N-soliton solutions of the Fokas-Lenells equation for the plasma ion-cyclotron waves: Inverse scattering transform approach

V. M. Lashkin^{a,b,*}

^aInstitute for Nuclear Research, Pr. Nauki 47, Kyiv 03028, Ukraine ^bSpace Research Institute, Pr. Glushkova 40 k.4/1, Kyiv 03187, Ukraine

Abstract

We present a simple and constructive method to find N-soliton solutions of the equation suggested by Davydova and Lashkin to describe the dynamics of nonlinear ion-cyclotron waves in a plasma and subsequently known (in a more general form and as applied to nonlinear optics) as the Fokas-Lenells equation. Using the classical inverse scattering transform approach, we find bright N-soliton solutions, rational N-soliton solutions, and N-soliton solutions in the form of a mixture of exponential and rational functions. Explicit breather solutions are presented as examples. Unlike purely algebraic constructions of the Hirota or Darboux type, we also give a general expression for arbitrary initial data decaying at infinity, which contains the contribution of the continuous spectrum (radiation).

Keywords: N-soliton solution, Fokas-Lenells equation, exponential-rational solution, inverse scattering transform, continuous spectrum, ion-cyclotron waves

1. Introduction

Equations integrable by the inverse scattering transform (IST) method, which arise in real physical situations and are important for practical applications, are of particular interest in nonlinear science [1]. Generally speaking, the number of such equations is very limited and not too large. In plasma physics, classical examples of completely integrable equations are the Korteweg-de Vries (KdV) equation (and the modified KdV equation) for the nonlinear ion-acoustic waves, the nonlinear Schrödinger (NLS) equation for the Langmuir waves (both of these equations are also derived for the cases of other branches of plasma oscillations using the reductive perturbation technique), the derivative nonlinear Schrödinger (DNLS) equation describing nonlinear Alfvén waves, and the

Email address: vlashkin62@gmail.com (V. M. Lashkin)

^{*}Corresponding author

two-dimensional Kadomtsev-Petviashvili equation, which is a two-dimensional generalization of the KdV equation [2, 3]. Slightly less known integrable models (in plasmas) are the Boussinesq equation for the beam instabilities [4] and the nonlinear string equation (elliptic Boussinesq) describing a nonlinear stage of the two-stream instability in quantum plasmas [5], and also the Yajima-Oikawa equations describing the interaction of Langmuir waves with ion-acoustic waves propagating in one direction [6].

Davydova and Lashkin [7] suggested a nonlinear equation governing the dynamics of short-wavelength ion-cyclotron waves in plasmas (the Bernstein modes) [8, 9], which in the one-dimensional case in dimensionless variables has the form

$$u_{xt} - u - i\sigma|u|^2 u_x = 0, (1)$$

where u(x,t) is the slowly varying complex envelope of the electrostatic potential at the ion-cyclotron frequency, and $\sigma=\pm 1$. Authors of [7] found the bright one-soliton solution of (1), and then the same authors (co-authored with A. I. Fishchuck) presented solutions of this equation in the form of bright algebraic soliton, dark and anti-dark solitons corresponding the nonvanishing boundary conditions, as well as solutions in the form of nonlinear periodic waves in elliptic Jacobi functions [10]. Later on, Fokas and Lenells showed [11, 12] that (1) is completely integrable and corresponds to the first negative flow of the Kaup–Newell hierarchy of the DNLS equation [13], and, therefore, can be solved by the IST [14]. Note that the original version of the equation considered in [11, 12] differs from (1) and has been derived as an integrable generalization of the NLS equation using bi-Hamiltonian methods [11], and then as a model for nonlinear pulse propagation in monomode optical fibers when certain higher-order nonlinear effects are taken into account [15]. Under this, the corresponding equation in dimensionless variables is

$$iu_t - \nu u_{xt} + \gamma u_{xx} + \sigma |u|^2 (u + i\nu u_x) = 0, \tag{2}$$

where ν and γ are real constants, $\sigma = \pm 1$, and by gauge transformation and a change of variables can be reduced to equation (1) suggested in [7]. Lenells rediscovered [15] the bright one-soliton solution of [7] without using the IST. Bright N-soliton solutions of (2) were obtained by Lenells in [16] with the dressing method, and for (1) by the Hirota bilinear method in [17]. Dark N-soliton solutions, which contain dark and anti-dark soliton solutions of [10], were found by the bilinearization method in [18, 19]. The N-order rogue wave solution of equations (1) and (2) were obtained using the N-fold Darboux transformation in [20, 21]. In what follows, we will refer to (1) as the Davydova-Lashkin-Fokas-Lenells (DLFL) equation. The DLFL equation (1) is universal in the sense that it contains only three terms of the second of which corresponds to weak dispersion ($\omega \sim 1/k \ll 1$) and the third to weak (cubic) nonlinearity. Here, ω and k are the frequency and wave number respectively, where in the linear part $u \sim \exp(i\omega t - ikx)$. The same situation holds for the NLS and DNLS equations with the weak dispersion $\omega \sim k^2 \ll 1$ (and cubic nonlinearity), and the KdV equation with $\omega \sim k^3 \ll 1$ (and quadratic nonlinearity). Note that the weak dispersion and nonlinearity in all these cases follow from the physical derivation of the corresponding equations [2].

Unlike equation (2) derived for a very special case of pulse propagation in nonlinear optics, the DLFL equation (1) describes nonlinear one-dimensional short-wavelength ion-cyclotron waves in plasmas. The importance of the theoretical study of nonlinear ion-cyclotron waves, in particular solitons, is due, first of all, to reliable experimental data on their observation in the Earth's magnetosphere [22, 23, 24]. In particular, the profile of the measured electric field is definitely a chain of several solitons [24].

The aim of this paper is to obtain the N-soliton solutions of the DLFL equation using the classical method of the IST in which it is possible to take into account the continuous spectrum of the spectral problem (radiation). Note that purely algebraic methods for finding the N-soliton solutions such as the Darboux transformation or the Hirota bilinear method are not suitable for this purpose by definition. Note that in practical situations, the initial perturbation almost never corresponds to a purely solitonic (reflectionless potential) and can be either quite close or very different from it. In any case, then taking into account the continuous spectrum and using IST (in one form or another) is necessary. From a physical point of view, the spectral parameter of the continuous spectrum λ in the IST for the DLFL equation is related to the wave number of emitted quasilinear ion-cyclotron waves k by a simple relation [25]. We use the IST in its classical form, but all results can be easily reformulated and reproduced using the Riemann-Hilbert problem.

The paper is organized as follows. In section 2 we review some results on the IST for the DLFL equation, and then present the general formal solution as the sum of the soliton and nonsoliton parts expressed in terms of the scattering data and Jost solutions corresponding to the discrete and continuous spectrum, respectively. In section 3 we find the N-soliton solutions and, in particular, algebraic N-soliton solutions and solutions in the form of a mixture of rational and exponential functions. The asymptotics of the N-soliton solution is considered in section 4. The conclusion is made in section 5. In the Appendix, we give a brief outline of the derivation of a two-dimensional nonlinear equation describing the dynamics of ion-cyclotron waves in plasmas, which in the one-dimensional case reduces to the DLFL equation (1).

2. Spectral problem for the DLFL equation

At the beginning of this section we will give a brief overview on the IST for the DLFL equation following [25]. The DLFL equation (1) can be written as the compatibility condition

$$\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0, \tag{3}$$

of two linear matrix equations [12, 25]

$$\mathbf{M}_x = \mathbf{U}\mathbf{M},\tag{4}$$

$$\mathbf{M}_t = \mathbf{V}\mathbf{M},\tag{5}$$

where

$$\mathbf{U} = -i\lambda^2 \sigma_3 + \lambda \mathbf{Q}_x, \quad \mathbf{Q} = \begin{pmatrix} 0 & u \\ \sigma u^* & 0 \end{pmatrix}$$
 (6)

$$\mathbf{V} = \frac{i}{4\lambda^2}\sigma_3 - \frac{i}{2\lambda}\sigma_3\mathbf{Q} + \frac{i}{2}\sigma_3\mathbf{Q}^2,\tag{7}$$

and where $\mathbf{M}(x,t,\lambda)$ is a 2×2 matrix-valued function, λ is a complex spectral parameter and σ_3 is the Pauli matrix. The Jost solutions $\mathbf{M}^{\pm}(x,t,\lambda)$ of (4) for real λ^2 and for some fixed t (t-dependence will be omitted for now) are defined by the boundary conditions

$$\mathbf{M}^{\pm}(x,\lambda) \to \exp(-i\lambda^2 \sigma_3 x)$$
 (8)

as $x \to \pm \infty$. The matrix Jost solutions \mathbf{M}^{\pm} are presented in the form

$$\mathbf{M}^{+}(x,\lambda) = \begin{pmatrix} \tilde{\psi}_{1}(x,\lambda) & \psi_{1}(x,\lambda) \\ \tilde{\psi}_{2}(x,\lambda) & \psi_{2}(x,\lambda) \end{pmatrix}, \ \mathbf{M}^{-}(x,\lambda) = \begin{pmatrix} \varphi_{1}(x,\lambda) & -\tilde{\varphi}_{1}(x,\lambda) \\ \varphi_{2}(x,\lambda) & -\tilde{\varphi}_{2}(x,\lambda) \end{pmatrix}.$$
(9)

The scattering matrix S

$$\mathbf{S}(\lambda) = \begin{pmatrix} a(\lambda) & -\tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda) \end{pmatrix}$$
 (10)

with $a\tilde{a} + b\tilde{b} = 1$ relates the two fundamental solutions \mathbf{M}^- and \mathbf{M}^+

$$\mathbf{M}^{-}(x,\lambda) = \mathbf{M}^{+}(x,\lambda)\mathbf{S}(\lambda),\tag{11}$$

so that

$$\varphi = a\tilde{\psi} + b\psi, \tag{12}$$

$$\tilde{\varphi} = -\tilde{a}\psi + \tilde{b}\tilde{\psi}.\tag{13}$$

Further we set $\sigma = -1$ without loss of generality. It follows from (4) and (11) that matrices \mathbf{M}^{\pm} and \mathbf{S} have the parity symmetry properties,

$$\mathbf{M}^{\pm}(x,\lambda) = \sigma_3 \mathbf{M}^{\pm}(x,-\lambda)\sigma_3, \ \mathbf{S}(\lambda) = \sigma_3 \mathbf{S}(-\lambda)\sigma_3, \tag{14}$$

and the conjugation symmetry properties

$$\mathbf{M}^{\pm}(x,\lambda) = \sigma_2 \mathbf{M}^{\pm *}(x,\lambda^*) \sigma_2, \tag{15}$$

$$\tilde{a}(\lambda) = a^*(\lambda^*), \quad \tilde{b}(\lambda) = b^*(\lambda^*),$$
(16)

where σ_2 and σ_3 are Pauli matrices. The coefficients $a(\lambda)$ and $b(\lambda)$ are

$$a(\lambda) = \det(\varphi, \psi), \ b(\lambda) = \det(\tilde{\psi}, \varphi).$$
 (17)

The zeros λ_j^2 $(j=1\dots N)$ of the function $a(\lambda)$ in the region of its analiticity ${\rm Im}\,\lambda^2>0$ (correspondingly, the zeros λ_j^{*2} of the function $\tilde{a}(\lambda)$ in the region

Im $\lambda^2 < 0$) give the discrete spectrum of the linear problem (4) and correspond to solitons. The zeros λ_j $(j=1\dots 2N)$ appear in pairs and one can choose λ_j $(j=1\dots N)$ in the first quadrant and $\lambda_{j+N}=-\lambda_j$ in the third quadrant. Then, as it follows from (12) and (13), the functions $\varphi(x,\lambda_j)$ and $\psi(x,\lambda_j)$ are linearly dependent

$$\varphi(x,\lambda_i) = b_i(\lambda_i)\psi(x,\lambda_i), \quad \tilde{\varphi}(x,\lambda_i^*) = b_i^*(\lambda_i^*)\tilde{\psi}(x,\lambda_i^*). \tag{18}$$

The coefficient $a(\lambda)$ can be expressed in terms of its zeros and the values of $b(\lambda)$ on the contour $\Gamma = (+\infty, 0) \bigcup (-\infty, 0) \bigcup (+i\infty, 0) \bigcup (-i\infty, 0)$ [25],

$$a(\lambda) = \prod_{j=1}^{N} \frac{\lambda_j^{*2}}{\lambda_j^2} \frac{(\lambda^2 - \lambda_j^2)}{(\lambda^2 - \lambda_j^{*2})} \exp\left\{\frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda^2 \ln(1 + \sigma \operatorname{sgn} \mu^2 |b(\mu)|^2)}{\mu(\mu^2 - \lambda^2)} d\mu\right\}. \quad (19)$$

An important particular case is that of the solitonic ("reflectionless") potentials u(x) when $b(\lambda, t) = 0$ as a function of λ for some fixed t. It then follows from (19) that

$$a(\lambda) = \prod_{j=1}^{N} \frac{\lambda_j^{*2}}{\lambda_j^2} \frac{(\lambda^2 - \lambda_j^2)}{(\lambda^2 - \lambda_j^{*2})}.$$
 (20)

The time evolution of the scattering data, as usual in the IST, turns out to be trivial,

$$\lambda_i(t) = \lambda_i(0), \tag{21}$$

$$b_j(t) = b_j(0) \exp[-i/(2\lambda_j^2)t],$$
 (22)

$$b(\lambda, t) = b(\lambda, 0) \exp[-i/(2\lambda^2)t]. \tag{23}$$

and in the following we denote $\lambda_j(t) \equiv \lambda_j$, $b_j(t) \equiv b_j$ and $b(\lambda, t) \equiv b(\lambda)$. Taking into account the boundary conditions (8), the corresponding integral equations for \mathbf{M}^+ can be obtained from (4):

$$\psi_1(x,\lambda) = -\lambda \int_x^\infty e^{-i\lambda^2(x-y)} u_y \psi_2(y,\lambda) \, dy, \tag{24}$$

$$\psi_2(x,\lambda) = e^{i\lambda^2 x} + \lambda \int_x^\infty e^{i\lambda^2 (x-y)} u_y^* \psi_1(y,\lambda) \, dy, \tag{25}$$

$$\tilde{\psi}_1(x,\lambda) = e^{-i\lambda^2 x} - \lambda \int_x^{\infty} e^{-i\lambda^2 (x-y)} u_y \tilde{\psi}_2(y,\lambda) \, dy, \tag{26}$$

$$\tilde{\psi}_2(x,\lambda) = \lambda \int_x^\infty e^{i\lambda^2(x-y)} u_y^* \tilde{\psi}_1(y,\lambda) \, dy. \tag{27}$$

Then, from (24)-(27) one can find the corresponding asymptotics at $\lambda \to 0$,

$$\psi_1(x,\lambda) = \lambda u + O(\lambda^2),\tag{28}$$

$$\psi_2(x,\lambda) = 1 + O(\lambda^2),\tag{29}$$

$$\tilde{\psi}_1(x,\lambda) = 1 + O(\lambda^2),\tag{30}$$

$$\tilde{\psi}_2(x,\lambda) = -\lambda u^* + O(\lambda^2). \tag{31}$$

Note that equation (4), that is the x part of the Lax pair (4)-(5) of the DLFL equation, is simply related to the x part of the Lax pair of the DNLS equation by the replacement $u \to u_x$. The revised Zakharov equations for the Jost functions ψ and $\tilde{\psi}$ of the DNLS equation were obtained in [26] and coincide with the corresponding equations of the DLFL equation, except that the time dependences of the coefficients $b_j(t)$ and $b(\lambda, t)$ are determined by equations (22) and (23) respectively. Following [26] and using this analogy, we can write the equations for $\tilde{\psi}_{1,2}$ in the form

$$\tilde{\psi}_{1}(x,\lambda) = e^{-i\lambda^{2}x} + \sum_{k=1}^{2N} \frac{\lambda^{2}}{\lambda_{k}^{2}(\lambda - \lambda_{k})} \frac{b_{k}(\lambda_{k})}{\dot{a}(\lambda_{k})} \psi_{1}(x,\lambda_{k}) e^{i(\lambda_{k}^{2} - \lambda^{2})x}
+ \frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda^{2}}{\lambda'^{2}(\lambda' - \lambda)} \frac{b(\lambda')}{a(\lambda')} \psi_{1}(x,\lambda') e^{i(\lambda'^{2} - \lambda^{2})x} d\lambda',$$

$$\tilde{\psi}_{2}(x,\lambda) = \sum_{k=1}^{2N} \frac{\lambda}{\lambda_{k}(\lambda - \lambda_{k})} \frac{b_{k}(\lambda_{k})}{\dot{a}(\lambda_{k})} \psi_{2}(x,\lambda_{k}) e^{i(\lambda_{k}^{2} - \lambda^{2})x}
+ \frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda}{\lambda'(\lambda' - \lambda)} \frac{b(\lambda')}{a(\lambda')} \psi_{2}(x,\lambda') e^{i(\lambda'^{2} - \lambda^{2})x} d\lambda'.$$
(32)

Then from (31) and (33) we have

$$u^* = -\lim_{\lambda \to 0} \frac{\tilde{\psi}_2(x,\lambda)}{\lambda} = u_s^* + u_{rad}^*, \tag{34}$$

where u_s^* corresponds to the discrete part of the spectrum (solitons),

$$u_s^* = \sum_{k=1}^N \frac{2}{\lambda_k^2} \frac{b_k(\lambda_k)}{\dot{a}(\lambda_k)} \psi_2(x, \lambda_k) e^{i\lambda_k^2 x}, \tag{35}$$

and we have taken into account the reduction properties (14) and (15), using which the sum over 2N terms in (33) is replaced by the sum over N. The term u_{rad}^* corresponds to the continuous spectrum (radiation field),

$$u_{rad}^* = \frac{1}{2\pi i} \int_{\Gamma} \frac{\psi_2(x,\lambda)}{\lambda^2} \frac{b(\lambda)}{a(\lambda)} e^{i\lambda^2 x} d\lambda.$$
 (36)

In the general case, as in other integrable models such as the NLS equation, KdV equation, etc., an arbitrary initial perturbation vanishing at infinity rapidly enough decays over time into dispersive quasilinear waves corresponding to u_{rad} and solitons corresponding to u_s (if any - depending on the initial conditions, the solitons may not occur at all). In contrast to the work of Matsuno [17], where the purely algebraic Hirota bilinear method was used to find N-soliton solutions, as well as the work of Lenells [16] using the dressing method, the expression (34) also contains the nonsoliton part u_{rad} associated with the nonzero coefficient $b(\lambda)$ (or, equivalently, to $r(\lambda) = b(\lambda)/a(\lambda)$ sometimes called the reflection coefficient in the IST). The initial conditions corresponding to purely

N-soliton solutions, as is known, correspond to $b(\lambda) = 0$ (the so-called reflectionless potentials). As was shown in [25], considering the radiative component as a superposition of free waves governed by the linearized equation (1) with the dispersion law $\omega = 1/k$, one can conclude that the spectral parameter λ is connected to the wave number of the emitted quasilinear waves k by the relation

$$k = 2\lambda^2. (37)$$

In the general case, when $b(\lambda) \neq 0$, as is known, the solution cannot be written in an explicit analytical form, however, if $b(\lambda)$ is not equal to zero, but small enough, $b(\lambda) \ll 1$, one can use the perturbation theory. Under this, the coefficient $a(\lambda)$ and the Jost function $\psi_2(x,\lambda)$ can be taken purely soliton (the calculation of the purely N-soliton $\psi_2(x,\lambda)$ is described in the next section). Continuous spectrum effects, in particular, the spectral distribution of ion-cyclotron wave radiation within the framework of the DLFL equation (1), were considered in [25]. An example of calculating of the radiation field u_{rad} in the physical space for the DNLS equation is given in [27].

3. N-soliton solutions

In the pure soliton case $b(\lambda) = 0$, using the parity and conjugation properties (14) and (15) in equations (32) and (33), we have for the Jost solutions $\tilde{\psi}_1(x,\lambda)$ and $\tilde{\psi}_2$,

$$\tilde{\psi}_1(x,\lambda) = e^{-i\lambda^2 x} + \sum_{k=1}^{N} \frac{2\lambda^2}{\lambda_k(\lambda^2 - \lambda_k^2)} \frac{b_k(\lambda_k)}{\dot{a}(\lambda_k)} \psi_1(x,\lambda_k) e^{i(\lambda_k^2 - \lambda^2)x}$$
(38)

$$\tilde{\psi}_2(x,\lambda) = \sum_{k=1}^N \frac{2\lambda}{(\lambda^2 - \lambda_k^2)} \frac{b_k(\lambda_k)}{\dot{a}(\lambda_k)} \psi_2(x,\lambda_k) e^{i(\lambda_k^2 - \lambda^2)x}.$$
 (39)

Evaluating (38) and (39) at λ_j^* and taking into account the conjugation properties (15), one can obtain

$$\psi_2^*(x,\lambda_j) = e^{-i\lambda_j^{*2}x} + \sum_{k=1}^N \frac{2\lambda_j^{*2}}{\lambda_k(\lambda_j^{*2} - \lambda_k^2)} \frac{b_k(\lambda_k)}{\dot{a}(\lambda_k)} \psi_1(x,\lambda_k) e^{i(\lambda_k^2 - \lambda_j^{*2})x}$$
(40)

$$\psi_1^*(x,\lambda_j) = -\sum_{k=1}^N \frac{2\lambda_j^*}{(\lambda_j^{*2} - \lambda_k^2)} \frac{b_k(\lambda_k)}{\dot{a}(\lambda_k)} \psi_2(x,\lambda_k) e^{i(\lambda_k^2 - \lambda_j^{*2})x}.$$
 (41)

Equations (40) (after its complex conjugation) and (41) are a system of 2N linear algebraic equations for the vector functions $\psi_1^*(x,\lambda_j)$ and $\psi_2(x,\lambda_j)$. This system can be solved in a standard way, and after obtaining $\psi_2(x,\lambda_j)$ and using (35) one can find a solution u^* through the corresponding determinants. A similar procedure was used to obtain N-soliton solutions of the DNLS equation [26]. Here, however, we present a simple alternative way of finding $\psi_2(x,\lambda_j)$, which leads to a much more compact formula for the N-soliton solution of

equation (1). An analogue of equation (38) for the function $\varphi_1(x,\lambda)$ can be written in the form

$$\varphi_1(x,\lambda) = e^{-i\lambda^2 x} + \sum_{k=1}^{N} \frac{2\lambda^2}{\lambda_k^* (\lambda^2 - \lambda_k^{*2})} \frac{b_k(\lambda_k^*)}{\dot{a}(\lambda_k^*)} \tilde{\varphi}_1(x,\lambda_k^*) e^{i(\lambda_k^{*2} - \lambda^2)x}.$$
(42)

Evaluating this at λ_i we have

$$\varphi_1(x,\lambda_j) = e^{-i\lambda_j^2 x} + \sum_{k=1}^N \frac{2\lambda_j^2}{\lambda_k^* (\lambda_j^2 - \lambda_k^{*2})} \frac{b_k(\lambda_k^*)}{\dot{a}(\lambda_k^*)} \tilde{\varphi}_1(x,\lambda_k^*) e^{i(\lambda_k^{*2} - \lambda_j^2)x}. \tag{43}$$

Using $\dot{\tilde{a}}(\lambda_k^*) = \dot{a}^*(\lambda_k)$ and $b_k(\lambda_k^*)b_k^*(\lambda_k) = 1$ we find

$$b_{j}(\lambda_{j})\psi_{1}(x,\lambda_{j}) = e^{-i\lambda_{j}^{2}x} + \sum_{k=1}^{N} \frac{2\lambda_{j}^{2}}{\lambda_{k}^{*}(\lambda_{j}^{2} - \lambda_{k}^{*2})} \frac{1}{\dot{a}^{*}(\lambda_{k})} \psi_{2}^{*}(x,\lambda_{k}) e^{i(\lambda_{k}^{*2} - \lambda_{j}^{2})x}.$$
(44)

On the other hand, taking complex conjugate of (41) and then multiplying it by $b_i(\lambda_i)$ we have

$$b_j(\lambda_j)\psi_1(x,\lambda_j) = -\sum_{k=1}^N \frac{2\lambda_j b_j(\lambda_j)}{(\lambda_j^2 - \lambda_k^{*2})} \frac{b_k^*(\lambda_k)}{\dot{a}^*(\lambda_k)} \psi_2^*(x,\lambda_k) e^{-i(\lambda_k^{*2} - \lambda_j^2)x}.$$
 (45)

Subtracting (45) from (44), one can readily get

$$1 + \sum_{k=1}^{N} \frac{(\lambda_j^2 \lambda_k^* + \lambda_j \lambda_k^{*2} c_j c_k^*)}{(\lambda_j^2 - \lambda_k^{*2})} F_k = 0, \tag{46}$$

where

$$F_k = \frac{2\psi_2^*(\lambda_k)}{\lambda_k^{*2}\dot{a}^*(\lambda_k)} e^{i\lambda_k^{*2}x},\tag{47}$$

and the time dependence of b_j in (22) is explicitly taken into account, so that

$$c_j = b_j \exp\left(2i\lambda_j^2 x - i/(2\lambda_j^2)t\right). \tag{48}$$

Using the expression for the N-soliton solution (35) and taking into account (47) we have for u

$$u = \sum_{k=1}^{N} c_k^* F_k. (49)$$

From (46) and (49) one can obtain the N-soliton solution u in a compact form

$$u = \sum_{k,j=1}^{N} c_k^* (\mathbf{K}^{-1})_{kj}, \tag{50}$$

where the elements of the $N \times N$ matrix **K** are

$$K_{jk} = \frac{\lambda_j \lambda_k^*}{\lambda_k^{*2} - \lambda_j^2} (\lambda_j + \lambda_k^* c_j c_k^*). \tag{51}$$

Equations (50) and (51) were previously obtained by Lenells [16] using the dressing method, but the solution u was not expressed in the determinant form. Note also that in [16], a more general equation than (1) was considered. Using (50) and the identity

$$\mathbf{A}_{1}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{A}_{2} = \frac{\det(\mathbf{A} + \mathbf{A}_{2}\mathbf{A}_{1}^{\mathrm{T}})}{\det(\mathbf{A})} - 1,$$
(52)

where **A** is an arbitrary $N \times N$ matrix, **A**₁ and **A**₂ are arbitrary $N \times 1$ matrices respectively, we can write the N-soliton solution of equation (1) as

$$u = \frac{\det(\tilde{\mathbf{K}}) - \det(\mathbf{K})}{\det(\mathbf{K})},\tag{53}$$

where the elements of the matrix $\tilde{\mathbf{K}}$ are

$$\tilde{K}_{jk} = K_{jk} + c_k^*. \tag{54}$$

In what follows, we parameterize the complex numbers λ_j and b_j in terms of four real parameters $\Delta_j > 0$, $0 < \gamma_j < \pi$, x_{0j} (the initial position of the soliton) and ϕ_{0j} (the initial phase) as

$$\lambda_j^2 = \Delta_j^2(\cos\gamma_j + i\sin\gamma_j),\tag{55}$$

$$b_j = \exp(2x_{0j}\Delta_j^2 \sin \gamma_j + i\phi_{0j}). \tag{56}$$

With this parametrization λ_j and $-\lambda_j$ lie in the 1-st and 3-rd quadrants respectively of the complex plane $(\pm \lambda_j^*$ – in the 2-st and 4-th quadrants respectively). Then c_j determined by (48) can be written as

$$c_i = \exp(-z_i + i\Phi_i), \tag{57}$$

where

$$z_j = 2\Delta_j^2 \left(x - x_{0j} + v_j t\right) \sin \gamma_j, \tag{58}$$

with $v_j = 1/(4\Delta_j^4)$, and

$$\Phi_j = 2\Delta_j^2 (x - v_j t) \cos \gamma_j + \phi_{0j}. \tag{59}$$

Using this parametrization and (51), for the elements K_{jk} of the matrix **K** one can obtain

$$K_{jk} = \frac{\Delta_j \Delta_k \left[\Delta_j e^{i(\gamma_j - \gamma_k/2)} + \Delta_k e^{i(\gamma_j/2 - \gamma_k)} e^{-z_j - z_k + i(\Phi_j - \Phi_k)} \right]}{\Delta_k^2 e^{-i\gamma_k} - \Delta_j^2 e^{i\gamma_j}}.$$
 (60)

In particular, for K_{jj} we have

$$K_{jj} = \frac{i\Delta_j e^{-z_j} \cosh(z_j + i\gamma_j/2)}{\sin \gamma_j}.$$
 (61)

3.1. N-soliton solutions: rational and a mixture of exponential and rational solutions

Equation (53) with matrix elements K_{jk} and c_j determined by (60) and (57) respectively is the N-soliton solution in exponential functions that decays exponentially at infinity. It is essential that for these solutions $\gamma_j < \pi$ and $\gamma_k < \pi$ in (60). However, the apparent singularity $\gamma_j = \pi$ for K_{jj} in (61) is fictitious and it is easy to show that in the limit $\gamma_j \to \pi$ the elements K_{jj} become rational functions of x. In the limit $\gamma_j \to \pi$ and $\gamma_k \to \pi$, for the values c_j^* in (57) and K_{jj} in (61) one can obtain

$$c_j^* = \exp\left[2i\Delta_j^2 \left(x - v_j t\right) - i\phi_{0j}\right],\tag{62}$$

and

$$K_{jj} = \frac{1}{2} \left[i\Delta_j - 4\Delta_j^3 \left(x - x_{0j} + v_j t \right) \right], \tag{63}$$

and for $j \neq k$

$$K_{jk} = \frac{i\Delta_j \Delta_k}{\Delta_k^2 - \Delta_j^2} \left\{ \Delta_k \exp\left[2i\left(\Delta_k^2 - \Delta_j^2\right) \left(x + \frac{t}{4\Delta_j^2 \Delta_k^2}\right)\right] - \Delta_j \right\}.$$
 (64)

Equation (53) with c_j and the matrix elements K_{jk} determined by (62) and (63), (64) respectively is the rational N-soliton solution decaying in power law at infinity.

An interesting situation arises if $\gamma_j \to \pi$ with $j = 1 \dots M$, where M < N and $\gamma_k \neq \pi$ with $k = M + 1 \dots N$. Then from (60) one can find

$$K_{jk} = \frac{\Delta_j \Delta_k}{\Delta_k^2 + \Delta_j^2 e^{i\gamma_k}} \left(i\Delta_k e^{-z_k + i\Psi_{jk}} - \Delta_j e^{i\gamma_k/2} \right), \tag{65}$$

where

$$\Psi_{jk} = -2(\Delta_j^2 + \Delta_k^2 \cos \gamma_k)x + \left(\frac{1}{\Delta_j^2} + \frac{\cos \gamma_k}{\Delta_k^2}\right)\frac{t}{2}.$$
 (66)

Similarly, if $\gamma_k \to \pi$ and $\gamma_j \neq \pi$ (note that the matrix K_{jk} is not symmetric) we have

$$K_{jk} = \frac{\Delta_j \Delta_k}{\Delta_k^2 e^{-i\gamma_j} + \Delta_j^2} \left(i\Delta_j + \Delta_k e^{-i\gamma_j/2} e^{-z_j + i\Psi_{jk}} \right), \tag{67}$$

where

$$\Psi_{jk} = 2(\Delta_k^2 + \Delta_j^2 \cos \gamma_j)x - \left(\frac{1}{\Delta_k^2} + \frac{\cos \gamma_j}{\Delta_j^2}\right)\frac{t}{2}.$$
 (68)

Equation (53) with the coefficients c_j and c_k determined by (48) and (62) respectively, and the matrix elements determined by (60) and (65), is an N-soliton solution consisting of a mixture of M rational and N-M exponential functions.

3.2. One-soliton solutions: exponential and rational

The case N=1 corresponds to the one-soliton solution of equation (1). Then from (51) and (54) we have

$$K_{11} = \frac{|\lambda_1|^2 (\lambda_1 + \lambda_1^* |c_1|^2)}{\lambda_1^{*2} - \lambda_1^2}, \quad \tilde{K}_{11} = K_{11} + c_1^*, \tag{69}$$

and from (53) one can readily get

$$u_1 = \frac{c_1^*}{K_{11}} = \frac{c_1^*(\lambda_1^{*2} - \lambda_1^2)}{|\lambda|^2(\lambda_1 + \lambda_1^*|c_1|^2)}.$$
 (70)

Using the parametrization (55) and (56) this solution takes the form

$$u_1 = \frac{\sin \gamma_1 \exp(-i\Phi_1)}{i\Delta_1 \cosh(z_1 + i\gamma_1/2)}.$$
(71)

An explicit expression for u_1 in terms of the soliton amplitude and phase is

$$u_{1} = \frac{\sin \gamma_{1} \exp\{-i\Phi_{1} - i \arctan[\tanh z_{1} \tan(\gamma_{1}/2)]\}}{i\Delta_{1} \sqrt{\cosh^{2} z_{1} - \sin^{2}(\gamma_{1}/2)}}.$$
 (72)

Earlier this solution was obtained by Davydova and Lashkin [7, 10] without using the IST. The soliton velocity (in the negative direction of x-axis) v_1 , amplitude A and the characteristic halfwidth of the soliton w are

$$v_1 = \frac{1}{4\Delta_1^4}, \quad A = \frac{\sin \gamma_1}{\Delta_1}, \quad w = \frac{1}{2\Delta_1^2 \sin \gamma_1}.$$
 (73)

It is seen that the soliton can not be motionless, and it moves only in the negative direction of x-axis. In the limit $\gamma_1 \to \pi$, from (71) (or, directly from (62), (63) and (70)) one can obtain the soliton with algebraic decay at infinity,

$$u_1 = \frac{2\exp(-i\Phi_1)}{\Delta_1(i-2y)},\tag{74}$$

where $y = 2\Delta_1(x - x_0 + v_1t)$ and $\Phi_1 = -2\Delta_1^2(x - v_1t) + \phi_{01}$. In terms of the amplitude and phase, the expression (74) takes the form

$$u_1 = \frac{2\exp[-i\Phi_1 + i\arccos(4\Delta_1^2 y)]}{\Delta_1\sqrt{1 + 16\Delta_1^2 y^2}}.$$
 (75)

This algebraic soliton solution of the DLFL equation (1) was first obtained in [10] and then rediscovered in [12].

3.3. Two-soliton solutions

In the case N=2 the corresponding matrix elements in (51) and (54) have the form

$$K_{12} = \frac{\lambda_1 \lambda_2^* (\lambda_1 + \lambda_2^* c_1 c_2^*)}{\lambda_2^{*2} - \lambda_1^2}, \quad \tilde{K}_{12} = K_{12} + c_2^*, \tag{76}$$

$$K_{21} = \frac{\lambda_2 \lambda_1^* (\lambda_2 + \lambda_1^* c_2 c_1^*)}{\lambda_1^{*2} - \lambda_2^2}, \quad \tilde{K}_{21} = K_{21} + c_1^*, \tag{77}$$

$$K_{22} = \frac{|\lambda_2|^2 (\lambda_2 + \lambda_2^* |c_2|^2)}{\lambda_2^{*2} - \lambda_2^2}, \quad \tilde{K}_{22} = K_{22} + c_2^*, \tag{78}$$

and K_{11} is determined by (69). Then, as one can see from (53), the corresponding general two-soliton solution is

$$u_2 = \frac{c_1^*(K_{22} - K_{12}) + c_2^*(K_{11} - K_{21})}{K_{11}K_{22} - K_{12}K_{21}},\tag{79}$$

where c_1^* and c_2^* are determined by (57).

In the particular case, when the eigenvalues $\lambda_{1,2}^2$ are purely imaginary (this corresponds to $\gamma_{1,2} = \pi/2$), we have

$$c_1 = e^{-y_1}, \quad c_2 = e^{-y_2},$$
 (80)

$$K_{11} = i\Delta_1 e^{-y_1} \cosh(y_1 + i\pi/4), \quad K_{22} = i\Delta_2 e^{-y_2} \cosh(y_2 + i\pi/4),$$
 (81)

$$K_{12} = i \frac{\Delta_1 \Delta_2}{\Delta_1^2 + \Delta_2^2} \left(\Delta_1 e^{i\pi/4} + \Delta_2 e^{-i\pi/4} e^{-y_1 - y_2} \right), \tag{82}$$

$$K_{21} = i \frac{\Delta_1 \Delta_2}{\Delta_1^2 + \Delta_2^2} \left(\Delta_2 e^{i\pi/4} + \Delta_1 e^{-i\pi/4} e^{-y_1 - y_2} \right), \tag{83}$$

where j = 1, 2 and $y_j = 2\Delta_j^2(x - x_{0j} + v_j t)$, and the corresponding two-soliton solution (79) has the simple form

$$u = \frac{i(\Delta_2^2 - \Delta_1^2)[\Delta_1 \cosh y_1^+ - \Delta_2 \cosh y_2^+]}{(\Delta_1^2 + \Delta_2^2)[\Delta_1 \Delta_2[\cosh y_1^+ \cosh y_2^+ + 2i \sinh(y_1 + y_2)] + \Delta_1^2 + \Delta_2^2\}}, \quad (84)$$

where $y_j^+ = y_j + i\pi/4$.

As another particular example, consider the two-soliton rational-exponential bound state. If the velocities v_1 and v_2 of the components in a two-soliton solution are equal, then the solution represents a bound state. Consider such a solution when one of the components is an algebraic soliton. Let $\gamma_1 \to \pi$ and $\gamma_2 \equiv \gamma < \pi$, and $v_1 = v_2 \equiv v$, $v_{01} = v_{02} = v$, $v_{01} = v_{02} = v$. Then the corresponding coefficients v_1^* and v_2^* are

$$c_1^* = \exp[2i\Delta^2(x - vt)], \quad c_2^* = \exp[-y\sin\gamma - 2i\Delta^2(x - vt)\cos\gamma],$$
 (85)

where $y = 2\Delta^2(x + vt)$. From (63) and (61) we have

$$K_{11} = \frac{\Delta}{2}(i - 2y), \quad K_{22} = \frac{i\Delta e^{-y\sin\gamma}\cosh(y\sin\gamma + i\gamma/2)}{\sin\gamma}, \tag{86}$$

and from (65) one can obtain

$$K_{12} = \frac{\Delta \left(i e^{-y \sin \gamma + i\Psi - i\gamma/2} - 1 \right)}{2 \cos(\gamma/2)}, \quad K_{21} = \frac{\Delta \left(i e^{i\gamma/2} + e^{-y \sin \gamma - i\Psi} \right)}{2 \cos(\gamma/2)}, \quad (87)$$

where $\Psi = qx - \Omega t$ with $\Omega = \cos^2(\gamma/2)/\Delta^2$ and $q = \Omega/v$. The solution (79) with such coefficients and matrix elements pulsates with two independent frequencies Ω and $1/(2\Delta^2)$. Note that, despite the presence of an exponential component, the solution decays at infinity by a power law $\sim 1/|y|$. In particular, for $\gamma = \pi/2$ (this corresponds to the largest amplitude of the exponential component), the solution simplified to

$$u = \frac{ie^{i\Psi}\cosh y^{+} - y + (3/2)i}{\Delta[2i(i-2y)\cosh y^{+} - \cosh y^{-} + \cos\Psi]},$$
 (88)

where $y^{\pm} = y \pm i\pi/4$. This solution is a breather (there is one independent frequency) and oscillates with a period $T = 4\pi\Delta^2$.

4. The asymptotic behavior of the N-soliton solution

Consider the time asymptotics of the two-soliton solution (79), assuming that the velocities of the two soliton components v_1 and v_2 are different. Assume $v_1 > v_2$ and let z_1 be fixed. Then at $t \to -\infty$ we have $z_2 \to \infty$ and $|c_1|$ is finite while $|c_2| \to 0$. Evaluating the corresponding K_{jk} from (69) and (76)-(78) and inserting into (79) one can obtain the leading term as

$$u_{2} \sim \frac{c_{1}^{*}(\lambda_{2}^{*2} - \lambda_{1})(\lambda_{1}^{*2} - \lambda_{2})[|\lambda_{2}|^{2}\lambda_{2}(\lambda_{2}^{*2} - \lambda_{1}^{2}) - \lambda_{1}^{2}\lambda_{2}^{*}(\lambda_{2}^{*2} - \lambda_{2}^{2})]}{\lambda_{2}|\lambda_{1}|^{2}|\lambda_{2}|^{2}[(\lambda_{2}^{*2} - \lambda_{1})(\lambda_{1}^{*2} - \lambda_{2})(\lambda_{1} + \lambda_{1}|c_{1}|^{2}) - \lambda_{1}(\lambda_{1}^{*2} - \lambda_{1})(\lambda_{2}^{*2} - \lambda_{2})]},$$
(89)

that can be written in the form

$$u_2 \sim \frac{\tilde{c}_1^*(\lambda_1^{*2} - \lambda_1^2)}{|\lambda|^2(\lambda_1 + \lambda_1^*|\tilde{c}_1|^2)},$$
 (90)

where

$$\tilde{c}_1 = c_1 \exp\left[-\ln\frac{\lambda_2^2(\lambda_2^{*2} - \lambda_1^2)}{\lambda_2^{*2}(\lambda_2^2 - \lambda_1^2)}\right]. \tag{91}$$

One can see that the asymptotic of u_2 determined by (90) has the same form as the one-soliton solution (70) except the phase shifts, so that we have

$$u_2 \sim u_1(z_1 + \Delta z_1^-, \Phi_1 + \Delta \Phi_1^-),$$
 (92)

where

$$\Delta z_1^- = \ln \left| \frac{\lambda_2^{*2} - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} \right|, \quad \Delta \Phi_1^- = -\arg \frac{\lambda_2^{*2} - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} - \arg \frac{\lambda_2^2}{\lambda_2^{*2}} + \pi.$$
 (93)

Similarly, if $t \to +\infty$ we have $z_2 \to -\infty$ and then $|c_1|$ is finite while $|c_2| \to \infty$. The leading term in that case is

$$u_2 \sim \frac{c_1^* \lambda_2^{*2} (\lambda_1^{*2} - \lambda_1^2)(\lambda_2^2 - \lambda_1^2)(\lambda_1^{*2} - \lambda_2^2)}{\lambda_2^2 |\lambda_1|^2 [\lambda_1(\lambda_1^2 - \lambda_2^{*2})(\lambda_1^{*2} - \lambda_2^{*2}) + \lambda_1^*(\lambda_1^2 - \lambda_2^{*2})(\lambda_1^{*2} - \lambda_1^2)|c_1|^2]}, \quad (94)$$

and it can be written as (90), where

$$\tilde{c}_1 = c_1 \exp\left[\ln \frac{\lambda_2^2(\lambda_2^{*2} - \lambda_1^2)}{\lambda_2^{*2}(\lambda_2^2 - \lambda_1^2)}\right]. \tag{95}$$

And

$$u_2 \sim u_1(z_1 + \Delta z_1^+, \Phi_1 + \Delta \Phi_1^+),$$
 (96)

where

$$\Delta z_1^+ = -\ln\left|\frac{\lambda_2^{*2} - \lambda_1^2}{\lambda_2^2 - \lambda_1^2}\right|, \quad \Delta \Phi_1^+ = \arg\frac{\lambda_2^{*2} - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} + \arg\frac{\lambda_2^2}{\lambda_2^{*2}} + \pi.$$
 (97)

The total shifts are determined by $\Delta z_1 = \Delta z_1^+ - \Delta z_1^-$ and $\Delta \Phi_1 = \Delta \Phi_1^+ - \Delta \Phi_1^-$. Then, taking into account (58), (93) and (97), we have for the position shift of the soliton j = 1,

$$\Delta x_1 = \frac{2i}{\lambda_1^2 - \lambda_1^{*2}} \ln \left| \frac{\lambda_2^{*2} - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} \right|$$
 (98)

and for the corresponding phase shift

$$\Delta\Phi_1 = 2\arg\frac{\lambda_2^{*2} - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} + 2\arg\frac{\lambda_2^2}{\lambda_2^{*2}}.$$
 (99)

If we now fix z_2 (as before $v_1 > v_2$), then after similar calculations one can obtain

$$\Delta x_2 = -\frac{2i}{\lambda_2^2 - \lambda_2^{*2}} \ln \left| \frac{\lambda_1^{*2} - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \right|, \tag{100}$$

and for the corresponding phase shift

$$\Delta\Phi_2 = -2\arg\frac{\lambda_1^{*2} - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} - 2\arg\frac{\lambda_1^2}{\lambda_1^{*2}}.$$
 (101)

Generalization to the N-soliton solution can be performed straightforwardly following the two-soliton case. Soliton velocities are assumed to be ordered as $v_1 > v_2 > \cdots > v_N$. As in the two-soliton case, we consider the limits $t \to -\infty$ and $t \to \infty$ and assume that z_n is fixed. Then in the first case we have $|c_j| \to \infty$ for j < n and $|c_j| \to 0$ for n < j, and in the second $|c_j| \to 0$ for j < n and $|c_j| \to \infty$ for n < j. In both cases, the leading-order asymptotics of the matrices $\tilde{\mathbf{K}}$ and \mathbf{K} are Cauchy matrices [28] and the corresponding determinants in (53) are factorized (that is, they are Cauchy determinants). Omitting the calculations, we present only the final results. The asymptotic form of the N-soliton solution is

$$u_N \sim u_1(z_n + \Delta z_n^-, \Phi_n + \Delta \Phi_n^-), \tag{102}$$

as
$$t \to -\infty$$
, and $u_N \sim u_1(z_n + \Delta z_n^+, \Phi_n + \Delta \Phi_n^+),$ (103)

as $t \to \infty$ with $\Delta z_n^+ = -\Delta z_n^-$ and $\Delta \Phi_n^+ = -\Delta \Phi_n^- + 2\pi N$. The total phase shifts of the *n*-th soliton

$$\Delta x_n = \frac{2i}{\lambda_n^2 - \lambda_n^{*2}} \left[\sum_{j=n+1}^N \ln \left| \frac{\lambda_j^{*2} - \lambda_n^2}{\lambda_j^2 - \lambda_n^2} \right| - \sum_{j=1}^{n-1} \ln \left| \frac{\lambda_j^{*2} - \lambda_n^2}{\lambda_j^2 - \lambda_n^2} \right| \right], \tag{104}$$

$$\Delta\Phi_{n} = 2\sum_{j=n+1}^{N} \left[\arg\left(\frac{\lambda_{j}^{*2} - \lambda_{n}^{2}}{\lambda_{j}^{2} - \lambda_{n}^{2}}\right) + \arg\left(\frac{\lambda_{j}^{2}}{\lambda_{j}^{*2}}\right) \right]$$

$$-2\sum_{j=1}^{n-1} \left[\arg\left(\frac{\lambda_{j}^{*2} - \lambda_{n}^{2}}{\lambda_{j}^{2} - \lambda_{n}^{2}}\right) + \arg\left(\frac{\lambda_{j}^{2}}{\lambda_{j}^{*2}}\right) \right].$$

$$(105)$$

In the general case, the asymptotic N-soliton solution is a superposition of N separate one-soliton solutions with the corresponding parameters Δ_j and γ_j where $j = 1 \dots N$. Note that, for a rational soliton $(\gamma_n \to \pi)$ we have $\lambda_n^2 = -\Delta_n^2$ and, using L'Hôpital's rule, we can obtain $\Delta x_n = 0$, so that the position shift of the of this soliton upon interaction with other solitons is equal to zero.

5. Conclusion

In this paper, we have presented a simple and constructive method for finding N-soliton solutions of the DLFL equation (1) to describe the dynamics of nonlinear ion-cyclotron waves in a plasma. The proposed method is based on the classical formulation of the IST and differs from the Hirota bilinear method used in [17] as well as the dressing method in [16] primarily in that it allows one to take into account the contribution of the continuous spectrum that is, the radiation field. The resulting general expression for arbitrary initial data decaying at infinity is written in terms of discrete and continuous scattering data and the corresponding Jost solutions and consists of soliton and nonsoliton (radiative) parts. The first of them corresponds to the discrete spectrum of the spectral problem (4) and the second part does to the continuous spectrum. The radiation part is represented as an integral over the spectral parameter, and depends on one of the Jost solutions and the reflection coefficient. Thus, the radiative part corresponding to quasilinear ion-cyclotron waves can, in principle, be determined explicitly if the corresponding Jost solution and the reflection coefficient are known. For example, under certain conditions, that is, using perturbation theory, the Jost solution and coefficient $a(\lambda)$ can be taken as purely soliton ones.

We have found two new types of N-soliton solutions the DLFL equation (1): an algebraic N-soliton solution in rational functions, and a solution in the form of a mixture of M rational and N-M exponential functions. Both solutions

are presented in determinant form. As an example, we write out two twosoliton solutions explicitly. The first of them corresponds to purely imaginary eigenvalues, and the second represents a solution in the form of a bound state of the usual bright soliton and the algebraic soliton, which pulsates with two independent frequencies.

6. Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

7. CRediT authorship contribution statement

V. M. Lashkin: Conceptualization, Methodology, Validation, Formal analysis, Investigation.

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9. Appendix

In this appendix we give a short outline of the derivation of the two-dimensional nonlinear equation describing the dynamics of ion-cyclotron waves in plasmas, first suggested by Davydova and Lashkin in [7] (see also [10]), which in the one-dimensional case reduces to the DLFL equation (1). For a plasma in an uniform external magnetic field $\mathbf{H}_0 = H_0 \hat{\mathbf{z}}$ oriented along the z-axis, the linear dispersion relation for the electrostatic ion-cyclotron waves (the Bernstein modes) in the short-wavelength limit $k_{\perp} \rho_i \gg 1$ under the conditions $k_{\perp} \rho_e \ll 1$ and $\omega \ll k_z v_{Te}$ is,

$$\omega(\mathbf{k}) = n\Omega_i \left[1 + \frac{1}{\sqrt{2\pi}(1 + T_i/T_e)k_{\perp}\rho_i} \right] \equiv n\Omega_i [1 + R(k_{\perp})], \tag{106}$$

where $R(k_{\perp}) \ll 1$ [9]. Here ω and \mathbf{k} are the frequency and wave vector respectively, $k_{\perp} = \sqrt{k_x^2 + k_y^2}$, Ω_i is the ion-cyclotron frequency, ρ_{α} , $v_{T\alpha}$ and T_{α} are the Larmor radius, thermal velocity and temperature of particle species α (e for electrons and i for ions) respectively, $n = 1, 2, \ldots$. Next, we consider the case of only the lowest harmonic n = 1. The Maxwell equation $\nabla \cdot \mathbf{D} = 0$ for the electrical displacement $\mathbf{D}(\omega, \mathbf{k}) = \varepsilon(\omega, \mathbf{k})\mathbf{E}(\omega, \mathbf{k})$, where ε and \mathbf{E} are the dielectric function and electric field in the Fourier space respectively, can be written in the physical two-dimensional space as

$$\nabla_{\perp} \cdot (\hat{\varepsilon} \nabla_{\perp} \varphi) = 0, \tag{107}$$

where $\hat{\varepsilon}$ is considered as a differential operator with $\omega \to i\partial/\partial t$ and $\mathbf{k} \to -i\nabla_{\perp} \equiv -i(\partial/\partial x, \partial/\partial y)$. The principal nonlinear effect for ion-cyclotron waves is the perturbation of the magnetic field δH_z [7, 10]. In this case, the nonlinear correction to the ion-cyclotron frequency in the expression for $\hat{\varepsilon}$ is taken into account, so that $\Omega_i \to \Omega_i(1+h)$, where the relative nonlinear perturbation h of the magnetic field H_0 is

$$h = \frac{\delta H_z}{H_0} = -\frac{\omega_{pe}^2 m |\Phi|^2}{4H_0^2 T_e},\tag{108}$$

where Φ is the envelope of the electrostatic potential $\tilde{\Phi}$ at the ion-cyclotron frequency,

$$\tilde{\Phi} = \frac{1}{2} [\Phi \exp(-i\Omega_i t) + c. c], \tag{109}$$

and ω_{pe} is the electron plasma frequency, m is the electron mass. In [7, 10], the anisotropy of electron temperatures was also taken into account, and then for the electron temperature T_e in (108) it would be $T_e \to T_{e,\parallel}^2/T_{e,\perp}$, where $T_{e,\parallel}$ (i.e., along the z-axis) and $T_{e,\perp}$ are parallel and transverse electron temperatures respectively. Expanding $\varepsilon(\omega, \mathbf{k})$ near the eigenfrequency $\omega_{\mathbf{k}}$ determined by (106) with the nonlinear correction (108) yields

$$\varepsilon(\omega, \mathbf{k}) = \varepsilon(\omega_{\mathbf{k}}, \mathbf{k}) + \varepsilon'(\omega_{\mathbf{k}}, \mathbf{k})(\omega - \omega_{\mathbf{k}}), \tag{110}$$

where $\varepsilon'(\omega_k) \equiv \partial \varepsilon(\omega)/\partial \omega \mid_{\omega=\omega_k}$. Substituting (110) into (107) along with (106) and (108), one can obtain the nonlinear equation [7, 10] in the form

$$\Delta_{\perp} \left(\frac{i}{\Omega_i} \frac{\partial \Phi}{\partial t} - \hat{R} \Phi \right) = \nabla_{\perp} \cdot (h \nabla \Phi), \tag{111}$$

where $\Delta_{\perp} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and the operator \hat{R} is defined by

$$\hat{R}\Phi(\mathbf{r},t) = \int R(k_{\perp})\hat{\Phi}(\mathbf{k}_{\perp},t) \exp(i\mathbf{k}_{\perp} \cdot \mathbf{r}) d\mathbf{k}_{\perp}.$$
 (112)

In the one-dimensional case, and in the dimensionless variables

$$x \to \frac{x}{\sqrt{2\pi}(1 + T_i/T_e)\rho_i}, \quad u \to \Phi \frac{\omega_{pe}}{2H_0} \sqrt{\frac{m}{T_e}},$$
 (113)

equation (111) reduces to the DLFL equation (1), where the signs $\sigma=\pm 1$ correspond to Φ^* and Φ respectively.

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