CLASSICAL MULTISEPARABLE HAMILTONIAN SYSTEMS, SUPERINTEGRABILITY AND HAANTJES GEOMETRY

DANIEL REYES NOZALEDA, PIERGIULIO TEMPESTA, AND GIORGIO TONDO

Abstract. We show that the theory of classical Hamiltonian systems admitting separation variables can be formulated in the context of $(\omega, \mathscr{H})$ structures. They are essentially symplectic manifolds endowed with a Haantjes algebra $\mathscr{H}$, namely an algebra of (1,1) tensor fields with vanishing Haantjes torsion. A special class of coordinates, called Darboux-Haantjes coordinates, will be constructed from the Haantjes algebras associated with a separable system. These coordinates enable the additive separation of variables of the corresponding Hamilton-Jacobi equation.

We shall prove that a multiseparable system admits as many $\omega\mathscr{H}$ structures as separation coordinate systems. In particular, we will show that a large class of multiseparable, superintegrable systems, including the Smorodinsky-Winternitz systems and some physically relevant systems with three degrees of freedom, possesses multiple Haantjes structures.

Contents

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1. Introduction

The prominence of integrable models in many areas of pure and applied mathematics and theoretical physics has motivated, in the last decades, a resurgence of interest in all aspects of the notion of integrability.

In particular, the algebraic and geometric structures underlying the notion of integrability have been intensively investigated. The study of the geometry of Hamiltonian integrable systems has a long history, dating back to the classical works by Liouville, Jacobi, Stäckel, Eisenhart, Arnold, etc. The approaches proposed in the literature are intimately related with the problem of the additive separation of the Hamilton-Jacobi (HJ) equation in a suitable coordinate system.

Recently, many new ideas coming from differential and algebraic geometry, topology and tensor analysis, have contributed to the formulation of important approaches as the theory of bi-Hamiltonian systems [22], the Lenard-Nijenhuis geometry [28], the theory of Dubrovin-Frobenius manifolds [10]. Besides, these theoretical developments shared new light on the multiple connections among integrability, topological field theories, singularity theory, co-isotropic deformations of associative algebras, etc.

At the same time, several integrable models, both classical and quantum, have been recently introduced, in particular in the domain of superintegrability. Superintegrable systems are a special class of integrable systems which possess algebraic and geometric properties surprisingly rich [31], [36], [30], [40], [51], [50]. Essentially, these systems possess more independent integrals of motion than degrees of freedom. In particular, the *maximally superintegrable* systems with n degrees of freedom admit $2n-1$ functionally independent integrals of motion. Among the most famous examples of superintegrable models we mention the classical harmonic oscillator, the Kepler potential, the Calogero-Moser potential, the Smorodinsky-Winternitz systems and the Euler top. The presence of hidden symmetries, expressed by integrals which are second (or higher) degree polynomials in the momenta, usually allows us to determine the dynamical behaviour of superintegrable models. In the maximal case, the bounded orbits are closed and periodic [36]. As is well known, the phase space topology is also very rich: it can be described in terms of a symplectic bifoliation, determined by the standard Liouville-Arnold invariant fibration [1] of Lagrangian tori, complemented by a coisotropic polar foliation [36], [14].

Although the present work focuses on classical systems, we point out that quantum superintegrable systems also possess many interesting properties. Exact quantum solvability of a Hamiltonian system, related with the existence of suitable Lie algebras of raising and lowering operators, is perhaps the quantum analog of the classical notion of maximal superintegrability [50].

A fundamental class of integrable models are the *separable* ones: they are characterized by the fact that one can find at least a coordinate system in which the corresponding Hamilton-Jacobi (HJ) equation takes a separated form.

The problem of finding separation variables for an integrable Hamiltonian system has also been largely investigated. In 1904, Levi-Civita proposed a test which permits to establish whether a given Hamiltonian is separable in an assigned coordinate system [21]. Another important result, due to Benenti [3], states that a family of Hamiltonian functions $\{H_i\}_{1\leq i\leq n}$ are separable in a set of canonical coordinates $(\boldsymbol{q},\boldsymbol{p})$ if and only if they are in separable involution, i.e. they satisfy the relations

$$\{H_i, H_j\}_{|k} = \frac{\partial H_i}{\partial q^k}\frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q^k} = 0 \ , \quad 1\leq k\leq n \tag{1}$$

where no summation over k is understood. However, such a theorem as well as the Levi–Civita test are not constructive, and do not allow us to determine a complete integral of the Hamilton–Jacobi equation. A constructive approach to separation variables was given by Sklyanin [45] within the framework of Lax systems.

The Hamiltonian functions $\{H_i\}_{1\leq i\leq n}$ are separable in a set of canonical coordinates $(\boldsymbol{q},\boldsymbol{p})$ if there exist n suitable equations, called the Jacobi-Sklyanin separation equations for $\{H_i\}_{1\leq i\leq n}$, of the form

$$\Phi_i(q^i, p_i; H_1,\ldots,H_n) = 0 \qquad \det\left[\frac{\partial \Phi_i}{\partial H_j}\right]\neq 0, \tag{2}$$

for $i = 1,\ldots,n$. These equations allow one to construct a solution of the HJ equation.

Nevertheless, the three above-mentioned criteria of separability are not intrinsic, since they require the explicit knowledge of the local chart $(\boldsymbol{q},\boldsymbol{p})$ in order to be applied. To overcome such a drawback, in the last decades the modern theory of separation of variables (SoV) has been conceived in the context of symplectic and Poisson geometry; in particular, the bi-hamiltonian theory has offered a fundamental geometric insight into the theory of integrable systems [13, 28].

The main purpose of the present work is to establish a novel relationship between the theory of separable Hamiltonian systems and the geometry of an important class of tensor fields, the Haantjes tensors, introduced in [19] as a relevant, natural generalization of the notion of Nijenhuis tensors [38, 39]. The class of Nijenhuis tensors plays a significant role in differential geometry and the theory of almost-complex structures, due to the celebrated Newlander-Nirenberg theorem [37].

Our approach is based on the notion of $\omega\mathscr{H}$ manifolds, introduced in [47] by analogy with the theory of ωN manifolds [28, 13] for finite-dimensional Hamiltonian systems (see also [15], [25] and [26] for a treatment of integrable hierarchies of PDEs from a different perspective). Essentially, an $\omega\mathscr{H}$ manifold is a symplectic manifold endowed with an algebra $\mathscr{H}$ of (1,1) tensor fields with vanishing Haantjes torsion, which are *compatible* with the symplectic structure. Under the hypotheses of the

Liouville-Haantjes (LH) theorem proved in [47], a non-degenerate Hamiltonian system is completely integrable in the Liouville-Arnold sense if and only if it admits a $\omega\mathscr{H}$ structure.

In our context, *Haantjes chains* represent in the Haantjes framework the generalization of the notion of Lenard-Magri chain [23] and of generalized Lenard chain [12, 27] defined previously for quasi-bi-hamiltonian systems [34], [35]. By means of these structures, one obtains a complete description of the integrals of the motion of a system in terms of the associated Haantjes operators.

The problem of SoV can also be recast and studied, in principle, in our approach. Precisely, as stated in Theorem 2 below, if an integrable system admits a *semisimple* $\omega\mathscr{H}$ structure, one can derive a set of coordinates, that we shall call the *Darboux-Haantjes (DH) coordinates*, representing separation coordinates for the Hamilton-Jacobi equation associated with the system. In these coordinates, the symplectic form takes a Darboux form, and the operators of the Haantjes algebra take all simultaneously a diagonal form. As we will show in Theorem 2, and in the examples of Sections 8 and 10, multiseparable systems possess different Haantjes structures associated in a nontrivial way with their separation coordinates.

The theory of ωN manifolds has proved to be a powerful tool for studying separable Hamiltonian systems. A comparison between the two approaches will be proposed in the final section.

This article is organized in two parts. In the first one, including Sections 2 and 3, for the sake of self-consistency we briefly summarize the notions necessary for the study of multiseparable systems: in Section 2, the basic definitions concerning Nijenhuis and Haantjes tensors are proposed; in Section 3, the Haantjes geometry is reviewed. In particular, the notions of Haantjes algebras, $\omega\mathscr{H}$ manifolds and Darboux-Haantjes coordinates are revised. In the second part, starting with Section 4, we shall propose the original results of our work. Precisely, in Section 4, we propose the main theorem concerning the existence of Haantjes structures for separable systems. In Section 5, as a direct application of the theory previously developed, we solve the problem of SoV for a family of Drach-Holt type systems, that were previously considered to be non-separable. Interestingly enough, the new separation variables we found are defined in the full phase space. In Section 6, this theorem is extended to the case of multiseparable (and superintegrable) models. We propose in Section 7 a novel geometrical construction: a lift of operators from the configuration space Q of dimension two to T^*Q, which generalizes the standard Yano lift [56]. By means of our procedure, a (Nijenhuis) Haantjes operator can be lifted into another (Nijenhuis) Haantjes operator (unlike the Yano lift, which only preserves Nijenhuis operators, but not the Haantjes ones). Section 8 is devoted to the study of the Haantjes structures for the Smorodinsky-Winternitz systems in the plane, whereas Section 9 deals with the study of the anisotropic oscillator. In Section 10, the $\omega\mathscr{H}$ manifolds associated with certain important multiseparable systems in 3 dimensions are determined. Future research perspectives are discussed in the final section 11.

2. Nijenhuis and Haantjes operators

In this section, we review some basic algebraic results concerning the theory of Nijenhuis and Haantjes tensors. For a more complete treatment, see the original papers [19, 38] and the related ones [39, 17].

2.1. **Geometric preliminaries.** Let M be a real differentiable n-dimensional manifold and $\boldsymbol{L} : TM \to TM$ a smooth $(1,1)$ tensor field, i.e., a field of linear operators on the tangent space at each point of M. In the following, all tensors will be assumed to be smooth.

Definition 1. *The Nijenhuis torsion of* $\boldsymbol{L}$ *is the vector-valued* 2*-form defined by*

$$\mathcal{T}_{\boldsymbol{L}}(X,Y) := \boldsymbol{L}^2[X,Y] + [\boldsymbol{L}X, \boldsymbol{L}Y] - \boldsymbol{L}\Big([X,\boldsymbol{L}Y] + [\boldsymbol{L}X, Y]\Big), \tag{3}$$

where $X, Y \in TM$ *and* $[\ ,\ ]$ *denotes the commutator of two vector fields.*

In local coordinates $\boldsymbol{x} = (x^1, \ldots, x^n)$, the Nijenhuis torsion can be written as the skew-symmetric $(1,2)$ tensor field

$$(\mathcal{T}_{\boldsymbol{L}})^i_{jk} = \sum_{\alpha=1}^{n} \left(\frac{\partial \boldsymbol{L}^i_k}{\partial x^\alpha} \boldsymbol{L}^\alpha_j - \frac{\partial \boldsymbol{L}^i_j}{\partial x^\alpha} \boldsymbol{L}^\alpha_k + \Big(\frac{\partial \boldsymbol{L}^\alpha_j}{\partial x^k} - \frac{\partial \boldsymbol{L}^\alpha_k}{\partial x^j} \Big) \boldsymbol{L}^i_\alpha \right) , \tag{4}$$

which possesses $n^2(n-1)/2$ independent components.

Definition 2. *The Haantjes torsion of* $\boldsymbol{L}$ *is the vector-valued* 2*-form defined by*

$$\mathcal{H}_{\boldsymbol{L}}(X,Y) := \boldsymbol{L}^2 \mathcal{T}_{\boldsymbol{L}}(X,Y) + \mathcal{T}_{\boldsymbol{L}}(\boldsymbol{L}X, \boldsymbol{L}Y) - \boldsymbol{L}\Big(\mathcal{T}_{\boldsymbol{L}}(X, \boldsymbol{L}Y) + \mathcal{T}_{\boldsymbol{L}}(\boldsymbol{L}X, Y)\Big). \tag{5}$$

The skew-symmetry of the Nijenhuis torsion implies that the Haantjes torsion is also skew-symmetric. Its local expression in explicit form is

$$\begin{aligned} (\mathcal{H}_{\boldsymbol{L}})^i_{jk} \ &= \ \sum_{\alpha=1}^{n} \bigg(-2(\boldsymbol{L}^3)^i_\alpha \partial_{\big[j} \boldsymbol{L}^\alpha_{k\big]} + (\boldsymbol{L}^2)^i_\alpha \Big(\partial_{\big[j} (\boldsymbol{L}^2)^\alpha_{k\big]} + 4 \sum_{\beta=1}^{n} \boldsymbol{L}^\beta_{\big[j} \partial_{|\beta|} \boldsymbol{L}^\alpha_{k\big]} \Big) \\ &- \ 2\boldsymbol{L}^i_\alpha \Big(\boldsymbol{L}^\beta_{\big[j} \partial_{|\beta|} (\boldsymbol{L}^2)^\alpha_{k\big]} + (\boldsymbol{L}^2)^\beta_{\big[j} \partial_{|\beta|} (\boldsymbol{L})^\alpha_{k\big]} \Big) + (\boldsymbol{L}^2)^\alpha_{\big[j} \partial_{|\alpha|} (\boldsymbol{L}^2)^i_{k\big]} \bigg) \ . \end{aligned} \tag{6}$$

Here, for the sake of brevity, we have used the notation $\partial_j := \frac{\partial}{\partial x^j}$; the indices between square brackets are to be skew-symmetrized, except those in $|\cdot|$.
In [49] the following notion was proposed.

Definition 3. *A Haantjes (Nijenhuis) operator is an operator field whose Haantjes (Nijenhuis) torsion identically vanishes.*

2.2. **General properties of Haantjes operators.** First, we shall consider some specific cases in which the construction of the Nijenhuis and Haantjes torsions is particularly simple.

Example 1. Let $dim\ M = 2$. Any operator field $\boldsymbol{L} : TM \to TM$ is a Haantjes operator. This can be proved by a straightforward calculation.

Example 2. Let $\dim M = n$, $n \geq 2$ and $\boldsymbol{L} : TM \to TM$ be an operator field. Assume that in a suitable local coordinate chart $(x^1, \ldots, x^n)$ the operator $\boldsymbol{L}$ takes the diagonal form

$$\boldsymbol{L}(\boldsymbol{x}) = \sum_{i=1}^{n} l_i(\boldsymbol{x}) \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i \ . \tag{7}$$

Then the Haantjes torsion of $\boldsymbol{L}$ identically vanishes.

Another interesting source of Nijenhuis and Haantjes operators is Classical Mechanics. Precisely, given a system of point masses in the n-dimensional affine euclidean space, the inertia tensor of this system is a Haantjes tensor, whereas the planar inertia tensor is a Nijenhuis one [49].
As is well known (see for instance [18]), given an invertible Nijenhuis operator, its inverse is also a Nijenhuis operator. The same property holds true for a Haantjes operator.
A crucial restriction in the Nijenhuis geometry is that, in general, the product of a Nijenhuis operator with an arbitrary $\mathcal{C}^\infty(M)$-function is no longer a Nijenhuis operator. Instead, this is the case for Haantjes operators: therefore, they allow us to define interesting algebraic structures, as we shall see in Section 3. The theory of these structures is based on the following

Proposition 1. [5], [6].
i) Let $\boldsymbol{L}$ be an operator field. The following identity holds

$$\mathcal{H}_{f\boldsymbol{I}+g\boldsymbol{L}}(X,Y) = g^4\,\mathcal{H}_{\boldsymbol{L}}(X,Y), \tag{8}$$

where $f, g : M \to \mathbb{R}$ are functions and $\boldsymbol{I}$ denotes the identity operator in TM.
ii) Let $\boldsymbol{L}$ be an Haantjes operator on M. Then for any polynomial in $\boldsymbol{L}$, with coefficients $a_j \in C^\infty(M)$, the associated Haantjes tensor also vanishes, i.e.

$$\mathcal{H}_{\boldsymbol{L}}(X,Y) = \mathbf{0} \implies \mathcal{H}_{(\sum_j a_j(\boldsymbol{x})\boldsymbol{L}^j)}(X,Y) = \mathbf{0}. \tag{9}$$

As proved in Ref. [49], a Haantjes operator generates a cyclic Haantjes algebra (i. e. a cyclic algebra of Haantjes operators) over the ring of smooth functions on M. Cyclic Haantjes algebras will play a special role in our theory, as we shall clarify in the coming sections.

3. Integrable frames, Haantjes algebras and $\omega\mathscr{H}$ manifolds

3.1. **Integrability.** In order to formulate our approach to separability, we shall review the relationship between Haantjes geometry and integrability. A more detailed treatment and a proof of the statements reviewed here is available in Refs. [49], [47].
We start recalling that a *reference frame* is a set of n vector fields $\{Y_1, \ldots, Y_n\}$ satisfying the following property: given an open set $U \subseteq M$, the frame represents a basis of the tangent space $T_{\boldsymbol{x}}U$ $\forall \boldsymbol{x} \in U$. Given two frames $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$, assume that n nowhere vanishing smooth functions f_i exist, such that

$$X_i = f_i(\boldsymbol{x}) Y_i \ , \qquad\qquad i = 1, \ldots, n \ .$$

Then we shall say that the two frames are *equivalent*. Let $\{U, (x^1, \ldots, x^n)\}$ be a local chart of U. The frame formed by the vector fields $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ will be said to be a *natural* frame.

Definition 4. *A reference frame equivalent to a natural frame will be said to be* integrable.

In the forthcoming considerations, given an operator $\boldsymbol{L}$, we shall denote by $Spec(\boldsymbol{L}) := \{l_1(\boldsymbol{x}), \ldots, l_s(\boldsymbol{x})\}$, $s \in \mathbb{N}\backslash\{0\}$, the set of the pointwise distinct eigenvalues of $\boldsymbol{L}$, assumed by default to be *real*. The distribution of all the generalized eigenvector fields associated with the eigenvalue $l_i = l_i(\boldsymbol{x})$ will be denoted by

$$\mathcal{D}_i := \ker\Big(\boldsymbol{L} - l_i \boldsymbol{I}\Big)^{\rho_i}, \qquad i = 1, \dots, s \tag{10}$$

where $\rho_i \in \mathbb{N}\backslash\{0\}$ is the Riesz index of l_i (which by hypothesis will always be taken to be independent of $\boldsymbol{x}$). The value $\rho = 1$ characterizes the *proper eigen-distributions*, namely the eigen-distributions of proper eigenvector fields of $\boldsymbol{L}$.

Definition 5. *An operator field $\boldsymbol{L}$ is semisimple if in each open neighborhood $U \subseteq M$ there exists a reference frame formed by proper eigenvector fields of $\boldsymbol{L}$. Moreover, $\boldsymbol{L}$ is simple if all of its eigenvalues are pointwise distinct, namely $l_i(\boldsymbol{x}) \neq l_j(\boldsymbol{x})$, $i, j = 1, \dots, n$, $\forall \boldsymbol{x} \in M$.*

A frame of proper eigenvectors will be said to be a proper eigen-frame of $\boldsymbol{L}$. If the frame contains generalized eigenvectors, it will be said to be a generalized eigen-frame.

Definition 6. *Given a set of distributions $\{\mathcal{D}_i, \mathcal{D}_j, \dots, \mathcal{D}_k\}$ of an operator $\boldsymbol{L}$, we shall say that such distributions are mutually integrable if*
(i) each of them is integrable;
(ii) any sum $\mathcal{D}_i + \mathcal{D}_j + \cdots + \mathcal{D}_k$ (where all indices $i, j, \dots, k$ are different) is also integrable.

In 1955, J. Haantjes proved a crucial result:

Theorem 1 ([19])**.** *Let $\boldsymbol{L} : TM \to TM$ be an operator field; assume that the rank of each generalized eigen-distribution $\mathcal{D}_i$, $i = 1, \dots, s$ is independent of $\boldsymbol{x} \in M$. The vanishing of the Haantjes torsion*

$$\mathcal{H}_{\boldsymbol{L}}(X, Y) = \mathbf{0} \qquad\qquad \forall\ X, Y \in TM \tag{11}$$

is a sufficient condition to ensure the mutual integrability of the generalized eigen-distributions $\{\mathcal{D}_1, \dots, \mathcal{D}_s\}$. In addition, if $\boldsymbol{L}$ is semisimple, condition (11) is also necessary.

Consequently, under the previous assumptions, one can select local coordinate charts in which $\boldsymbol{L}$ takes a block-diagonal form. An equivalent statement can be formulated in terms of the existence of integrable generalized eigen-frames of $\boldsymbol{L}$.

Proposition 2. *The vanishing of the Haantjes torsion of an operator field $\boldsymbol{L}$ is a sufficient condition to ensure that $\boldsymbol{L}$ admits an equivalence class of integrable generalized eigen-frames, where $\boldsymbol{L}$ takes a block-diagonal form. Furthermore, if $\boldsymbol{L}$ is semisimple, the condition is also necessary and $\boldsymbol{L}$ takes a diagonal form; if $\boldsymbol{L}$ is* simple *each of its proper eigen-frames is integrable.*

3.2. **Haantjes algebras.** The notion of Haantjes algebra, introduced and discussed in [49], is a crucial piece of the geometrical construction we wish to propose for the analysis of separable systems.

Definition 7. *A Haantjes algebra of rank m is a pair $(M, \mathscr{H})$ with the following properties:*

- *M is a differentiable manifold of dimension* n*;*
- *$\mathscr{H}$ is a set of Haantjes operators $\boldsymbol{K} : TM \to TM$ that generate*
  - *a free module of rank* m *over the ring of smooth functions on M:*

$$\mathcal{H}_{\big(f\boldsymbol{K}_1 + g\boldsymbol{K}_2\big)}(X, Y) = \mathbf{0}\ , \qquad \forall\, X, Y \in TM\ , \quad f, g \in C^\infty(M)\ , \quad \forall\ \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}; \tag{12}$$

– a ring w.r.t. the composition operation

(13)
$$\mathcal{H}_{\left(\boldsymbol{K}_1\,\boldsymbol{K}_2\right)}(X,Y)=\mathcal{H}_{\left(\boldsymbol{K}_2\,\boldsymbol{K}_1\right)}(X,Y)=\mathbf{0}\ , \qquad \forall\,\boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H}, \quad \forall\,X,Y\in TM\ .$$

If

$$\boldsymbol{K}_1\,\boldsymbol{K}_2=\boldsymbol{K}_2\,\boldsymbol{K}_1\ , \qquad \boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H}\ , \tag{14}$$

the algebra $(M,\mathscr{H})$ will be said to be an Abelian Haantjes algebra. Moreover, if the identity operator $\boldsymbol{I}\in\mathscr{H}$, then $(M,\mathscr{H})$ will be said to be a Haantjes algebra with identity.

In other words, the set $\mathscr{H}$ can be regarded as an associative algebra of Haantjes operators. Observe that if $\boldsymbol{K}\in\mathscr{H}$, then the powers $\boldsymbol{K}^i\in\mathscr{H}\ \forall\,i\in\mathbb{N}\backslash\{0\}$.
Haantjes algebras possess several important properties. Among them, we recall that for a given Abelian Haantjes algebra $\mathscr{H}$ there exists associated a set of coordinates, called *Haantjes coordinates*, by means of which all $\boldsymbol{K}\in\mathscr{H}$ can be written simultaneously in a block-diagonal form. In particular, if $\mathscr{H}$ is also semisimple, on each set of Haantjes coordinates all $\boldsymbol{K}\in\mathscr{H}$ can be written simultaneously in a diagonal form [49].

3.3. **Haantjes chains.** The notion of Haantjes chains, which generalizes that of Lenard-Magri chains [22, 23] has been proposed in [47]. In the forthcoming analysis, it will enable us to build a bridge between the Haantjes geometry and the theory of separable systems. Other generalizations have also been proposed in the literature of the last decades [32, 33, 54, 12, 13].

Definition 8. *Let $(M,\mathscr{H})$ be a Haantjes algebra of rank* m. *A function $H\in C^\infty(M)$ is said to generate a Haantjes chain of 1-forms of length* m *if there exist a distinguished basis $\{\boldsymbol{K}_1,\ldots,\boldsymbol{K}_m\}$ of $\mathscr{H}$ such that*

$$\mathrm{d}(\boldsymbol{K}_\alpha^T\,\mathrm{d}H)=\mathbf{0}\ , \qquad \alpha=1,\ldots,m \tag{15}$$

where $\boldsymbol{K}_\alpha^T:T^*M\to T^*M$ is the transposed operator of $\boldsymbol{K}_\alpha$. The (locally) exact 1-forms $\mathrm{d}H_i$ such that

$$\mathrm{d}H_\alpha=\boldsymbol{K}_\alpha^T\,\mathrm{d}H\ ,$$

supposed to be linearly independent, are called the elements of the Haantjes chain of length m *generated by H and the functions $H_\alpha\in C^\infty(M)$ are their potential functions.*

Given a basis $\{\boldsymbol{K}_1,\boldsymbol{K}_2,\ldots,\boldsymbol{K}_m\}$ of $\mathscr{H}$, let us denote by

$$\mathcal{D}_H^\circ:=\langle\boldsymbol{K}_1^T\mathrm{d}H,\boldsymbol{K}_2^T\mathrm{d}H,\ldots,\boldsymbol{K}_m^T\,\mathrm{d}H\rangle \tag{16}$$

the co-distribution generated by a function H, and $\mathcal{D}_H$ the distribution of the vector fields annihilated by them (of rank $(n-m)$). A result proved in [47] states that the function H generates a Haantjes chain (15) if and only if $\mathcal{D}_H^\circ$ (or equivalently $\mathcal{D}_H$) is Frobenius-integrable.
Now, we shall briefly review the theory of $\omega\mathscr{H}$ or symplectic-Haantjes manifolds, firstly introduced in [47]. They offer a natural theoretical framework for the formulation of the theory of Hamiltonian integrable systems. We shall limit ourselves to illustrate some basic facts of the theory; for the proofs of the statements and further details, the reader can consult Ref. [47].

3.4. $\omega\mathscr{H}$ **manifolds.**

Definition 9. *A symplectic–Haantjes (or $\omega\mathscr{H}$) manifold of class* m *is a triple* $(M, \omega, \mathscr{H})$ *which satisfies the following properties:*

i) (M, ω) *is a symplectic manifold of dimension* 2n*;*
ii) $\mathscr{H}$ *is a Haantjes algebra of rank* m*;*
iii) $(\omega, \mathscr{H})$ *are algebraically compatible, that is*

$$\omega(X, \boldsymbol{K}Y) = \omega(\boldsymbol{K}X, Y) \qquad \forall \boldsymbol{K} \in \mathscr{H} \ ,$$

or equivalently

$$\boldsymbol{\Omega}\,\boldsymbol{K} = \boldsymbol{K}^T\boldsymbol{\Omega}, \qquad \forall \boldsymbol{K} \in \mathscr{H} \ . \tag{17}$$

Hereafter $\boldsymbol{\Omega} := \omega^\flat : TM \to T^*M$ *denotes the fiber bundles isomorphism defined by*

$$\omega(X, Y) = \langle \boldsymbol{\Omega} X, Y\rangle \qquad \qquad \forall X, Y \in TM \ ,$$

and the map $\boldsymbol{P} := \boldsymbol{\Omega}^{-1} : T^*M \to TM$ *is the Poisson bivector induced by the symplectic structure* ω*.*
If the identity operator $\boldsymbol{I}$ *belongs to* $\mathscr{H}$*, then* $(M, \omega, \mathscr{H})$ *will be said to be a* $\omega\mathscr{H}$ *manifold with identity. If* $\mathscr{H}$ *is an Abelian Haantjes algebra, we shall say that the resulting* $\omega\mathscr{H}$ *manifold is Abelian.*

Definition 10. *An* $\omega\mathscr{H}$ *manifold* $(M, \omega, \mathscr{H})$ *is semisimple if* $\mathscr{H}$ *is a semisimple Haantjes algebra.*

Observe that a simple realization of the notion of Haantjes algebra is given in a Darboux chart $\{\boldsymbol{x} = (q^1, \dots q^n, p_1, \dots, p_n)\}$ by

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} l_i^{(\alpha)}(\boldsymbol{x})\Big(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i\Big), \qquad \qquad \alpha = 1, \dots, m \ , \tag{18}$$

where $l_i^{(\alpha)} = \lambda_i^{(\alpha)}(\boldsymbol{x}) = \lambda_{n+i}^{(\alpha)}(\boldsymbol{x})$, $i = 1, \dots, n$.
We proved in [47] that there exists a spectral decomposition of the tangent spaces $T_{\boldsymbol{x}}M = \bigoplus_{i=1}^{s} \mathcal{D}_i(\boldsymbol{x})$ realized in terms of (generalized) eigenspaces $\mathcal{D}_i(\boldsymbol{x})$ of *even rank*. Consequently, the number of the distinct eigenvalues of any Haantjes operator $\boldsymbol{K}$ of a $\omega\mathscr{H}$ structure is not greater than n. In particular, if the number of distinct eigenvalues of an operator $\boldsymbol{K} \in \mathscr{H}$ is just n, the operator will be said to be *maximal*. This is equivalent to require that the minimal polynomial of $\boldsymbol{K}$ $m_{\boldsymbol{K}}(\boldsymbol{x}, \lambda) = \prod_{i=1}^{n} \Big(\lambda - l_i(\boldsymbol{x})\Big)^{\rho_i}$ has degree $m = n$.
Several other interesting results can be stated in the $\omega\mathscr{H}$ geometry. In particular, we recall that given a $\omega\mathscr{H}$ manifold, the distributions $\mathcal{D}_j$, $j = 1, \dots, s$ of each $\boldsymbol{K} \in \mathscr{H}$ are integrable and of even rank. Besides, their integral leaves are symplectic submanifolds of M and are symplectically orthogonal to each other, namely $\omega(\mathcal{D}_j, \mathcal{D}_k) = \boldsymbol{0}$, $j \neq k$.

3.5. **Darboux–Haantjes coordinates for** $\omega\mathscr{H}$ **manifolds.** Assume that $(M, \omega, \mathscr{H})$ is an Abelian $\omega\mathscr{H}$ manifold of class m. In [47] it was proved that there exist local charts in $U \subset M$ which are Darboux coordinates for ω; besides, all Haantjes operators take simultaneosly a block-diagonal form. Due to their twofold role, they will be called *Darboux–Haantjes* (DH) coordinates.

Corollary 1. *Given a semisimple Abelian* $\omega\mathscr{H}$ *manifold* $(M, \omega, \mathscr{H})$*, on a set of Darboux–Haantjes coordinates each* $\boldsymbol{K} \in \mathscr{H}$ *takes the diagonal form* (18)*.*

The relevance of Haantjes chains in the theory of $\omega\mathscr{H}$ manifolds is due to the following

Lemma 1. *Let $(M, \omega, \mathscr{H})$ be an Abelian $\omega\mathscr{H}$ manifold. Then the potential functions $H_\alpha \in C^\infty(M)$ of the Haantjes chain generated by a distinguished function $H \in C^\infty(M)$ are in involution among each others and with H, w.r.t. the Poisson bracket defined by the Poisson operator $\boldsymbol{P} = \boldsymbol{\Omega}^{-1}$.*

3.6. **Cyclic $\omega\mathscr{H}$ manifolds.** Cyclic Haantjes algebras are a simple and particularly interesting instance of Haantjes algebras: they are cyclically generated by a suitable Haantjes operator $\boldsymbol{L}$. In other words, all the operators of the algebra are of the form $\boldsymbol{K}_\alpha = p_\alpha(\boldsymbol{L})$, where $p_\alpha(\boldsymbol{x})$ is a suitable polynomial with coefficients in $C^\infty(M)$. A *cyclic $\omega\mathscr{H}$* manifold is a $\omega\mathscr{H}$ manifold endowed with a cyclic Haantjes algebra.

We mention that an interesting family of cyclic $\omega\mathscr{H}$ manifolds is represented by ωN manifolds [28, 24]. In that context, cyclic Haantjes chains specialize into Nijenhuis chains, as in [12], or generalized Lenard chains as in [46, 52].

As a consequence of Proposition 47 in [47], semisimple Abelian $\omega\mathscr{H}$ manifolds are always cyclic ones. Besides, their generator can be chosen to be a Nijenhuis operator.

4. Separation of variables in $\omega\mathscr{H}$ manifolds

4.1. **Main Theorem.** The next theorem represents our main result concerning the existence of separation variables in the theory of $\omega\mathscr{H}$ manifolds.

Theorem 2. *Let M be a semisimple $\omega\mathscr{H}$ manifold of class n and $\{H_1, H_2, \ldots, H_n\}$ be a set of $C^\infty(M)$ functions belonging to a Haantjes chain generated by a function $H \in C^\infty(M)$ via the basis of operators $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_n\} \in \mathscr{H}$. Then, each set $(\boldsymbol{q}, \boldsymbol{p})$ of* DH *coordinates provides us with separation variables for the Hamilton–Jacobi equation associated with each function H_j.*

Conversely, if $\{H_1, H_2, \ldots, H_n\}$ are n independent, $C^\infty(M)$ functions separable in a set of Darboux coordinates $(\boldsymbol{q}, \boldsymbol{p})$, then they belong to the Haantjes chain generated by the operators

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H}{\partial p_i}} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q_i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right) \qquad \alpha = 1, \ldots, n \ , \tag{19}$$

where H is any of the functions $\{H_1, \ldots, H_n\}$, with $\frac{\partial H}{\partial p_i} \neq 0$, $i = 1, \ldots, n$. These operators generate a semisimple $\omega\mathscr{H}$ structure on M.

Proof. Theorem 25 of [47] guarantees the existence of sets of DH coordinates for a semisimple $\omega\mathscr{H}$ manifold. Therefore, it suffices to show that the functions H_j in such coordinates are in separable involution, according to eq. (1). To this aim, let us note that, due to the diagonal form of $\boldsymbol{K}_\alpha^T$ in a DH local chart, the relations

$$\frac{\partial H_j}{\partial q^k} = l_k^{(j)} \frac{\partial H}{\partial q^k}, \tag{20}$$

$$j, k = 1, \ldots, n \ ,$$

$$\frac{\partial H_j}{\partial p_k} = l_k^{(j)} \frac{\partial H}{\partial p_k}, \tag{21}$$

hold. Here $l_k^{(j)}$ denotes the eigenvalues of the Haantjes operator $\boldsymbol{K}_j^T$. Therefore,

$$\{H_i, H_j\}_{|k} = l_k^{(i)} \frac{\partial H}{\partial q^k} l_k^{(j)} \frac{\partial H}{\partial p_k} - l_k^{(j)} \frac{\partial H}{\partial q^k} l_k^{(i)} \frac{\partial H}{\partial p_k} = 0 \ .$$

In order to prove the converse statement, without loss of generality we can assume that $\frac{\partial H}{\partial p_i} \neq 0$, $i = 1, \ldots, n$. The operators (19), being diagonal in the separated coordinates, are Haantjes operators. Also, they commute with each others and generate an Abelian, semisimple Haantjes algebra $\mathscr{H}$. The algebraic compatibility conditions (17) of the operators (19) with the symplectic form are equivalent to the conditions

$$l_{n+i}^{(\alpha)} = l_i^{(\alpha)} \qquad i = 1, \ldots, n \ . \tag{22}$$

Thus, the Haantjes operators (19) must possess at least double eigenvalues.
Finally, we impose that the integrals of motion $\{H_1, H_2, \ldots, H_n\}$ form a Haantjes chain, which will be generated by any of these functions, denoted by H. Since $\boldsymbol{K}_\alpha$ $(\alpha = 1, \ldots, n)$ is diagonal in the $(\boldsymbol{q}, \boldsymbol{p})$ variables, such conditions are equivalent, for each α, to the overdetermined system of $2n$ algebraic equations in the n functions $l_i^{(\alpha)}$

$$l_i^{(\alpha)} \frac{\partial H}{\partial q^i} = \frac{\partial H_\alpha}{\partial q^i} \ , \tag{23}$$

$$l_i^{(\alpha)} \frac{\partial H}{\partial p_i} = \frac{\partial H_\alpha}{\partial p_i} \ , \tag{24}$$

$i = 1, \ldots, n$. However, the above equations are compatible, because the Benenti conditions (1) of separate involution ensure that

$$\frac{\partial H}{\partial q^i} \frac{\partial H_\alpha}{\partial p_i} = \frac{\partial H}{\partial p_i} \frac{\partial H_\alpha}{\partial q^i} \ , \quad 1 \leq i \leq n \ .$$

Consequently, the equations (24) provide us with the unique solution (19). □

Remark 1. Given n arbitrary smooth functions $\{H_1, \ldots, H_n\}$ on a $2n$-dimensional manifold M, it is always possible to determine n diagonal Haantjes operators $\boldsymbol{K}_i$ which satisfy the chain equations $\boldsymbol{K}_i^T dH = dH_i$, where H is any of the previous functions. However, if M is a symplectic manifold, the compatibility condition (17) of the Haantjes operators $\boldsymbol{K}_i$ with the symplectic structure defined on M imposes n additional constraints on the eigenvalues of these operators. Thus, the systems of $2n$ algebraic equations (23) and (24) can be solved if and only if n of these equations are automatically satisfied. This requirement is equivalent to the Benenti conditions (1). Consequently, the existence of a Haantjes chain for an integrable system constructed with semisimple operators (and consequently of a $\omega\mathscr{H}$ structure), is not at all a trivial property. The case of the superintegrable Post-Winternitz system is illustrative of this aspect: no separation variables are known for this system, and the unique known $\omega\mathscr{H}$ structures are *non-semisimple ones*.

4.2. **A general procedure.** In order to determine the $\omega\mathscr{H}$ structures admitted by a separable system, we need to construct the Haantjes chains associated with it. Precisely, we wish to solve the chain equations

$$\mathrm{d}(\boldsymbol{K}_\alpha^T \, \mathrm{d}H) = \mathbf{0} \qquad \alpha = 1, \ldots, m \tag{25}$$

for suitable Haantjes operators $\boldsymbol{K}_\alpha^T$. Generally speaking, these equations do not admit a unique solution. Notice that the operators $\boldsymbol{K}_\alpha^T$ we are interested in must be compatible with the symplectic form ω (see eq. (17)). The most general operator $\boldsymbol{M}$ compatible with the symplectic form, in Darboux coordinates $(\boldsymbol{q}, \boldsymbol{p})$ reads

$$\boldsymbol{M} = \left[\begin{array}{c|c} \boldsymbol{A}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{B}(\boldsymbol{q},\boldsymbol{p}) \\ \hline \boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{A}^T(\boldsymbol{q},\boldsymbol{p}) \end{array}\right] , \quad \boldsymbol{B} + \boldsymbol{B}^T = \boldsymbol{0} , \quad \boldsymbol{C} + \boldsymbol{C}^T = \boldsymbol{0} , \tag{26}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are $n \times n$ matrices with coefficients smoothly depending on the Darboux coordinates. However, the operator (26) in general is not a Haantjes operator, unless some specific choices for its arbitrary functions are made. Therefore, our task is to determine solutions of eqs. (25) in the subclass of Haantjes operators of the form (26). For $n = 2$, the operators of the form (26) can depend up to 6 arbitrary functions; for $n = 3$, up to 15 arbitrary functions are admissible.
The procedure for the determination of the $\omega\mathscr{H}$ structure associated with a n-dimensional integrable Hamiltonian system can be summarized in the following steps.
(1) Determine the operators $\boldsymbol{K}$ of the class (26) that solve eqs. (25).
(2) Among the solutions found, choose the operators satisfying the vanishing condition for their Haantjes torsion:

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) = \boldsymbol{0}, \qquad \forall\ X, Y \in TM\ .$$

(3) Find all the semisimple Abelian $\omega\mathscr{H}$ structures admitted by the system and their generators.
(4) Find the DH coordinates associated with each $\omega\mathscr{H}$ structure.

We remind that in these coordinates the generator of $\mathscr{H}$ (and consequently the whole Haantjes algebra) takes a diagonal form.

This procedure is completely general.

An alternative strategy is to use a suitable ansatz for the form of the operators $\boldsymbol{K}$ that are supposed to solve eqs. (25). Precisely, one can start from a sub-family of operators depending on arbitrary functions, which are both compatible with the symplectic form (i.e., they belong to the class (26)) and have vanishing Haantjes torsion. In this way, the requirement of step (2) is already fulfilled. Then, one can try to fix the arbitrary functions available by solving eqs. (25) (step 1). This approach is less general, since it presupposes the a priori determination of special subclasses of operators of the form (26) which are Haantjes's as well. This task, which in general is quite hard, is quite affordable in the case $n = 2$; consequently, this procedure has been adopted, for instance, in Section 9.

5. Separation of variables for a Drach-Holt type system

The approach proposed in this paper offers an effective procedure to construct algorithmically separation variables admitted by Hamiltonian integrable systems. As a paradigmatic example, we shall study the case of a system showing an irrational dependence on its coordinates, namely a three-parametric deformation of the Holt potential, that has been introduced in [8]:

$$H_1 = \frac{1}{2}(p_x^2 + p_y^2) + k_1 \frac{4x^2 + 3y^2}{y^{2/3}} + k_2 \frac{x}{y^{2/3}} + \frac{k_3}{y^{2/3}}. \tag{27}$$

It is integrable in the manifold $M = T^*(E^2 \setminus \{y = 0\})$, with a third-order integral

$$\begin{aligned} H_2 \quad &= \quad 2p_x^3 + 3p_x p_y^2 + 12k_1\left(\frac{2x^2 - 3y^2}{y^{2/3}} + 6xy^{1/3}p_y\right) + \\ &+ \quad k_2\left(\frac{6x}{y^{2/3}}p_x + 9y^{1/3}p_y\right) + 6\frac{k_3}{y^{2/3}}p_x \ . \end{aligned} \tag{28}$$

When $k_1 \to 0$, the Hamiltonian H_1 converts into the (generalized) Post-Winternitz (gPW) superintegrable potential [40]. A crucial aspect is that H_1 has been considered in the literature to be an example of *nonseparable system*, since, not being of Stäckel type, it is not separable by means of an extended–point transformation. A natural question is whether there is a full canonical transformation (in general difficult to find) redeeming its separability. Our theory of integrability *à la Haantjes* enables us to solve this problem, since it provides us with a set of DH separation coordinates, according to Theorem 2.

5.1. **The $\omega\mathscr{H}$ structure.** Solving the equations for the Haantjes chains

$$\mathrm{d}H_2 = (\boldsymbol{K}^{(DH)})^T \ \mathrm{d}H_1 \ , \tag{29}$$

we get a Haantjes operator linear in the momenta, which reads

$$\boldsymbol{K}^{(DH)} = 3\left[\begin{array}{cc|cc} 2p_x & p_y & 0 & 3y \\ 0 & 2p_x & -3y & 0 \\ \hline 0 & -24k_1y^{1/3} & 2p_x & 0 \\ 24k_1y^{1/3} & 0 & p_y & 2p_x \end{array}\right]. \tag{30}$$

This operator is a generator of the algebra $\langle \boldsymbol{I}, \boldsymbol{K^{(DH)}} \rangle$, since its minimal polynomial is of degree two. According to the general theory developed above, the potential functions of the exact one-forms belonging to the eigen-distributions of $(\boldsymbol{K}^{(DH)})^T$ provide the separation coordinates.

Remark 2. For the large class of quasi-bi-Hamiltonian systems [7, 32, 33, 54] in two degrees of freedom, one can show that the eigenvalues of the Haantjes operators are themselves characteristic functions of the web [49], in involution w.r.t. the Poisson bracket. In particular, the eigenvalues of the Haantjes operator (30) are just characteristic functions of the web associated with $\boldsymbol{K}^{(DH)}$.

5.2. **Separation Coordinates.** These functions read

$$\lambda_1 = -6(p_x + 3\sqrt{2k_1}y^{2/3}), \ \lambda_2 = -6(p_x - 3\sqrt{2k_1}y^{2/3}) \ . \tag{31}$$

In order to get a system of DH coordinates, they can be completed with a pair of conjugate momenta that have the following, non trivial expressions

$$\begin{aligned} \mu_1 = -\frac{1}{576}\sqrt{\frac{2}{k_1}} \quad &\Big(p_x^4 + 12\sqrt{2k_1}p_x^3y^{2/3} + 108k_1p_x^2y^{4/3} + 216\sqrt{2k_1^3}p_xy^2 \\ &+12p_yy^{1/3} - 24\sqrt{2k_1}x + 324k_1^2y^{8/3}\Big) \ , \end{aligned} \tag{32}$$

$$\mu_2 = -\frac{1}{576}\sqrt{\frac{2}{k_1}}\quad\Big(p_x^4 - 12\sqrt{2k_1}p_x^3y^{2/3} + 108k_1p_x^2y^{4/3} - 216\sqrt{2k_1^3}p_xy^2 \\ -12p_yy^{1/3} - 24\sqrt{2k_1}x + 324k_1^2y^{8/3}\Big)\ ,$$

and have been computed as potential functions of two exact 1-forms belonging to the two complementary eigen-distributions of $(\boldsymbol{K}^{(DH)})^T$.

5.3. **Separation Equations of Jacobi–Sklyanin.** The approach of Jacobi-Sklyanin represents a fundamental piece in the theory of separable systems. Here we will establish a connection between the Haantjes geometry and the Jacobi-Sklyanin separation equations for the case of the Drach-Holt system.
These equations allow one to construct a solution (W, E) of the Hamilton-Jacobi equation. In fact, by solving (2) with respect to $p_k = \frac{\partial W_k}{\partial q_k}$, we get

$$W = \sum \int p_k(q_k'; H_1, \dots, H_n)_{|_{H_i=a_i}} dq_k'. \tag{33}$$

The set of coordinates (31), (32), being DH coordinates, are separation variables for both the Hamiltonian functions H_1 and H_2; besides, they fulfill the Jacobi-Sklyanin separation equations

$$\begin{aligned} b_1\mu_1^2 + (b_2\lambda_1^4 + b_3)\mu_1 + b_4\lambda_1^8 + b_5\lambda_1^4 + b_6\lambda_1^3 + \lambda_1 H_1 + H_2 + b_7 \quad &= 0\ , \\ b_1\mu_2^2 + (b_2\lambda_2^4 + b_3)\mu_2 + b_4\lambda_2^8 + b_5\lambda_2^4 - b_6\lambda_2^3 - \lambda_2 H_1 - H_2 + b_7 \quad &= 0\ , \end{aligned}$$

where b_i, $i = 1, \dots, 7$ are the constants given by

$$\begin{aligned} b_1 = &\quad 10368\sqrt{2k_1^3},\ b_2 = \tfrac{k_1}{18},\ b_3 = 216\sqrt{2k_1}\,k_2,\ b4 = \tfrac{\sqrt{2k_1}}{26873856}\ , \\ b_5 = &\quad \tfrac{k_2}{1728},\ b_6 = \tfrac{-1}{216},\ b_7 = 18\sqrt{2k_1}\,k_3\ . \end{aligned}$$

We arrive therefore at the separated solutions of the Hamilton-Jacobi equation

$$\begin{aligned} W_1(\lambda_1; h_1, h_2) &= \tfrac{1}{2b_1}\int^{\lambda_1}\Big(-(b_2\lambda_1^4 + b_3) \pm \sqrt{(b_2\lambda_1^4 + b_3)^2 - 4b_1P_8(\lambda_1)}\Big)\mathrm{d}\lambda_1, \\ W_2(\lambda_2; h_1, h_2) &= \tfrac{1}{2b_1}\int^{\lambda_2}\Big(-(b_2\lambda_2^4 + b_3) \pm \sqrt{(b_2\lambda_2^4 + b_3)^2 - 4b_1Q_8(\lambda_2)}\Big)\mathrm{d}\lambda_2, \end{aligned}$$

where h_1, h_2 are the values of H_1, H_2 on the lagrangian tori, and

$$\begin{aligned} P_8(\lambda_1; h_1, h_2) &:= b_4\lambda_1^8 + b_5\lambda_1^4 + b_6\lambda_1^3 + \lambda_1 h_1 + h_2 + b_7, \\ Q_8(\lambda_2; h_1, h_2) &= b_4\lambda_2^8 + b_5\lambda_2^4 - b_6\lambda_2^3 - \lambda_2 h_1 - h_2 + b_7. \end{aligned}$$

6. Multiseparable systems and Haantjes geometry

A particularly interesting instance of the previous theory is offered by the case of multiseparable systems. They are Hamiltonian systems which can be separated in more than one coordinate system in the associated phase space. Fundamental physical examples of multiseparable systems are the n-dimensional harmonic oscillator and the Kepler system. Another important class is represented by the four Smorodinsky-Winternitz systems, which are the only systems in the Euclidean plane admitting orthogonal separation variables.
Multiseparable systems are superintegrable ones, provided that the sets of separation functions in the Hamilton-Jacobi equation related to different separation coordinates are functionally independent. However, to our knowledge, there is no general theoretical result establishing a relation between the two notions of superintegrability and separability. For instance, for the classical Post-Winternitz

(PW) system [40], which is maximally superintegrable, no separation coordinates are known in phase space. Besides, due to the presence of integrals of motion of degree higher than two as polynomials in the momenta, the PW system does not admit orthogonal separation coordinates in its configuration space. A superintegrable system can also be simply separable, without being a multiseparable one. This is the case for the anisotropic oscillator discussed in Section 9. In this article, we shall focus mainly on the case of superintegrable multiseparable systems, which is perhaps the most interesting one from a geometrical and physical point of view. In order to extend the Haantjes geometry to the case of multiseparable systems, an important, preliminar aspect should be pointed out.

Let us consider an integrable Hamiltonian system with Hamiltonian function H and $\{(J_k, \phi_k)\}$, $k = 1, \dots, n$, be a set of action-angle variables for the system, with associated frequencies $\nu_k(\boldsymbol{J}) := \frac{\partial H}{\partial J_k}$. In [47] the Liouville-Haantjes theorem was proved. Under the hypothesis of *nondegeneracy* for H, that is

$$\det\left(\frac{\partial \nu_k}{\partial J_i}\right) = \det\left(\frac{\partial^2 H}{\partial J_i \partial J_k}\right) \neq 0 \ , \tag{34}$$

this theorem states that one can define a semisimple $\omega\mathscr{H}$ manifold in any tubular neighbourhood of an Arnold torus. However, superintegrable systems just violate the condition (34); therefore, the LH theorem cannot be applied to them. Nevertheless, this does not imply that $\omega\mathscr{H}$ structures can not exist for superintegrable systems. Indeed, we can construct such structures by means of a different approach, based on a simple consequence of Theorem 2.

Corollary 2. *An integrable Hamiltonian system admits as many inequivalent separation coordinate systems as the number of its independent semisimple Abelian $\omega\mathscr{H}$ structures of class n.*

Proof. It suffices to observe that, according to Theorem 2, for each semisimple $\omega\mathscr{H}$ structure of class n admitted by the Hamiltonian system there exists a set of Darboux-Haantjes coordinates, which play the role of separating coordinates for the corresponding Hamilton-Jacobi equation. □

The previous result represents, jointly with Theorem 2, the main theoretical contribution of this work.

In the following sections, we shall exhibit explicitly the Haantjes structures associated with celebrated examples of maximally superintegrable and multiseparable Hamiltonian systems.

7. Construction of Haantjes operators in T^*Q: a novel geometric lift

The problem of constructing Haantjes operators on an n-dimensional manifold M is, in general, a hard one, as it entails to solve a sistem of $(n^2(n-1)/2)$ nonlinear PDE of the first order in the n^2 unknown components of the operators we wish to determine.

However, when $M \equiv T^*Q$, being Q the configuration space of a mechanical system, one can plan to simplify the problem of constructing a $(T^*Q, \omega, \mathscr{H})$ manifold by lifting a given Haantjes operator $\boldsymbol{A} : TQ \to TQ$ to an Haantjes operator

$\hat{\boldsymbol{A}} : T(T^*Q \to T(T^*Q)$ via a suitable geometric procedure that, in order to be effective, should preserve the vanishing condition of Haantjes tensors. Our procedure is inspired by the one introduced in [20] for the construction of a suitable Nijenhuis operator for the Benenti systems and it has been successfully interpreted and applied in [13] in the context of the ωN geometry. In those works, technically the *complete lift* from a manifold to its cotangent bundle introduced by Yano [56] has been adopted. Although the Yano lift preserves the vanishing of Nijenhuis tensors, unfortunately this property does not hold true for the Haantjes case: the Yano lift of a Haantjes operator need not be another Haantjes operator.

In order to overcome this drawback, we propose a novel geometric lifting procedure, which generalizes the Yano's one. To this aim, let us consider an operator $\boldsymbol{A} : TQ \to TQ$ and the canonical projection map $\pi : T^*Q \to Q$. Let us denote by $\hat{\boldsymbol{A}}$ a lift of $\boldsymbol{A}$ to the cotangent bundle T^*Q which is required to be *projectable* onto $\boldsymbol{A}$, that is to say, $\boldsymbol{A}\,\pi_* = \pi_*\,\hat{\boldsymbol{A}}$. Such a condition is fulfilled if and only if $\hat{\boldsymbol{A}}$ takes the following block-matrix form in a Darboux chart $(\mathbf{q}, \mathbf{p})$:

$$\hat{\boldsymbol{A}} = \left[\begin{array}{c|c} \boldsymbol{A}(\boldsymbol{q}) & \mathbf{0} \\ \hline \boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{D}(\boldsymbol{q},\boldsymbol{p}) \end{array}\right] \ , \tag{35}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are $(n \times n)$ matrices depending possibly on all coordinates. In addition, we must impose the compatibility condition with the symplectic form (26), which reduces the form of $\hat{\boldsymbol{A}}$ to

$$\hat{\boldsymbol{A}} = \left[\begin{array}{c|c} \boldsymbol{A}(\boldsymbol{q}) & \mathbf{0} \\ \hline \boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{A}^T(\boldsymbol{q}) \end{array}\right] \qquad \boldsymbol{C} + \boldsymbol{C}^T = \mathbf{0} \ . \tag{36}$$

Now, we assume that $\boldsymbol{A}$ is a Haantjes operator and we wish to determine the form of the matrix $\boldsymbol{C}$ in such a way that $\hat{\boldsymbol{A}}$ is still a Haantjes operator. In the subsequent discussion, we shall prove our result for configuration spaces of dimension $n = 2$ only.

As clarified in Example 1, when dim $Q = 2$, any operator $\boldsymbol{A}(\boldsymbol{q})$ is a Haantjes operator. Then, its lifted operator takes the form

$$\hat{\boldsymbol{A}} = \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & r & a & c \\ -r & 0 & b & d \end{array}\right] \ , \tag{37}$$

where all the entries are functions of (q^1, q^2) only, except for the smooth function $r = r(q^1, q^2, p_1, p_2)$. Let us assume that the eigenvalues of $\boldsymbol{A}(\boldsymbol{q})$ are pointwise distinct; by requiring that $\hat{\boldsymbol{A}}$ is a Haantjes operator we find that the most general solution for r is an affine function of (p_1, p_2). Precisely,

$$r = f(q^1, q^2)p_1 + g(q^1, q^2)p_2 + h(q^1, q^2),$$

where

$$f \;=\; \frac{\partial a}{\partial q^2} - \frac{\partial b}{\partial q^1} - \frac{a-d}{\Delta}\tau^1_{12} - \frac{2b}{\Delta}\tau^2_{12} \tag{38}$$

$$g \;=\; \frac{\partial c}{\partial q^2} - \frac{\partial d}{\partial q^1} + \frac{a-d}{\Delta}\tau^2_{12} - \frac{2c}{\Delta}\tau^1_{12} \ , \tag{39}$$

$\Delta := (a-d)^2 + 4bc$ is the discriminant of the minimal polynomial of $\boldsymbol{A}(\boldsymbol{q})$, τ^1_{12} and τ^2_{12} are the two independent components of the Nijenhuis torsion of $\boldsymbol{A}(\boldsymbol{q})$ and $h(q^1, q^2)$ is an arbitrary smooth function. In particular, when $\boldsymbol{A}(\boldsymbol{q})$ is a Nijenhuis operator and $h(q^1, q^2) = 0$, the operator $\hat{\boldsymbol{A}}$ coincides with the Yano complete lift of $\boldsymbol{A}(\boldsymbol{q})$, therefore it is still a Nijenhuis operator.

The lifting procedure presented here can be extended to the n-dimensional case, $n \geq 3$. The details concerning this extension will be discussed elsewhere.

8. Haantjes structures for multiseparable systems in E_2

The Smorodinsky-Winternitz (SW) systems are a family of superintegrable systems defined in the Euclidean plane E_2, which were introduced first as quantum-mechanical systems in [16, 29, 57] and later studied from a group theoretical point of view in [44, 50]. They are all multiseparable in E_2 and admit three independent integrals of motion, expressed in terms of second degree polynomials in the momenta. Also, they are separable in at least two different orthogonal coordinate systems in their configuration space.
In [46], the SW systems were analyzed in the context of Nijenhuis geometry. Precisely, it was shown that a ωN structure can be associated with each of them; this can be achieved by renouncing to the standard notion of Lenard chain, and using a generalized version of it. In this section, we will show that a natural and more general framework for studying the geometry of SW systems is offered by the Haantjes geometry. As we will show, one can introduce Haantjes chains and construct two different $\omega\mathscr{H}$ structures for each of the SW systems. In turn, according to Theorem 2 and Corollary 2, these structures guarantee the existence of separation variables for the SW systems.

8.1. **Smorodinsky-Winternitz system SWI.** The Hamiltonian function is

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}a(x^2 + y^2) + \frac{c_1}{x^2} + \frac{c_2}{y^2}. \tag{40}$$

This system is separable in *cartesian*, *polar* and *elliptic* coordinates, and admits the integrals

$$H_2 = \frac{p_y^2}{2} + \frac{a}{2}y^2 + \frac{c_2}{y^2}, \tag{41}$$

$$H_3 = 2\left(c^2 p_x^2 + (xp_y - yp_x)^2 + ac^2x^2 + 2c_1\frac{y^2 + c^2}{x^2} + 2c_2\left(\frac{x}{y}\right)^2\right), \tag{42}$$

with a, c, c_1 and $c_2 \in \mathbb{R}$. We obtain a first Haantjes structure $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(SWI)})$ where $\boldsymbol{I}$ is the identity operator, and

$$\mathbf{K}_2^{(SWI)} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{43}$$

The algebra $\langle \boldsymbol{I}, \boldsymbol{K}_2^{(SWI)} \rangle$ admits the cartesian coordinates as DH coordinates: therefore they are obviously separation coordinates for the SWI system.

We can also obtain a second Haantjes structure $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(SWI)})$ with

$$\boldsymbol{K}_3^{(SWI)} = 4 \left[\begin{array}{cc|cc} y^2+c^2 & -xy & 0 & 0 \\ -xy & x^2 & 0 & 0 \\ \hline 0 & -(xp_y - yp_x) & y^2+c^2 & -xy \\ xp_y - yp_x & 0 & -xy & x^2 \end{array}\right]. \tag{44}$$

The algebra $\langle \boldsymbol{I}, \boldsymbol{K}_3^{(SWI)} \rangle$ diagonalizes in elliptic coordinates; therefore, they are separation coordinates. If $c = 0$, this algebra diagonalizes in polar coordinates.

8.2. **The Smorodinsky-Winternitz system SWII.** The Hamiltonian function reads:

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2) + a(4x^2 + y^2) + c_1 x + \frac{c_2}{y^2} \ , \tag{45}$$

with a, c_1, $c_2 \in \mathbb{R}$. The associated integrals are

$$H_2 = \frac{p_y^2}{2} + ay^2 + \frac{c_2}{y^2}, \tag{46}$$

$$H_3 = p_y(yp_x - xp_y) + 2axy^2 + \frac{c_1}{2}y^2 - 2c_2\frac{x}{y^2} \ . \tag{47}$$

We obtain the structures $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(SWII)})$ and $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(SWII)})$, where

$$\boldsymbol{K}_2^{(SWII)} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \tag{48}$$

and

$$\boldsymbol{K}_3^{(SWII)} = \left[\begin{array}{cc|cc} 0 & y & 0 & 0 \\ y & -2x & 0 & 0 \\ \hline 0 & p_y & 0 & y \\ -p_y & 0 & y & -2x \end{array}\right] \ . \tag{49}$$

The first structure admits the cartesian coordinates as DH coordinates, whereas the second one diagonalizes in parabolic coordinates.

8.3. **The Smorodinsky-Winternitz system SWIII.** The system is defined in polar coordinates $(r, \theta, p_r, p_\theta)$ by the Hamiltonian

$$H = H_1 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{\alpha}{r} + \frac{1}{r^2}\frac{\beta + \gamma\cos\theta}{\sin^2\theta}, \tag{50}$$

where α, β, $\gamma \in \mathbb{R}$. The integrals of motion are

$$H_2 = \frac{p_\theta^2}{2} + \frac{\beta + \gamma\cos\theta}{\sin^2\theta} \ , \tag{51}$$

$$H_3 = -p_\theta\left(\frac{p_\theta\cos\theta}{r} + p_r\sin\theta\right) - \alpha\cos\theta - \frac{\gamma + 2\beta\cos\theta + \gamma\cos^2\theta}{r\sin^2\theta} \ . \tag{52}$$

We can construct the Haantjes structures $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(SWIII)})$ and $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(SWIII)})$, with

$$\boldsymbol{K}_2^{(SWIII)} = r^2 \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \tag{53}$$

$$\boldsymbol{K}_3^{(SWIII)} = - \left[\begin{array}{cc|cc} 0 & r^2 \sin\theta & 0 & 0 \\ \sin\theta & 2r\cos\theta & 0 & 0 \\ \hline 0 & p_\theta \cos\theta & 0 & \sin\theta \\ -p_\theta \cos\theta & 0 & r^2 \sin\theta & 2r\cos\theta \end{array}\right]. \tag{54}$$

The algebra $\langle \boldsymbol{I}, \boldsymbol{K}_2^{(SWIII)} \rangle$ ensures the separability of the system in polar coordinates, which are indeed DH coordinates. Notice that if we re-write the operator $\boldsymbol{K}_2^{(SWIII)}$ in cartesian coordinates, it coincides (up to an irrelevant multiplicative factor) with the operator $\boldsymbol{K}_3^{(SWI)}$, for $c = 0$. Besides, the algebra $\langle \boldsymbol{I}, \boldsymbol{K}_3^{(SWIII)} \rangle$ diagonalizes in parabolic coordinates. If we write the expression of $\boldsymbol{K}_3^{(SWIII)}$ in cartesian coordinates, it converts into the form of the operator $\boldsymbol{K}_3^{(SWII)}$.

8.4. **The Smorodinsky-Winternitz system SWIV.** The Hamiltonian function of this system is

$$H = H_1 = \frac{1}{2} \frac{p_\xi^2 + p_\eta^2}{\xi^2 + \eta^2} + \frac{2\alpha + \beta\xi + \gamma\eta}{\xi^2 + \eta^2}, \tag{55}$$

where we have used the parabolic coordinates

$$x = \frac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta. \tag{56}$$

The corresponding integrals read

$$H_2 = \frac{\gamma\xi^3 + \xi^2(p_\xi p_\eta - \beta\eta) - \xi\eta(p_\xi^2 + p_\eta^2 + 4\alpha + \gamma\eta) + \eta^2(p_\xi p_\eta + \beta\eta)}{\xi^2 + \eta^2}, \tag{57}$$

$$H_3 = \frac{\xi^2 \left(p_\eta^2 + 2(\alpha + \gamma\eta) - 2\beta\xi\eta^2 - \eta^2(p_\xi^2 + 2\alpha)\right)}{\xi^2 + \eta^2}. \tag{58}$$

In the coordinates $(\xi, \eta, p_\xi, p_\eta)$ we get the Haantjes structures $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(SWIV)})$ and $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(SWIV)})$, with

$$\boldsymbol{K}_2^{(SWIV)} = \left[\begin{array}{cc|cc} -2\xi\eta & \xi^2 + \eta^2 & 0 & 0 \\ \xi^2 + \eta^2 & -2\xi\eta & 0 & 0 \\ \hline 0 & 0 & -2\xi\eta & \xi^2 + \eta^2 \\ 0 & 0 & \xi^2 + \eta^2 & -2\xi\eta \end{array}\right], \tag{59}$$

$$\boldsymbol{K}_3^{(SWIV)} = \left[\begin{array}{cc|cc} -2\eta^2 & 0 & 0 & 0 \\ 0 & 2\xi^2 & 0 & 0 \\ \hline 0 & 0 & -2\eta^2 & 0 \\ 0 & 0 & 0 & 2\xi^2 \end{array}\right]. \tag{60}$$

These two structures are related with two different parabolic coordinate systems, with distinct axes, which are separation coordinates for the systems SWIV. Notice that the operator $\boldsymbol{K}_3^{(SWIV)}$ in cartesian coordinates converts into the operator $-2\boldsymbol{K}_3^{(SWII)}$.

Remark 3. The form of the Haantjes operators presented above can be geometrically interpreted, in all cases, as the application of the generalized lifting procedure described in Section 7 to a suitable Haantjes operator on the configuration space E_2, related to a specific coordinate system in which they diagonalize. Its lifted version to the 4-dimensional phase space is again a Haantjes operator, at most linearly depending on the momenta.

9. Anisotropic oscillator with Rosochatius terms

We shall determine now the $\omega\mathscr{H}$ structures of an important physical model: the two-dimensional anisotropic oscillator with Rosochatius terms [43]. This system in the general case with n degrees of freedom has been studied in [41, 42], where its maximal superintegrability was established. In particular, in [41] the higher order (missing) integral was determined by means of a geometric approach based on the Marsden-Weinstein reduction procedure. The anisotropic oscillator on curved spaces has been introduced and studied in [2].

The Hamiltonian function of the system reads

$$H_{AO} = \frac{1}{2}\left(p_x^2 + p_y^2 + \nu^2(n_1x^2 + n_2y^2) + \frac{c_1}{x^2} + \frac{c_2}{y^2}\right) , \tag{61}$$

where $n_1, n_2 \in \mathbb{N}\backslash\{0\}$, $c_1, c_2, \nu \in \mathbb{R}$. The integrals of motion corresponding to the one-dimensional energies are

$$E_1 = \frac{1}{2}\left(p_x^2 + \nu^2 n_1 x^2 + \frac{c_1}{x^2}\right) , \tag{62}$$

$$E_2 = \frac{1}{2}\left(p_y^2 + \nu^2 n_2 y^2 + \frac{c_2}{y^2}\right) . \tag{63}$$

The system (61) is separable in Cartesian coordinates, and admits a Haantjes algebra $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_1^{(AO)}\rangle$, with

$$\boldsymbol{K}_1^{(AO)} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] . \tag{64}$$

However, the system admits a further integral, not related with separating coordinates. Let us introduce the complex quantities

$$\Delta_j = p_j^2 + \frac{c_j}{x_j^2} - \nu^2 n_j^2 x_j^2 - 2i\nu n_j p_j x_j , \qquad \Psi = \Delta_1^{n_2}\bar{\Delta}_2^{n_1} .$$

The additional, real integral is given by

$$\Re(\Psi) = \Re(\Delta_1^{n_2}\bar{\Delta}_2^{n_1}), \tag{65}$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$.

We can obtain a second Haantjes algebra in the following way. First, we shall consider the family of operators of the form

$$\boldsymbol{K} = \left[\begin{array}{cc|cc} m_d & 0 & 0 & 0 \\ m_{21} & m_d & 0 & 0 \\ \hline 0 & m_{32} & m_d & m_{21} \\ -m_{32} & 0 & 0 & m_d \end{array}\right], \tag{66}$$

which are Haantjes operators for any choice of the three arbitrary functions m_d, m_{21}, m_{32} of (x, y, p_x, p_y). This family generalizes (up to a transposition of coordinates with momenta) the form of the two Haantjes operators admitted by the Post-Winternitz system

$$H^{(PW)} = \frac{1}{2}(p_x^2 + p_y^2) + a\frac{x}{y^{2/3}}, \qquad a \in \mathbb{R}. \tag{67}$$

Notice that the operators of the form (66) generically are non-semisimple ones. We shall show that the class (66) also contains the explicit form of the specific Haantjes operator related to the integral of motion (65).
Now, let us impose the chain equation $(\boldsymbol{K}_2^{(AO)})^T \; dH_{AO} = d\Re(\Psi)$, where $\boldsymbol{K}_2^{(AO)}$ has the form (66). We find the unique solution

$$\begin{aligned} \big(\boldsymbol{K}_2^{(AO)}\big)^T \; dH &= \big(\nu^2(m_d n_1^2 x + m_{21} n_2^2 y) - m_{32} p_y - (\frac{m_d c_1}{x^3} + \frac{m_{21} c_2}{y^3})\big)dx \\ &+ (\nu^2 m_d n_2^2 y + m_{32} p_x - \frac{m_d c_2}{y^3})dy + (m_d p_x)dp_x \\ &+ (m_{21} p_x + m_d p_y)dp_y \;, \end{aligned}$$

where

$$m_d = \frac{1}{p_x}\frac{\partial \Re(\Psi)}{\partial p_x} \;; \tag{68}$$

$$m_{21} = \frac{1}{p_x}\left(-\frac{p_y}{p_x}\frac{\partial \Re(\Psi)}{\partial p_x} + \frac{\partial \Re(\Psi)}{\partial p_y}\right) \;; \tag{69}$$

$$m_{32} = \frac{1}{p_x}\left(-\nu^2\big(n_2^2 y + \frac{c_2}{y^3}\big)\frac{1}{p_x}\frac{\partial \Re(\Psi)}{\partial p_x} + \frac{\partial \Re(\Psi)}{\partial y}\right) \;. \tag{70}$$

The coefficients m_d, m_{21}, m_{32} are explicitly reported in the Appendix A. Thus, we have proved that the system (61) admits a second, non-semisimple $\omega\mathscr{H}$ structure, with $\mathscr{H}_2 = \langle \boldsymbol{I}, \boldsymbol{K}_2^{(AO)} \rangle$. It is an open problem to ascertain whether the system (61) admits other separating coordinates in phase space, apart the Cartesian ones.

10. $\omega\mathscr{H}$ structures for multiseparable systems in E_3

We shall study in detail three relevant examples of multiseparable systems in the Euclidean space E_3, in the context of Haantjes geometry. One of them is maximally superintegrable, the other two are minimally superintegrable ones [11]. For a different treatment, in the framework of the Killing-Stäckel theory, see [4], [9]. In the following analysis, $\mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{L}_z$ denote the components of the angular momentum in the cartesian frame.

10.1. **The Kepler system with a Rosochatius-type term.** We shall consider the Hamiltonian function

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2+y^2+z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} \ , \tag{71}$$

whose integrals of motion are

$$H_2 = \frac{1}{2}|\mathcal{L}|^2 + (x^2+y^2+z^2)\left(\frac{k_1}{x^2} + \frac{k_2}{y^2}\right), \tag{72}$$

$$H_3 = \frac{1}{2}\mathcal{L}_z^2 + (x^2+y^2)\left(\frac{k_1}{x^2} + \frac{k_2}{y^2}\right), \tag{73}$$

$$H_4 = \frac{1}{2}\mathcal{L}_y^2 + \frac{k_1 z^2}{x^2}, \tag{74}$$

$$H_5 = \mathcal{L}_x p_y - p_x \mathcal{L}_y - 2z\left(-\frac{k}{2\sqrt{x^2+y^2+z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2}\right). \tag{75}$$

These integrals form three families of functions in involution: $\{H_1, H_2, H_3\}$, $\{H_1, H_2, H_4\}$, $\{H_1, H_3, H_5\}$. The equations of the Haantjes chains associated are: $\boldsymbol{K}_i^T dH = dH_i$ $(i = 1, \ldots, 5)$, with the Haantjes operators

$$\boldsymbol{K}_2 = \left[\begin{array}{ccc|ccc} y^2+z^2 & -xy & -xz & 0 & 0 & 0 \\ -xy & x^2+z^2 & -yz & 0 & 0 & 0 \\ -xz & -yz & x^2+y^2 & 0 & 0 & 0 \\ \hline 0 & -(xp_y - yp_x) & zp_x - xp_z & y^2+z^2 & -xy & -xz \\ xp_y - yp_x & 0 & -(yp_z - zp_y) & -xy & x^2+z^2 & -yz \\ -(zp_x - xp_z) & yp_z - zp_y & 0 & -xz & -yz & x^2+y^2 \end{array}\right], \tag{76}$$

$$\boldsymbol{K}_3 = \left[\begin{array}{ccc|ccc} y^2 & -xy & 0 & 0 & 0 & 0 \\ -xy & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -(xp_y - yp_x) & 0 & y^2 & -xy & 0 \\ xp_y - yp_x & 0 & 0 & -xy & x^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{77}$$

$$\boldsymbol{K}_4 = \left[\begin{array}{ccc|ccc} z^2 & 0 & -xz & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -xz & 0 & x^2 & 0 & 0 & 0 \\ \hline 0 & 0 & zp_x - xp_z & z^2 & 0 & -xz \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -(zp_x - xp_z) & 0 & 0 & -xz & 0 & x^2 \end{array}\right] \tag{78}$$

and

$$\boldsymbol{K}_5 = \left[\begin{array}{ccc|ccc} -2z & 0 & x & 0 & 0 & 0 \\ 0 & -2z & y & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -p_x & -2z & 0 & x \\ 0 & 0 & -p_y & 0 & -2z & y \\ p_x & p_y & 0 & x & y & 0 \end{array}\right]. \tag{79}$$

We shall also take into account the integral

$$H_6 := \frac{1}{2}\mathcal{L}_x^2 + \frac{k_2 z^2}{y^2} \ ,$$

which is not functionally independent with respect to the other ones; however, it will play a relevant role in the construction of separation coordinates. Imposing the chain equation $\boldsymbol{K}_6^T dH = dH_6$, we get the further Haantjes operator

$$\boldsymbol{K}_6 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2 & -yz & 0 & 0 & 0 \\ 0 & -yz & y^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & zp_y - yp_z & 0 & z^2 & -yz \\ 0 & yp_z - zp_y & 0 & 0 & -yz & y^2 \end{array}\right] . \tag{80}$$

Let us discuss now the separation coordinates admitted by the system (71). First, we observe that there exist three Abelian semisimple Haantjes algebras related with the operators (76)-(79): $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$, $\mathscr{H}_2 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_4 \rangle$ and $\mathscr{H}_3 := \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_6 \rangle$. The algebra $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$, as was discussed in ref. [49], diagonalizes in the *spherical polar* coordinates with the z-axis as the polar axis; therefore, according to the previous discussion, they are DH coordinates and separation coordinates for the Kepler system (71). The algebra $\mathscr{H}_2$ diagonalizes in spherical polar coordinates with the y-axis as the polar axis; besides, the algebra $\mathscr{H}_3$ ensures the separability of the system in spherical polar coordinates with the x-axis as the polar axis.

We also have the algebra $\mathscr{H}_4 := \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$, which diagonalizes in the *rotational parabolic* coordinates.

Finally, the system (71) separates in *spherical conical* coordinates. Precisely, the algebra $\mathscr{H}_5 := \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_7 := a\boldsymbol{K}_6 + b\boldsymbol{K}_4 + c\boldsymbol{K}_3 \rangle$ diagonalizes in the spherical conical coordinates $u_1 \in \mathbb{R}$, $u_2 \in \mathbb{R}$, $u_3 > 0$, where $a \le u_1 \le b \le u_2 \le c$ and $a, b, c \in \mathbb{R}$.

10.1.1. *Generators of the cyclic Haantjes algebras.* In order to find a generator of a cyclic semisimple Abelian Haantjes algebra of rank n, it is sufficient to choose inside the algebra a Haantjes operator whose minimal polynomial is of degree n, as explained in subsection 3.6. As a consequence of this observation, we obtain that a simple (not unique) choice for the generator for the algebra $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$ reads

$$\boldsymbol{L}_1 = \boldsymbol{K}_2 + \boldsymbol{K}_3 \ . \tag{81}$$

We have

$$\boldsymbol{K}_2 = \alpha_1^{(1)} \boldsymbol{L}_1 + \alpha_2^{(1)} \boldsymbol{L}_1^2 \ , \qquad \alpha_1^{(1)} = \frac{3x^2 + 3y^2 + 2z^2}{2x^2 + 2y^2 + z^2} \ , \qquad \alpha_2^{(1)} = -\frac{1}{2x^2 + 2y^2 + z^2} \ , \tag{82}$$

and

$$\boldsymbol{K}_3 = \beta_1^{(1)} \boldsymbol{L}_1 + \beta_2^{(1)} \boldsymbol{L}_1^2 \ , \qquad \beta_1^{(1)} = -\frac{x^2 + y^2 + z^2}{2x^2 + 2y^2 + z^2}, \qquad \beta_2^{(1)} = \frac{1}{2x^2 + 2y^2 + z^2} \ . \tag{83}$$

A generator of the algebra $\mathscr{H}_2 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_4 \rangle$ is given by

(84)
$$\boldsymbol{L}_2 = \boldsymbol{K}_2 + \boldsymbol{K}_4 \ .$$

We have

(85)
$$\boldsymbol{K}_2 = \alpha_1^{(2)} \boldsymbol{L}_2 + \alpha_2^{(2)} \boldsymbol{L}_2^2 \,, \qquad \alpha_1^{(2)} = \frac{3x^2+2y^2+3z^2}{2x^2+y^2+2z^2} \,, \qquad \alpha_2^{(2)} = -\frac{1}{2x^2+y^2+2z^2} \,,$$

and

(86)
$$\boldsymbol{K}_4 = \beta_1^{(2)} \boldsymbol{L}_2 + \beta_2^{(2)} \boldsymbol{L}_2^2 \,, \qquad \beta_1^{(2)} = -\frac{x^2+y^2+z^2}{2x^2+y^2+2z^2} \,, \qquad \beta_2^{(2)} = \frac{1}{2x^2+y^2+2z^2} \ .$$

A generator of the algebra $\mathscr{H}_3 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_6 \rangle$ is $\boldsymbol{L}_3 = \boldsymbol{K}_2 + \boldsymbol{K}_6$, with

(87)
$$\boldsymbol{K}_2 = \alpha_1^{(3)} \boldsymbol{L}_3 + \alpha_2^{(3)} \boldsymbol{L}_3^2 \,, \qquad \alpha_1^{(3)} = \frac{2x^2+3y^2+3z^2}{x^2+2y^2+2z^2} \,, \qquad \alpha_2^{(3)} = -\frac{1}{x^2+2y^2+2z^2} \,,$$

and

(88)
$$\boldsymbol{K}_6 = \beta_1^{(3)} \boldsymbol{L}_3 + \beta_2^{(3)} \boldsymbol{L}_3^2 \,, \qquad \beta_1^{(3)} = -\frac{x^2+y^2+z^2}{x^2+2y^2+2z^2} \,, \qquad \beta_2^{(3)} = \frac{1}{x^2+2y^2+2z^2} \ .$$

A generator of the algebra $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$ is just $\boldsymbol{L}_4 = \boldsymbol{K}_5$, since the minimal polynomial of $\boldsymbol{K}_5$ is of degree 3. We have

$$\boldsymbol{K}_3 = \alpha_0^{(4)} \boldsymbol{I} + \alpha_1^{(4)} \boldsymbol{L}_4 + \alpha_2^{(4)} \boldsymbol{L}_4^2 \ ,$$

with

$$\alpha_0^{(4)} = x^2+y^2, \qquad \alpha_1^{(4)} = -2z, \qquad \alpha_3^{(4)} = -1 \ .$$

A generator of the algebra $\mathscr{H}_5 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_7 \rangle$ is simply $\boldsymbol{L}_5 = \boldsymbol{K}_7$, with

$$\boldsymbol{K}_2 = \alpha_1^{(5)} \boldsymbol{L}_5 + \alpha_2^{(5)} \boldsymbol{L}_5^2 \ ,$$

where

$$\alpha_1^{(5)} = \frac{(b+c)x^2+(a+c)y^2+(a+b)z^2}{bcx^2+acy^2+abz^2} \qquad \alpha_2^{(5)} = -\frac{1}{c(bx^2+ay^2)+abz^2} \ .$$

10.2. **A class of generalized Kepler systems.** We consider a family of deformations of the Kepler system which depends on an arbitrary function $F\left(\frac{y}{x}\right)$. The Hamiltonian function of this family has the form

(89)
$$H = H_1 = \frac{1}{2}(p_x^2+p_y^2+p_z^2) - \frac{k}{\sqrt{x^2+y^2+z^2}} + \frac{k_1 z}{\sqrt{x^2+y^2+z^2}(x^2+y^2)} + \frac{F\left(\frac{y}{x}\right)}{x^2+y^2}.$$

The associated integrals of motion are

(90)
$$H_2 = \frac{1}{2}|\mathcal{L}|^2 + \frac{k_1 z\sqrt{x^2+y^2+z^2} + (x^2+y^2+z^2)\ F\left(\frac{y}{x}\right)}{x^2+y^2},$$

(91)
$$H_3 = \frac{1}{2}\mathcal{L}_z^2 + F\left(\frac{y}{x}\right),$$

(92)
$$H_4 = \mathcal{L}_x p_y - p_x \mathcal{L}_y + \frac{kz}{\sqrt{x^2+y^2+z^2}} - \frac{k_1(x^2+y^2+2z^2)}{\sqrt{x^2+y^2+z^2}(x^2+y^2)} - 2z\frac{F\left(\frac{y}{x}\right)}{x^2+y^2}.$$

These integrals form two families of functions in involution: $\{H_1, H_2, H_3\}$ and $\{H_1, H_3, H_4\}$. The equations of the Haantjes chains are: $\boldsymbol{K}_2^T dH = dH_2$, $\boldsymbol{K}_3^T dH = dH_3$, $\boldsymbol{K}_5^T dH = dH_4$. The Haantjes operators associated are $\boldsymbol{K}_2$, eq. (76), $\boldsymbol{K}_3$, eq. (77) and $\boldsymbol{K}_5$, eq. (79). We obtain again the two Abelian, cyclic algebras $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$ and $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$, whose cyclic generators have been determined above. They provide us with the two coordinate systems admitted by the class of Hamiltonian systems (89), namely the spherical polar and the rotational parabolic ones.

10.3. **A class of anisotropic oscillators.** An interesting family of deformations of the anisotropic oscillator is given by the Hamiltonian function

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + k(x^2+y^2) + 4kz^2 + \frac{F\left(\frac{y}{x}\right)}{x^2+y^2}. \tag{93}$$

where again F is an arbitrary function of its argument. It admits the following integrals of motion:

$$H_2 = \frac{1}{2}p_z^2 + 4kz^2, \tag{94}$$

$$H_3 = \frac{1}{2}\mathcal{L}_z^2 + F\left(\frac{y}{x}\right), \tag{95}$$

$$H_4 = \mathcal{L}_x p_y - p_x \mathcal{L}_y + 2kz(x^2+y^2) - 2z\frac{F\left(\frac{y}{x}\right)}{x^2+y^2}. \tag{96}$$

Therefore, this class of systems is minimally superintegrable. These integrals form two families of functions in involution: $\{H_1, H_2, H_3\}$ and $\{H_1, H_3, H_4\}$. The equations for the Haantjes chains are: $\boldsymbol{K}_1^T dH = dH_2$, $\boldsymbol{K}_3^T dH = dH_3$, $\boldsymbol{K}_5^T dH = dH_4$. The corresponding Haantjes operators are:

$$\boldsymbol{K}_1 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{97}$$

$\boldsymbol{K}_3$, eq. (77) and $\boldsymbol{K}_5$, eq. (79). The family of systems (93) admits the two Abelian algebras $\mathscr{H}_6 := \langle \boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_3 \rangle$ and $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$. It separates in polar cylindrical and rotational parabolic coordinates.

The algebra $\mathscr{H}_6 = \langle \boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_3 \rangle$ admits the generator

$$\boldsymbol{L}_6 = 2\boldsymbol{K}_1 + \frac{1}{x^2+y^2}\boldsymbol{K}_3 \ . \tag{98}$$

We have

$$\boldsymbol{K}_1 = \alpha_1^{(6)}\boldsymbol{L}_6 + \alpha_2^{(6)}\boldsymbol{L}_6^2, \qquad \alpha_1^{(6)} = -\frac{1}{2}, \qquad \alpha_2^{(6)} = \frac{1}{2} \ , \tag{99}$$

and

$$\boldsymbol{K}_3 = \beta_1^{(6)} \boldsymbol{L}_6 + \beta_2^{(6)} \boldsymbol{L}_6^2, \qquad \beta_1^{(6)} = 2(x^2+y^2), \qquad \beta_2^{(6)} = -(x^2+y^2) \ . \tag{100}$$

11. Future perspectives

The problem of finding separation variables for integrable Hamiltonian systems is certainly among the most relevant ones of Classical Mechanics. The theory of $\omega\mathscr{H}$ manifolds offers a twofold contribution to this fundamental problem, being of both conceptual and applicative nature.

From a conceptual point of view, we have shown the relevance of $\omega\mathscr{H}$ structures for the construction of separation variables. Precisely, as stated in Theorem 2, if an integrable system admits a semisimple Abelian $\omega\mathscr{H}$ structure, then one can construct a privileged set of coordinates, the Darboux-Haantjes (DH) coordinates, which are separation coordinates for the Hamilton-Jacobi equation associated with the system. Besides, in these coordinates the symplectic form takes a Darboux form, and the operators of the Haantjes algebra diagonalize simultaneously.

Vice versa, any separable system admits a semisimple Abelian $\omega\mathscr{H}$ structure. In particular, an integrable Hamiltonian system admits at least as many separation systems as the number of its distinct $\omega\mathscr{H}$ structures.

From an applicative point of view, the $\omega\mathscr{H}$ structures represent a very flexible tool, which can be used either to construct in a consistent way separation coordinates, or to enhance the applicability of Nijenhuis geometry. In this regard, it is useful to clarify the relationship between $\omega\mathscr{H}$ and ωN structures. As we said, the approach *à la* Nihenjuis represents a powerful theoretical framework, allowing for the construction of separating variables via the eigenvalues of a suitable semisimple Nijenhuis operator associated with a given integrable system [13]. From this point of view, the Haantjes geometry can complement and integrate, for practical purposes, the Nijenhuis approach. Indeed, as we have shown in [49], a semisimple Haantjes algebra always admits a Haantjes cyclic generator; in addition, such a generator can be chosed to be a Nijenhuis one. Therefore, a possible strategy for finding separation variables is the following: given a Hamiltonian system, first we can construct in a natural way a $\omega\mathscr{H}$ manifold constructing its Haantjes chains (without the need for generalized ones, as is often necessary in Nijenhuis geometry [46]). Then, assuming that the algebra $\mathscr{H}$ is semisimple and Abelian, we can always select in $\mathscr{H}$ a Nijenhuis generator, providing us with separation variables (the so called Darboux-Nijenhuis (DN) coordinates). In the multiseparable case, this procedure can be repeated for all the structures allowed by the considered system.

Instead, finding a Nihenjuis operator without the help of the $\omega\mathscr{H}$ structure can be computationally cumbersome. Usually, one tries to construct Nijenhuis operators as reductions of bi-Hamiltonian structures associated to systems defined in a higher-dimensional space. Another possibility is to generate ωN structures as Yano liftings of Nijenhuis operators defined in the configuration space. However, these procedures are not generally applicable: only in some cases one can determine a system (easily tractable within the Nijenhuis approach), that by reduction gives the original system under study (and consequently, its ωN structure). Also, there are systems (as the Drach-Holt and the Post-Winternitz ones) whose Nijenhuis structure is not obtainable as a Yano complete lift.

Nevertheless, determining a $\omega\mathscr{H}$, as shown in this article (and in many examples of [47], [55]) is a direct procedure, computationally affordable, since the differential equations for the Haantjes chains are usually quite manageable.

In summary, given an integrable Hamiltonian system, separating coordinates can be found in two different ways: as DH coordinates associated with its $\omega\mathscr{H}$ semisimple structures or, equivalently, via the eigenvalues of the Nijenhuis operator generating the same $\omega\mathscr{H}$ structures (DN coordinates).
Interestingly enough, $\omega\mathscr{H}$ manifolds appear to be a quite ubiquitous geometrical structure in the class of superintegrable systems: indeed, they exist also in cases where orthogonal separating variables are not allowed in T^*Q and are realized by *non-semisimple* Haantjes operators. This is the case for the anisotropic harmonic oscillator described above, and for the Post-Winternitz system [40], which admits a *non-Abelian* Haantjes algebra, possessing two Abelian subalgebras of non-semisimple Haantjes operators.
We mention that a generalization of the notion of $\omega\mathscr{H}$ manifolds is that of $P\mathscr{H}$ manifolds, introduced in [53], where P is a Poisson bivector compatible with the Haantjes algebra $\mathscr{H}$. When P is invertible, a $P\mathscr{H}$ structure reduces to a $\omega\mathscr{H}$ one.

A fundamental open problem, in both the Haantjes and the Nijenhuis scenarios, is to deepen into the geometry of non-semisimple structures and to ascertain their possible relevance from the point of view of the general problem of SoV.
Another interesting open problem is the geometrical interpretation of the new class of generalized tensors introduced in [48] from the perspective of classical Hamiltonian systems. They represent an infinite "tower" of new tensors generalizing the Nijenhuis and Haantjes's ones. A crucial result, proved in [48], states that if the generalized torsion of an operator field vanishes, then its eigendistributions are mutually integrable. The ultimate implications of this property in the context of the theory of Hamiltonian integrable systems is presently under investigation.

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Appendix A. On the Haantjes structure of the system (61)

As an illustrative example of the form of the integral of motion $\Re(\Psi)$, eq. (65), admitted by the system (61), we shall consider the case $n_1 = 1, n_2 = 3$. The integral (65) reads explicitly

(101)

$$\Re(\Psi) = 9\,\nu^8 x^6 y^2 + 27\,\frac{{c_1}^2 \nu^4 y^2}{x^2} - 27\,c_1\,x^2 \nu^6 y^2 + \frac{{c_1}^3 c_2}{x^6 y^2}$$

$$
\begin{aligned}
&- 9\,\frac{{c_1}^3\nu^2y^2}{x^6} - \frac{\nu^6x^6c_2}{y^2} - 3\,\frac{{c_1}^2\nu^2c_2}{x^2y^2} + 3\,\frac{c_1\,x^2\nu^4c_2}{y^2}\\
&+ \left(162\,c_1\,\nu^4y^2 - 135\,\nu^6x^4y^2 + 3\,\frac{{c_1}^2c_2}{x^4y^2} - 27\,\frac{{c_1}^2\nu^2y^2}{x^4} - 18\,\frac{c_1\,\nu^2c_2}{y^2} + 15\,\frac{\nu^4x^4c_2}{y^2}\right){p_x}^2\\
&+ \left(\frac{{c_1}^3}{x^6} - \nu^6x^6 - 3\,\frac{{c_1}^2\nu^2}{x^2} + 3\,c_1\,x^2\nu^4\right){p_y}^2 + \left(3\,\frac{{c_1}^2}{x^4} - 18\,c_1\,\nu^2 + 15\,\nu^4x^4\right){p_x}^2{p_y}^2\\
&+ {p_x}^6{p_y}^2 + \left(36\,\frac{{c_1}^2\nu^2y}{x^3} - 72\,c_1\,x\nu^4y + 36\,\nu^6x^5y\right)p_y\,p_x\\
&+ \left(135\,\nu^4x^2y^2 + 3\,\frac{c_1\,c_2}{x^2y^2} - 27\,\frac{c_1\,\nu^2y^2}{x^2} - 15\,\frac{\nu^2x^2c_2}{y^2}\right){p_x}^4\\
&+ \left(\frac{c_2}{y^2} - 9\,\nu^2y^2\right){p_x}^6 + \left(72\,\frac{c_1\,\nu^2y}{x} - 120\,\nu^4x^3y\right)p_y\,{p_x}^3 + \left(3\,\frac{c_1}{x^2} - 15\,\nu^2x^2\right){p_x}^4{p_y}^2\\
&+ 36\,\nu^2{p_x}^5xp_y\,y
\end{aligned}
$$

and the elements of the Haantjes operator are

(102)

$$
\begin{aligned}
m_d =& - 270\,\nu^6x^4y^2 + 324\,c_1\,\nu^4y^2 + 30\,\frac{\nu^4x^4c_2}{y^2} - 54\,\frac{{c_1}^2\nu^2y^2}{x^4} - 36\,\frac{c_1\,\nu^2c_2}{y^2}\\
&+ 6\,\frac{{c_1}^2c_2}{x^4y^2} + \left(540\,\nu^4x^2y^2 - 108\,\frac{c_1\,\nu^2y^2}{x^2} - 60\,\frac{\nu^2x^2c_2}{y^2} + 12\,\frac{c_1\,c_2}{x^2y^2}\right){p_x}^2\\
&+ \left(30\,\nu^4x^4 - 36\,c_1\,\nu^2 + 6\,\frac{{c_1}^2}{x^4}\right){p_y}^2 + \left(-60\,\nu^2x^2 + 12\,\frac{c_1}{x^2}\right){p_x}^2{p_y}^2\\
&+ \frac{p_y}{p_x}\left(36\,\frac{\nu^2{c_1}^2y}{x^3} - 72\,c_1\,x\nu^4y + 36\,\nu^6x^5y\right) + 180\,{p_x}^3xy\nu^2p_y\\
&+ \left(-360\,\nu^4x^3y + 216\,\frac{c_1\,\nu^2y}{x}\right)p_x\,p_y + \left(-54\,\nu^2y^2 + 6\,\frac{c_2}{y^2}\right){p_x}^4 + 6\,{p_x}^4{p_y}^2,
\end{aligned}
$$

(103)

$$
\begin{aligned}
m_{21} =& 36\,\frac{\nu^2{c_1}^2y}{x^3} - 72\,c_1\,x\nu^4y + 36\,\nu^6x^5y + \left(72\,\frac{c_1\,\nu^2y}{x} - 120\,\nu^4x^3y\right){p_x}^2\\
&+ \left(360\,\nu^4x^3y - 216\,\frac{c_1\,\nu^2y}{x}\right){p_y}^2\\
&+ \frac{p_y}{p_x}\Bigg(36\,\frac{c_1\,\nu^2c_2}{y^2} - 2\,\nu^6x^6 + 270\,\nu^6x^4y^2 + 2\,\frac{{c_1}^3}{x^6} + 6\,c_1\,x^2\nu^4\\
&\qquad -324\,c_1\,\nu^4y^2 - 30\,\frac{\nu^4x^4c_2}{y^2} - 6\,\frac{{c_1}^2\nu^2}{x^2} + 54\,\frac{{c_1}^2\nu^2y^2}{x^4} - 6\,\frac{{c_1}^2c_2}{x^4y^2}\Bigg)\\
&+ \left(-30\,\nu^2x^2 + 54\,\nu^2y^2 + 6\,\frac{c_1}{x^2} - 6\,\frac{c_2}{y^2}\right)p_y\,{p_x}^3 + \left(60\,\nu^2x^2 - 12\,\frac{c_1}{x^2}\right){p_y}^3p_x\\
&- 6\,{p_y}^3{p_x}^3 + 2\,{p_x}^5p_y - 180\,{p_x}^2yx\nu^2{p_y}^2\\
&+ \Bigg(30\,\nu^4x^4 - 540\,\nu^4x^2y^2 + 6\,\frac{{c_1}^2}{x^4} - 36\,c_1\,\nu^2\\
&\qquad +108\,\frac{c_1\,\nu^2y^2}{x^2} + 60\,\frac{\nu^2x^2c_2}{y^2} - 12\,\frac{c_1\,c_2}{x^2y^2}\Bigg)p_x\,p_y
\end{aligned}
$$

$$+\frac{{p_y}^3}{p_x}\left(-30\,\nu^4x^4-6\,\frac{{c_1}^2}{x^4}+36\,c_1\,\nu^2\right)+36\,{p_x}^4yx\nu^2$$
$$+\frac{{p_y}^2}{{p_x}^2}\left(-36\,\nu^6x^5y+72\,c_1\,x\nu^4y-36\,\frac{\nu^2{c_1}^2y}{x^3}\right),$$

(104)

$$\begin{aligned}
m_{32}=&\left(324\,y\nu^4c_1+972\,\frac{y^3\nu^4c_1}{x^2}-30\,\frac{\nu^4x^4c_2}{y^3}\right.\\
&\left.+1080\,\frac{\nu^4x^2c_2}{y}-54\,\frac{y\nu^2{c_1}^2}{x^4}-60\,\frac{\nu^2x^2{c_2}^2}{y^5}-6\,\frac{{c_1}^2c_2}{x^4y^3}\right)p_x\\
&+\left(12\,\frac{c_1\,{c_2}^2}{x^2y^5}+36\,\frac{c_1\,\nu^2c_2}{y^3}-216\,\frac{c_1\,\nu^2c_2}{x^2y}-270\,x^4y\nu^6-4860\,x^2y^3\nu^6\right)p_x\\
&+\left(216\,\frac{c_1\,\nu^2c_2}{xy^2}+36\,\nu^6x^5+3240\,x^3y^2\nu^6\right.\\
&\left.+36\,\frac{\nu^2{c_1}^2}{x^3}-72\,c_1\,x\nu^4-1944\,\frac{c_1\,\nu^4y^2}{x}-360\,\frac{\nu^4x^3c_2}{y^2}\right)p_y\\
&+\left(-54\,\frac{c_1\,\nu^2y}{x^2}+30\,\frac{\nu^2x^2c_2}{y^3}-108\,\frac{c_2\,\nu^2}{y}\right.\\
&\left.-6\,\frac{c_1\,c_2}{x^2y^3}+270\,\nu^4x^2y+486\,y^3\nu^4+6\,\frac{{c_2}^2}{y^5}\right){p_x}^3\\
&+\frac{1}{p_x}\left(-54\,c_1\,x^2\nu^6y-2916\,y^3\nu^6c_1+2\,\frac{\nu^6x^6c_2}{y^3}-540\,\frac{\nu^6x^4c_2}{y}\right.\\
&\left.+54\,\frac{{c_1}^2\nu^4y}{x^2}+486\,\frac{y^3\nu^4{c_1}^2}{x^4}+30\,\frac{\nu^4x^4{c_2}^2}{y^5}-18\,\frac{{c_1}^3\nu^2y}{x^6}\right)\\
&+\frac{1}{p_x}\left(-2\,\frac{{c_1}^3c_2}{x^6y^3}+6\,\frac{{c_1}^2{c_2}^2}{x^4y^5}-6\,\frac{c_1\,x^2\nu^4c_2}{y^3}+648\,\frac{c_1\,\nu^4c_2}{y}+6\,\frac{\nu^2{c_1}^2c_2}{x^2y^3}\right.\\
&\left.-108\,\frac{\nu^2{c_1}^2c_2}{x^4y}-36\,\frac{c_1\,\nu^2{c_2}^2}{y^5}+18\,\nu^8x^6y+2430\,x^4y^3\nu^8\right)\\
&+\left(-18\,y\nu^2-2\,\frac{c_2}{y^3}\right){p_x}^5+\left(-54\,y\nu^2+6\,\frac{c_2}{y^3}\right){p_y}^2{p_x}^3\\
&+\left(540\,\nu^4x^2y-60\,\frac{\nu^2x^2c_2}{y^3}+12\,\frac{c_1\,c_2}{x^2y^3}-108\,\frac{c_1\,\nu^2y}{x^2}\right)p_x\,{p_y}^2\\
&+\left(-120\,\nu^4x^3-1620\,\nu^4xy^2+72\,\frac{c_1\,\nu^2}{x}+180\,\frac{\nu^2xc_2}{y^2}\right){p_x}^2p_y\\
&+\frac{p_y}{{p_x}^2}\left(-72\,\frac{c_1\,x\nu^4c_2}{y^2}+36\,\frac{\nu^2{c_1}^2c_2}{x^3y^2}-324\,\nu^8x^5y^2\right.\\
&\left.-324\,\frac{{c_1}^2\nu^4y^2}{x^3}+648\,c_1\,x\nu^6y^2+36\,\frac{\nu^6x^5c_2}{y^2}\right)\\
&+\frac{{p_y}^2}{px}\left(-36\,\frac{c_1\,\nu^2c_2}{y^3}-270\,x^4y\nu^6-54\,\frac{y\nu^2{c_1}^2}{x^4}\right.\\
&\left.+6\,\frac{{c_1}^2c_2}{x^4y^3}+324\,y\nu^4c_1+30\,\frac{\nu^4x^4c_2}{y^3}\right)+36\,{p_x}^4x\nu^2p_y.
\end{aligned}$$

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Departamento de Física Teórica, Facultad de Ciencias Físicas, Universidad Complutense de Madrid, 28040 – Madrid, Spain, and Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain
Email address: danreyes@ucm.es, daniel.reyes@icmat.es

Departamento de Física Teórica, Facultad de Ciencias Físicas, Universidad Complutense de Madrid, 28040 – Madrid, Spain, and Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain
Email address: piergiulio.tempesta@icmat.es, ptempest@ucm.es

Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, piaz.le Europa 1, I–34127 Trieste, Italy.
Email address: tondo@units.it