# ON THE EQUILATERAL PENTAGONAL CENTRAL CONFIGURATIONS 

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#### Abstract

An equilateral pentagon is a polygon in the plane with five sides of equal length. In this paper we classify the central configurations of the 5 -body problem having the five bodies at the vertices of an equilateral pentagon with an axis of symmetry. We prove that there are two unique classes of such equilateral pentagons providing central configurations, one concave equilateral pentagon and one convex equilateral pentagon, the regular one. A key point of our proof is the use of rational parameterizations to transform the corresponding equations, which involve square roots, into polynomial equations.


## 1. Introduction and statement of the result

The Newtonian planar 5 -body problem describes the dynamics of five point particles of positive masses $m_{i}$ at positions $\mathbf{q}_{i} \in \mathbb{R}^{2}$ moving according to the Newton's laws under their mutual gravitational forces. The equations of motion of this 5 -body problem are

$$
m_{i} \ddot{\mathbf{q}}_{i}=-\sum_{j=1, j \neq i}^{5} G m_{i} m_{j} \frac{\mathbf{q}_{i}-\mathbf{q}_{j}}{r_{i j}^{3}}, \quad 1 \leq i \leq 5
$$

where $r_{i j}=\left|\mathbf{q}_{i}-\mathbf{q}_{j}\right|$ is the mutual distances between the masses $m_{i}$ and $m_{j}$, and $G$ is the gravitational constant. We take conveniently the time unit so that $G=1$.

The configuration space is defined by

$$
\mathcal{E}=\left\{\mathbf{q}=\left(\mathbf{q}_{1}, \ldots, \mathbf{q}_{5}\right) \in\left(\mathbb{R}^{2}\right)^{5}: \mathbf{q}_{i} \neq \mathbf{q}_{j}, \quad i \neq j\right\}
$$

The configuration $\mathbf{q}=\left(\mathbf{q}_{1}, \ldots, \mathbf{q}_{5}\right)$ is called central if the position vector of each body with respect to the center of mass is proportional to the corresponding acceleration vector. In other words, if there exists a positive constant $\lambda$ such that

$$
\ddot{\mathbf{q}}_{i}=\lambda\left(\mathbf{q}_{i}-\mathbf{c}_{m}\right), \quad i=1, \ldots, 5,
$$

where $\mathbf{c}_{m}=\left(m_{1} \mathbf{q}_{1}+\cdots+m_{5} \mathbf{q}_{5}\right) / M$ and $M=m_{1}+\cdots+m_{5}$, being $\mathbf{c}_{m}$ and $M$ the center of mass of the five bodies and the total mass, respectively. Hence a given configuration $\left(\mathbf{q}_{1}, \ldots, \mathbf{q}_{5}\right) \in \mathcal{E}$ of the 5 -body problem with positive masses $m_{1}, \ldots, m_{5}$, is central if there exists a $\lambda$ such that $\left(\lambda, \mathbf{q}_{1}, \ldots, \mathbf{q}_{5}\right)$ is a solution of the system

$$
\begin{equation*}
\sum_{j=1, j \neq i}^{5} m_{j} \frac{\mathbf{q}_{i}-\mathbf{q}_{j}}{r_{i j}^{3}}=\lambda\left(\mathbf{q}_{i}-\mathbf{c}_{m}\right), \quad 1 \leq i \leq 5 \tag{1}
\end{equation*}
$$

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A central configuration is convex if no body belongs to the convex hull of the other four bodies; otherwise it is called concave. A planar central configuration is called a relative equilibrium when they become equilibrium solutions in a rotating coordinate system [17].

Since equations (1) are invariant under rotations, translations and dilations, when we consider the number of central configurations, this will be restricted to count the classes of central configurations modulo these mentioned transformations.

The central configurations are of special importance in Celestial Mechanics for several reasons. For instance, central configurations are the initial conditions for the homographic orbits of the $n$-body problem. Central configurations play an important role in the description of the topology of the integral manifolds in the $n$-body problem. Moreover, in the planar case the central configurations are initial positions for periodic solutions. For more information on this subject, recent advances and open questions, the reader is addressed to [18] and references therein.

In this paper we will investigate some central configurations of the 5 -body problem, for which there are few known results. The first results concerning this issue take us back decades ago to the work due to Williams [23], who settled necessary and sufficient conditions for any plane central configuration of five bodies. In what follows we offer a non-exhaustive list of some interesting works concerning this topic, which have been published more recently.

Albouy and Kaloshin [3] proved that for a choice of five positive masses in the complement of a codimension-two algebraic variety in the mass space, there are only a finite number of equivalence classes of central configurations of the Newtonian 5-body problem. In [4] Chen and Hsiao provided necessary conditions for strictly convex central configurations of the planar 5-body problem.

In the last times the interest in stacked central configurations has grown a lot, that is, central configurations in which some subset of three or more masses also forms a central configuration. This concept was introduced by Hampton [12], who was arguably the first to find stacked central configurations in the 5 -body problem, where two bodies can be removed and the remaining three bodies are already in a central configuration. After, several papers have been published showing the existence of other stacked central configurations in the planar 5 -body problem; see, among others, [5], $[6,8,11,16,14]$.

Other studies have focused on restricting the problem to a particular shape, in [15] it was proved that the unique co-circular central configuration in the planar 5 -body problem is the regular 5-gon with equal masses, while in [14] was proved the existence of three families of planar central configurations where three bodies are at the vertices of an equilateral triangle and the other two bodies are on a perpendicular bisector. Later on in [21] was studied the central configuration in a symmetric 5 -body problem with three masses on an axis of symmetry and two other masses outside this axis, placed in symmetric positions. A complete classification of the isolated central configurations of the planar 5-body problem with equal masses was given in [13]. Recently in [7] were studied the central configurations of the planar 5-body problem having four bodies at the vertices of a rhombus.


Figure 1. Pentagonal configurations: a convex one in the left hand picture and two different type of concave ones

For the reader's convenience, we summarize here some basic facts about the pentagon, which is a polygon with five sides and five angles. It is said that it is convex if all its vertices are pointing outwards, otherwise it is concave; see Figure 1. A pentagon with five sides of equal length is named equilateral. Moreover, a pentagon is called regular when all the sides are equal in length, and five angles are of equal measures. If the pentagon does not have equal side length and angle measure, then it is called irregular. The regular pentagon is unique, up to similarity transformations, because it is equilateral and its five angles are equal.

The goal in this paper is to characterize the equilateral pentagonal central configurations with an axis of symmetry for the planar 5-body problem whose five positive masses are at the vertices of an equilateral pentagon.

(a) Convex

(b) Concave

Figure 2. The two equilateral pentagonal central configurations with an axis of symmetry of the 5 -body problem.

Then our main result is the following one, which will be proved in the next section.
Theorem 1. There are two classes of equilateral pentagonal central configuration having an axis of symmetry for the 5-body problem.
(a) The convex regular pentagon with equal five masses, see Figure 2(a).
(b) The equilateral concave pentagon with the masses normalized, i.e. $\sum_{i=1}^{5} m_{i}=$ 1, equal to $m_{1}=m_{2} \approx 0.0922539749, m_{3}=m_{4} \approx 0.3860948766$ and $m_{5} \approx$ 0.04330242730. In Figure 2(b) we show a representative of its class where the bodies of masses $m_{1}$ and $m_{2}$ are fixed at $(0,1 / 2)$ and $(0,-1 / 2)$, respectively. The other bodies of masses $m_{3}, m_{4}$ and $m_{5}$ are located at $\left(x_{3}, y_{3}\right),\left(-x_{3}, y_{3}\right)$ and
$\left(0, y_{5}\right)$, respectively, where $x_{3} \approx 0.5402091568, y_{3} \approx 0.9991912848$ and $y_{5} \approx$ 0.1576604970 . See Remark 2 for some comments of how these values are obtained.

Remark 2. From the proof of Theorem 1 and the results of Appendix B we obtain a value $t=t^{*} \approx 0.7332148086$ which is the smallest root of the quadratic polynomial $t^{2}-u^{*} t+1=0$, where $u=u^{*} \approx 2.0970716051$ is the unique root in the interval [205/100, 210/100] of the polynomial of degree 60 with integer coefficients, $R_{60}(u)$, given in Appendix B. This polynomial is constructed from a reciprocal polynomial of degree 120 and also with integer coefficients that appears in (10). Then $y_{5}=\left(1-\left(t^{*}\right)^{2}\right) /\left(4 t^{*}\right)$ and all the other values in the theorem, $x_{3}, y_{3}, m_{j}, j=1,2, \ldots, 5$, can be obtained from this $t^{*}$ by elementary computations: sums, subtractions, multiplications, divisions and square roots. Recall that, although we do not know the exact value of $u^{*}$, the classical use of Sturm sequences allows to obtain explicit intervals, with rational endpoints, containing $u^{*}$ and with arbitrarily small length.

At this stage the reader should be warned that there is a previous work by Perko and Walter [19], who showed that $n$ equal masses at the vertices of a regular polygon, for $n \geq 4$, forms a central configuration if and only if the masses are equal. Therefore it was known that the regular pentagon with equal masses is a central configuration for the 5 -body problem, but it was unknown that it is the unique equilateral convex pentagonal central configuration with an axis of symmetry.

One of the key points of our approach is the use of rational parameterizations to eliminate some of the square roots that appear in the equations governing the central configurations, converting in this way these equations into polynomial ones. Then these equations can be treated analytically by using some classical tools, like for instance the Sturm sequences or the computation of resultants.

## 2. Preliminaries

Central configurations are invariant under composition of translations, rotations, and scaling through its center of mass, hence without loss of generality we can assume that the position of the masses $m_{i}>0$ for $i=1 \ldots, 5$ at the vertices of an equilateral pentagon with an axis of symmetry are $p_{k}=\left(x_{k}, y_{k}\right)$ for $k=1,2,3,4,5$, where $y_{1}=0$, $x_{2}=-x_{1}, y_{2}=0, x_{4}=-x_{3}, y_{4}=y_{3}$ and $x_{5}=0$. Note that we can assume that $x_{1}>0$, $x_{3}>0, y_{3}>0, y_{5}>0$ and $y_{3} \neq y_{5}$, because we want that the points $\left(x_{i}, y_{i}\right)$ be the vertices of a pentagon.

Next we will obtain the coordinates for the equilateral pentagon vertices. By a suitable scaling we may assume that $r_{12}=1$, so $x_{1}=1 / 2$. Now we substitute this value into the equation $r_{13}=1$, obtaining that $y_{3}^{2}=\left(3+4 x_{3}-4 x_{3}^{2}\right) / 4$. These values replaced in the equation $r_{35}=1$ provides $x_{3}$ in terms of $y_{5}$,

$$
\begin{equation*}
x_{3}=\Psi^{ \pm}\left(y_{5}\right)=\frac{1}{4} \pm \frac{y_{5}}{2} \Phi\left(y_{5}\right), \quad \text { where } \quad \Phi(y)=\sqrt{\frac{15-4 y^{2}}{1+4 y^{2}}} . \tag{2}
\end{equation*}
$$

Since we are interested in central configurations, with an axis of symmetry, modulus rotations and homothetic transformations, without loss of generality, we have the next result.

Proposition 3. Suppose that $\mathbf{q}_{1}, \ldots, \mathbf{q}_{5}$ form an equilateral pentagon, with $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$ fixed on the $x$-axis and with the $y$-axis as an axis of symmetry. Then

$$
\left(x_{1}, y_{1}\right)=(0,1 / 2), \quad\left(x_{2}, y_{2}\right)=(-1 / 2,0), \quad\left(x_{3}, y_{3}\right), \quad\left(-x_{3}, y_{3}\right) \quad \text { and } \quad\left(0, y_{5}\right)
$$

where
(a) either $x_{3}=\Psi^{+}\left(y_{5}\right)=\frac{1}{4}+\frac{y_{5}}{2} \Phi\left(y_{5}\right), \quad y_{3}=\frac{y_{5}}{2}+\frac{1}{4} \Phi\left(y_{5}\right), \quad$ and $\quad y_{5} \in(0, \sqrt{15} / 2)$,
(b) or $x_{3}=\Psi^{-}\left(y_{5}\right)=\frac{1}{4}-\frac{y_{5}}{2} \Phi\left(y_{5}\right), \quad y_{3}=\frac{y_{5}}{2}-\frac{1}{4} \Phi\left(y_{5}\right), \quad$ and $\quad y_{5} \in(1+$ $\sqrt{3} / 2, \sqrt{15} / 2)$.

The geometrically distinct equilateral pentagon are shown in Figure 3. It follows from the case (a) of Proposition 3 that the equilateral pentagon is concave when $y_{5} \in$ $(0, \sqrt{3} / 2)$, and convex if $y_{5} \in(\sqrt{3} / 2, \sqrt{15} / 2)$. While in case (b) of Proposition 3 the equilateral pentagon is always concave.


$$
A=\sqrt{3} / 2 \approx 0.866, B=1+\sqrt{3} / 2 \approx 1.866, C=\sqrt{15} / 2 \approx 1.936 \text {. }
$$

Figure 3. Gallery of possible pentagon equilateral configurations, according whether $x_{3}=\Psi^{+}\left(y_{5}\right)$ or $x_{3}=\Psi^{-}\left(y_{5}\right)$ and the value of $y_{5}$, together with the boundary cases, that are no more pentagons.

In the next section we will see that the only values of $y_{5}$ that give rise to central configurations will be $x_{3}=\Psi^{+}\left(y_{5}\right)$, where $y_{5}=(\sqrt{5+2 \sqrt{5}}) / 2 \approx 1.539$ is associated to the regular pentagon, while $y_{5} \approx 0.1576605$ gives a concave central configuration.

## 3. Proof of Theorem 1

By a suitable scaling we may assume that $m_{1}+m_{2}+m_{3}+m_{4}+m_{5}=1$. Then the center of mass of the five bodies is

$$
\mathbf{c}_{m}=\left(x_{m}, y_{m}\right)=\left(\frac{m_{1}-m_{2}+2\left(m_{3}-m_{4}\right) x_{3}}{2}, \frac{\left(m_{3}+m_{4}\right) \sqrt{3+4 x_{3}-4 x_{3}^{2}}+2 m_{5} y_{5}}{2}\right)
$$

Since we are studying equilateral pentagons we can assume that $r_{12}=r_{13}=r_{35}=$ $r_{45}=r_{24}=1$. Then the others mutual distances are

$$
\begin{equation*}
r_{14}=r_{2,3}=\sqrt{1+2 x_{3}}, \quad r_{1,5}=r_{2,5}=\sqrt{y_{5}^{2}+1 / 4}, \quad r_{3,4}=2 x_{3} \tag{3}
\end{equation*}
$$

From (1) we obtain the ten equations for the central configurations of the 5-body problem in the plane:

$$
\begin{align*}
& e_{j}=\sum_{j=1, j \neq i}^{5} \frac{m_{j}\left(x_{i}-x_{j}\right)}{r_{i, j}^{3}}-\lambda\left(x_{j}-x_{m}\right)=0, \quad 1 \leq j \leq 5,  \tag{4}\\
& e_{j+5}=\sum_{j=1, j \neq i}^{5} \frac{m_{j}\left(y_{i}-y_{j}\right)}{r_{i, j}^{3}}-\lambda\left(y_{j}-y_{m}\right)=0, \quad 1 \leq j \leq 5 .
\end{align*}
$$

Substituting into (4) the values and expressions of the mutual distances and taking $m_{5}=1-m_{1}-m_{2}-m_{3}-m_{4}$, it is seen that

$$
\begin{equation*}
e_{8}-e_{9}=-\frac{\left(m_{1}-m_{2}\right) \sqrt{3+4 x_{3}-4 x_{3}^{2}}\left(-1+\sqrt{1+2 x_{3}}+2 x_{3} \sqrt{1+2 x_{3}}\right)}{2\left(1+2 x_{3}\right)^{3 / 2}}=0 . \tag{5}
\end{equation*}
$$

A straightforward computation shows that $\sqrt{3+4 x_{3}-4 x_{3}^{2}}\left(-1+\sqrt{1+2 x_{3}}+2 x_{3} \sqrt{1+2 x_{3}}\right)=$ 0 for $x_{3}=-1 / 2,0,3 / 2$. However, any of these values is good, because $x_{3}>0$ and $x_{3}=3 / 2$ implies that $r_{1,3}=\sqrt{1+y_{3}^{2}}$, but this is impossible because we have assumed that $y_{3} \neq 0$ and $r_{1,3}=1$. So $m_{2}=m_{1}$.

Since $e_{3}+e_{4}=\left(m_{3}-m_{4}\right)\left(1+8 \lambda x_{3}^{3}\right) /\left(4 x_{3}^{2}\right)=0$. It follows that either $m_{4}=m_{3}$, or $\lambda=-1 /\left(8 x_{3}^{3}\right)$. In this last case we obtain

$$
e_{6}-e_{7}=\frac{\left(m_{3}-m_{4}\right) \sqrt{3+4 x_{3}-4 x_{3}^{2}}\left(-1+\sqrt{1+2 x_{3}}+2 x_{3} \sqrt{1+2 x_{3}}\right)}{2\left(1+2 x_{3}\right)^{3 / 2}}=0 .
$$

Hence as in (5) we have that $m_{4}=m_{3}$. Therefore we do not need to consider $\lambda=$ $-1 /\left(8 x_{3}^{3}\right)$, and in what follows we consider that $m_{4}=m_{3}$, such that, $e_{6}-e_{7}=0$.

In summary, we have that $e_{1}+e_{2}=0, e_{5}=0$ and $e_{6}-e_{7}=0$. Hence we conclude that from the ten equations (4) only $e_{1}, e_{3}, e_{6}, e_{8}, e_{10}$ remain independent. These equations
are

$$
\begin{aligned}
f_{1}= & -\frac{\lambda}{2}-\frac{4}{E_{3}}+m_{1}\left(-1+\frac{8}{E_{3}}\right)+m_{3}\left(-\frac{1}{2}+x_{3}-\frac{1}{2 E_{1}}+\frac{8}{E_{3}}\right) \\
f_{2}= & -(1+\lambda) x_{3}+\frac{1}{4} m_{3}\left(8-\frac{1}{x_{3}^{3}}\right) x_{3}+m_{1}\left(\frac{1}{2}+x_{3}-\frac{1}{2 E_{1}}\right) \\
f_{3}= & y_{5}\left(\lambda+\frac{8}{E_{3}}\right)+m_{1}\left(-2 \lambda y_{5}-\frac{16 y_{5}}{E_{3}}\right) \\
& +m_{3}\left(\frac{1}{2} E_{2}\left(1+\frac{1}{\left(1+2 x_{3}\right) E_{1}}\right)+\lambda\left(E_{2}-2 y_{5}\right)-\frac{16 y_{5}}{E_{3}}\right) \\
f_{4}= & -\frac{1}{2}(1+\lambda)\left(E_{2}-2 y_{5}\right)+(1+\lambda) m_{3}\left(E_{2}-2 y_{5}\right) \\
& +m_{1}\left(\frac{1}{2} E_{2}\left(1-\frac{1}{\left(1+2 x_{3}\right) E_{1}}\right)-2(1+\lambda) y_{5}\right) \\
f_{5}= & (1+\lambda) m_{3}\left(E_{2}-2 y_{5}\right)+m_{1}\left(-2 \lambda y_{5}-\frac{16 y_{5}}{E_{3}}\right)
\end{aligned}
$$

where $E_{1}=\sqrt{1+2 x_{3}}, E_{2}=\sqrt{3+4 x_{3}-4 x_{3}^{2}}$ and $E_{3}=\left(1+4 y_{5}^{2}\right)^{3 / 2}$.
Solving $f_{2}=0$ and $f_{5}=0$, we obtain the following expressions for $m_{1}$ and $m_{3}$

$$
\begin{align*}
m_{1} & =\left(2 E_{1} E_{3}(1+\lambda)^{2} x_{3}^{3}\left(E_{2}-2 y_{5}\right)\right) / m  \tag{6}\\
m_{3} & \left.=\left(4 E_{1}(1+\lambda)\left(8+E_{3} \lambda\right) x_{3}^{3} y_{5}\right)\right) / m
\end{align*}
$$

where

$$
\begin{aligned}
m=2 E_{1}\left(E_{2} E_{3} \lambda+2 \lambda y_{5} E_{3}+\right. & \left.E_{2} E_{3}-2 E_{3} y_{5}+32 y_{5}\right) x_{3}^{3} \\
& +E_{3}(1+\lambda)\left(E_{2}-2 y_{5}\right)\left(E_{1}-1\right) x_{3}^{2}-E_{1} y_{5}\left(\lambda E_{3}+8\right) .
\end{aligned}
$$

We substitute the values of $m_{1}$ and $m_{3}$ into the equations $f_{1}=0, f_{3}=0$ and $f_{4}=0$, and taking only the numerators of these three equations because the denominators do not vanish, the former system, reduce to

$$
\begin{aligned}
g_{1}= & 8 E_{1} E_{3}(1+\lambda)\left(\lambda E_{3}+8\right) y_{5} x_{3}^{4}+\left(\left(16 E_{1} E_{3}^{2} \lambda-4 \lambda^{2} E_{3}^{2}+8 E_{1} E_{3}^{2}-192 \lambda E_{1} E_{3}-4 \lambda E_{3}^{2}\right.\right. \\
& \left.\left.-64 E_{1} E_{3}+512 E_{1} \lambda-32 \lambda E_{3}-32 E_{3}\right) y_{5}-2 E_{1} E_{2} E_{3}(1+\lambda)\left(3 \lambda E_{3}+2 E_{3}-16 \lambda-8\right)\right) x_{3}^{3} \\
& +\left(2 E_{3}(1+\lambda)\left(\lambda E_{3}+8\right)\left(E_{1}-1\right) y_{5}-E_{2} E_{3}(1+\lambda)\left(\lambda E_{3}+8\right)\left(E_{1}-1\right)\right) x_{3}^{2} \\
& +E_{1}\left(\lambda E_{3}+8\right)^{2} y_{5}, \\
g_{3}= & \left(8+E_{3} \lambda\right) y_{5}\left(L_{2} \lambda-L_{1}\right), \\
g_{4}= & (1+\lambda)\left(2 y_{5}-E_{2}\right)\left(L_{2} \lambda-L_{1}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
L_{1}= & -2\left(1+2 x_{3}\right)\left(2 E_{1} E_{3} x_{3}^{3}-E_{1} E_{3} x_{3}^{2}+x_{3}^{2} E_{3}-4 E_{1}\right) y_{5}-E_{2} E_{3} x_{3}^{2}\left(2 x_{3} E_{1}+E_{1}-1\right), \\
L_{2}= & \left(1+2 x_{3}\right)\left(12 E_{1} E_{3} x_{3}^{3}-2 E_{1} E_{3} x_{3}^{2}-64 x_{3}^{3} E_{1}+2 x_{3}^{2} E_{3}-E_{1} E_{3}\right) y_{5} \\
& +E_{2} E_{3} x_{3}^{2}\left(2 x_{3} E_{1}+E_{1}-1\right),
\end{aligned}
$$

Computing $\lambda$ from equation $g_{3}=0$ we obtain the two solutions

$$
\begin{equation*}
\lambda_{1}=-\frac{8}{E_{3}}, \quad \lambda_{2}=\frac{L_{1}}{L_{2}} . \tag{7}
\end{equation*}
$$

The solution $\lambda=\lambda_{1}$ is not suitable because then $m_{3}=0$. Substituting the solution $\lambda=\lambda_{2}$ in the equations $g_{1}=0$ and $g_{4}=0$, we get that $g_{4} \equiv 0$, and the equation $g_{1}=0$ reduces to

$$
\bar{h}_{1}=\left(2 x_{3}^{3} y_{5} E_{1} E_{3}\left(E_{3}-8\right)^{2}\left(2 x_{3}-1\right)\left(1+2 x_{3}+4 x_{3}^{2}\right) h_{1}=0,\right.
$$

where

$$
\begin{aligned}
h_{1}= & -4\left(1+2 x_{3}\right)^{2}\left(4 E_{1}^{2}\left(E_{3}-16\right) x_{3}^{4}-8 E_{1}\left(E_{1} E_{3}-4 E_{1}-4\right) x_{3}^{3}-E_{3}\left(E_{1}-1\right)^{2} x_{3}^{2}\right. \\
& \left.+E_{1}^{2}\left(E_{3}-8\right)\right) y_{5}^{2}-2 E_{2}\left(1+2 x_{3}\right)\left(16 E_{1}^{2}\left(E_{3}-4\right) x_{3}^{4}+4 E_{1}\left(3 E_{1} E_{3}-8 E_{1}-E_{3}-8\right) x_{3}^{3}\right. \\
& \left.+2 E_{3}\left(E_{1}-1\right)^{2} x_{3}^{2}-2 E_{1}^{2}\left(E_{3}-8\right) x_{3}-E_{1}^{2}\left(E_{3}-8\right)\right) y_{5}+E_{2}^{2} E_{3} x_{3}^{2}\left(2 x_{3} E_{1}+E_{1}-1\right)^{2} .
\end{aligned}
$$

Notice that $2 x_{3}^{3} y_{5} E_{1} E_{3}\left(1+2 x_{3}+4 x_{3}^{2}\right)$ does not vanish because $x_{3} \in(0,1), y_{5}>0$ and $E_{1}=r_{1,4}>0$.

At this step we shall prove that condition $\left(E_{3}-8\right)\left(2 x_{3}-1\right)=0$ implies that $m_{5}=0$. This is so, because

$$
\begin{aligned}
m_{5}= & 1-\left(m_{1}+m_{2}+m_{3}+m_{4}\right)=1-2\left(m_{1}+m_{3}\right) \\
= & \frac{2 E_{1} y_{5} x_{3}{ }^{3} E_{2} E_{3}\left(2 x_{3}-1\right)\left(4 x_{3}^{2}+2 x_{3}+1\right)\left(E_{3}-8\right)}{L_{2}^{2}} \\
& \times\left(-\left(1+2 x_{3}\right)\left[8 E_{1}^{2}\left(E_{3}-16\right) x_{3}^{4}+8 E_{1}\left(E_{3}-8\right)\left(E_{1}+1\right) x_{3}^{3}+2 E_{3}\left(E_{1}-1\right)^{2} x_{3}^{2}\right.\right. \\
& \left.\left.-2 E_{1}^{2}\left(E_{3}-16\right) x_{3}-E_{1}\left(E_{1} E_{3}-16 E_{1}+E_{3}\right)\right] y_{5}+E_{2} E_{3} x_{3}^{2}\left(2 E_{1} x_{3}+E_{1}-1\right)^{2}\right)
\end{aligned}
$$

where we have used the expressions (6) for $m_{1}, m_{3}$ and substituted $\lambda=\lambda_{2}$ where $\lambda_{2}$ is given in equation (7). In short we have proved that the equations $\bar{h}_{1}=0$ and $h_{1}=0$ are equivalent.

From Proposition 3 we have that

$$
x_{3}=\frac{1}{4} \pm \frac{y_{5}}{2} \sqrt{\frac{15-4 y_{5}^{2}}{1+4 y_{5}^{2}}}
$$

or equivalently,

$$
\begin{equation*}
h_{2}=\left(1-4 x_{3}\right)^{2}\left(1+4 y_{5}^{2}\right)+4 y_{5}^{2}\left(4 y_{5}^{2}-15\right)=0 \tag{8}
\end{equation*}
$$

Hence the central configurations are the solutions of the simultaneous solution of both equations $h_{1}=0$ and $h_{2}=0$, with the two unknowns $x_{3}$ and $y_{5}$. Indeed it provides positive masses $m_{j}$. In order to avoid the square roots which appear in $E_{1}, E_{2}$ and $E_{3}$ in $h_{1}$, we do a change of variables such that the expressions appearing inside each square root are equal to some new squared expressions. These changes of variables are given by the so called rational parameterizations and correspond to parameterizations of planar algebraic curves given by rational functions. Due to the famous Cayley-Riemann's Theorem [1, 2] they exist if and only if the corresponding surfaces have genus zero. There are effective methods to find one of these parameterizations see for instance [20, Chap. $4 \& 5]$. In fact, many programs of symbolic calculus have implemented some
methods and algorithms for obtaining them. For more information on this subject the reader is addressed to [10] and references therein, where there are several examples of applications of this approach.

In our case, for instance, we have that $E_{1}=\sqrt{1+2 x_{3}}, E_{2}=\sqrt{3+4 x_{3}-4 x_{3}^{2}}$. Hence if we write $x_{3}=\left(u^{2}-1\right) / 2$ we get that $E_{1}=\sqrt{u^{2}}$. Then

$$
3+4 x_{3}-\left.4 x_{3}^{2}\right|_{x_{3}=\left(u^{2}-1\right) / 2}=u^{2}\left(4-u^{2}\right) .
$$

Consider now the algebraic curve $F(u, v)=u^{2}\left(4-u^{2}\right)-v^{2}=0$. It has genus 0 , and by the Cayley-Riemann's theorem it admits a rational parameterization. For example, for all $s$,

$$
F\left(\frac{4(2 s-1) s}{5 s^{2}-4 s+1}, \frac{8 s(s-1)(3 s-1)(2 s-1)}{\left(5 s^{2}-4 s+1\right)^{2}}\right)=0
$$

As a consequence, by taking

$$
x_{3}=\left.u^{2}\left(4-u^{2}\right)\right|_{u=\frac{4(2 s-1) s}{5 s^{2}-4 s+1}}=\frac{\left(3 s^{2}-1\right)\left(13 s^{2}-8 s+1\right)}{2\left(5 s^{2}-4 s+1\right)^{2}}
$$

we obtain that

$$
E_{2}=\sqrt{v^{2}}=\sqrt{\frac{(8 s(s-1)(3 s-1)(2 s-1))^{2}}{\left(5 s^{2}-4 s+1\right)^{4}}}
$$

The rational parameterization of $E_{3}=\left(1+4 y_{5}^{2}\right)^{3 / 2}$ is much simpler and can be obtained similarly. It suffices to consider the algebraic curve of genus $0, G\left(y_{5}, w\right)=1+4 y_{5}^{2}-w^{2}=$ 0 . A good parameterization for $y_{5}$ is $y_{5}=\left(1-t^{2}\right) /(4 t)$.

We are interested on values $y_{5} \in(0, \sqrt{15} / 2)$ and $x_{3} \in(0,1)$. In short, we do the change of variables $\left(x_{3}, y_{5}\right) \rightarrow(s, t)$ where

$$
x_{3}=\frac{\left(3 s^{2}-1\right)\left(13 s^{2}-8 s+1\right)}{2\left(5 s^{2}-4 s+1\right)^{2}}, \quad y_{5}=\frac{1-t^{2}}{4 t} .
$$

Then we have that $t$ varies in the interval $T:=(4-\sqrt{15}, 1)$ and similarly, the values of $x_{3} \in(0,1)$ are covered for instance for $s \in S:=(\sqrt{3} / 3,(6+\sqrt{3}) / 11)$. Hence,

$$
E_{1}=\frac{4 s(2 s-1)}{5 s^{2}-4 s+1}, \quad E_{2}=\frac{-8 s(s-1)(3 s-1)(2 s-1)}{\left(5 s^{2}-4 s+1\right)^{2}}, \quad E_{3}=\frac{\left(t^{2}+1\right)^{3}}{8 t^{3}} .
$$

In the variables $(s, t)$ the two equations $h_{1}=0$ and $h_{2}=0$ become

$$
\begin{align*}
& h_{1}=\frac{8 s^{2}(2 s-1)^{2}}{\left(5 s^{2}-4 s+1\right)^{2} t^{3}} H_{1}=0  \tag{9}\\
& h_{2}=\frac{1}{16\left(5 s^{2}-4 s+1\right)^{4} t^{4}} H_{2}=0
\end{align*}
$$

respectively, where

$$
\begin{aligned}
H_{2}= & \left(5 s^{2}-4 s+1\right)^{4}\left(t^{8}+1\right)-4\left(47 s^{4}-152 s^{3}+150 s^{2}-56 s+7\right) \\
& \times\left(153 s^{4}-168 s^{3}+58 s^{2}-8 s+1\right)\left(t^{6}+t^{2}\right)+\left(101222 s^{8}-258784 s^{7}+326904 s^{6}\right. \\
& \left.-286240 s^{5}+183428 s^{4}-79776 s^{3}+21432 s^{2}-3168 s+198\right) t^{4}
\end{aligned}
$$

and $H_{1}$ ia a huge polynomial of total degree 34 and with $\operatorname{deg}_{s}\left(H_{1}\right)=24$ and $\operatorname{deg}_{t}\left(H_{1}\right)=$ 10, which is given in Appendix A.

Taking into account that $t \in T$ and $s \in S$ to solve system (9), we see that this can be reduced to solve the system $H_{1}=0, H_{2}=0$, because $s(2 s-1)\left(s^{2}+2 s-1\right) \neq 0$. Hence, to find the real solutions of the systems $h_{1}=0, h_{2}=0$, is equivalent to find the real solutions, $(s, t) \in S \times T$, of the polynomial system $H_{1}=0, H_{2}=0$.

To study the above planar systems of equations we will use a mixture of the classical approach applying resultants together with simple inequalities in the original variables $x_{3}$ and $y_{5}$. For our problem this approach is very suitable because of the simplicity of the equation $h_{2}=0$, given in (8).

We start with the polynomials system $H_{1}(s, t)=0, H_{2}(s, t)=0$. Recall that if $(\hat{s}, \hat{t})$ is one of its solutions (real or complex), then $t=\hat{t}$ must be a zero of the one variable polynomial

$$
P(t)=\operatorname{Res}_{s}\left(H_{1}, H_{2}\right),
$$

where $\operatorname{Res}_{s}(\cdot, \cdot)$ denotes the resultant of two polynomials with respect the variable $s$, see for instance [22]. After some computations (implemented for instance in Maple or Mathematica) we get that

$$
\begin{equation*}
P(t)=\left(1+t^{2}\right)^{6} p_{4}(t) q_{4}(t) p_{120}(t) p_{132}(t) \tag{10}
\end{equation*}
$$

where $p_{4}(t)=1+4 t-14 t^{2}+4 t^{3}+t^{4}, q_{4}(t)=1-4 t-14 t^{2}-4 t^{3}+t^{4}$ and $p_{k}$ denotes a polynomial with integer coefficients of degree $k$ that we do not detail. We only remark that precisely $p_{120}$ is the polynomial that gives rise to the polynomial $R_{60}$, detailed in Appendix B and that gives rise to the values $t^{*}$ and $u^{*}$ that appear in Remark 2. Hence, by computing the Sturm sequences of each of the four polynomials, $p_{4}, q_{4}, p_{120}$ and $p_{132}$, we get that they have respectively, $4,4,28$ and 32 real roots and, moreover, that all them are simple. Furthermore, since we are only interested on the roots $t \in T \sim(0.127,1)$, we consider a slightly bigger interval $T \subset T^{\prime}=(3 / 25,100)$, with rational endpoints. Again, the corresponding Sturm sequences allow to prove that their number of roots in $T^{\prime}$ are $1,1,7$ and 9 , respectively. We will denote them by $t_{1} ; t_{2} ; t_{3}, \ldots, t_{9}$ and $t_{10}, \ldots t_{18}$, where for each $p_{k}$ the roots are ordered. Their approximated ordered value are

$$
\begin{aligned}
& t_{10} \approx 0.1278827, \quad t_{3} \approx 0.1296657, \quad t_{11} \approx 0.1318307, \quad t_{12} \approx 0.1535285, \quad t_{2} \approx 0.1583844, \\
& t_{13} \approx 0.1690804, \quad t_{4} \approx 0.1818971, \quad t_{5} \approx 0.1871837, \quad t_{14} \approx 0.4693713, \quad t_{1} \approx 0.5095254, \\
& t_{15} \approx 0.5490528, \quad t_{16} \approx 0.5930556, \quad t_{6} \approx 0.7095411, \quad t_{7} \approx 0.7332148, \quad t_{8} \approx 0.9432977, \\
& t_{17} \approx 0.9681690, \quad t_{9} \approx 0.9958185, \quad t_{18} \approx 0.9962499 .
\end{aligned}
$$

In fact, the roots $t_{1}$ and $t_{2}$ are $t_{1}=-1+\sqrt{5}-\sqrt{5-2 \sqrt{5}} \approx 0.5095254$ and $t_{2}=$ $1+\sqrt{5}-\sqrt{5+2 \sqrt{5}} \approx 0.1583844$, while each of the other sixteen roots can be obtained, again using the Sturm sequences, with any desired error.

Therefore, each $t_{j}, j=3, \ldots, 18$, can be bounded by $\underline{t}_{j}<t_{j}<\bar{t}_{j}$, with $\underline{t}_{j}, \bar{t}_{j} \in \mathbb{Q}$ and $0<\bar{t}_{j}-\underline{t}_{j}$ as small, as desired. For each of these values of $t_{j}$ we can use that the function $t \rightarrow\left(1-t^{2}\right) /(4 t)$ is decreasing in $(0,1)$ and that $y_{5}=\left(1-t^{2}\right) /(4 t)$ to get that if $\left(1-t_{j}^{2}\right) /\left(4 t_{j}\right)=y_{5}(j)$, then

$$
\underline{y}_{5}(j)=\frac{1-\bar{t}_{j}^{2}}{4 \bar{t}_{j}}<y_{5}(j)<\frac{1-\underline{t}_{j}^{2}}{4 \underline{t}_{j}}=\bar{y}_{5}(j), \quad \text { with } \quad \underline{y}_{5}(j), \bar{y}_{5}(j) \in \mathbb{Q}
$$

Recall that in (2) we have introduced the functions

$$
\Psi^{ \pm}(y)=\frac{1}{4} \pm \frac{y}{2} \sqrt{\frac{15-4 y^{2}}{1+4 y^{2}}}
$$

By simple derivation we get that $\Psi^{+}$(resp. $\Psi^{-}$) is increasing (resp. decreasing) for $y \in(0, \sqrt{3} / 2)$ and decreasing (resp. increasing) for $y \in(\sqrt{3} / 2,1)$. Hence, if we define $x_{3}^{ \pm}=\Psi^{ \pm}\left(y_{5}\right)$ it holds that

- If $j$ is such that $y_{5}(j)<\sqrt{3} / 2$ then

$$
\Psi^{+}\left(\underline{y}_{5}(j)\right)<x_{3}^{+}(j)<\Psi^{+}\left(\bar{y}_{5}(j)\right), \quad \Psi^{-}\left(\bar{y}_{5}(j)\right)<x_{3}^{-}(j)<\Psi^{-}\left(\underline{y}_{5}(j)\right) .
$$

- If $j$ is such that $y_{5}(j)>\sqrt{3} / 2$ then

$$
\Psi^{+}\left(\bar{y}_{5}(j)\right)<x_{3}^{+}(j)<\Psi^{+}\left(\underline{y}_{5}(j)\right), \quad \Psi^{-}\left(\underline{y}_{5}(j)\right)<x_{3}^{-}(j)<\Psi^{-}\left(\bar{y}_{5}(j)\right) .
$$

From these inequalities it is easy to find rational values $\underline{x}_{3}^{ \pm}(j)$ and $\bar{x}_{3}^{ \pm}(j)$ such that

$$
\underline{x}_{3}^{ \pm}(j)<x_{3}^{ \pm}(j)<\bar{x}_{3}^{ \pm}(j)
$$

and with $0<\bar{x}_{3}^{ \pm}(j)-\underline{x}_{3}^{ \pm}(j)$ as small as desired. Finally, the functions that define $E_{1}, E_{2}$ and $E_{3}$, given by $\psi_{1}(x)=\sqrt{1+2 x}, \psi_{2}(x)=\sqrt{3+4 x-4 x^{2}}$ and $\Psi_{3}(y)=\left(1+4 y^{2}\right)^{3 / 2}$ are increasing, increasing for $x \in(0, \sqrt{3} / 3)$ and decreasing for $x \in(\sqrt{3} / 3,1)$, and increasing, respectively. Similarly that for $x^{ \pm}(j)$ we can find rational bounds, $\underline{E}_{j}$ and $\bar{E}_{j}, j=1,2,3$, as sharp as desired and satisfying

$$
\underline{E}_{1}<E_{1}<\bar{E}_{1}, \quad \underline{E}_{2}<E_{2}<\bar{E}_{2}, \quad \underline{E}_{3}<E_{3}<\bar{E}_{3} .
$$

In short, we have found sharp rational upper and lower bound of any possible solution $y_{5}=y_{5}(j), x_{3}^{ \pm}(j)$, corresponding to each $t=t_{j}, j=1,2, \ldots, 18$. These rational bounds give also rational bounds for $E_{1}, E_{2}$ and $E_{3}$. Gluing these bounds we prove that some candidates to be solutions of our system can be discarded. That other candidates are actual solution of our problem can be easily proved from Bolzano's theorem. We detail two examples, one of each type and skip the computations for all the rest.

A suitable way is to write $h_{1}$ in function of the variables $x_{3}, y_{5}, E_{1}, E_{2}$ and $E_{3}$, namely

$$
\begin{aligned}
& h_{1}=-128 x_{3}{ }^{3} y_{5}{ }^{2}\left(4 x_{3}{ }^{2}+4 x_{3}+1\right) E_{1}+4 x_{3}{ }^{2} y_{5}{ }^{2}\left(4 x_{3}{ }^{2}+4 x_{3}+1\right) E_{3} \\
& +32 y_{5}{ }^{2}\left(32 x_{3}{ }^{6}+16 x_{3}{ }^{5}-8 x_{3}{ }^{4}-4 x_{3}{ }^{3}+4 x_{3}{ }^{2}+4 x_{3}+1\right) E_{1}{ }^{2}+64 x_{3}{ }^{3} y_{5}\left(1+2 x_{3}\right) E_{1} E_{2} \\
& -8 x_{3}{ }^{2} y_{5}{ }^{2}\left(4 x_{3}{ }^{2}+4 x_{3}+1\right) E_{1} E_{3}-4 x_{3}{ }^{2} y_{5}\left(1+2 x_{3}\right) E_{3} E_{2} \\
& +16 y_{5}\left(16 x_{3}{ }^{5}+16 x_{3}{ }^{4}+4 x_{3}{ }^{3}-4 x_{3}{ }^{2}-4 x_{3}-1\right) E_{1}{ }^{2} E_{2} \\
& -4 y_{5}{ }^{2}\left(16 x_{3}{ }^{6}-16 x_{3}{ }^{5}-32 x_{3}{ }^{4}-12 x_{3}{ }^{3}+3 x_{3}{ }^{2}+4 x_{3}+1\right) E_{1}{ }^{2} E_{3} \\
& +8 x_{3}{ }^{2} y_{5}\left(2 x_{3}{ }^{2}+3 x_{3}+1\right) E_{1} E_{2} E_{3}+E_{2}{ }^{2} E_{3} x_{3}{ }^{2} \\
& -2 y_{5}\left(32 x_{3}{ }^{5}+40 x_{3}{ }^{4}+16 x_{3}{ }^{3}-2 x_{3}{ }^{2}-4 x_{3}-1\right) E_{1}{ }^{2} E_{2} E_{3} \\
& -2 x_{3}{ }^{2}\left(1+2 x_{3}\right) E_{1} E_{2}^{2} E_{3}+x_{3}{ }^{2}\left(4 x_{3}{ }^{2}+4 x_{3}+1\right) E_{1}{ }^{2} E_{2}{ }^{2} E_{3} .
\end{aligned}
$$

We remark that the addends of $h_{1}$ can be bounded using the inequalities given above. For instance

$$
\underline{E}_{2}^{2} \underline{E}_{3} \underline{x}_{3}^{2}<E_{2}^{2} E_{3} x_{3}^{2}<\bar{E}_{2}^{2} \bar{E}_{3} \bar{x}_{3}^{2}
$$

and

$$
-4 \bar{x}_{3}{ }^{2} \bar{y}_{5}\left(1+2 \bar{x}_{3}\right) \bar{E}_{3} \bar{E}_{2}<-4 x_{3}^{2} y_{5}\left(1+2 x_{3}\right) E_{3} E_{2}<-4 \underline{x}_{3}{ }^{2} \underline{y}_{5}\left(1+2 \underline{x}_{3}\right) \underline{E}_{3} \underline{E}_{2} .
$$

Let us prove for instance that the value $y_{5}=y_{5}(5)$ corresponding to $t=t_{5} \approx$ 0.1871837 , together with $x_{3}=x_{3}^{+}(5)=\Psi^{+}\left(y_{5}(5)\right)$ does not provide a solution of our system. From the Sturm sequence we get that

$$
\frac{1871}{10000}<t_{5}<\frac{1872}{10000}
$$

From these inequalities and all the above considerations we obtain that

$$
\begin{aligned}
1.2886701 \approx & \frac{94234}{73125}<y_{5}<\frac{96499359}{74840000} \approx 1.2894088, \quad \frac{9235}{10000}<x_{3}^{+}<\frac{9238}{10000} \\
& \frac{16873}{10000}<E_{1}<\frac{16873}{10000}, \quad \frac{18118}{10000}<E_{2}<\frac{18115}{10000}, \quad \frac{21128}{1000}<E_{3}<\frac{21161}{1000}
\end{aligned}
$$

By using all these inequalities we obtain that the corresponding value of $h_{1}>242$ and the system has no solution for the value of $y_{5}$, and its corresponding $x_{3}^{+}(5)$ associated to $t=t_{5}$.

On the other hand, let us prove that the value $y_{5}=y_{5}(7)$ corresponding to $t=t_{7}=$ $t^{*} \approx 0.7332148$ together with $x_{3}=x_{3}^{+}(7)=\Psi^{+}\left(y_{5}(7)\right)$ does provide an actual solution. This is a simple consequence of Bolzano's theorem, because if we denote as $h_{1}(\tau)$ the value of the expression of $h_{1}$ when all the values $y_{5}, x_{5}^{+}, E_{1}, E_{2}$ and $E_{3}$ are obtained when $t=\tau$ we get for instance that

$$
h_{1}\left(\frac{7332}{10000}\right) h_{1}\left(\frac{7333}{10000}\right)<0 .
$$

We carry out similar computations for $y_{5}=y_{5}(j)$, corresponding to $t=t_{j}$, and $x_{3}=x_{3}^{ \pm}(j)=\Psi^{ \pm}\left(y_{5}(j)\right)$. We conclude that among the 36 candidates that could be a solution of the system $h_{1}\left(x_{3}, y_{5}\right)=0, h_{2}\left(x_{3}, y_{5}\right)=0$ the only couples $\left(x_{3}, y_{5}\right)$ that do solve it are:
(I) $\left(x_{3}^{+}(2), y_{5}(2)\right)=\left(\frac{1+\sqrt{5}}{4}, \frac{\sqrt{5+2 \sqrt{5}}}{2}\right) \approx(0.8090170,1.5388418)$ corresponding to $t=t_{2}$,
(II) $\left(x_{3}^{+}(7), y_{5}(7)\right) \approx(0.5402091,0.1576605)$ corresponding to $t=t_{7}=t^{*}$,
(III) $\left(x_{3}^{+}(9), y_{5}(9)\right) \approx(0.2540572,0.0020951)$ corresponding to $t=t_{9}$,
(IV) $\left(x_{3}^{-}(4), y_{5}(4)\right) \approx(-0.4091526,1.3289291)$ corresponding to $t=t_{4}$,
(V) $\left(x_{3}^{-}(14), y_{5}(14)\right) \approx(-0.3542470,0.4152845)$ corresponding to $t=t_{14}$,
(VI) $\left(x_{3}^{-}(18), y_{5}(18)\right) \approx(0.2463622,0.0018786)$ corresponding to $t=t_{18}$.

Clearly, solutions in items (IV) and (V) can be discarded because the corresponding values of $x_{3}$ are negative. The solutions given in items (III) and (VI) are not good either, since both options result in negative $m_{5}$ values. In short the central configurations are:
(I) $\left(x_{3}, y_{3}\right)=\left(\frac{1+\sqrt{5}}{4}, \frac{\sqrt{10+2 \sqrt{5}}}{4}\right)$ and $y_{5}=\frac{\sqrt{5+2 \sqrt{5}}}{2}$, with masses $m_{j}=\frac{1}{5}$, for all $j$.
(II) $\left(x_{3}, y_{3}\right)=\left(x_{3}^{+}(7), y_{3}^{+}(7)\right) \approx(0.5402091,0.9991913)$ and $y_{5}=y_{5}(7) \approx 0.1576605$ with masses $m_{1}=m_{2} \approx 0.0922539, m_{3}=m_{4} \approx 0.3860949$ and $m_{5} \approx 0.04330243$.

The values of $y_{3}$ and $m_{j}$ are obtained from Proposition 3 and the expressions (6) and (7).

The first solution provides the regular pentagon of Figure 2(a) as a convex central configuration of the 5 -body problem with masses equal to $1 / 5$. While the second one provides the equilateral concave pentagon of Figure 2(b) as a concave central configuration of the 5 -body problem with the masses given in the statement (b) of the theorem. This completes the proof of Theorem 1.
3.1. Alternative approaches to solve system $h_{1}=0, h_{2}=0$. In this section we comment about to alternative approaches two solve this system and its equivalent one $H_{1}=0, H_{2}=0$.

A first one consists on computing the Gröbner basis of the two polynomials $H_{1}$ and $H_{2}$ with respect to the two variables $s$ and $t$. Doing this we obtain three polynomial equations whose common solutions are also the solutions of system $H_{1}=0, H_{2}=0$. We do not provide explicitly these three polynomials, but only comment that they are huge and their expressions need many pages. The first one essentially coincides with $P(t)$ given in (10). The second one $P_{2}(s, t)=\left(1+t^{2}\right)^{2} p_{259}(s, t)$. depends on both variables $s$ and $t$, and $P_{2}$ is linear in the variable $s$. Consequently each root $t=t_{j}$ of the polynomial $P$ provides a single value of $s$ from $P_{2}\left(s, t_{j}\right)=0$, say $s=s_{j}$. Then we only need to keep the $j$ 's such that $s_{j} \in S$. Finally, the third polynomial $P_{3}(s, t)$ of the Gröbner basis has degree 262 but it is only cubic in the variable $s$. By keeping only the values $s_{j} \in S$ that also satisfy $P_{3}\left(s_{j}, t_{j}\right)=0$ we arrive to the actual solutions. We have used our approach instead of this one because it is not easy to check analytically all the above facts because only two of the eighteen roots of $P$ are known analytically. Moreover, we prefer our point of view because the computation of a resultant is simple and self contained while the computation of a Gröbner basis is implemented in the computer algebra systems but the user has no control on what the algorithm is doing.

A second alternative approach would consist on computing also $Q(s)=\operatorname{Res}_{t}\left(H_{1}, H_{2}\right)$. In this case we arrive to

$$
Q(s)=q_{2}^{6}(s) q_{4}^{2}(s) q_{120}(s) q_{132}(s)
$$

for some polynomials $q_{k}$ of degree $k$, where here $q_{4}$ is different to the one given in (10). Their respective number of real roots are $0,4,28$ and 32 . Moreover, only $1,0,4$ and 6 of them are in $S$. Call them $s_{m}, m=1,2, \ldots, 11$. Hence all the possible solutions of system $H_{1}=0, H_{2}=0$ in $S \times T$ are given by $11 \times 18$ values $\left(s_{m}, t_{j}\right)$. Then, a discarding process, similar to the one done in our proof of Theorem 1 can be done. On the other hand, a proof that the non discarded candidates to be solutions are actual solutions can be done for instance by using the nice Poincaré-Miranda theorem. See for instance [9] to have more details of how to utilize this approach. We have not used it in our work because the expression of $h_{2}$ is much simpler that the one of $Q$.

## Appendix A. The expression of $H_{1}$

The polynomial $H_{1}$ in (9) writes as

$$
H_{1}(s, t)=R_{0}(s)\left(t^{10}+1\right)-R_{1}(s)\left(t^{9}+4 t^{7}-t\right)+R_{2}(s)\left(t^{8}+t^{2}\right)+R_{3}(s)\left(t^{7}+t^{3}\right)+\sum_{j=4}^{6} R_{j}(s) t^{j}
$$

where

$$
\begin{aligned}
& R_{0}=s^{2}(2 s-1)^{2}\left(140137001 s^{20}-473336800 s^{19}-77771662 s^{18}+3288160224 s^{17}\right. \\
& -8637659443 s^{16}+12537556864 s^{15}-12225124968 s^{14}+8691543680 s^{13} \\
& -4785798270 s^{12}+2180211392 s^{11}-880204628 s^{10}+324883264 s^{9}-105646862 s^{8} \\
& \left.+28078976 s^{7}-5656040 s^{6}+796288 s^{5}-67435 s^{4}+1568 s^{3}+306 s^{2}-32 s+1\right) \text {, } \\
& R_{1}=2 s(s-1)(3 s-1)(2 s-1)\left(402088273 s^{20}-1176961940 s^{19}+169440330 s^{18}\right. \\
& +4244422908 s^{17}-9214008723 s^{16}+10491664368 s^{15}-7879610248 s^{14}+4350977648 s^{13} \\
& -1966298574 s^{12}+810673640 s^{11}-313304452 s^{10}+104139528 s^{9}-26338798 s^{8} \\
& \left.+4329392 s^{7}-227208 s^{6}-97872 s^{5}+33277 s^{4}-5620 s^{3}+586 s^{2}-36 s+1\right), \\
& R_{2}=2610735845 s^{24}-15402964740 s^{23}+36242909513 s^{22}-34656361080 s^{21}-19279444736 s^{20} \\
& +100341955724 s^{19}-143714126267 s^{18}+121841849904 s^{17}-68503362257 s^{16} \\
& +27259846072 s^{15}-9818466526 s^{14}+5237546480 s^{13}-3390994016 s^{12}+1681689656 s^{11} \\
& -535976190 s^{10}+78574400 s^{9}+17032379 s^{8}-14081204 s^{7}+4575253 s^{6} \\
& -963768 s^{5}+142992 s^{4}-15044 s^{3}+1081 s^{2}-48 s+1, \\
& R_{3}=4 s(2 s-1)\left(4011437555 s^{22}-17997365760 s^{21}+47815887103 s^{20}-119864245704 s^{19}\right. \\
& +262594976801 s^{18}-427430598888 s^{17}+498867338773 s^{16}-424920266400 s^{15} \\
& +272105947982 s^{14}-136189975712 s^{13}+56438427910 s^{12}-20882782960 s^{11} \\
& +7255673746 s^{10}-2308064816 s^{9}+623396538 s^{8}-133508384 s^{7}+21674927 s^{6} \\
& \left.-2595360 s^{5}+227627 s^{4}-15240 s^{3}+861 s^{2}-40 s+1\right), \\
& R_{4}=-128824963717 s^{24}+476911508392 s^{23}+165019719846 s^{22}-4572358588576 s^{21} \\
& +13580634656322 s^{20}-22953475876792 s^{19}+26679405170926 s^{18}-23043609894160 s^{17} \\
& +15476921278317 s^{16}-8354767360048 s^{15}+3726249539948 s^{14}-1406580864896 s^{13} \\
& +458981618588 s^{12}-132125593264 s^{11}+34342034540 s^{10}-8246853600 s^{9}+1837269525 s^{8} \\
& -367787384 s^{7}+62442990 s^{6}-8479328 s^{5}+879394 s^{4}-67480 s^{3}+3750 s^{2}-144 s+3,
\end{aligned}
$$

$$
\begin{aligned}
R_{5}= & -1024 s^{3}(2 s-1)^{3}\left(5478853 s^{18}-19634676 s^{17}+61412857 s^{16}-189035488 s^{15}\right. \\
& +395309060 s^{14}-534006384 s^{13}+487648452 s^{12}-316317344 s^{11}+154130630 s^{10} \\
& -61395832 s^{9}+22312398 s^{8}-7766560 s^{7}+2406420 s^{6}-594416 s^{5} \\
& \left.+108148 s^{4}-13664 s^{3}+1117 s^{2}-52 s+1\right) \\
R_{6}= & 138883898747 s^{24}-544790344792 s^{23}+35213761702 s^{22}+4234516514656 s^{21} \\
& -13218500821438 s^{20}+22675637356616 s^{19}-26464806060818 s^{18}+22812618679024 s^{17} \\
& -15226596524179 s^{16}+8152770613200 s^{15}-3614885715604 s^{14}+1368000514176 s^{13} \\
& -453304331364 s^{12}+133617865040 s^{11}-35152022420 s^{10}+8179417120 s^{9}-1643965931 s^{8} \\
& +272577160 s^{7}-34266642 s^{6}+2694560 s^{5}-25822 s^{4}-22424 s^{3}+2726 s^{2}-144 s+3
\end{aligned}
$$

## Appendix B. The polynomial $R_{60}$

The polynomial $p_{120}(t)$ is reciprocal, that is $p_{120}(t)-t^{120} p_{120}(1 / t) \equiv 0$. Notice that if $\hat{t}$ is one of its roots, $1 / \hat{t}$ is another one. Hence there is a standard trick to "reduce" its degree to the half. Consider the numerator of $t+1 / t=u$, that is $t^{2}+t u-1$ and compute the resultant between it and $p_{120}(t)$. We obtain that
$\operatorname{Res}_{t}\left(p_{120}(t), t^{2}-u t+1\right)=\left(R_{60}(u)\right)^{2}$,
where $R_{60}$ is the polynomial of degree $120 / 2=60$,

$$
\begin{aligned}
& u^{60}+4 u^{59}-480 u^{58}-2368 u^{57}+102656 u^{56}+661504 u^{55}-12378112 u^{54}-114393088 u^{53} \\
& +813367296 u^{52}+13487570944 u^{51}-6779043840 u^{50}-1119258411008 u^{49} \\
& -4587041849344 u^{48}+63761809408000 u^{47}+556458915659776 u^{46}-2085902406909952 u^{45} \\
& -38793866899357696 u^{44}-14062083704356864 u^{43}+1871701598900584448 u^{42} \\
& +6613561341561012224 u^{41}-63439436081954553856 u^{40}-468879147034843545600 u^{39} \\
& +1328797291115756650496 u^{38}+20993318838230010822656 u^{37} \\
& -2305632465931114381312 u^{36}-680737782622703312699392 u^{35} \\
& -1102380299141445066948608 u^{34}+16433802069777919820955648 u^{33} \\
& +51705311821812917239545856 u^{32}-287454839290286351637807104 u^{31} \\
& -1467660179422186654016733184 u^{30}+3159833116868066015124127744 u^{29} \\
& +30235824601650376596023934976 u^{28}-3647821652057127970278473728 u^{27} \\
& -478786881979744683498227630080 u^{26}-692146369158869263721793847296 u^{25} \\
& +6031919651486804311277903544320 u^{24}+17009050671171345412955758919680 u^{23} \\
& -62657388272839173181632407404544 u^{22}-249363510060237095878177991950336 u^{21} \\
& +558394430418773435280130962358272 u^{20}+2603575445697403174765988761567232 u^{19} \\
& -4366163704989486187475739888058368 u^{18}-20183899906055516645882016026329088 u^{17} \\
& +29417136038642440588505535383339008 u^{16}+117121419223047038546206624994820096 u^{15} \\
& -162680894447793178205603154016337920 u^{14}-505682422731087446635760756082081792 u^{13} \\
& +700532798421762302278975021920550912 u^{12}+1597276835067625721595839817342517248 u^{11} \\
& -2240245382048536583959836766075092992 u^{10}-3574566797945169482598857579834638336 u^{9} \\
& +5053220375882588124433490027151884288 u^{8}+5336383096359169095222217602355953664 u^{7} \\
& -7423686433490169801891477126702432256 u^{6}-4673067172681344865677446696298086400 u^{5} \\
& +5934795309305307979410357304298569728 u^{4}+1599227432428726909587392869399789568 u^{3} \\
& -996920996838686904677855295210258432 u^{2}+332306998946228968225951765070086144 u \\
& -1329227995784915872903807060280344576 .
\end{aligned}
$$

Then all the roots of $p_{120}$ can be obtained simply by computing the roots of $R_{60}$ and then for each one of them, say $u=\hat{u}$, two roots of $p_{120}$ are given by the solutions of the quadratic equation $t^{2}-\hat{u} t+1=0$.

By computing its Sturm sequence we get that $R_{60}$ has exactly 14 real roots, all them simple (of course half the number the real roots of $p_{120}$ ). One of them is $u=$ $u^{*} \approx 2.0970716051$ and the value $t=t^{*}$ appearing in Remark 2 is the smallest root of $t^{2}-u^{*} t+1=0$.

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