Symmetries, conservation laws and difference schemes of the (1+2)-dimensional shallow water equations in Lagrangian coordinates

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Abstract

The two-dimensional shallow water equations in Eulerian and Lagrangian coordinates are considered. Lagrangian and Hamiltonian formalism of the equations is given. The transformations mapping the two-dimensional shallow water equations with a circular or plane bottom into the gas dynamics equations of a polytropic gas with polytropic exponent $\gamma = 2$ is represented.

Group properties of the equations are considered, and the group classification for the case of the elliptic paraboloid bottom topography is performed.

The properties of the two-dimensional shallow water equations in Lagrangian coordinates are discussed from the discretization point of view. New invariant conservative finite-difference schemes for the equations and their one-dimensional reductions are constructed. The schemes are derived either by extending the known one-dimensional schemes or by direct algebraic construction based on some assumptions on the form of the energy conservation law. Among the proposed schemes there are schemes possessing conservation laws of mass and energy.

Keywords: shallow water, Lagrangian coordinates, Lie point symmetries, numerical scheme

1. Introduction

Shallow water equations are widely used to describe various physical phenomena, for example, to study large-scale atmospheric and ocean currents, to describe currents in the coastal zones of the seas and oceans, to simulate tsunamis, the propagation of breakthrough waves and tidal bores in rivers, the distribution of heavy gases and impurities in the Earth's atmosphere.

One of the approaches of the analysis of nonlinear wave fluid motions in rotating basins of various shapes is carried out in the framework of the theory of shallow water [1]. The rotating shallow water model is a well-known nonlinear approximation used to describe largescale atmospheric and ocean currents. These equations make it possible to provide important qualitative properties of the currents. It should be mention here that there are many different approaches for deriving shallow water equations.

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1.1. The shallow water equations in Eulerian and Lagrangian coordinates

The hyperbolic shallow water equations were derived as the first-order approximation with respect to the depth of the equations obtained from the averaged inviscid incompressible fluid dynamics equations. This approximation allows one to describe incompressible heavy fluid flow with a free surface. These equations have the following form

$$u_t + uu_x + vu_y + h_x = H_x,\tag{1}$$

$$v_t + uv_x + vv_y + h_y = H_y, (2)$$

$$h_t + (uh)_x + (vh)_y = 0, (3)$$

where H is the height of the lower surface (the bottom topography), $h = \eta + H$ is the total fluid thickness, η is the height of the upper free surface, (u, v) is the velocity, t is time, (x, y) are the Eulerian coordinates.



Figure 1: The notation in Eulerian coordinates

Introducing Lagrangian coordinates $(t, \tilde{a}, \tilde{b})$, where the labels \tilde{a} and \tilde{b} denote initial coordinates of a particle, the variables x and y become dependent

$$x = \phi^1(t, \tilde{a}, \tilde{b}), \qquad y = \phi^2(t, \tilde{a}, \tilde{b}),$$

and the relations between the Lagrangian $(t, \tilde{a}, \tilde{b})$ and Eulerian (t, x, y) coordinates are defined by the equations

$$\phi_t^1(t, \tilde{a}, \tilde{b}) = u(t, \phi^1(t, \tilde{a}, \tilde{b}), \phi^2(t, \tilde{a}, \tilde{b})), \qquad \phi_t^2(t, \tilde{a}, \tilde{b}) = v(t, \phi^1(t, \tilde{a}, \tilde{b}), \phi^2(t, \tilde{a}, \tilde{b})).$$
(4)

The conservation law of mass (3) provides the relation [2]

$$h = h_0 / \hat{J},\tag{5}$$

where $h_0(\tilde{a}, \tilde{b}) > 0$ is the function of integration, and

$$\tilde{J} = \phi_{\tilde{a}}^1 \phi_{\tilde{b}}^2 - \phi_{\tilde{b}}^1 \phi_{\tilde{a}}^2 \neq 0.$$
(6)

Applying the change

$$a = f^{1}(\tilde{a}, \tilde{b}), \qquad b = f^{2}(\tilde{a}, \tilde{b}), \tag{7}$$

one finds that

$$\tilde{J} = (f_{\tilde{a}}^1 f_{\tilde{b}}^2 - f_{\tilde{b}}^1 f_{\tilde{a}}^2) J.$$
(8)

where

$$J = \phi_a^1 \phi_b^2 - \phi_b^1 \phi_a^2 \neq 0.$$
(9)

Hence, choosing $f^1(\tilde{a}, \tilde{b})$ and $f^2(\tilde{a}, \tilde{b})$ such that

$$f_{\tilde{a}}^{1}f_{\tilde{b}}^{2} - f_{\tilde{b}}^{1}f_{\tilde{a}}^{2} = 2h_{0}, \tag{10}$$

one derives that

$$h(t, a, b) = H + \eta = 2J^{-1}(t, a, b).$$
(11)

Following the one-dimensional case, the coordinates (t, a, b) are called the mass Lagrangian coordinates. As there is no ambiguity, the sign $\tilde{}$ is further omitted.

Finally, we rewrite (1) and (2) in Lagrangian coordinates as

$$x_{tt} + 2J^{-3}(x_a y_a y_{bb} - (y_b x_a + x_b y_a)y_{ab} + x_b y_{aa} y_b - x_{aa} y_b^2 + 2x_{ab} y_a y_b - x_{bb} y_a^2) = H_x,$$

$$y_{tt} + 2J^{-3}(x_b y_b x_{aa} - (x_a y_b + x_b y_a)x_{ab} + x_a y_a x_{bb} - x_b^2 y_{aa} + 2x_a x_b y_{ab} - x_a^2 y_{bb}) = H_y.$$
(12)

1.2. Commutativity of Eulerian and Lagrangian derivatives

The Lagrangian derivatives D_a , D_b and D_t^L can be defined through the Eulerian ones D_t^E , D_x and D_y as follows [3]

$$D_t = D_t^E + uD_x + vD_y, \qquad D_a = \phi_a^1 D_x + \phi_a^2 D_y, \qquad D_b = \phi_b^1 D_x + \phi_b^2 D_y, \tag{13}$$

where the total derivatives in Eulerian coordinates are defined as follows

$$D_t^E = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + \dots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \dots,$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{ty} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + \dots.$$

From the latter relations it follows that the Eulerian derivatives with respect to x and y are

$$D_x = \frac{\phi_b^2 D_a - \phi_a^2 D_b}{J}, \qquad D_y = \frac{\phi_a^1 D_b - \phi_b^1 D_a}{J}, \tag{14}$$

where J is given by equation (9).

Here and further on for the sake of brevity we write $\phi^1 \equiv x$ and $\phi^2 \equiv y$ keeping in mind that x and y are the coordinates of a Lagrangian particle. Also we denote Lagrangian derivatives of a quantity f as f_t , f_a and f_b , and its Eulerian derivatives as f_t^E , f_x and f_y .

Notice that the Lagrangian derivative D_t^L does not commute with the Eulerian derivatives D_x and D_y :

$$[D_t^L, D_x] = D_x D_t^L - D_t^L D_x = u_x D_x + v_x D_y \neq 0,$$

$$[D_t^L, D_y] = D_y D_t^L - D_t^L D_y = u_y D_x + v_y D_y \neq 0.$$
(15)

We now show that the Lagrangian derivatives (13) do commute in any order. First,

$$[D_a, D_b] = [(x_b)_y y_a + (x_b)_x x_a - (x_a)_x x_b - (x_a)_y y_b] D_x$$

$$-[(y_a)_x x_b + (y_a)_y y_b - (y_b)_x x_a - (y_b)_y y_a] D_y.$$
(16)

By means of (14),

$$(x_b)_y y_a + (x_b)_x x_a - (x_a)_x x_b - (x_a)_y y_b = \frac{1}{J} [(x_a x_{bb} - x_b x_{ab})y_a + (x_{ab} y_b - x_{bb} y_a)x_a - (x_{aa} y_b - x_{ab} y_a)x_b - (x_a x_{ab} - x_b x_{aa})y_b] = 0.$$
(17)

The similar way one shows that

$$(y_a)_x x_b + (y_a)_y y_b - (y_b)_x x_a - (y_b)_y y_a = 0.$$

Thus, the operators D_a and D_b commute on smooth enough solutions of the considered equations.

Next,

$$\begin{aligned} [D_t^L, D_a] &= (x_a u_x + y_a u_y - u(x_a)_x - v(x_a)_y - (x_a)_t^E) D_x + (x_a v_x + y_a v_y - u(y_a)_x - v(y_a)_y - (y_a)_t^E) D_y \\ &= (x_a(x_t)_x + y_a(x_t)_y - x_{at}) D_x + (x_a(y_t)_x + y_a(y_t)_y - y_{at}) D_y \\ &= \left[\frac{1}{J} \left(x_a(x_{ta}y_b - x_{tb}y_a) + y_a(x_a x_{tb} - x_b x_{ta})\right) - x_{at}\right] D_x \\ &+ \left[\frac{1}{J} \left(x_a(y_{ta}y_b - y_{tb}y_a) + y_a(x_a y_{tb} - x_b y_{ta})\right) - y_{at}\right] D_y \\ &= \left[\frac{1}{J} x_{ta}(x_a y_b - x_b y_a) - x_{at}\right] D_x + \left[\frac{1}{J} y_{ta}(x_a y_b - x_b y_a) - y_{at}\right] D_y \\ &= \left(x_{ta} - x_{at}\right) D_x + (y_{ta} - y_{at}) D_y = 0. \end{aligned}$$
(18)

The similar way one shows that $[D_t, D_b] = 0$.

Thus, it was shown that the operators D_t^L , D_a and D_b commute on smooth enough solutions of the system.

1.3. The group analysis of the shallow water equations (1)–(3)

Group analysis of the one-dimensional shallow water equations has been applied in numerous papers. A comprehensive review of these results can be found in $[4]^1$. Among these, we mention the papers [4, 5], where variable bottom topography was considered. In [5] the study was performed in Eulerian coordinates. The conservation laws were derived by the direct method, which had been applied earlier to the gas dynamics equations [6, 7]. On the other hand, in [4] the shallow water equations were studied in Lagrangian coordinates.

The group analysis method has been applied to the two-dimensional shallow water equations in [8–16].

In [13–15] group classification and conservation laws of the two-dimensional shallow-water equations over an uneven bottom in the absence of a Coriolis force (f = 0) were studied. For finding conservation laws the authors of [13–15] used the same approach as in [5].

In the papers [8–12] the Coriolis parameter was assumed to be constant, f = const, whereas in [16] $f = f_0 + \beta y$ ($\beta \neq 0$). The authors of [8] studied group properties of the shallow water

¹See also literature therein.

equations with the elliptic paraboloid

$$H = Ax^2 + By^2$$
, $(A > 0, B > 0)$,

and it was shown that if $A \neq B$, then the admitted Lie algebra is six-dimensional, while if the bottom is a circular paraboloid, then it is nine-dimensional. It was noted in [12] that for a circular bottom the admitted Lie algebra is isomorphic to the Lie algebra admitted by the classical shallow water equations with flat bottom H = const. This allowed to guess and then to prove that there is a change of the dependent and independent variables such that these two systems of equations are locally equivalent. For the particular $(f \neq 0)$ case A = B = 0, this property had been proven by the same author earlier in [10, 11]. In [8, 10–12], the study was performed in Eulerian coordinates, whereas in [9, 16], the two-dimensional shallow water equations were considered in Lagrangian coordinates. In [9], group properties of the twodimensional shallow water equations over a flat bottom (H = const) were studied. Using a Lagrangian of the form presented in [17], the authors of [9] constructed conservation laws by applying Noether's theorem. According to [10, 11] the shallow water equations (H = const)analysed in [9] are equivalent to the gas dynamics equations of an isentropic flow of a polytropic gas for the exponent $\gamma = 2$. The group properties and conservation laws of the twodimensional gas dynamics equations in Lagrangian coordinates were studied in [18].

One advantage of choosing Lagragian coordinates for the study of the shallow water equations is that equations (1)-(3) have a variational structure: choosing the Lagrangian²

$$\mathcal{L} = \frac{1}{2}(x_t^2 + y_t^2) - (J^{-1} - H), \tag{19}$$

the shallow water equations (1)–(3) turn to be the Euler-Lagrange equations $\frac{\delta \mathcal{L}}{\delta x} = 0$ and $\frac{\delta \mathcal{L}}{\delta y} = 0$, where $\frac{\delta}{\delta x}$ and $\frac{\delta}{\delta y}$ are variational derivatives. This variational structure allowed the authors of [9, 18] to apply Noether's theorem [19] for deriving conservation laws.

Application of the Hamiltonian principle to fluid dynamics in Eulerian coordinates can be found in [20, 21].

The group classification of the shallow water equations with constant Coriolis parameter and a variable bottom topography was studied in [22].

The group classification of the shallow water equations without Coriolis force in Eulerian coordinates and variable bottom topography was studied in [23].

This paper is organized as follows. Lagrangian and Hamiltonian formalism of equations (1)-(3) is given in the next section. Preliminary analysis of equations (1)-(3) is presented in Section 3, where the transformations mapping the two-dimensional shallow water equations with a circular or plane bottom into the gas dynamics equations of a polytropic gas with polytropic exponent $\gamma = 2$ (equations (1)-(3)) with H = const) are found. Group properties of equations (1)-(3) are described in Section 4.

The shallow water equations in Lagrangian coordinates (12) are discussed from the discretization point of view in Section 5. Their symmetry properties are considered in more detail and some conservation laws of the equations are provided. It is also shown that equations (12)

²The authors of [9] used a different Lagrangian [17].

can be rewritten in conservative form. In Section 6, invariant finite-difference schemes for the two-dimensional shallow water equations in Lagrangian coordinates and their reductions are constructed. Two different approaches for construction of such schemes are proposed: 1) construction by extending the known one-dimensional schemes, and 2) direct algebraic construction assuming the general form of the conservation law of energy is known. The results are summarized in Conclusion.

2. Lagrangian and Hamiltonian formalism of equations (1)-(3)

The Lagrangian (19) has the form

$$\mathcal{L}(x_t, y_t, x, y, x_a, x_b, y_a, y_b) = \frac{1}{2}(x_t^2 + y_t^2) + g(x, y, x_a, x_b, y_a, y_b).$$
(20)

The Euler-Lagrange equations are

$$x_{tt} = \frac{\delta g}{\delta x}, \quad y_{tt} = \frac{\delta g}{\delta y}.$$
 (21)

Introducing the variables

$$c_1 = x_t, \ c_2 = y_t,$$

the Lagrangian \mathcal{L} becomes

$$\mathcal{L}(c_1, c_2, x, y, x_a, x_b, y_a, y_b) = \frac{1}{2}(c_1^2 + c_2^2) + g(x, y, x_a, x_b, y_a, y_b).$$

The Euler-Lagrange equations (22) in the Lagrangian formalism can be rewritten in the form

$$(a_1)_t = \frac{\delta \mathcal{L}}{\delta x}, \ (a_2)_t = \frac{\delta \mathcal{L}}{\delta y}, \ x_t = c_1, \ y_t = c_2,$$
(22)

where c_1 and c_2 are found from the equations

$$a_1 = \frac{\delta \mathcal{L}}{\delta c_1} = c_1, \ a_2 = \frac{\delta \mathcal{L}}{\delta c_2} = c_2.$$

As the Lagrangian (20) is nonsingular [24], then one can derive the Hamiltonian form as follows [25].

Using the Legendre transformation

$$H = x_t \mathcal{L}_{x_t} + y_t \mathcal{L}_{y_t} - \mathcal{L} = \frac{1}{2}(x_t^2 + y_t^2) - g,$$

the Hamiltonian becomes

$$H(a_1, a_2, x, y, x_a, x_b, y_a, y_b) = \frac{1}{2}(a_1^2 + a_2^2) - g(x, y, x_a, x_b, y_a, y_b).$$
(23)

The Hamiltonian equations are

$$x_t = \frac{\delta H}{\delta a_1}, \quad y_t = \frac{\delta H}{\delta a_2}, \quad (a_1)_t = -\frac{\delta H}{\delta x}, \quad (a_2)_t = -\frac{\delta H}{\delta y}.$$
 (24)

Substituting the Hamiltonian (23) into (24), the Hamiltonian equations become

$$x_t = a_1, \ y_t = a_2, \ (a_1)_t = \frac{\delta g}{\delta x}, \ (a_2)_t = \frac{\delta g}{\delta y}.$$

Hence,

 $a_1 = x_t, \quad a_2 = y_t,$

and one notes that equations (24) coincide with (21).

3. Preliminary consideration

As the gas dynamics equations have been extensively studied, before proceeding to the group classification, we show that for particular bottoms the shallow water equations (1)–(3) can be reduced to the gas dynamics equations (f = 0, H = const) of a polytropic gas with the exponent $\gamma = 2$. In [22]³ it was found transformations mapping the shallow water equations (1)–(3) with $H = p(x^2 + y^2)$ and $H = q_1x + q_2y$ into equations (1)–(3) with horizontal bottom H = 0.

Remark 3.1. Particular case of transformations found in [22] is the change

$$f^{t} = t^{-1}, \quad f^{x} = 2t^{-1}(x+y), \quad f^{y} = 2t^{-1}(x-y),$$

$$f^{h} = 8ht^{2}, \quad f^{u} = -2t(u+v) + 2(x+y), \quad f^{v} = 2t(-u+v) + 2(x-y),$$

which leaves equations (1)-(3) invariant.

4. Group classification

There is vast literature dedicated to the group classification of classes of differential equations. A comprehensive review can be found, for example, in $[26, 27]^4$. In the present paper we use the classical approaches [28].

Equations (1)–(3) contain the arbitrary function H(x, y). The first step in group classification is to find transformations that change the arbitrary elements while preserving the differential structure of the equations themselves. Such transformations are called equivalence transformations. The group classification is considered with respect to equivalence transformations.

4.1. Equivalence group

A generator of an equivalence Lie group [28] is assumed to be in the form [29]

$$X^e = \zeta \partial_t + \zeta^a \partial_a + \zeta^b \partial_b + \zeta^x \partial_x + \zeta^y \partial_y + \zeta^{f_0} \partial_{f_0} + \zeta^\beta \partial_\beta + \zeta^H \partial_H$$

where all coefficients of the generator depend on (t, a, b, x, y, f, H). Applying the prolonged generator to the system consisting of equation (1)–(3) and the equations

$$H_t = 0, \ H_a = 0, \ H_b = 0,$$

See also literature therein

³In [22], the author considered the shallow water equations with a constant Coriolis parameter $f \neq 0$. However, one can check that the found there transformations are also valid and for equations with f = 0. ⁴See also literature therein.

and splitting them with respect to the parametric derivatives, one obtains an overdetermined system of partial differential equations. Solving this system, one finds the equivalence group. The equivalence group corresponds to the generators:

$$\begin{aligned} X_1^e &= \partial_x, \ X_2^e &= \partial_y, \ X_3^e &= y\partial_x - x\partial_y, \ X_{\psi}^e &= \psi_b\partial_a - \psi_a\partial_b, \\ X_4^e &= \partial_t, \ X_5^e &= 2a\partial_a + t\partial_t + x\partial_x + y\partial_y, \\ X_6^e &= \partial_H, \ X_7^e &= 2t\partial_t + x\partial_x + y\partial_y - 2H\partial_H. \end{aligned}$$

There are also two involutions

$$E_1: \quad t \to -t; \\ E_2: \quad x \to -x, \quad y \to -y;$$

4.2. Classification of Admitted Lie groups

An admitted generator is sought in the form

$$X = \zeta \partial_t + \zeta^a \partial_a + \zeta^b \partial_b + \zeta^x \partial_x + \zeta^y \partial_y,$$

where all coefficients depend on (t, a, b, x, y).

Applying the prolonged generator to equations (1)-(3), the determining equations are reduced to the study of the classifying equation

$$(\alpha x + \beta y + \gamma_1)H_x + (-\beta x + \alpha y + \gamma_2)H_y = 2\gamma H + q(x^2 + y^2) + q_1x + q_2y + q_0.$$
(25)

where

$$\alpha = \zeta' + 2k_1, \quad \beta = 2k_4, \quad \gamma = 2(\zeta' - 2k_1), \quad q = -\frac{1}{2}\zeta''',$$

$$\gamma_1 = 2\zeta_1, \quad \gamma_2 = 2\zeta_2, \quad q_1 = -2\zeta''_1, \quad q_2 = -2\zeta''_2, \quad q_0 = g,$$

$$\zeta^a = -\psi_b + 4k_1a, \quad \zeta^b = \psi_a,$$

$$\zeta^x = \frac{1}{2}\zeta'x + k_1x + k_4y + \zeta_1, \quad \zeta^y = \frac{1}{2}\zeta'y + k_1y - k_4x + \zeta_2,$$

and the functions $\zeta(t)$, $\zeta_1(t)$, $\zeta_2(t)$, g(t) and $\psi = \psi(a, b)$ are arbitrary functions of their arguments.

The kernel of admitted Lie algebras, which is admitted for all cases of the function H(x, y), consists of the generators

$$X_1 = \partial_t, \qquad X_2 = -\psi_b \partial_a + \psi_a \partial_b. \tag{26}$$

Extensions of the kernel occur for specific functions H(x, y). Consideration of these cases leads to the analysis similar to applied in $[22]^5$.

We restrict ourselves in this paper with the case $H = px^2 + 2cxy + by^2 + q_1x + q_2y + q_0$.

Using an orthogonal transformation one can reduce the function H(x, y) to the canonical form:

$$H(x,y) = \lambda_1 x^2 + \lambda_2 y^2 + q_1 x + q_2 y + q_0.$$
(27)

Notice that for $\lambda_1 \neq 0$, by virtue of the equivalence transformations corresponding to the

⁵In Eulerian coordinates group classification of equations (1)–(3) was done in [23].

shifts of x and H, one can assume that $q_1 = 0$. Similar, one can assume that $q_2 = 0$ if $\lambda_2 \neq 0$. Using the shift H, one can assume that $q_0 = 0$. By virtue of the preliminary study, this allows us to state the following theorem.

Theorem 4.1. If the bottom has the form

$$H(x,y) = p(x^{2} + y^{2}) + q_{1}x + q_{2}y + q_{0},$$
(28)

where p and q_i (i = 0, 1, 2) are constant, then system of the shallow water equations (1)–(3) can be reduced to the gas dynamics equations⁶ of a polytropic gas with the exponent $\gamma = 2$.

Thus, the group classification of equations (1)–(3) with the bottom (27) is restricted to the following cases

- a) $\lambda_1 \neq \lambda_2;$
- b) $\lambda_1 = \lambda_2 = q_1 = q_2 = q_0 = 0.$

Calculations in the first case are straight forward and the classification results are listed in Table 1. In the first column of the table different forms of the function H are given. The corresponding extensions of the kernel (31) and constraints on the constants and arbitrary functions are presented in the second and the third columns of the table.

#	Н	Extension	Conditions
1.	$\lambda_1 x^2 + \lambda_2 y^2$	$x\partial_x + y\partial_y + 4a\partial_a,$ $\zeta_1(t)\partial_x + \zeta_2(t)\partial_y$	$\lambda_1 \lambda_2 (\lambda_1 - \lambda_2) \neq 0,$ $\zeta_1'' = 2\lambda_1 \zeta_1,$ $\zeta_2'' = 2\lambda_2 \zeta_2$
2.	$\lambda_1 x^2 + q_2 y$	$4x\partial_x + 16a\partial_a + (2y - q_2t^2)\partial_y,$ $t\partial_y, \partial_y, \zeta(t)\partial_x,$	$\lambda_1 \neq 0,$ $\zeta_1'' = 2\lambda_1 \zeta_1$

Table 1: Classification results H(x, y)

5. Preliminary analysis of the shallow water equations for constructing finitedifference schemes

5.1. The general case H = H(x, y)

For the further discretization of equations (12) it is more useful to consider them in the following form

$$F_1 = D_t(x_t) + D_a(y_b J^{-2}) - D_b(y_a J^{-2}) - H_x = 0,$$
(29)

⁶Equations (1)–(3) with f = 0 and H = const.

$$F_2 = D_t(y_t) - D_a(x_b J^{-2}) + D_b(x_a J^{-2}) - H_y = 0.$$
(30)

Recall that x and y denote the coordinates of a Lagrangian particle.

As it was shown above, the kernel of admitted Lie algebras, which is admitted by equations (29) and (30) for all cases of the function H(x, y), consists of the generators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \psi_b \frac{\partial}{\partial a} - \psi_a \frac{\partial}{\partial b}, \tag{31}$$

where $\psi(a, b)$ is an arbitrary function corresponding to relabelling of Lagrangian variables.

Notice that there are shifts $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$, inhomogeneous scaling $a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b}$ and rotation $b\frac{\partial}{\partial a} - a\frac{\partial}{\partial b}$ among the particular forms of the generator X_2 .

Equations (29), (30) possess the local conservation laws of mass, energy and momentum.

By means of equations (4) and (11) one can write the Eulerian conservation law of mass (3) in Lagrangian coordinates as

$$D_t(J) = D_t(x_a y_b - x_b y_a) = D_a(x_t y_b - y_t x_b) + D_b(y_t x_a - x_t y_a).$$
(32)

The conservation law of energy which corresponds to the generator $X_1 = \frac{\partial}{\partial t}$ can be obtained with the help of Noether's theorem [19, 30]

$$x_{t} \left[D_{t}(x_{t}) + D_{a} \left(y_{b} J^{-2} \right) - D_{b} \left(y_{a} J^{-2} \right) - H_{x} \right] + y_{t} \left[D_{t}(y_{t}) - D_{a} \left(x_{b} J^{-2} \right) + D_{b} \left(x_{a} J^{-2} \right) - H_{y} \right] = D_{t} \left[\frac{1}{2} (x_{t}^{2} + y_{t}^{2}) + J^{-1} - H \right] + D_{a} \left[(x_{t} y_{b} - y_{t} x_{b}) J^{-2} \right] + D_{b} \left[(y_{t} x_{a} - x_{t} y_{a}) J^{-2} \right] = 0.$$
(33)

The conservation law of momentum which corresponds to the generators $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ is

$$(x_{a} + x_{b}) \left[D_{t}(x_{t}) + D_{a} \left(y_{b} J^{-2} \right) - D_{b} \left(y_{a} J^{-2} \right) - H_{x} \right] + (y_{a} + y_{b}) \left[D_{t}(y_{t}) - D_{a} \left(x_{b} J^{-2} \right) + D_{b} \left(x_{a} J^{-2} \right) - H_{y} \right] = D_{t} \left[(x_{a} + x_{b}) x_{t} + (y_{a} + y_{b}) y_{t} \right] + D_{a} \left[2J^{-1} - \frac{x_{t}^{2} + y_{t}^{2}}{2} - H \right] + D_{b} \left[2J^{-1} - \frac{x_{t}^{2} + y_{t}^{2}}{2} - H \right] = 0.$$
(34)

Remark 5.1. Equations (29)–(30) can be reduced to the one-dimensional shallow water equation

$$x_{tt} - \frac{2x_{aa}}{x_a^3} - \tilde{H}'(x) = 0,$$
(35)

by means of the relations $x(t, a, b) = x(t, a), y(t, a, b) \equiv b$.

The identity (32) becomes

$$D_t(x_a) - D_a(x_t) = 0, (36)$$

and the Jacobian J is just reduced to x_a in this case.

5.2. The case of a horizontal bottom H = const

For the further discretization purposes, here we consider the case of a horizontal bottom topography (H = const) in more detail.

In case H = const, the shallow water equations in Lagrangian coordinates are

$$F_1^0 = x_{tt} + D_a \left(y_b J^{-2} \right) - D_b \left(y_a J^{-2} \right) = 0, \tag{37}$$

$$F_2^0 = y_{tt} - D_a \left(x_b J^{-2} \right) + D_b \left(x_a J^{-2} \right) = 0, \tag{38}$$

and the admitted Lie algebra is the same as for the two-dimensional gas isentropic flows for the polytropic constant $\gamma = 2$ [30, 31], i.e., the extension of the kernel (31) is

$$Y_{1} = \frac{\partial}{\partial x}, \qquad Y_{2} = \frac{\partial}{\partial y}, \qquad Y_{3} = t\frac{\partial}{\partial x}, \qquad Y_{4} = t\frac{\partial}{\partial y},$$
$$Y_{5} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad Y_{6} = t^{2}\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y},$$
$$Y_{7} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y_{8} = 2a\frac{\partial}{\partial a} + 2b\frac{\partial}{\partial b} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$
(39)

Equations (37), (38) are "weak-invariant" with respect to the generators Y_5-Y_8 , i.e., they satisfy the infinitesimal criterion on solutions only:

$$Y_k(F_j^0)|_{F_1^0=0, F_2^0=0} = 0, \qquad j = 1, 2, \quad k = 5, 6, \dots, 8.$$

Remark 5.2. Jacobian (9) is a differential invariant of the generators $X_1, X_2, Y_1 - Y_5$.

The conservation laws for the case H = const are listed in [4]. Among them there is the center-of-mass law

$$D_t(t(x_t + y_t) - x - y) + D_a(t(y_b - x_b)J^{-2}) + D_b(t(x_a - y_a)J^{-2}) = 0.$$
(40)

6. Discretization of the shallow water equations in Lagrangian coordinates

It is often more suitable to construct invariant finite-difference schemes for equations of continuum mechanics in Lagrangian coordinates then in Eulerian ones [32], as the generators admitted by the equations in Lagrangian coordinates typically preserve uniformness and orthogonality of the corresponding finite difference meshes. Such invariant conservative schemes have been successfully constructed by the authors in [33–35].

6.1. Notation

Following this approach, we consider discretizations in Lagrangian coordinates in 3 + 3 + 3 + 27 + 27 = 63 variables on 27-point stencil (see Figure 2), i.e.,

$$t_{n+i} \equiv t_{m+j,s+k}^{n+i}, \quad a_{m+j} \equiv a_{m+j,s+k}^{n+i}, \quad b_{s+k} \equiv b_{m+j,s+k}^{n+i}, \qquad (41)$$
$$x_{m+j,s+k}^{n+i}, \quad y_{m+j,s+k}^{n+i}, \quad i, j, k = -1, 0, 1,$$

or, in alternative notation,

$$\dot{t}, t, \dot{t}, \quad a_{-}, a, a_{+}, \quad {}^{-}b, b, {}^{+}b,
{}^{-}\check{x}_{-}, \check{x}_{-}, \check{x}, \check{x}_{+}, {}^{+}\check{x}, \dots, {}^{+}\hat{x}_{+},
{}^{-}\check{y}_{-}, \check{y}_{-}, \check{y}, \check{y}_{+}, {}^{+}\check{y}, \dots, {}^{+}\hat{y}_{+},$$
(42)



Figure 2: The 27-point stencil, Lagrangian coordinates

where

$$\hat{z} = z_{m,s}^{n+1}, \quad \check{z} = z_{m,s}^{n-1}, \quad z_{\pm} = z_{m\pm 1,s}^{n}, \quad {}^{\pm}z = z_{m,s\pm 1}^{n} \quad \text{for} \quad z \in \{t, a, b, x, y\}.$$

The standard finite-difference shift operators are defined as follows

$$S_{\pm t}^{k}(f(t_{n}, a_{m}, b_{s}, x_{m,s}^{n}, y_{m,s}^{n})) = f(t_{n\pm k}, a_{m}, b_{s}, x_{m,s}^{n\pm k}, y_{m,s}^{n\pm k}),$$

$$S_{\pm a}^{k}(f(t_{n}, a_{m}, b_{s}, x_{m,s}^{n}, y_{m,s}^{n})) = f(t_{n}, a_{m\pm k}, b_{s}, x_{m\pm k,s}^{n}, y_{m\pm k,s}^{n}),$$

$$S_{\pm b}^{k}(f(t_{n}, a_{m}, b_{s}, x_{m,s}^{n}, y_{m,s}^{n})) = f(t_{n}, a_{m}, b_{s\pm k}, x_{m,s\pm k}^{n}, y_{m,s\pm k}^{n}).$$
(43)

The total differentiation operators are defined through the shifts as

$$D_{+\tau} = \frac{S_{+\tau}}{t_{n+1} - t_n}, \qquad D_{-\tau} = \frac{1 - S_{-\tau}}{t_n - t_{n-1}},$$

$$D_{+\alpha} = \frac{S_{+\alpha}}{a_{m+1} - a_m}, \qquad D_{-\alpha} = \frac{1 - S_{-\tau}}{a_m - a_{m-1}}, \qquad D_{+b} = \frac{S_{-1}}{b_{s+1} - b_s}, \qquad D_{-b} = \frac{1 - S_{-b}}{b_s - b_{s-1}},$$

$$(44)$$

and the following notation is used for difference derivatives

$$x_{t} = \underset{+\tau}{D}(x), \quad \check{x}_{t} = \underset{-\tau}{D}(x), \quad x_{t\check{t}} = \underset{-\tau}{D} \underset{+\tau}{D}(x), \quad x_{a} = \underset{+a}{D}(x), \quad \check{x}_{a} = \underset{-a}{D}(x), \quad x_{a\bar{a}} = \underset{-a}{D} \underset{-a}{D}(x), \quad \text{etc.}$$

6.2. Invariance of uniform orthogonal meshes

Notice that all the shift and differentiation operators commute in any order on uniform orthogonal mesh

$$\hat{t} - t = t - \check{t} = \tau, \qquad a_{+} - a = a - a_{-} = h^{a}, \qquad {}^{+}b - b = b - {}^{-}b = h^{b},$$
(45)

where $\tau > 0$, $h^a > 0$ and $h^b > 0$ are small enough constant values.

In the finite-difference case, in order to preserve uniform orthogonal meshes the generator $X = \xi^t \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial a} + \xi^b \frac{\partial}{\partial b}$ must satisfy the following criteria [32, 36]

$$\underbrace{D}_{+\tau - \tau} D(\xi^t) = 0, \qquad \underbrace{D}_{+a - a} D(\xi^a) = 0, \qquad \underbrace{D}_{+b - b} D(\xi^b) = 0, \tag{46}$$

$$D_{\pm b}(\xi^{a}) = -D_{\pm a}(\xi^{b}), \quad D_{\pm \tau}(\xi^{a}) = -D_{\pm a}(\xi^{t}), \quad D_{\pm \tau}(\xi^{b}) = -D_{\pm b}(\xi^{t}),$$
(47)

The transformation corresponding to X_2 is related to the freedom of the Lagrangian coordinates parametrization [4] which should be restricted in the difference case due to (46) and (47).

One can check that the generator X_2 in its general form which depend on an arbitrary function ψ does not satisfy (46), (47). As a particular example, consider

$$\psi(a,b) = a^2 - b^2.$$

From (31) it follows that $\xi^a = -2b$ and $\xi^b = -2a$ in this case, and the orthogonality condition (47) does not hold

$$D_{+b}(\xi^a) + D_{+a}(\xi^b) = D_{+b}(-2b) + D_{+a}(-2a) = -4 \neq 0.$$
(48)

For simplicity, we restrict our consideration to generators with coefficients of the form

$$\xi^{t} = 0, \qquad \xi^{a} = \alpha_{1}a + \beta_{1}b + \gamma_{1}, \qquad \xi^{b} = \alpha_{2}a + \beta_{2}b + \gamma_{2}, \qquad \alpha_{i}, \beta_{i}, \gamma_{i} = \text{const}, \quad i = 1, 2,$$
(49)

which satisfy the uniformness conditions (46). Substituting ξ^a and ξ^b into the orthogonality condition (47), one obtains $\beta_1 = -\alpha_2$, i.e.,

$$\xi^a = \alpha_1 a - \alpha_2 b + \gamma_1, \qquad \xi^b = \alpha_2 a + \beta_2 b + \gamma_2. \tag{50}$$

According to (31), one has the following restrictions on the function ψ

$$\xi^{a} = \alpha_{1}a - \alpha_{2}b + \gamma_{1} = \psi_{b}, \qquad \xi^{b} = \alpha_{2}a + \beta_{2}b + \gamma_{2} = -\psi_{a}.$$
(51)

Integrating the latter equations, one gets

$$\psi(a,b) = \alpha_1 a b - \frac{1}{2} \alpha_2 b^2 + \gamma_1 b + \chi_1(a) = -\beta_2 a b - \frac{1}{2} \alpha_2 a^2 - \gamma_2 a + \chi_2(b),$$
(52)

where χ_1 and χ_2 are some functions of their arguments. Comparing the latter expressions for the function ψ , one obtains that $\beta_2 = -\alpha_1$, and the function ψ corresponding to the chosen particular solution is the following

$$\psi(a,b) = \alpha_1 a b - \frac{\alpha_2}{2} (a^2 + b^2) - \gamma_2 a + \gamma_1 b + \delta, \qquad \delta = \text{const.}$$
(53)

Substituting (53) into (31), one obtains the following particular form of the generator X_2

$$X_2^0 = (\alpha_1 a - \alpha_2 b + \gamma_1) \frac{\partial}{\partial a} + (\alpha_2 a - \alpha_1 b + \gamma_2) \frac{\partial}{\partial b},$$

which results in the following set of shifting, inhomogeneous scaling and rotation generators

$$X_2^1 = \frac{\partial}{\partial a}, \qquad X_2^2 = \frac{\partial}{\partial b}, \qquad X_2^3 = a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b}, \qquad X_2^4 = b\frac{\partial}{\partial a} - a\frac{\partial}{\partial b}.$$
 (54)

The generators X_2^3 and X_2^4 form Lie algebras with the generators $X_1, X_2^1, X_2^2, Y_1-Y_8$, but their

commutator

$$[X_2^3, X_2^4] = 2b\frac{\partial}{\partial a} + 2a\frac{\partial}{\partial b}$$

does not satisfy orthogonality conditions (47). One has to choose either the generator X_2^3 or the generator X_2^4 . Further we prefer the generator X_2^3 , and we restrict ourselves to the Lie algebra $X_1, X_2^1 - X_2^3, Y_1 - Y_8$.

The generators X_1 , $X_2^1 - X_2^3$, $Y_1 - Y_5$, Y_7 and Y_8 satisfy conditions (46) and (47), and the uniform orthogonal mesh (45) is invariant with respect to these generators.

6.3. Jacobian invariance and mass conservation

While construction of conservative schemes, it is important to preserve a finite-difference analogue of the conservation of mass identity (32). Considering approximations for (32) on the 8-point stencil⁷ on a uniform orthogonal mesh in the form

where $i_1, ..., i_{12} \in \{0, 1\}$ are unknown indices, one states by direct computation that there are only two possible difference analogues of equation (32) that identically hold, namely

$$D_{+\tau}({}^{+}x_ay_b - x_b^{+}y_a) - D_{+a}({}^{+}x_ty_b - \hat{x}_by_t) - D_{+b}(\hat{x}_ay_t - x_t^{+}y_a) = 0,$$
 (56)

and

$$D_{+\tau}(x_a y_b^+ - {}^+y_a x_b) - D_{+a}(x_t y_b - {}^+y_t \hat{x}_b) - D_{+b}(\hat{x}_a y_t^+ - x_t y_a) = 0.$$
 (57)

We consider linear combinations

$$\theta({}^{+}x_{a}y_{b} - x_{b}^{+}y_{a}) + (1 - \theta)(x_{a}y_{b}^{+} - {}^{+}y_{a}x_{b}), \qquad 0 \le \theta \le 1$$
(58)

as approximations for Jacobian (9). In order to choose the value of θ we notice that Jacobian (9) is invariant with respect to the generators X_1 , X_2 , Y_1-Y_5 (see Remark 5.2). Jacobian (9) is a fundamental generating differential invariant of the particle relabelling symmetry [?], and in the finite-difference case we would like to hold as much its geometric properties as possible. Recall that in the previous sections we posed restrictions on the generator X_2 due to the mesh uniformness and orthogonality conditions, so the chosen particle relabelling symmetries of our interest are X_2^1 , X_2^2 and X_2^3 . Applying the generators X_1 , $X_2^1-X_2^3$ and Y_1-Y_5 to (58), one finds that the only value of θ that preserves invariance with respect to all the considered symmetries is $\theta = 1/2$. Thus, we prefer the following approximation for Jacobian (9)

$$J_{h} = \frac{1}{2} \left(x_{a} y_{b}^{+} + {}^{+} x_{a} y_{b} - y_{a} x_{b}^{+} - {}^{+} y_{a} x_{b} \right).$$
(59)

⁷Approximations on the chosen 8-point stencil can be then shifted to the left by the operators $D_{-\tau}$, D_{-a} and D_{-b} .

In this case, the conservation law of mass is the following identity

$$D_{+\tau}({}^{+}x_{a}y_{b} + x_{a}y_{b}^{+} - x_{b}^{+}y_{a} - {}^{+}y_{a}x_{b}) - D_{+a}({}^{+}x_{t}y_{b} - \hat{x}_{b}y_{t} + x_{t}y_{b} - {}^{+}y_{t}\hat{x}_{b}) - D_{+b}(\hat{x}_{a}y_{t} - x_{t}^{+}y_{a} + \hat{x}_{a}y_{t}^{+} - x_{t}y_{a}) = 0,$$
(60)

or

$$D_{+\tau}({}^{+}x_{a}y_{b} + x_{a}y_{b}^{+} - x_{b}^{+}y_{a} - {}^{+}y_{a}x_{b}) - D_{+a}((x_{t} + {}^{+}x_{t})y_{b} - \hat{x}_{b}(y_{t} + {}^{+}y_{t})\hat{x}_{b}) - D_{+b}(\hat{x}_{a}(y_{t} + y_{t}^{+}) - y_{a}(x_{t} + x_{t}^{+})) = 0$$

$$(61)$$

6.4. Approximations for the derivatives of the function H(x, y)

Before proceeding any further, we consider the problem of approximating the differential derivatives H_x and H_y . No finite-difference derivatives by the dependant variables x and y have been defined yet. Generalizing the approach introduced by the authors in [33], we notice that from the obvious differential relations

$$H_a = H_x x_a + H_y y_a, \qquad H_b = H_x x_b + H_y y_b, \qquad H_t = H_x x_t + H_y y_t$$
 (62)

it follows that

$$H_{x} = \frac{H_{b}y_{t} - H_{t}y_{b}}{x_{b}y_{t} - x_{t}y_{b}} = \frac{H_{a}y_{t} - H_{t}y_{a}}{x_{a}y_{t} - x_{t}y_{a}}, \qquad H_{y} = \frac{H_{t}x_{b} - H_{b}x_{t}}{x_{b}y_{t} - x_{t}y_{b}} = \frac{H_{t}x_{a} - H_{a}x_{t}}{x_{a}y_{t} - x_{t}y_{a}}.$$
 (63)

Further we consider some approximation Θ for the function H. For definiteness, we choose

$$\Theta = \frac{1}{2}(H + \check{H}). \tag{64}$$

According to (63), it seems natural to define approximations Θ_x and Θ_y for the continuous derivatives H_x and H_y through the known finite-difference derivatives Θ_t and Θ_a . We choose the following approximations for the derivatives of (64)

$$\Theta_{x} = \frac{\Theta_{a}(y_{t} + \check{y}_{t}) - 2\Theta_{t}y_{a}}{x_{a}(y_{t} + \check{y}_{t}) - (x_{t} + \check{x}_{t})y_{a}} = \frac{\Theta_{b}(y_{t} + \check{y}_{t}) - 2\Theta_{t}y_{b}}{x_{b}(y_{t} + \check{y}_{t}) - (x_{t} + \check{x}_{t})y_{b}},$$

$$\Theta_{y} = \frac{2\Theta_{t}x_{a} - \Theta_{a}(x_{t} + \check{x}_{t})}{x_{a}(y_{t} + \check{y}_{t}) - (x_{t} + \check{x}_{t})y_{a}} = \frac{2\Theta_{t}x_{b} - \Theta_{b}(x_{t} + \check{x}_{t})}{x_{b}(y_{t} + \check{y}_{t}) - (x_{t} + \check{x}_{t})y_{b}}.$$
(65)

Notice that from the latter relations one derives that

$$D_{+\tau}(\Theta) = \Theta_t = \frac{x_t + \check{x}_t}{2}\Theta_x + \frac{y_t + \check{y}_t}{2}\Theta_y.$$
(66)

Below we consider two approaches to the construction of invariant conservative schemes for the two-dimensional shallow water equations.

6.5. Approach 1: Construction of two-dimensional schemes by extending the known one-dimensional scheme

Recall that the case of an arbitrary bottom topography H = H(x, y) is considered.

In case the bottom is arbitrary, the only generators that should be admitted by invariant schemes are $X_1 = \frac{\partial}{\partial t}$ and $X_2^1 - X_2^3$ which is given by (54). Such a poor set of generators makes the method of differential invariants [32] barely applicable as almost any arbitrary given discretization would be an invariant one. As an alternative approach, it seems natural to choose schemes that reduce to the invariant conservative scheme

$$x_{t\check{t}} + D_{-a} \left(\frac{1}{\hat{x}_a \check{x}_a}\right) - \frac{2}{x_t + \check{x}_t} \Theta_t = 0, \qquad (67)$$

$$\tau = \text{const}, \qquad h^a = \text{const},$$

previously constructed by the authors in [33] for the one-dimensional shallow water equations (35). Here Θ is given by (64).

There still a broad class of such schemes, and as an example one can choose the following one

$$F_1 = \underset{-\tau}{D}(x_t) + \underset{-a}{D}\left(\frac{y_{\bar{b}}}{-\check{J}-\hat{J}}\right) - \underset{-b}{D}\left(\frac{y_{\bar{a}}}{\hat{J}-\check{J}-}\right) - \Theta_x = 0,$$
(68)

$$F_2 = \underset{-\tau}{D}(y_t) - \underset{-a}{D}\left(\frac{x_{\bar{b}}}{-\check{J}-\hat{J}}_{h-h}\right) + \underset{-b}{D}\left(\frac{x_{\bar{a}}}{\hat{J}-\check{J}-}_{h-h}\right) - \Theta_y = 0,$$
(69)

where J_{h} is given by equation (59), and Θ_{x} and Θ_{y} are given by (65).

The sum of equations (68) and (69) is a inhomogenious (see e.g. [37]) conservation law

$$D_{-\tau}(x_t + y_t) + D_{-a}\left(\frac{y_{\bar{b}} - x_{\bar{b}}}{-\check{J} - \hat{J}}_{h-h}\right) + D_{-b}\left(\frac{x_{\bar{a}} - y_{\bar{a}}}{\hat{J} - \check{J}}_{h-h}\right) - \Theta_x - \Theta_y = 0$$
(70)

which becomes a homogenious one in case H = const, namely

$$D_{-\tau}(x_t + y_t) + D_{-a}\left(\frac{y_{\bar{b}} - x_{\bar{b}}}{-\check{J} - \hat{J}}_{h \ h}\right) + D_{-b}\left(\frac{x_{\bar{a}} - y_{\bar{a}}}{\hat{J} - \check{J}^-}_{h \ h}\right) = 0.$$
(71)

In addition, in case H = const, the following finite-difference analogue of the center of mass conservation (40) is possessed by system (68), (69)

$$D_{-\tau}(t(x_t + y_t) - x - y) + D_{-a}\left(t \frac{y_{\bar{b}} - x_{\bar{b}}}{-\check{J} - \hat{J}}\right) + D_{-b}\left(t \frac{x_{\bar{a}} - y_{\bar{a}}}{\hat{J} - \check{J} - \check{J}}\right) = 0.$$
(72)

6.6. Approach 2: Direct algebraic construction of conservative schemes

An alternative approach is to consider the conservation law of energy in some divergent form, e.g.

$$D_{+\tau}\left(\frac{\check{x}_t^2 + \check{y}_t^2}{2} + J_0^{-1} - \Theta\right) + D_{+a}\left(A \underset{-a}{S}(N^A(J_0\hat{J}_0)^{-1})\right) + D_{+b}\left(B \underset{-b}{S}(N^B(J_0\hat{J}_0)^{-1})\right) = 0, \quad (73)$$

where

$$A = \frac{x_t + \check{x}_t}{2} \tilde{y}_b - \frac{y_t + \check{y}_t}{2} \tilde{x}_b, \qquad B = -\frac{x_t + \check{x}_t}{2} \tilde{y}_a + \frac{y_t + \check{y}_t}{2} \tilde{x}_a, \tag{74}$$

 $\tilde{x}_a, \tilde{x}_b, \tilde{y}_a, \tilde{y}_b$ are some approximations for the corresponding partial derivatives, J_0 is some approximation for Jacobian (9).

In order to approximate the conservation law of energy (33) there should be $N^A \to 1$ and $N^B \to 1$. The exact form of the finite-difference terms N^A and N^B will be stated later.

Notice that there is a reason for choosing approximations precisely of the form (74). It was mentioned by the authors [33, 38] that conservation law multipliers corresponding to energy conservation laws of schemes for one-dimensional equations often have the form $\frac{1}{2}(x_t + \check{x}_t)$, and, thus, in the two-dimensional case we assume them to be

$$\frac{x_t + \check{x}_t}{2} \quad \text{and} \quad \frac{y_t + \check{y}_t}{2}. \tag{75}$$

As it is shown below, by means of the finite-difference Leibniz rule (see e.g. [32]) equation (73) can be brought in such a form where the terms (75) of the approximations (74) become conservation law multipliers.

By applying the finite-difference Leibniz rule, one rewrites (73) in the following form

$$\frac{D}{+\tau} \left(\frac{\check{x}_{t}^{2} + \check{y}_{t}^{2}}{2} + J_{0}^{-1} - \Theta \right) + \left(\frac{x_{t} + \check{x}_{t}}{2} \tilde{y}_{b} - \frac{y_{t} + \check{y}_{t}}{2} \tilde{x}_{b} \right) \underbrace{D}_{-a} (N^{A} (J_{0} \hat{J}_{0})^{-1}) + (J_{0} \hat{J}_{0})^{-1} N^{A} \underbrace{D}_{+a} (A) \\
+ \left(-\frac{x_{t} + \check{x}_{t}}{2} \tilde{y}_{a} + \frac{y_{t} + \check{y}_{t}}{2} \tilde{x}_{a} \right) \underbrace{D}_{-b} (N^{B} (J_{0} \hat{J}_{0})^{-1}) + (J_{0} \hat{J}_{0})^{-1} N^{B} \underbrace{D}_{+b} (B) = 0. \quad (76)$$

Expanding the first term $D_{+\tau}(\cdots)$ of the latter equation, one gets

$$\frac{(x_t + \check{x}_t)(x_t - \check{x}_t)}{2\tau} + \frac{(y_t + \check{y}_t)(y_t - \check{y}_t)}{2\tau} - D_{+\tau}(J_0)(J_0\hat{J}_0)^{-1} - \Theta_t + \left(\frac{x_t + \check{x}_t}{2}\tilde{y}_b - \frac{y_t + \check{y}_t}{2}\tilde{x}_b\right) D_{-a}(N^A(J_0\hat{J}_0)^{-1}) + (J_0\hat{J}_0)^{-1}N^A D_{+a}(A) + \left(-\frac{x_t + \check{x}_t}{2}\tilde{y}_a + \frac{y_t + \check{y}_t}{2}\tilde{x}_a\right) D_{-b}(N^B(J_0\hat{J}_0)^{-1}) + (J_0\hat{J}_0)^{-1}N^B D_{+b}(B) = 0.$$
(77)

Taking (66) into account and collecting the terms with respect to $(x_t + \check{x}_t)$ and $(y_t + \check{y}_t)$, one derives the following equation

$$\frac{x_t + \check{x}_t}{2} \left[x_{t\check{t}} + \check{y}_b \mathop{D}_{-a} (N^A (J_0 \hat{J}_0)^{-1}) - \check{y}_a \mathop{D}_{-b} (N^B (J_0 \hat{J}_0)^{-1}) - \Theta_x \right] + \frac{y_t + \check{y}_t}{2} \left[y_{t\check{t}} - \check{x}_b \mathop{D}_{-a} (N^A (J_0 \hat{J}_0)^{-1}) + \check{x}_a \mathop{D}_{-b} (N^B (J_0 \hat{J}_0)^{-1}) - \Theta_y \right] + \left(\mathop{D}_{+\tau} J_0 - N^A \mathop{D}_{+a} A - N^B \mathop{D}_{+b} B \right) (J_0 \hat{J}_0)^{-1} = 0.$$
(78)

To eliminate the term

$$D_{+\tau} J_0 - N^A D_{+a} A - N^B D_{+b} B,$$
 (79)

one can put

$$N^{A} = \frac{D}{\frac{A_{0}}{D}}_{+a}^{A} \to 1, \qquad N^{B} = \frac{D}{\frac{B_{0}}{D}}_{+a}^{B} \to 1,$$
(80)

where $A_0 \neq 0$ and $B_0 \neq 0$ are some approximations alternative to A and B. Thus, one brings (79) to the divergent form

$$\sum_{+\tau} J_0 - \sum_{+a} A_0 - \sum_{+b} B_0.$$
(81)

The latter expression approximates the conservation law of mass and vanishes in the continuous limit. Thus, the final step is to choose some specific approximations J_0 , A_0 and B_0 to make *identically* hold the equation

$$\underset{+\tau}{D} J_0 - \underset{+a}{D} A_0 - \underset{+b}{D} B_0 = 0.
 \tag{82}$$

We choose, for example, the invariant approximation (59) as J_0 , and, according to (61), the following approximations as A_0 and B_0

$$A_0 = \frac{1}{2} \left((x_t + x_t)y_b - (y_t + y_t)\hat{x}_b \right), \qquad B_0 = \frac{1}{2} \left((y_t + y_t^+)\hat{x}_a - (x_t + x_t^+)y_a \right).$$
(83)

Substituting the latter approximations into (78), one obtains the scheme

$$F_{1} = x_{t\tilde{t}} + \tilde{y}_{b} D_{-a} \left[\frac{1}{J_{0}\hat{J}_{0}} \frac{\frac{D}{A}((x_{t} + x_{t})y_{b} - (y_{t} + y_{t})\hat{x}_{b})}{D((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + \check{y}_{t})\tilde{x}_{b})} \right] \\ - \tilde{y}_{a} D_{-b} \left[\frac{1}{J_{0}\hat{J}_{0}} \frac{\frac{D}{A}((y_{t} + y_{t}^{+})\hat{x}_{a} - (x_{t} + x_{t}^{+})y_{a})}{D((y_{t} + \check{y}_{t})\tilde{x}_{a} - (x_{t} + \check{x}_{t})\tilde{y}_{a})} \right] - \Theta_{x} = 0,$$

$$F_{2} = y_{t\tilde{t}} - \tilde{x}_{b} D_{-a} \left[\frac{1}{J_{0}\hat{J}_{0}} \frac{\frac{D}{A}((x_{t} + x_{t})y_{b} - (y_{t} + y_{t})\hat{x}_{b})}{D((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + y_{t})\hat{x}_{b})} \right] \\ + \tilde{x}_{a} D_{-b} \left[\frac{1}{J_{0}\hat{J}_{0}} \frac{\frac{D((y_{t} + y_{t}^{+})\hat{x}_{a} - (x_{t} + x_{t}^{+})y_{a})}{D((y_{t} + \check{y}_{t})\tilde{x}_{a} - (x_{t} + x_{t}^{+})y_{a})} \right] - \Theta_{y} = 0,$$

$$(84)$$

on uniform orthogonal mesh (45), where

$$J_0 = J_b = \frac{1}{2} \left(x_a y_b^+ + x_a y_b - y_a x_b^+ - y_a x_b \right),$$
(85)

and Θ_x , Θ_y are given by equations (65).

The conservation law of energy for scheme (84) is

$$\frac{x_t + \check{x}_t}{2}F_1 + \frac{y_t + \check{y}_t}{2}F_2 = \sum_{+\tau} \left(\frac{\check{x}_t^2 + \check{y}_t^2}{2} + J_0^{-1} - \Theta \right) \\ + \frac{1}{2} \sum_{+a} \left(((x_t + \check{x}_t)\tilde{y}_b - (y_t + \check{y}_t)\tilde{x}_b) \frac{D((x_t + +x_t)y_b - (y_t + +y_t)\hat{x}_b)}{D((x_t + \check{x}_t)\tilde{y}_b - (y_t + \check{y}_t)\tilde{x}_b)} \sum_{-a}^{-a} ((J_0\hat{J}_0)^{-1}) \right)$$

$$+\frac{1}{2} \frac{D}{+b} \left(((y_t + \check{y}_t) \tilde{x}_a - (x_t + \check{x}_t) \tilde{y}_a) \frac{D((y_t + y_t^+) \hat{x}_a - (x_t + x_t^+) y_a)}{D((y_t + \check{y}_t) \tilde{x}_a - (x_t + \check{x}_t) \tilde{y}_a)} \frac{S((J_0 \hat{J}_0)^{-1})}{-b} \right) = 0, \quad (86)$$

and the conservation law of mass is just an identity

$$D_{+\tau} J_0 - \frac{1}{2} D_{+a} ((x_t + x_t) y_b - (y_t + y_t) \hat{x}_b) - \frac{1}{2} D_{+b} ((y_t + y_t^+) \hat{x}_a - (x_t + x_t^+) y_a) = 0.$$
(87)

Thus, we have the conservation laws of mass and energy by construction.

6.7. Reductions of scheme (84) to the one-dimensional case

In the present section we consider one-dimensional reductions for scheme (84) and its modifications.

In the one-dimensional case, Jacobian (85) becomes $J_0 = x_a$. From equations (65) and the condition $H_y = 0$ (or $\Theta_y = 0$) it follows that

$$\Theta_t = \frac{x_t + \check{x}_t}{2} \frac{\Theta_a}{x_a}.$$
(88)

Taking the latter and conditions (80) into account, one reduces scheme (84) to the form

$$F = x_{t\check{t}} + D_{-a} \left(\frac{2x_{ta}}{x_a \hat{x}_a (x_{ta} + \check{x}_{ta})} \right) - \Theta_x = 0.$$

$$(89)$$

The conservation law of energy (86) possesses the following form

$$\frac{x_t + \check{x}_t}{2}F = D_{+\tau}\left(\frac{\check{x}_t^2}{2} + \frac{1}{x_a} - \Theta\right) + D_{+a}\left((x_t + \check{x}_t)\frac{x_{t\bar{a}}}{x_{\bar{a}}\hat{x}_{\bar{a}}(x_{t\bar{a}} + \check{x}_{t\bar{a}})}\right) = 0,$$
(90)

and the conservation law of mass (87) is

$$D_{+\tau}(x_a) - D_{+a}(x_t) = 0.$$
 (91)

Notice that the reduction (89) depend on the third difference derivatives $x_{ta\bar{a}}$ and $\check{x}_{ta\bar{a}}$. The one-dimensional schemes constructed by the authors in [32] are defined on 9-point finitedifference stencil, and the higher difference derivatives they depend on are of the second order. To obtain a better reduction, we modify scheme (84) as follows.

By shifting Jacobian (85) along the time axis, we consider its modified version on the extended stencil

$$J_1 = \frac{1}{2} \left({}^+x_a \check{y}_b + \check{x}_a y_b^+ - \check{y}_a x_b^+ - {}^+y_a \check{x}_b \right).$$
(92)

One can check that J_1 admits all the generators X_1 , $X_2^1 - X_2^3$ and $Y_1 - Y_5$.

According to the latter changes, the conservation law of mass (87) becomes

$$D_{+\tau}J_1 - \frac{1}{2}D_{+a}(\check{x}_ty_b + {}^+x_t\check{y}_b - \check{y}_t\hat{x}_b - {}^+y_tx_b) - \frac{1}{2}D_{+b}(\check{y}_t\hat{x}_a + y_t^+x_a - \check{x}_ty_a - x_t^+\check{y}_a) = 0, \quad (93)$$

and scheme (84) possesses the form

$$F_{1} = x_{t\check{t}} + \tilde{y}_{b} D_{-a} \left[\frac{1}{J_{1}\hat{J}_{1}} \frac{D(\check{x}_{t}y_{b} + {}^{+}x_{t}\check{y}_{b} - \check{y}_{t}\hat{x}_{b} - {}^{+}y_{t}x_{b})}{D((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + \check{y}_{t})\tilde{x}_{b})} \right]$$

$$-\tilde{y}_{a} D_{-b} \left[\frac{1}{J_{1}\hat{J}_{1}} \frac{D(\check{y}_{t}\hat{x}_{a} + y_{t}^{+}x_{a} - \check{x}_{t}y_{a} - x_{t}^{+}\check{y}_{a})}{D((y_{t} + \check{y}_{t})\tilde{x}_{a} - (x_{t} + \check{x}_{t})\tilde{y}_{a})} \right] - \Theta_{x} = 0,$$

$$F_{2} = y_{t\check{t}} - \tilde{x}_{b} D_{-a} \left[\frac{1}{J_{1}\hat{J}_{1}} \frac{D(\check{x}_{t}y_{b} + {}^{+}x_{t}\check{y}_{b} - \check{y}_{t}\hat{x}_{b} - {}^{+}y_{t}x_{b})}{D((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + \check{y}_{t})\tilde{x}_{b})} \right]$$

$$+\tilde{x}_{a} D_{-b} \left[\frac{1}{J_{1}\hat{J}_{1}} \frac{D(\check{y}_{t}\hat{x}_{a} + y_{t}^{+}x_{a} - \check{x}_{t}y_{a} - x_{t}^{+}\check{y}_{a})}{D((y_{t} + \check{y}_{t})\tilde{x}_{a} - (x_{t} + \check{x}_{t})\tilde{y}_{a})} \right] - \Theta_{y} = 0.$$

$$(94)$$

The conservation law of energy (86) becomes

$$\frac{D}{+\tau} \left(\frac{\check{x}_{t}^{2} + \check{y}_{t}^{2}}{2} + J_{1}^{-1} - \Theta \right) \\
+ \frac{1}{2} \frac{D}{+a} \left(((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + \check{y}_{t})\tilde{x}_{b}) \frac{D(\check{x}_{t}y_{b} + {}^{+}x_{t}\check{y}_{b} - \check{y}_{t}\hat{x}_{b} - {}^{+}y_{t}x_{b})}{D((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + \check{y}_{t})\tilde{x}_{b})} \frac{S((J_{1}\hat{J}_{1})^{-1})}{D((x_{t} + \check{x}_{t})\tilde{y}_{b} - (y_{t} + \check{y}_{t})\tilde{x}_{b})} \frac{S((J_{1}\hat{J}_{1})^{-1})}{-a} \right) \\
+ \frac{1}{2} \frac{D}{+b} \left(((y_{t} + \check{y}_{t})\tilde{x}_{a} - (x_{t} + \check{x}_{t})\tilde{y}_{a}) \frac{D(\check{y}_{t}\hat{x}_{a} + y_{t}^{+}x_{a} - \check{x}_{t}y_{a} - x_{t}^{+}\check{y}_{a})}{D((y_{t} + \check{y}_{t})\tilde{x}_{a} - (x_{t} + \check{x}_{t})\tilde{y}_{a})} \frac{S((J_{1}\hat{J}_{1})^{-1})}{-b} \right) = 0. \quad (95)$$

Finally, the reduced one-dimensional scheme is

$$F = x_{t\bar{t}} + D_{-a} \left(\frac{4}{(x_a + \check{x}_a)(x_a + \hat{x}_a)} \right) - \Theta_x = 0,$$
(96)
$$h^a = \text{const}, \qquad \tau = \text{const},$$

and the corresponding conservation laws of mass and energy have the following forms

$$D_{+\tau}(x_a + \check{x}_a) - D_{+a}(x_t + \check{x}_t) = 0,$$
(97)

$$D_{+\tau}\left(\frac{\check{x}_{t}^{2}}{2} + \frac{2}{x_{a} + \check{x}_{a}} - \Theta\right) + D_{+a}\left(\frac{2(x_{t} + \check{x}_{t})}{(x_{\bar{a}} + \check{x}_{\bar{a}})(x_{\bar{a}} + \hat{x}_{\bar{a}})}\right) = 0.$$
(98)

In [32], with the help of the finite-difference analogue of the direct method [39], the authors have obtained a family of invariant conservative schemes for the one-dimensional shallow water equations. As a simplest example of such a scheme, the authors considered scheme (67). One can check that scheme (96) is found among the obtained family of the one-dimensional conservative schemes as well.

7. Conclusion

The group classification of the two-dimensional shallow water equations with variable bottom topography $H = px^2 + 2cxy + by^2 + q_1x + q_2y + q_0$ in mass Lagrangian coordinates is performed. The advantage of studying the shallow water equations in Lagrangian coordinates is that in Lagrangian coordinates they have a variational structure. This variational representation allows one to apply Noether's theorem for constructing conservation laws. The classification results of the considered case are presented in Table 1 and formulated in Theorem 4.1. If the function H(x, y) is either of the form $H(x, y) = p(x^2 + y^2)$ (corresponding to a circular paraboloid bottom) or H(x, y) = const (corresponding to a plane bottom), then system of the shallow water equations (1)–(3) can be reduced to the gas dynamics equations of a polytropic gas with the exponent $\gamma = 2$ [22].

Notice that the admitted Lie algebra of the original two-dimensional shallow water equations contains infinite algebra of relabelling operators. We restrict this algebra to preserve a difference mesh orthogonality and uniformness, and keep all the rest symmetry of the equations. New invariant schemes for the two-dimensional shallow water equations with arbitrary bottom topography H(x, y) in Lagrangian coordinates on uniform orthogonal meshes are proposed. The schemes are constructed either by extending the known one-dimensional schemes or by direct algebraic construction. As it was mentioned, in case the bottom is arbitrary, such schemes can be constructed on uniform orthogonal meshes. At the same time, there is a rather complicated problem of approximation of the derivatives H_x and H_y of the arbitrary bottom H(x, y) for which there are no obvious representations of difference differentiation operators. This problem is discussed in a separate section of the paper.

Among the proposed schemes there are schemes possessing the conservation laws of mass and energy. In case of a horizontal bottom H(x, y) = const, some of the schemes have conservative form and possess conservation laws of momentum and the center-of-mass law.

Finally, it is shown that the proposed schemes can be reduced to the known one-dimensional schemes previously constructed by the authors in [33].

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