) are mutually exclusive and collectively exhaustive. Owing to Proposition 9.9 (iv), if Inequalities (87) are assumed, then Inequalities (88), (89), and $\neg(88) \wedge \neg(89)$ are mutually exclusive and collectively exhaustive.

Finally, the hysteresis loop does not depend on the initial condition x_0 owing to (i)–(vii). □

9.3. Numerical examples

In this section we illustrate the results of this paper by means of numerical simulations. We choose the following values for the input u: $u_{\min} = 0$ and $u_{\max} = 1$, so that $\varrho_1 = 1$ and $\varrho_2 = 2$. Since the hysteresis loop does not depend directly on the input u but rather on the normalized input ψ_u, we provide only the plot for ψ_u, see Figure 3.

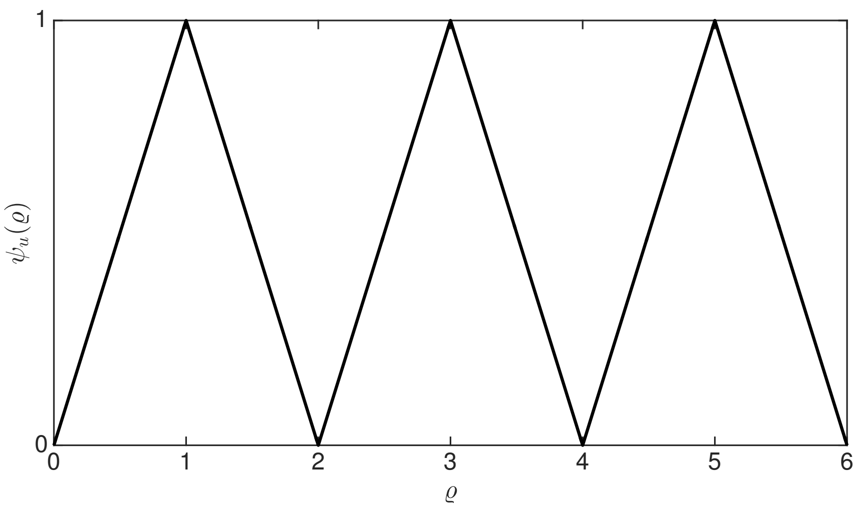

Figure 3: Normalized input $\psi_u(\varrho)$ versus ϱ.

Following [14] we consider that $g_1(b) = c_1 e^{\alpha_1 b}$ and $g_2(b) = c_2 e^{\alpha_2 b}$, where $c_1, c_2 > 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Thus

$$
\begin{aligned}
G_1(z) &= \int_0^z c_1 e^{\alpha_1 \nu} \mathrm{d}\nu = \frac{c_1}{\alpha_1}\big(e^{\alpha_1 z} - 1\big),\\
G_2(z) &= \int_0^z c_2 e^{\alpha_2 \nu} \mathrm{d}\nu = \frac{c_2}{\alpha_2}\big(e^{\alpha_2 z} - 1\big),\\
G_1^{-1}(\nu) &= \frac{1}{\alpha_1} \ln\left(\frac{\alpha_1 \nu}{c_1} + 1\right),\\
G_2^{-1}(\nu) &= \frac{1}{\alpha_2} \ln\left(\frac{\alpha_2 \nu}{c_2} + 1\right).
\end{aligned}
$$

For the functions h_1 and h_2 we consider that

$$
\begin{cases} h_1(a) = (D_1 - a)^{\beta_1}, & a \le D_1\\ h_1(a) = 0, & a \ge D_1, \end{cases}
$$

and

$$
\begin{cases} h_2(a) = (D_2 + a)^{\beta_2}, & a \ge -D_2\\ h_2(a) = 0, & a \le -D_2, \end{cases}
$$

where $D_1, D_2, \beta_1, \beta_2 > 0$. We study different examples which illustrate the different cases that appear in Theorem 9.5.

9.3.1. $\beta_1 = 1$ *and* $\beta_2 = 1$

We get

$$\ell_1(z) = \int_0^z (D_1 - \nu)^{-1} \mathrm{d}\nu = \ln(D_1) - \ln(D_1 - z),$$
$$\ell_2(z) = \int_0^z (D_2 + \nu)^{-1} \mathrm{d}\nu = \ln(D_2 + z) - \ln(D_2).$$

We have $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) = -\infty$ so that by Proposition 9.10 Case κ_4 applies which is given by Equations (77), (78), and (79).

We have $\ell_1^{-1}(\nu) = D_1(1-e^{-\nu})$ and $\ell_2^{-1}(\nu) = D_2(e^{\nu}-1)$ so that Equation (79) becomes

$$(D_2 + \zeta)\exp\left[\frac{c_2}{\alpha_2}\left(e^{\alpha_2 u_{\max}} - e^{\alpha_2 u_{\min}}\right)\right] - D_2 = D_1 - (D_1 - \zeta)\exp\left[\frac{c_1}{\alpha_1}\left(e^{\alpha_1 u_{\min}} - e^{\alpha_1 u_{\max}}\right)\right]$$

which leads to

$$\zeta = \frac{D_1(1-E_1) + D_2(1-E_2)}{E_2 - E_1}, \tag{101a}$$
$$E_2 = \exp\left[\frac{c_2}{\alpha_2}\left(e^{\alpha_2 u_{\max}} - e^{\alpha_2 u_{\min}}\right)\right], \tag{101b}$$
$$E_1 = \exp\left[\frac{c_1}{\alpha_1}\left(e^{\alpha_1 u_{\min}} - e^{\alpha_1 u_{\max}}\right)\right]. \tag{101c}$$

If we choose $\alpha_1 > 0$ and $\alpha_2 > 0$ we get $E_2 > 1$ and $E_1 < 1$ so that the denominator of (101a) is not zero.

We take $c_1 = c_2 = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, $D_1 = 1$, $D_2 = 0.5$, $x_0 = 0$. We use Matlab solver ode23s to solve the differential equation (52a)–(52c) which gives x_u and thus $x_{u,k}$ (see Definition 5.1). Figure 4 shows $x_{u,k}$ for different values of k. The function $\bar{x}_u$ is calculated from Equations (77), (78), and (101). It can be seen in Figure 4 that $\lim_{k\to\infty} \|x_{u,k} - \bar{x}_u|_{[0,2]}\| = 0$ as predicted by Proposition 8.1.

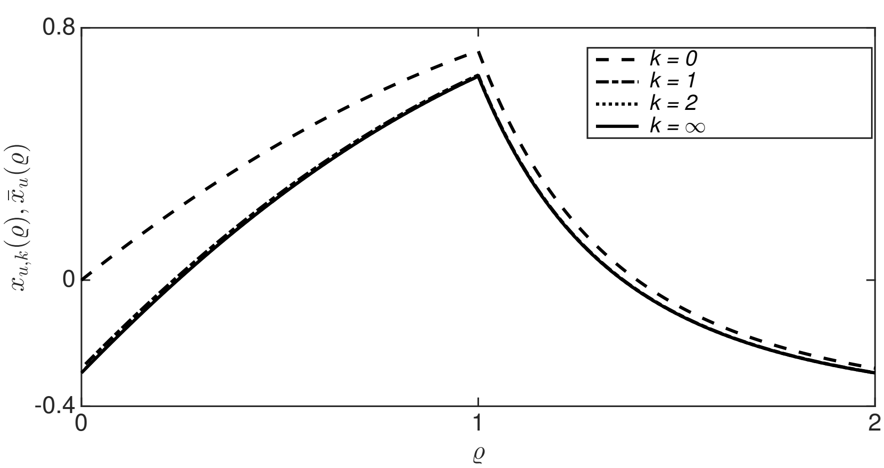

Figure 4: $x_{u,k}(\varrho)$ versus ϱ for $k = 0$ (dashed), $k = 1$ (dash-dotted), and $k = 2$ (dotted). Solid: $\bar{x}_u(\varrho)$ versus ϱ labeled as $k = \infty$. The dotted line is practically the same as the solid one.

The hysteresis loop $\mathcal{G}_u$ is provided in Figure 5 (solid). As predicted by Proposition 8.1 we can see that the point $\big(x_u(\varrho), \psi_u(\varrho)\big)$ -dotted- gets closer and closer to the set $\mathcal{G}_u$ as ϱ increases.

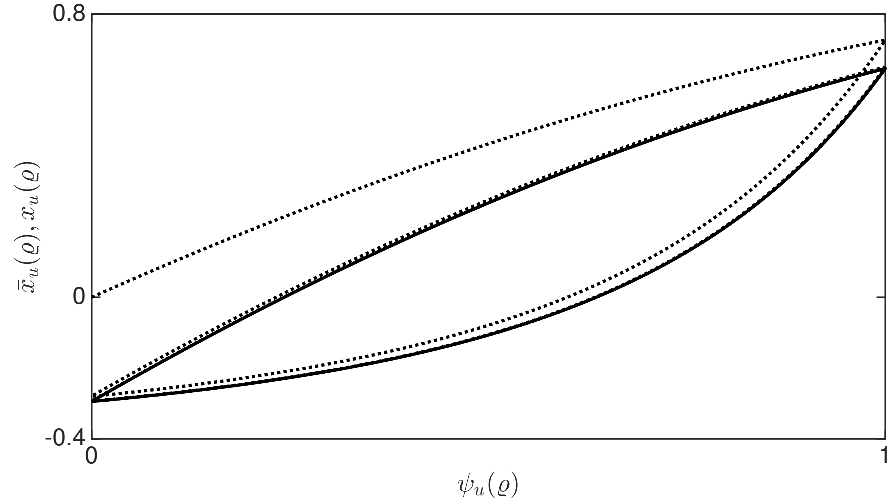

Figure 5: Dotted $x_u(\varrho)$ versus $\psi_u(\varrho)$. Solid: hysteresis loop $\mathcal{G}_u$, that is $\bar{x}_u(\varrho)$ versus $\psi_u(\varrho)$.

9.3.2. $\beta_1 = 1$ *and* $0 < \beta_2 < 1$

We get

$$\begin{aligned}
\ell_1(z) &= \ln(D_1) - \ln(D_1 - z),\\
\ell_2(z) &= \int_0^z (D_2+\nu)^{-\beta_2}\mathrm{d}\nu = \frac{1}{1-\beta_2}\left[(D_2+\nu)^{1-\beta_2}\right]_0^z = \frac{1}{1-\beta_2}\left((D_2+z)^{1-\beta_2} - D_2^{1-\beta_2}\right),\\
\ell_1^{-1}(\nu) &= D_1(1-e^{-\nu}),\\
\ell_2^{-1}(\nu) &= \left((1-\beta_2)\nu + D_2^{1-\beta_2}\right)^{\frac{1}{1-\beta_2}} - D_2.
\end{aligned}$$

We have $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) \neq -\infty$ so that Cases κ_1 and κ_2 cannot occur by Proposition 9.10. We take $c_1 = 1$, $c_2 = 2$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, $D_1 = 1$, $D_2 = 0.5$, $\beta_2 = 0.5$. In this case Inequalities (86) hold so that Case κ_3 applies by Proposition 9.8. Then Equations (74)–(76) hold. Figure 6 provides the corresponding hysteresis loop.

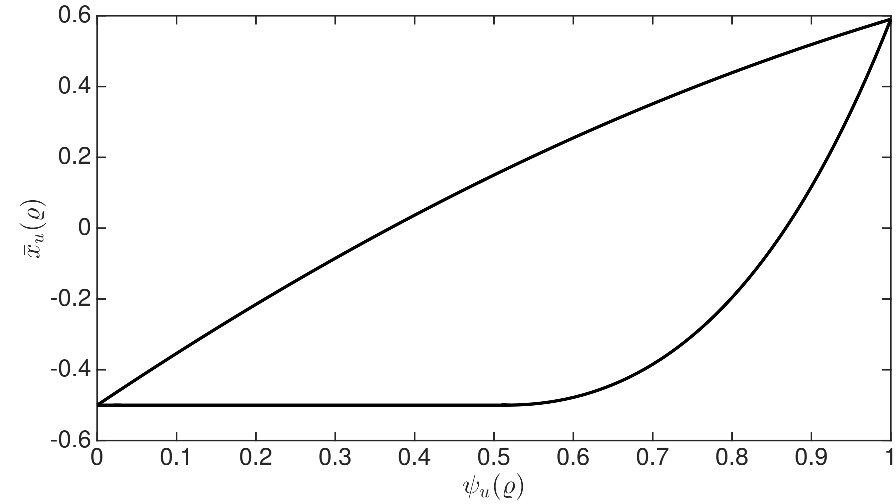

Figure 6: Hysteresis loop $\mathcal{G}_u$, that is $\bar{x}_u(\varrho)$ versus $\psi_u(\varrho)$.

9.3.3. $\beta_1 = 1$ *and* $\beta_2 > 1$

The functions ℓ_1 and ℓ_2 have the same expressions as in Section 9.3.2. We have $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) = -\infty$ so that Case κ_4 applies by Proposition 9.10. The equations that describe the hysteresis loop and the general shape of that hysteresis loop are the same as for Section 9.3.1. To find ζ we determine the function $H_{\nu_1}(\zeta)$ of Equation (90) as

$$H_{\nu_1}(\zeta) = \frac{1}{1-\beta_2}\left[\left(D_2 + D_1 - (D_1 - \zeta)e^{-\nu_1}\right)^{1-\beta_2} - (D_2 + \zeta)^{1-\beta_2}\right].$$

Then by Proposition (9.9) (vi) the function H_{ν_1} is invertible and ζ is given by Equation (91) where

$$\nu_1 = \frac{c_1}{\alpha_1}\left(e^{\alpha_1 u_{\max}} - 1\right) - \frac{c_1}{\alpha_1}\left(e^{\alpha_1 u_{\min}} - 1\right),$$
$$\nu_2 = \frac{c_2}{\alpha_2}\left(e^{\alpha_2 u_{\max}} - 1\right) - \frac{c_2}{\alpha_2}\left(e^{\alpha_2 u_{\min}} - 1\right).$$

9.3.4. $0 < \beta_1 < 1$ *and* $\beta_2 = 1$

We get

$$\ell_1(z) = \int_0^z (D_1 - \nu)^{-\beta_1} \mathrm{d}\nu = \frac{1}{1-\beta_1}\left(D_1^{1-\beta_1} - (D_1 - z)^{1-\beta_1}\right),$$
$$\ell_2(z) = \ln(D_2 + z) - \ln(D_2),$$
$$\ell_1^{-1}(\nu) = D_1 - \left(D_1^{1-\beta_1} - (1-\beta_1)\nu\right)^{\frac{1}{1-\beta_1}},$$
$$\ell_2^{-1}(\nu) = D_2(e^{\nu} - 1).$$

We have $\ell_1(D_1) \neq \infty$ and $\ell_2(-D_2) = -\infty$ so that Cases κ_1 and κ_3 cannot occur by Proposition 9.10. We take $c_1 = 3$, $c_2 = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, $D_1 = 1$, $D_2 = 0.5$, $\beta_1 = 0.5$ so that Inequalities (85) hold, which corresponds to Case κ_2 by Proposition 9.7. Thus Equations (71)–(73) apply and the corresponding hysteresis loop is shown in Figure 7.

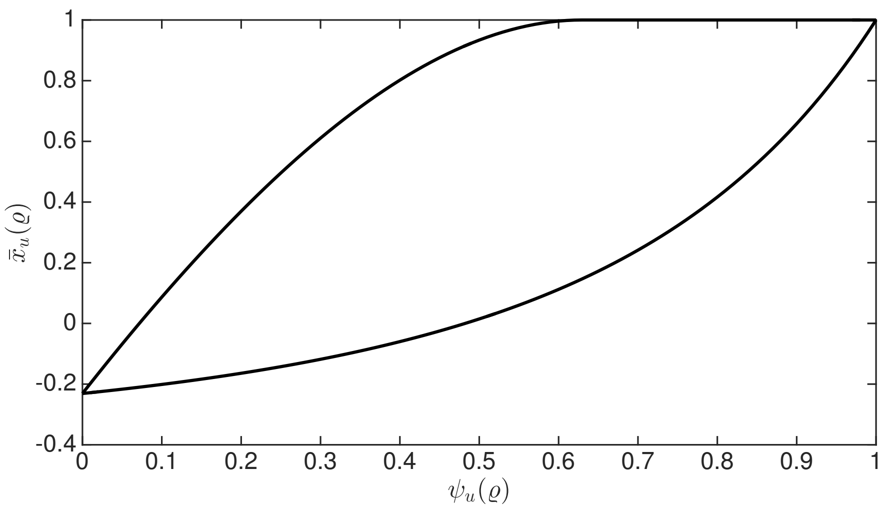

Figure 7: Hysteresis loop $\mathcal{G}_u$, that is $\bar{x}_u(\varrho)$ versus $\psi_u(\varrho)$.

9.3.5. $0 < \beta_1 < 1$ *and* $0 < \beta_2 < 1$

We get

$$\ell_1(z) = \frac{1}{1-\beta_1}\left(D_1^{1-\beta_1} - (D_1 - z)^{1-\beta_1}\right), \tag{102a}$$

$$\ell_2(z) = \frac{1}{1-\beta_2}\left((D_2 + z)^{1-\beta_2} - D_2^{1-\beta_2}\right), \tag{102b}$$

$$\ell_1^{-1}(\nu) = D_1 - \left(D_1^{1-\beta_1} - (1-\beta_1)\nu\right)^{\frac{1}{1-\beta_1}}, \tag{102c}$$

$$\ell_2^{-1}(\nu) = \left((1-\beta_2)\nu + D_2^{1-\beta_2}\right)^{\frac{1}{1-\beta_2}} - D_2. \tag{102d}$$

Then $\ell_1(D_1) \neq \infty$ and $\ell_2(-D_2) \neq -\infty$. Any of the cases κ_1, κ_2, κ_3, or κ_4 may occur depending on the values of the parameters. For example, taking $c_1 = 2.5$, $c_2 = 2$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, $D_1 = 1$, $D_2 = 0.5$, $\beta_1 = 0.5$, and $\beta_2 = 0.4$, Inequalities (84) hold so that Case κ_1 applies by Proposition 9.6. The corresponding hysteresis loop is provided in Figure 8.

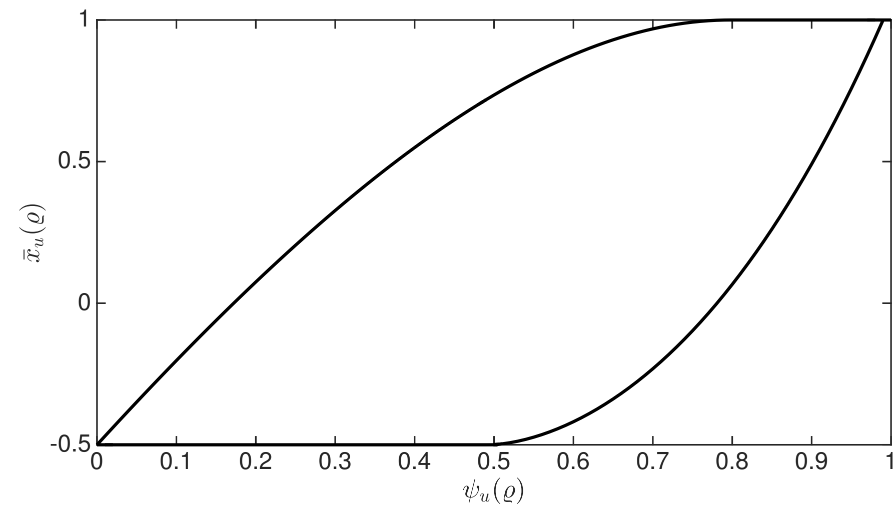

Figure 8: Hysteresis loop $\mathcal{G}_u$, that is $\bar{x}_u(\varrho)$ versus $\psi_u(\varrho)$.

9.3.6. $0 < \beta_1 < 1$ *and* $\beta_2 > 1$

The formulas for ℓ_1 and ℓ_2 are the same as in Section 9.3.5. However we have $\ell_1(D_1) \neq \infty$ and $\ell_2(-D_2) = -\infty$ which implies by Proposition 9.10 that Cases κ_1 and κ_3 cannot occur.

9.3.7. $\beta_1 > 1$ *and* $\beta_2 = 1$

We get

$$\begin{aligned} \ell_1(z) &= \frac{1}{1-\beta_1}\left(D_1^{1-\beta_1} - (D_1 - z)^{1-\beta_1}\right), \\ \ell_2(z) &= \ln(D_2 + z) - \ln(D_2), \\ \ell_1^{-1}(\nu) &= D_1 - \left(D_1^{1-\beta_1} - (1-\beta_1)\nu\right)^{\frac{1}{1-\beta_1}}, \\ \ell_2^{-1}(\nu) &= D_2(e^{\nu} - 1). \end{aligned}$$

Then $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) = -\infty$ so that Case κ_4 applies by Proposition 9.10. The calculations are similar to those of Section 9.3.3 mutatis mutandis.

9.3.8. $\beta_1 > 1$ *and* $0 < \beta_2 < 1$

The formulas for ℓ_1 and ℓ_2 are the same as in Section 9.3.5. However we have $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) \neq -\infty$ which implies by Proposition 9.10 that Cases κ_1 and κ_2 cannot occur.

9.3.9. $\beta_1 > 1$ *and* $\beta_2 > 1$

The formulas for ℓ_1 and ℓ_2 are the same as in Section 9.3.5. However we have $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) = -\infty$ which implies by Proposition 9.10 that Cases κ_4 applies. The determination of ζ is done as in Section 9.3.3 mutatis mutandis.

10. Application to friction modeling

10.1. The Coulomb model for dry friction

Consider the cube of Figure 9 resting on an inclined plane with slope θ.

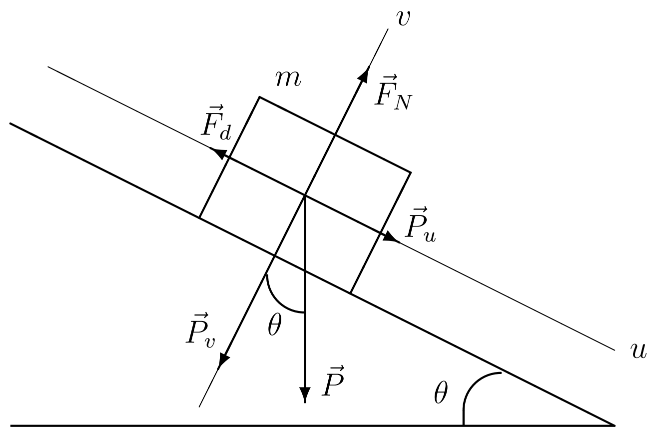

Figure 9: Cube on an inclined plane.

Using Newton's second law we get

$$m\ddot{u} = P_u - F_d \tag{103}$$

where the u–axis is parallel to the slope of the plane, $P_u = mg\sin(\theta)$ is the projection of the weight $\vec{P}$ on the u–axis, m the mass of the cube, g is gravity, $\ddot{u} = \dfrac{d^2u}{dt^2}$ (being u the displacement of the cube and t the time), $-F_d$ the tangential friction force, and θ is the angle that provides the inclination of the plane.

We observe experimentally that for small values of θ the cube does not move. This can be explained by the existence of a force equal to $-\vec{P}_u$: friction. The friction force $\vec{F}_d$ is called dry because the cube and the inclined plane are both solid objects, and there is no lubricant in between. To complete the description of Equation (103), it is necessary to find a description of the force F_d. The simplest way to describe dry friction is through the Coulomb model [32, pp. 41–42] (see Figure 10):

$$F_d = F_c \quad \text{for} \quad \dot{u} > 0, \tag{104a}$$
$$F_d = -F_c \quad \text{for} \quad \dot{u} < 0, \tag{104b}$$
$$-F_c \leq F_d \leq F_c \quad \text{for} \quad \dot{u} = 0, \tag{104c}$$

where $F_c > 0$ is the Coulomb friction level.

10.2. On the physics of friction

Quoting from [33, p. 133]: "Friction between solid bodies is an extremely complicated physical phenomenon... What is astonishing is the fact that it is possible to formulate a very simple law for dry friction... The frictional force is proportional to the normal force and as good as independent from the speed."

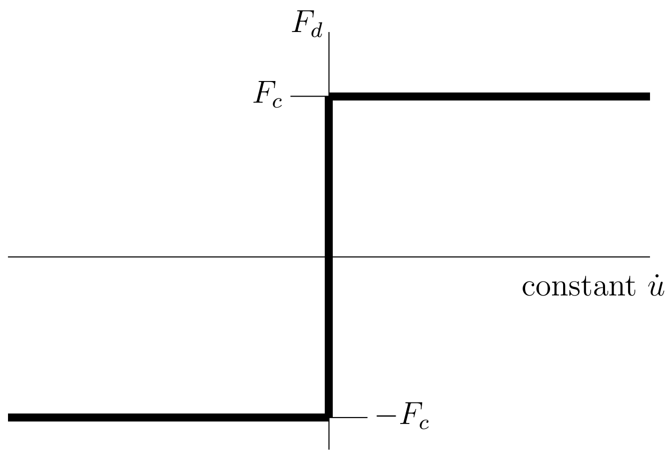

Figure 10: The Coulomb model for dry friction.

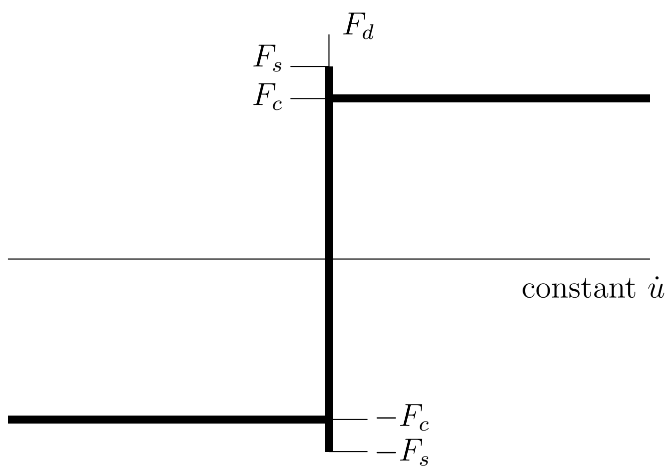

Figure 11: The Coulomb model with stiction.

The following properties that characterize dry friction are taken from [33, Chapter 10]:

(i) In order to set in motion a body lying on an even surface in a state of rest, a critical force, *the force of static friction* F_s , must be overcome. We have $F_s = \mu_s F_N$ where F_N is the normal force and μ_s the *coefficient of static friction.* It is dependent on the pairing of the contacting materials but shows almost no dependence on contact area or roughness.

(ii) The *kinetic friction* F_c is the resisting force which acts on a body after the force of static friction has been overcome. We have $F_c = \mu_k F_N$ where μ_k is the *coefficient of kinetic friction.* It shows no considerable dependence on the contact area or roughness of the surface. The kinetic friction is very weakly dependent on the sliding velocity.

There are deviations from the simple laws described above. In particular, Coulomb discovered that the static frictional force increases with the amount of time an object is at rest: his data shows that this increase is logarithmic [33, Section 10.4]. Also, the linear dependence of the frictional force–either static or kinetic–on the normal force is only met in a specific force domain: for not too large or too small a normal force. This linear dependence is no longer valid when the real contact area is comparable to the apparent contact area as for polymers and elastomers. Finally, it is often assumed that the coefficient of kinetic friction is independent of sliding speed. This is a good but rough approximation that is valid for not too high and not too low speeds [33, Section 10.6].

Frequently, the origins of friction are explained through the roughness of the surfaces. However, in a large domain of roughnesses, the frictional force is independent or only very slightly dependent on the roughness [33, Section 10.7]. An alternative explanation for the physical origin of friction has been proposed by Bowden and Tabor through the formation of cold weld junctions [33, Section 10.9].

Despite the efforts to understand friction, we are not aware of any widely accepted model that would describe quantitatively friction dynamics. The available models combine basic physical laws with fictitious equations that aim to emulate the macroscopic behavior of friction observed experimentally. These models are called *phenomenological*.

An example of such models is the one proposed in [34]. In this model, Equation [34, (1)] is derived from Newton's second law whilst [34, (4)] is a fictitious equation that governs the rupture and formation of bonds. Another example of a phenomenological model is that of Prandtl-Tomlinson studied in [33, Chapter 11].

10.3. Static and dynamic models for friction

In this section we consider the friction models that are presented in the survey paper [35]. The emphasis in that paper is made on the pure dry sliding friction, stick-slip effect, viscous friction and Stribeck effect. The models are classified as static when they describe the steady-state behavior of friction force, and as dynamic when they use state variables to capture more properties.

The simplest static model is the Coulomb model (104). Stiction can be included to the model to state that $F_s > F_c$, see Figure 11; a Stribeck curve can be incorporated to ensure that the decrease from F_s to F_c is continuous, see Figure 12; and a linear term in $\dot{x}$ can be added to take into account viscous friction. More variations on the Coulomb model are presented in [35, Section 2] along with a discussion of the pros and cons of these variations.

The simplest dynamic model for friction is the Dahl model [30]:

$$F(t) = F_d(t) + F_{\text{viscous}}(t), \tag{105a}$$
$$F_{\text{viscous}}(t) = f\big(\dot{u}(t)\big), \tag{105b}$$
$$F_d(t) = F_c\, x(t), \tag{105c}$$
$$\dot{x}(t) = \rho\big(\dot{u}(t) - |\dot{u}(t)|x(t)\big), \tag{105d}$$
$$x(0) = x_0, \quad -1 \le x_0 \le 1, \tag{105e}$$

where $t \geq 0$ denotes time, F_{viscous} refers to viscous friction, x is an internal state, $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ is the relative displacement and is the input of the model, F is the total friction and is the output of the model, $f \in C^0(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$, $x_0 \in \mathbb{R}$ is the initial state, and $\rho > 0$ is a parameter. The compatibility of the Dahl model and the Coulomb model is studied in [31].

Other dynamic models of friction are reviewed in [35, Section 3] along with a discussion of the pros and cons of these models. In particular, the LuGre model is an extension of the Dahl model that takes into account the Stribeck and stiction effects. The LuGre model is given by [28]:

$$\begin{aligned}
\dot{x}(t) &= -\sigma_0 \frac{|\dot{u}(t)|}{g\big(\dot{u}(t)\big)} x(t) + \dot{u}(t), && \text{(106a)}\\
x(0) &= x_0, && \text{(106b)}\\
F_d(t) &= \sigma_0 x(t) + \sigma_1 \dot{x}(t), && \text{(106c)}\\
F(t) &= F_d(t) + f\big(\dot{u}(t)\big), && \text{(106d)}
\end{aligned}$$

where the parameters $\sigma_0 > 0$ and $\sigma_1 > 0$ are respectively the stiffness and the microscopic damping friction coefficients, and the function $g \in C^0(\mathbb{R}, \mathbb{R})$ represents the macrodamping friction with $g(\vartheta) > 0, \forall \vartheta \in \mathbb{R}$. In [36] the expression for g is taken to be

$$g(\vartheta) = F_c + (F_s - F_c)e^{-|\vartheta/v_s|^\alpha} \tag{107}$$

where $v_s > 0$ and $\alpha > 0$ are parameters.

When $\dot{u}$ is constant, integrating (106a) shows that F_d reaches a steady state given by $g\big(\dot{u}\big)$ for $\dot{u} > 0$ and $-g\big(\dot{u}\big)$ for $\dot{u} < 0$, see Figure 12.

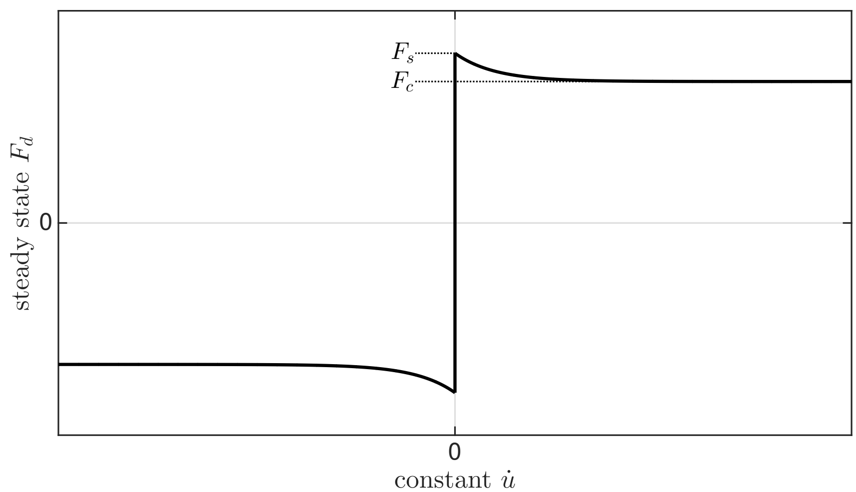

Figure 12: Steady state of the dry friction force versus constant velocity.

For $\dot{u} \equiv 0$ we have $-F_s \le F_d \le F_s$ which translates into

$$-\frac{F_s}{\sigma_0} \le x_0 \le \frac{F_s}{\sigma_0}. \tag{108}$$

10.4. Compatibility of Babuška's model with the Coulomb model

In this section we consider Babuška's model (51) augmented by the following equations:

$$\begin{aligned}
F(t) &= F_d(t) + F_{\text{viscous}}(t), && \text{(109a)}\\
F_{\text{viscous}}(t) &= f\big(\dot{u}(t)\big), && \text{(109b)}\\
F_d(t) &= \sigma_0 x(t), && \text{(109c)}
\end{aligned}$$

where $\sigma_0 > 0$ and $f \in C^0(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$. We call Π the model composed of Equations (51) and (109). The input of Π is the relative displacement $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ and its output is the friction force F.

We have first to ensure that Π is compatible with the Coulomb model. To this end suppose that $u(t) = u(0)$ for all $t \geq 0$. Then $x(t) = x_0$ for all $t \geq 0$ so that $F_d(t) = \sigma_0 x_0$ for all $t \geq 0$. To be compatible with Inequalities (104c) we need to have $-F_c \leq \sigma_0 x_0 \leq F_c$, that is

$$-\frac{F_c}{\sigma_0} \leq x_0 \leq \frac{F_c}{\sigma_0}. \tag{110}$$

We choose

$$D_1 = D_2 = \frac{F_c}{\sigma_0} \tag{111}$$

so that Equation (110) can be written as $-D_2 \leq x_0 \leq D_1$. Then, by Proposition 4.1 we get $-D_2 \leq x(t) \leq D_1$ for all $t \geq 0$ so that by Equation (109c) we have

$$-F_c \leq F_d(t) \leq F_c \text{ for all } t \geq 0.$$

Proposition 10.1. *Let $u(t) = \beta t, t \geq 0$ so that $\dot{u}(t) = \beta, t > 0$ where $\beta \in \mathbb{R}$. Suppose that*

$$\lim_{z \to \infty} G_1(z) > \ell_1(D_1) - \ell_1(-D_2), \tag{112a}$$

$$\lim_{z \to \infty} G_2(-z) < \ell_2(-D_2) - \ell_2(D_1). \tag{112b}$$

Then $\lim_{t \to \infty} F_d(t) = F_c$ if $\beta > 0$ and $\lim_{t \to \infty} F_d(t) = -F_c$ if $\beta < 0$.

Proof. Suppose that $\beta > 0$. By Equation (51a) we have $\dot{x}(t) \geq 0$ for almost all $t \geq 0$ so that $x(t) \geq x_0$ for all $t \geq 0$. If $x_0 = D_1$ then $x(t) = D_1$ for all $t \geq 0$ so that $F_d(t) = F_c$ for all $t \geq 0$.

Suppose now that $-D_2 \leq x_0 < D_1$. Then, owing to Inequality (112a) there exists a unique $t_1 > 0$ such that $G_1(\beta t_1) = \ell_1(D_1) - \ell_1(x_0)$. Define the function $z_1 : [0, t_1] \to \mathbb{R}$ by $z_1(t) = \ell_1^{-1}\big(\ell_1(x_0) + G_1(\beta t)\big)$. Then z_1 is continuous, C^1 on the interval $]0, t_1[$, and satisfies Equation (51a). Owing to the uniqueness of solutions we have $x = z_1$ on the interval $[0, t_1]$ so that $x(t_1) = z_1(t_1) = D_1$. Then $x(t) = D_1$ for all $t \geq t_1$ so that $F_d(t) = F_c$ for all $t \geq t_1$.

A similar proof holds for the case $\beta < 0$. □

To sum up, Babuška's model Π is compatible with the Coulomb model when Equations (110)–(112) hold.

10.5. A comparative study

In this section we compare some characteristics of the LuGre model, the Dahl model, and Babuška's model Π.

Consider a periodic input $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ which has the form of Figure 1. The corresponding normalized function is also periodic and given by Equations (11), see Figure 2.

The hysteresis loop of the LuGre model has been determined in [29, Equations (25)-(26)]. This hysteresis loop is independent of the initial condition x_0 so that it can be denoted $\mathcal{G}_{u,\text{LuGre}}$, that is

$$\mathcal{G}_{u,\text{LuGre}} = \left\{ \left(\psi_u(\varrho), F^{\circ}_{\text{LuGre}}(\varrho)\right), \varrho \in [0, \varrho_2] \right\}, \tag{113}$$

where

$$F^{\circ}_{\text{LuGre}}(\varrho) = g(0)\left(1 - \alpha e^{-\frac{\sigma_0}{g(0)}[\psi_u(\varrho) - u_{\min}]}\right), \forall \varrho \in [0, \varrho_1], \tag{114a}$$

$$F^{\circ}_{\text{LuGre}}(\varrho) = g(0)\left(-1 + \alpha e^{-\frac{\sigma_0}{g(0)}[u_{\max} - \psi_u(\varrho)]}\right), \forall \varrho \in [\varrho_1, \varrho_2], \tag{114b}$$

$$\alpha = \frac{2}{1 + e^{-c}}; \quad c = \frac{\sigma_0 \varrho_1}{g(0)}. \tag{114c}$$

Observe the following:

- The hysteresis loop of the LuGre model is symmetric with respect to its center [29, Comment 2].
- The viscous friction term $f(\dot{u})$ does not contribute to the hysteresis loop. This means that even if we choose an asymmetric form for f which is different for positive and negative velocities, the hysteresis loop of the LuGre model remains symmetric.
- For the validity of Equations (114) the function g does not need to be of the form (107). Equations (114) are valid for any continuous and strictly positive function g.
- The hysteresis loop depends solely on $g(0)$. It does not depend on any other value of g. This means that even if we choose an asymmetric form for g which is different for positive and negative velocities, the hysteresis loop of the LuGre model remains symmetric.
- The Dahl model is a particular case of the LuGre model obtained when g is constant and $\sigma_1 = 0$. This means that the hysteresis loop of the Dahl model is also given by Equations (114). Thus the hysteresis loop of the Dahl model is symmetric.

On the other hand, the hysteresis loop of Babuška's model Π may be symmetric or asymmetric, see for instance Figure 8. As a matter of fact, even when $D_1 = D_2$, we can choose the functions h_1, h_2, g_1, g_2 to obtain a symmetric or an asymmetric hysteresis loop, see for example Equations (102).

However, Babuška's model Π does not include stiction or the Stribeck effect so that its steady state behavior is the same as the Coulomb model of Figure 10 owing to Proposition 10.1.

11. Application to hysteresis loop matching

A search in the Web of Science Core Collection provides for the 2010s alone 5888 research articles whose title contains the word "hysteresis". Whilst this fact highlights the interest of the scientific community into the study of hysteresis, it also makes it difficult to fairly review the field. Indeed, more than 10 survey papers on hysteresis related issues have been published between 2003 and 2020 leaving important items like control, identification or estimation of hysteresis systems without a thorough specific review. In the next we present a brief overview of hysteresis loop matching.

Materials, composites, and structural assemblies may present inelasticities that generate hysteresis: for example Figure 2.2 in [23, p. 19] shows six shapes of hysteresis loops in which the loading and unloading curves are both strictly increasing and intersect only at maximum and minimum loads. On the other hand, magnetostrictive materials and ferromagnetic shape memory alloys present hysteresis loops that have the shape of a butterfly, see Figures 1.1 and 4.14 in [38].

The first question to be answered is the following: which model generates a hysteresis loop that matches the experimental one?

Mayergoyz classifies hysteresis models into two categories: models with local memories and models with nonlocal memories [39, pp. xvii]. In a hysteresis with a local memory, the state at time $t \geq t_0$ is completely defined by the state at the initial time t_0 and the input on $[t_0, t]$. This is the case for example of a hysteresis given by a differential equation. Hysteresis with a nonlocal memory is a hysteresis which is not with local memory. This is the case for example of the Preisach model. Both classes of models have been studied in–amongst other references–[40, 41] from the point of view of the existence and mathematical properties of operators.

When the aim of the study is to get a dynamical model that generates a given hysteresis loop, it is in general irrelevant whether the model is with local memory or with nonlocal memory. This issue is relevant only when the hysteresis loop includes a major loop and minor loops that are closed since closed minor loops can be obtained by models with nonlocal memory but not with models with local memory, see [5, Section 10] for instance.

A nice example of this point of view is [42] in which Drinčić et al. show that the butterfly shape of [38, Fig. 4.14, p. 176] can be generated from the simple hysteresis loops of [23, Figure 2.2, p. 19] through a unimodal map and vice-versa. No assumption about the type of hysteresis model that generates the simple hysteresis loop is needed.

This said, hysteresis models with a finite number of parameters are likely to produce a limited range of shapes for the hysteresis loop. A good example of this case is the Bouc-Wen model which is a first-order differential equation with three parameters whose hysteresis loop has a specific shape given in [15, Fig. 4.3].

On the other hand, models with an infinite number of parameters, or that include functions, are likely to produce a wider range of shapes for the hysteresis loop. A nice example of this case is [43] in which Jayawardhana et al. propose sufficient conditions on the weighing function of a Preisach operator to obtain a hysteresis loop with a butterfly shape. The obtained model is then used to generate an experimental butterfly hysteresis loop obtained from a piezoelectric material. The asymptotic stability of Lur'e systems that incorporate the Preisach operator with butterfly hysteresis is then analyzed in [44].

In this section we show that asymmetric and symmetric hysteresis loops with a strictly increasing loading and unloading curves can be generated by Babuška's model. This is the case in particular of eight-shaped hysteresis loops.

To this end, suppose that a hysteresis loop of some material is described by

$$\mathcal{G}_u = S_\uparrow \cup S_\downarrow, \tag{115a}$$
$$S_\uparrow = \{(\psi_u(\varrho), y_\uparrow(\varrho)), \varrho \in [0, \varrho_1]\}, \tag{115b}$$
$$S_\downarrow = \{(\psi_u(\varrho), y_\downarrow(\varrho)), \varrho \in [\varrho_1, \varrho_2]\}, \tag{115c}$$

when the input u is the one of Figure 1, and $y_\uparrow, y_\downarrow \in C^1(\mathbb{R}, \mathbb{R})$ correspond respectively to loading (that is an increasing u) and unloading (that is a decreasing u). Suppose that $\dot{y}_\uparrow(\varrho) > 0$ for all $\varrho \in [0, \varrho_1]$ and that $\dot{y}_\downarrow(\varrho) < 0$ for all $\varrho \in [\varrho_1, \varrho_2]$. Suppose, moreover, that $y_\uparrow(\varrho_1) = y_\downarrow(\varrho_1)$ and $y_\uparrow(0) = y_\downarrow(\varrho_2)$. Our aim is to find h_1, h_2, g_1, g_2 so that the hysteresis loop of the corresponding Babuška's model is precisely $\mathcal{G}_u$ of Equation (115a).

The conditions above show that Case κ_4 of Proposition 9.5 applies. This means by Proposition 9.9(v) that Inequalities (87) hold, and Inequality (88) does not hold, and Inequality (89) does not hold. A sufficient condition for this to happen is to have $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) = -\infty$. A sufficient condition to obtain these two equalities is to choose

$$h_1(a) = (D_1 - a)^{\beta_1}, \text{ for } a \le D_1, \tag{116a}$$
$$h_1(a) = 0, \text{ for } a \ge D_1, \tag{116b}$$
$$h_2(a) = (D_2 + a)^{\beta_2}, \text{ for } a \ge -D_2, \tag{116c}$$
$$h_2(a) = 0, \text{ for } a \le -D_2, \tag{116d}$$

where $D_1 > y_\uparrow(\varrho_1)$, $-D_2 < y_\downarrow(\varrho_2)$, $\beta_1 \ge 1$, and $\beta_2 \ge 1$.

On the other hand, by Equations (52), (116), and (11) we have

$$\dot{y}_\uparrow(\varrho) = \big(D_1 - y_\uparrow(\varrho)\big)^{\beta_1} g_1(\varrho + u_{\min}), \forall \varrho \in [0, \varrho_1], \tag{117a}$$
$$\dot{y}_\downarrow(\varrho) = -\big(D_2 + y_\downarrow(\varrho)\big)^{\beta_2} g_2(-\varrho + 2u_{\max} - u_{\min}), \forall \varrho \in [\varrho_1, \varrho_2]. \tag{117b}$$

Equations (117) give g_1 and g_2 as

$$g_1(\varrho + u_{\min}) = \frac{\dot{y}_\uparrow(\varrho)}{\big(D_1 - y_\uparrow(\varrho)\big)^{\beta_1}}, \forall \varrho \in [0, \varrho_1], \tag{118a}$$
$$g_2(-\varrho + 2u_{\max} - u_{\min}) = -\frac{\dot{y}_\downarrow(\varrho)}{\big(D_2 + y_\downarrow(\varrho)\big)^{\beta_2}}, \forall \varrho \in [\varrho_1, \varrho_2]. \tag{118b}$$

The functions g_1 and g_2 can be completed with constant parts to comply with Conditions (A) and (E) of Section 9.1.

To illustrate the process above we consider the following example. Consider a hysteretic process with input u and output y. Suppose that the input u is given by $u(t) = -\cos(2\pi t), t \ge 0$, see Figure 13.

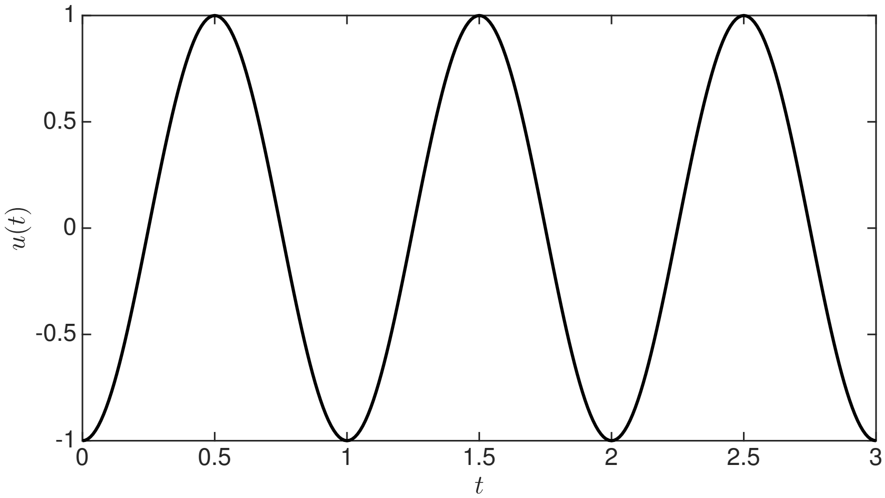

Figure 13: $u(t)$ versus t.

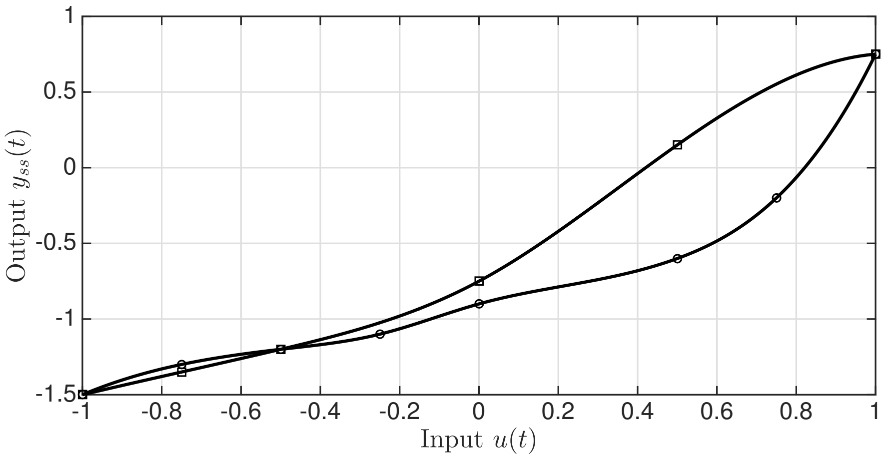

Figure 14: Given hysteresis loop $\mathcal{G}_u$.

Let the corresponding hysteresis loop be $\mathcal{G}_u = \{\big(u(t), y_{\mathrm{ss}}(t)\big), t \in [0,1]\}$ of Figure 14, where y_{ss} corresponds to the steady state of y when $u(t) = -\cos(2\pi t), t \geq 0$. We have constructed the plot of Figure 14 as follows: we chose arbitrarily the points marked with a circle to be the loading curve and the points marked with a rectangle to be the unloading curve. Then we used the Matlab command spline to generate the points in between to make both curves smooth.

To make precise the corresponding notations, the restriction of y_{ss} to the time interval $[0, 1/2]$ is denoted y_{loading} and corresponds to an increasing u, whilst the restriction of y_{ss} to the time interval $[1/2, 1]$ is denoted $y_{\mathrm{unloading}}$ and corresponds to a decreasing u.

The hysteresis loop of Figure 14 will be considered an "experimental" curve, meaning that the aim of Babuška's model is to match that "experimental" curve. We would like to mention that we got the idea of using this eight-shape hysteresis loop from [37, Figure 7].

The first step is to find the functions $y_\uparrow$ and $y_\downarrow$. The loading part of the hysteresis loop satisfies

$$y_\uparrow \circ \rho_u = y_{\text{loading}}, \tag{119}$$

where

$$\rho_u(t) = \int_0^t |\dot{u}(\nu)| \mathrm{d}\nu. \tag{120}$$

For $0 \le t \le \frac{1}{2}$ we have $\rho_u(t) = \int_0^t \dot{u}(\nu)\mathrm{d}\nu = u(t) - u(0) = -\cos(2\pi t) + 1$. For $\frac{1}{2} \le t \le 1$ we have $\rho_u(t) = \int_0^{\frac{1}{2}} \dot{u}(\nu)\mathrm{d}\nu - \int_{\frac{1}{2}}^t \dot{u}(\nu)\mathrm{d}\nu = 2 - \big(u(t) - u(1/2)\big) = 3 + \cos(2\pi t)$. To sum up

$$\begin{cases} \rho_u(t) = -\cos(2\pi t) + 1, & t \in \left[0, \dfrac{1}{2}\right], \\ \rho_u(t) = 3 + \cos(2\pi t), & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{121}$$

Note that ρ_u is strictly increasing so that it is invertible. Then Equation (119) gives $y_\uparrow$ as

$$y_\uparrow = y_{\text{loading}} \circ \rho_u^{-1}. \tag{122}$$

From Equations (121), (8), and (10) we find $\varrho_1 = \rho_u(1/2) = 2$ and $\varrho_2 = \rho_u(1) = 4$. The loading part of the "experimental" hysteresis loop provides $y_\uparrow(\varrho_1) = 0.75$ so that we choose $D_1 = 1$ to ensure that $D_1 > y_\uparrow(\varrho_1)$.

Similarly, we find that $y_\downarrow = y_{\text{unloading}} \circ \rho_u^{-1}$. We choose $D_2 = 2$ to ensure that $-D_2 < y_\downarrow(\varrho_2) = -1.5$. Finally we choose $\beta_1 = \beta_2 = 1$ to get $\ell_1(D_1) = \infty$ and $\ell_2(-D_2) = -\infty$. These values provide h_1 and h_2 using Equations (116).

From Equation (118a) we take the following form for g_1:

$$\begin{cases} g_1(\varrho) = \dfrac{\dot{y}_\uparrow(\varrho+1)}{D_1 - y_\uparrow(\varrho+1)}, & \varrho \in [-1, 1], \\ g_1(\varrho) = \dfrac{\dot{y}_\uparrow(0)}{D_1 - y_\uparrow(0)}, & \varrho \le -1, \\ g_1(\varrho) = \dfrac{\dot{y}_\uparrow(2)}{D_1 - y_\uparrow(2)}, & \varrho \ge 1. \end{cases} \tag{123}$$

Also, from Equation (118b) we take the following form for g_2:

$$\begin{cases} g_2(\varrho) = -\dfrac{\dot{y}_\downarrow(3-\varrho)}{D_2 + y_\downarrow(3-\varrho)}, & \varrho \in [-1, 1], \\ g_2(\varrho) = -\dfrac{\dot{y}_\downarrow(4)}{D_2 + y_\downarrow(4)}, & \varrho \le -1, \\ g_2(\varrho) = -\dfrac{\dot{y}_\downarrow(2)}{D_2 + y_\downarrow(2)}, & \varrho \ge 1. \end{cases} \tag{124}$$

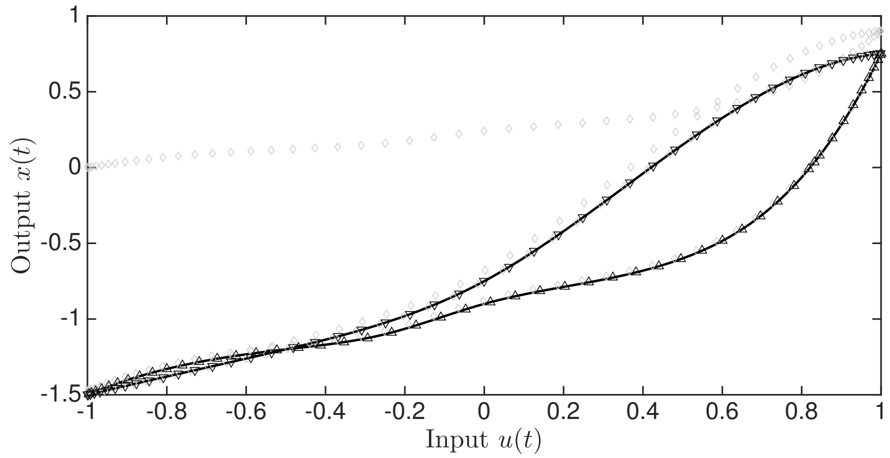

Figure 15: $x(t)$ versus $u(t)$.

Now that the functions h_1, h_2, g_1, g_2 have been determined, the differential equation (51) is solved Using Matlab solver `ode23s` for an initial condition $x_0 = 0$. The obtained solution x of (51) is plotted in Figure 15: the grey points with the shape of a diamond correspond to the transient of x, that is the time interval $[0, 2]$; the black points with a shape of a triangle that points upwards correspond to the steady state of x in the loading part, which is the time interval $[2, 2.5]$; and the black points with a shape of a triangle that points downwards correspond to the steady state of x in the unloading part, which is the time interval $[2.5, 3]$.

In the same figure we have plotted in solid line the "experimental" hysteresis loop of Figure 14 without rectangles or circles.

We can see that the steady state points of the numerical solution x are practically on the "experimental" hysteresis loop.

12. Conclusion

Experimental evidence shows that real hysteretic processes exhibit hysteresis loops that may be symmetric or asymmetric. It is thus of great interest to have mathematical models that are able to match this experimentally observed behavior. The Preisach or the Prandtl models of hysteresis can be designed or modified in order to obtain an asymmetric hysteresis loop. However, these models are not simple to study or program, and thus the interest of Babuška's hysteresis model which consists of a first order scalar differential equation that can be programmed easily using a Matlab solver. This model can match a wide range of asymmetric and symmetric hysteresis loops which makes it of particular interest for the practitioner.

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Compliance with Ethical Standards

Conflict of Interest: The author declares that they have no conflict of interest.

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