

Robust globally divergence-free Weak Galerkin finite element method for incompressible Magnetohydrodynamics flow *

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Abstract

This paper develops a weak Galerkin (WG) finite element method of arbitrary order for the steady incompressible Magnetohydrodynamics equations. The WG scheme uses piecewise polynomials of degrees $k (k \geq 1)$, k , $k - 1$ and $k - 1$ respectively for the approximations of the velocity, the magnetic field, the pressure, and the magnetic pseudo-pressure in the interior of elements, and uses piecewise polynomials of degree k for their numerical traces on the interfaces of elements. The method is shown to yield globally divergence-free approximations of the velocity and magnetic fields. We give existence and uniqueness results for the discrete scheme and derive optimal a priori error estimates. We also present a convergent linearized iterative algorithm. Numerical experiments are provided to verify the obtained theoretical results.

Keywords: incompressible Magnetohydrodynamics flow, Weak Galerkin method, globally divergence-free, error estimate

MSC: 65N55, 65F10, 65N22, 65N30.

1 Introduction

Magnetohydrodynamics (MHD) equations describe the basic physics laws of electrically conducting fluid flow interacting with magnetic fields, and are widely used in engineering areas [16, 23, 47, 56, 61] such as magnetic propulsion devices, optical modulation and switch, continuous metal casting, semi-conductor manufacture, and nuclear reactor technology. This paper is to consider a finite element analysis of a steady incompressible MHD flow model.

The incompressible MHD flow is described by a coupling system of Navier-Stokes equations and Maxwell equations. Some early research on the finite element analysis of MHD can be found in [26, 50, 68]. In particular, in [26] Gunzburger et al. considered a steady incompressible MHD model in three dimensions, showed the existence and uniqueness of a weak solution, and proved an optimal estimate for a mixed finite element discretization. In recent twenty years there have developed many finite element methods for the incompressible MHD equations; see, e.g. [22, 24, 25, 30, 31, 54, 57, 59, 67, 69, 73, 76] for steady models and [18, 19, 21, 29, 51, 71, 74] for unsteady models.

There are two divergence constraints in the incompressible MHD equations, i.e. the velocity and magnetic fields are both divergence-free, which correspond to the conservation of mass and magnetic flux, respectively. How to obtain exactly divergence-free approximations is an important issue in numerically solving the related problems, since numerical methods with poor conservation may lead to instabilities [1, 4, 34, 35, 43, 48, 60]. In particular, for incompressible fluid flows the exactly divergence-free discretizations automatically lead to pressure-robustness in the sense that the velocity approximation error is independent of the pressure approximation [35, 44]. We refer to [8, 11, 13, 27, 28, 70, 77] for some divergence-free finite element methods for the incompressible fluid flows, and to [5, 14, 32] for several divergence-free finite element methods for Maxwell equations.

For the incompressible MHD equations, there are considerable efforts devoted to divergence-free finite element approaches [24, 30, 31, 40–42]. Greif et al. [24] proposed a mixed interior-penalty discontinuous Galerkin (DG) method with the exactly divergence-free velocity, where the velocity field is discretized

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by $H(\text{div}; \Omega)$ -conforming Brezzi-Douglas-Marini elements, the pressure by fully discontinuous finite elements, and the magnetic field by $H(\text{curl}; \Omega)$ -conforming Nedelec elements. Li et al. [40, 41] and Hu et al. [31] developed central DG methods and stable finite element methods with the exactly divergence-free magnetic field, respectively. Hiptmair et al. [30] developed a mixed DG method with the exactly divergence-free velocity and magnetic field for three-dimensional transient incompressible magnetohydrodynamic equations, where the velocity and pressure approximations are as same as those in [24], and the divergence-free property of magnetic field is realized by means of a magnetic vector potential. Li et al. [42] presented a constrained transport finite element method with the exactly divergence-free velocity and magnetic field for three-dimensional incompressible resistive MHD equations, by following the same ideas as in [13, 24].

This paper is to develop an arbitrary order weak Galerkin (WG) scheme with exactly divergence-free velocity and magnetic approximations for the following steady incompressible MHD equations:

$$-\frac{1}{H_a^2} \Delta \mathbf{u} + \frac{1}{N} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p - \frac{1}{R_m} \nabla \times \mathbf{B} \times \mathbf{B} = \mathbf{f}, \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad (1.2)$$

$$\frac{1}{R_m} \nabla \times \nabla \times \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla r = \mathbf{g}, \quad \text{in } \Omega, \quad (1.3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \text{in } \Omega, \quad (1.4)$$

with the homogeneous boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega, \quad (1.5)$$

$$\mathbf{B} \times \mathbf{n} = \mathbf{0}, \quad \text{on } \partial\Omega, \quad (1.6)$$

$$r = 0, \quad \text{on } \partial\Omega. \quad (1.7)$$

Here $\Omega \in \mathbb{R}^d (d = 2, 3)$ is a polygonal or polyhedral domain, $\mathbf{u} = (u_1, u_2, \dots, u_d)^T$ is the velocity vector, p the pressure, $\mathbf{B} = (B_1, B_2, \dots, B_d)^T$ the magnetic field, and r the magnetic pseudo-pressure. The right-hand side terms $\mathbf{f}, \mathbf{g} \in [L^2(\Omega)]^d$ are the forcing functions. H_a , N and R_m are the Hartmann number, the interaction parameter and the magnetic Reynolds number, respectively.

The WG method was first proposed by Wang and Ye for second-order elliptic problems [62, 63]. Due to the use of weakly defined gradient/divergence operators over functions with discontinuity, this method allows the use of totally discontinuous functions on finite element partitions with arbitrary shape of polygons/polyhedra. It also has the local elimination property, i.e. the unknowns defined in the interior of elements can be locally eliminated by using the numerical traces defined on the interfaces of elements. We refer to [8, 27, 28, 46, 64, 65, 75, 77, 78] for some applications of the WG method to the incompressible fluid flows and Maxwell equations.

In this paper, we consider the WG discretization of the MHD model (1.1)-(1.7). The main features of our scheme are as follows.

- We apply piecewise polynomials of degrees $k (k \geq 1), k, k-1$ and $k-1$, respectively to approximate the velocity, the magnetic field, the pressure, and the magnetic pseudo-pressure in the interior of elements, and apply piecewise polynomials of degree k to approximate their numerical traces on the interfaces of elements.
- The scheme is “parameter-friendly” in the sense that it does not require the stabilization parameters to be “sufficiently large”.
- The scheme gives globally divergence-free approximations of the velocity and magnetic fields.
- The unknowns of the velocity, the magnetic field, the pressure and the magnetic pseudo-pressure in the interior of elements can be locally eliminated so as to obtain a reduced discrete system of smaller size.
- The obtained error estimates are optimal.

The rest of this paper is arranged as follows. Section 2 gives weak formulations of the model problem. Section 3 is devoted to the WG finite element scheme and some preliminary results. In Section 4 we discuss the existence and uniqueness of the discrete solution. Section 5 derives a priori error estimates. Section 6 shows the local elimination property and proposes an iteration algorithm for the nonlinear WG scheme. Finally, We provide some numerical results.

2 Weak problem

2.1 Notation

For any bounded domain $D \in R^s$ ($s = d, d - 1$), nonnegative integer m and real number $1 \leq q < \infty$, let $W^{m,q}(D)$ and $W_0^{m,q}(D)$ be the usual Sobolev spaces defined on D with norm $\|\cdot\|_{m,q,D}$ and semi-norm $|\cdot|_{m,q,D}$. In particular, $H^m(D) := W^{m,2}(D)$ and $H_0^m(D) := W_0^{m,2}(D)$, with $\|\cdot\|_{m,D} := \|\cdot\|_{m,2,D}$ and $|\cdot|_{m,D} := |\cdot|_{m,2,D}$. We use $(\cdot, \cdot)_{m,D}$ to denote the inner product of $H^m(D)$, with $(\cdot, \cdot)_D := (\cdot, \cdot)_{0,D}$. When $D = \Omega$, we set $\|\cdot\|_m := \|\cdot\|_{m,\Omega}$, $|\cdot|_m := |\cdot|_{m,\Omega}$, and $(\cdot, \cdot) := (\cdot, \cdot)_\Omega$. Especially, when $D \subset R^{d-1}$ we use $\langle \cdot, \cdot \rangle_D$ to replace $(\cdot, \cdot)_D$. For any integer $k \geq 0$, let $P_k(D)$ denote the set of all polynomials on D with degree no more than k . We also need the following spaces:

$$\begin{aligned} L_0^2(\Omega) &:= \{v \in L^2(\Omega) : (v, 1) = 0\}, \\ \mathbf{H}(\text{div}, \Omega) &:= \{\mathbf{v} \in L^2(\mathcal{D})^d : \nabla \cdot \mathbf{v} \in L^2(\mathcal{D})\}, \\ \mathbf{H}(\text{curl}; \Omega) &:= \{\mathbf{v} \in L^2(\mathcal{D})^d : \nabla \times \mathbf{v} \in L^2(\Omega)^{2d-3}\}, \\ \mathbf{H}_0(\text{curl}; \Omega) &:= \{\mathbf{v} \in \mathbf{H}(\text{curl}; \Omega) : \mathbf{n} \times \mathbf{v} = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

where the cross product \times of two vectors is defined as following: for $\mathbf{v} = (v_1, \dots, v_d)^T$, $\mathbf{w} = (w_1, \dots, w_d)^T$,

$$\mathbf{v} \times \mathbf{w} = \begin{cases} v_1 w_2 - v_2 w_1, & \text{if } d = 2, \\ (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)^T, & \text{if } d = 3. \end{cases}$$

Let \mathcal{T}_h be a shape regular partition of Ω into closed simplexes, and let ε_h be the set of all edges(faces) of all the elements in Ω . For any $K \in \mathcal{T}_h$, $e \in \varepsilon_h$, we denote by h_K the diameter of K and by h_e the diameter of e . Let \mathbf{n}_K and \mathbf{n}_e denote the outward unit normal vectors along the boundary ∂K and e , respectively. Sometimes we may abbreviate \mathbf{n}_K as \mathbf{n} . We use ∇_h , $\nabla_h \cdot$ and $\nabla_h \times$ to denote respectively the operators of piecewise-defined gradient, divergence and curl with respect to the decomposition \mathcal{T}_h .

Throughout this paper, we use $\alpha \lesssim \beta$ to denote $\alpha \leq C\beta$, where C is a positive constant independent of the mesh size h .

2.2 Weak form

For simplicity, we set

$$\mathbf{V} := [H_0^1(\Omega)]^d, \quad \mathbf{W} := \mathbf{H}_0(\text{curl}; \Omega).$$

For all $\mathbf{u}, \mathbf{v}, \Phi \in \mathbf{V}$, $\mathbf{B}, \mathbf{w} \in \mathbf{W}$, $q \in L_0^2(\Omega)$, $\theta \in H_0^1(\Omega)$, we define the following bilinear and trilinear forms:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \frac{1}{H_a^2}(\nabla \mathbf{u}, \nabla \mathbf{v}), & b(\mathbf{v}, q) &:= (q, \nabla \cdot \mathbf{v}), \\ \tilde{a}(\mathbf{B}, \mathbf{w}) &:= \frac{1}{R_m^2}(\nabla \times \mathbf{B}, \nabla \times \mathbf{w}), & \tilde{b}(\mathbf{w}, \theta) &:= \frac{1}{R_m}(\nabla \theta, \mathbf{w}), \\ c(\Phi; \mathbf{u}, \mathbf{v}) &:= \frac{1}{N} \left\{ \frac{1}{2}(\nabla \cdot (\Phi \otimes \mathbf{u}), \mathbf{v}) - \frac{1}{2}(\nabla \cdot (\Phi \otimes \mathbf{v}), \mathbf{u}) \right\}, \\ \tilde{c}(\mathbf{v}; \mathbf{B}, \mathbf{w}) &:= \frac{1}{R_m}(\nabla \times \mathbf{w}, \mathbf{v} \times \mathbf{B}). \end{aligned}$$

It is easy to see that $c(\Phi; \mathbf{v}, \mathbf{v}) = 0$.

Then the weak form of the problem (1.1)-(1.7) reads: find $\mathbf{u} \in \mathbf{V}$, $\mathbf{B} \in \mathbf{W}$, $p \in L_0^2(\Omega)$, $r \in H_0^1(\Omega)$ such that

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}) + \tilde{a}(\mathbf{B}, \mathbf{w}) + b(\mathbf{u}, q) - b(\mathbf{v}, p) + \tilde{b}(\mathbf{w}, r) - \tilde{b}(\mathbf{B}, \theta) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \tilde{c}(\mathbf{v}; \mathbf{B}, \mathbf{B}) - \tilde{c}(\mathbf{u}; \mathbf{B}, \mathbf{w}) \\ &= (\mathbf{f}, \mathbf{v}) + \frac{1}{R_m}(\mathbf{g}, \mathbf{w}), \quad \forall \mathbf{v} \in \mathbf{V}, \mathbf{w} \in \mathbf{W}, q \in L_0^2(\Omega), \theta \in H_0^1(\Omega). \end{aligned} \tag{2.1}$$

Remark 2.1. From [54, Corollary 2.18] for $d = 3$, if $\mathbf{f} \in H^{-1}(\Omega)^3$ and $\mathbf{g} \in L^2(\Omega)^3$, then the weak problem (2.1) admits at least one solution, and under a certain smallness condition the solution is unique.

3 Weak Galerkin finite element method

3.1 WG scheme

To establish the WG method for the problem (1.1)-(1.7), We firstly introduce, for integer $m \geq 0$, the discrete weak gradient operator $\nabla_{w,m}$, the discrete weak divergence operator $\nabla_{w,m}^\cdot$ and the discrete weak curl operator $\nabla_{w,m} \times$ as follows:

Definition 3.1. For any $\mathbf{v} \in \mathbf{V}(K) := \{\mathbf{v} = \{\mathbf{v}_o, \mathbf{v}_b\} : \mathbf{v}_o \in L^2(K), \mathbf{v}_b \in H^{1/2}(\partial K)\}$ and $K \in \mathcal{T}_h$, the discrete weak gradient, $\nabla_{w,m,K}\mathbf{v} \in [P_m(K)]^d$, of \mathbf{v} on K is defined by

$$(\nabla_{w,m,K}\mathbf{v}, \phi)_K = -(\mathbf{v}_0, \nabla \cdot \phi)_K + \langle \mathbf{v}_b, \phi \cdot \mathbf{n}_K \rangle_{\partial K}, \quad \forall \phi \in [\mathcal{P}_m(K)]^d. \quad (3.1)$$

Then the global discrete weak gradient operator $\nabla_{w,m}$ is defined by

$$\nabla_{w,m}|_K := \nabla_{w,m,K}, \quad \forall K \in \mathcal{T}_h.$$

Moreover, for a vector $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_d)^T$, the discrete weak gradient $\nabla_{w,m}\mathbf{v}$ is defined by

$$\nabla_{w,m}\mathbf{v} := (\nabla_{w,m}\mathbf{v}_1, \dots, \nabla_{w,m}\mathbf{v}_d)^T.$$

Definition 3.2. For any $\mathbf{w} \in \mathbf{W}(K) := \{\mathbf{w} = \{\mathbf{w}_o, \mathbf{w}_b\} : \mathbf{w}_o \in [L^2(K)]^d, \mathbf{w}_b \cdot \mathbf{n}_K \in H^{-1/2}(\partial K)\}$ and $K \in \mathcal{T}_h$, the discrete weak divergence, $\nabla_{w,m,K} \cdot \mathbf{w} \in \mathcal{P}_m(K)$, of \mathbf{w} on K is defined by

$$(\nabla_{w,m,K} \cdot \mathbf{w}, \phi)_K = -(\mathbf{w}_o, \nabla \phi)_K + \langle \mathbf{w}_b \cdot \mathbf{n}_K, \phi \rangle_{\partial K}, \quad \forall \phi \in \mathcal{P}_m(K).$$

Then the global discrete weak divergence operator $\nabla_{w,m}^\cdot$ is defined by

$$\nabla_{w,m}^\cdot|_K := \nabla_{w,m,K}^\cdot, \quad \forall K \in \mathcal{T}_h.$$

Moreover, for a tensor $\hat{\mathbf{w}} = (\mathbf{w}_1, \dots, \mathbf{w}_d)$, the discrete weak divergence $\nabla_{w,m} \cdot \hat{\mathbf{w}}$ is defined by

$$\nabla_{w,m} \cdot \hat{\mathbf{w}} := (\nabla_{w,m} \cdot \mathbf{w}_1, \dots, \nabla_{w,m} \cdot \mathbf{w}_d)^T.$$

Definition 3.3. For any $\mathbf{w} \in \mathbf{W}(K) := \{\mathbf{w} = \{\mathbf{w}_o, \mathbf{w}_b\} : \mathbf{w}_o \in [L^2(K)]^d, \mathbf{w}_b \times \mathbf{n}_K \in [H^{-1/2}(\partial K)]^{2d-3}\}$ and $K \in \mathcal{T}_h$, the discrete weak curl $\nabla_{w,m,K} \times \mathbf{w} \in [\mathcal{P}_m(K)]^{2d-3}$ on K is defined by

$$(\nabla_{w,m,K} \times \mathbf{w}, \phi)_K = (\mathbf{w}_o, \nabla \times \phi)_K + \langle \mathbf{w}_b \times \mathbf{n}_K, \phi \rangle_{\partial K}, \quad \forall \phi \in [\mathcal{P}_m(K)]^{2d-3}, \quad (3.2)$$

where

Then the global discrete weak curl operator $\nabla_{w,m} \times$ is defined by

$$\nabla_{w,m} \times|_K := \nabla_{w,m,K} \times, \quad \forall K \in \mathcal{T}_h.$$

For any integer $k \geq 1$, we introduce the following finite dimensional spaces:

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\} : \mathbf{v}_{ho}|_K \in [\mathcal{P}_k(K)]^d, \mathbf{v}_{hb}|_e \in [\mathcal{P}_k(e)]^d, \forall K \in \mathcal{T}_h, \forall e \in \mathcal{E}_h\}, \\ \mathbf{V}_h^0 &= \{\mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\} \in \mathbf{V}_h; \mathbf{v}_{hb}|_{\partial\Omega} = 0\}, \\ \mathbf{W}_h^0 &= \{\mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \mathbf{V}_h; \mathbf{w}_{hb} \times \mathbf{n}|_{\partial\Omega} = 0\}, \\ Q_h &= \{q_h = \{q_{ho}, q_{hb}\} : q_{ho}|_K \in \mathcal{P}_{k-1}(K), q_{hb}|_e \in \mathcal{P}_k(e), \forall K \in \mathcal{T}_h, \forall e \in \mathcal{E}_h\}, \\ Q_h^0 &= \{q_h = \{q_{ho}, q_{hb}\} \in Q_h : q_{ho} \in L_0^2(\Omega)\}, \\ R_h^0 &= \{\theta_h = \{\theta_{ho}, \theta_{hb}\} \in Q_h; \theta_{hb}|_{\partial\Omega} = 0\}. \end{aligned}$$

We also define the following bilinear forms and trilinear terms:

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) &:= \frac{1}{H_a^2} (\nabla_{w,k-1} \mathbf{u}_h, \nabla_{w,k-1} \mathbf{v}_h) + s_h(\mathbf{u}_h, \mathbf{v}_h), \\
s_h(\mathbf{u}_h, \mathbf{v}_h) &:= \frac{1}{H_a^2} \langle \tau(\mathbf{u}_{ho} - \mathbf{u}_{hb}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial\mathcal{T}_h}, \\
\tilde{a}_h(\mathbf{B}_h, \mathbf{w}_h) &:= \frac{1}{R_m^2} (\nabla_{w,k-1} \times \mathbf{B}_h, \nabla_{w,k-1} \times \mathbf{w}_h) + \tilde{s}_h(\mathbf{B}_h, \mathbf{w}_h), \\
\tilde{s}_h(\mathbf{B}_h, \mathbf{w}_h) &:= \frac{1}{R_m^2} \langle \tau(\mathbf{B}_{ho} - \mathbf{B}_{hb}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h}, \\
b_h(\mathbf{v}_h, q_h) &:= (\nabla_{w,k} q_h, \mathbf{v}_{ho}), \quad \tilde{b}_h(\mathbf{w}_h, \theta_h) := \frac{1}{R_m} (\nabla_{w,k} \theta_h, \mathbf{w}_{ho}), \\
c_h(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) &:= \frac{1}{2N} ((\nabla_{w,k} \cdot \{\mathbf{u}_{ho} \otimes \Phi_{ho}, \mathbf{u}_{hb} \otimes \Phi_{hb}\}, \mathbf{v}_{ho}) - (\nabla_{w,k} \cdot \{\mathbf{v}_{ho} \otimes \Phi_{ho}, \mathbf{v}_{hb} \otimes \Phi_{hb}\}, \mathbf{u}_{ho})), \\
\tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) &:= \frac{1}{R_m} (\nabla_{w,k} \times \mathbf{w}_h, \mathbf{v}_{ho} \times \mathbf{B}_{ho}),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{u}_h &= \{\mathbf{u}_{ho}, \mathbf{u}_{hb}\}, \mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\}, \Phi_h = \{\Phi_{ho}, \Phi_{hb}\} \in \mathbf{V}_h^0, \\
\mathbf{B}_h &= \{\mathbf{B}_{ho}, \mathbf{B}_{hb}\}, \mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \mathbf{W}_h^0, \\
q_h &= \{q_{ho}, q_{hb}\} \in Q_h^0, \quad \theta_h = \{\theta_{ho}, \theta_{hb}\} \in R_h^0,
\end{aligned}$$

and the stabilization parameter τ in $s_h(\cdot; \cdot, \cdot)$ and $\tilde{s}_h(\cdot; \cdot, \cdot)$ is given by

$$\tau|_{\partial K} = h_K^{-1}, \quad \forall K \in \mathcal{T}_h.$$

We easily see that

$$c_h(\Phi_h; \mathbf{v}_h, \mathbf{v}_h) = 0, \quad \forall \Phi_h, \mathbf{v}_h. \quad (3.3)$$

With the above definitions, the WG scheme for the problem (1.1)-(1.7) reads as follows: find $\mathbf{u}_h = \{\mathbf{u}_{ho}, \mathbf{u}_{hb}\} \in \mathbf{V}_h^0$, $\mathbf{B}_h = \{\mathbf{B}_{ho}, \mathbf{B}_{hb}\} \in \mathbf{W}_h^0$, $p_h = \{p_{ho}, p_{hb}\} \in Q_h^0$, $r_h = \{r_{ho}, r_{hb}\} \in R_h^0$, such that

$$\begin{aligned}
&a_h(\mathbf{u}_h, \mathbf{v}_h) + \tilde{a}_h(\mathbf{B}_h, \mathbf{w}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) + \tilde{b}_h(\mathbf{w}_h, r_h) - \tilde{b}_h(\mathbf{B}_h, \theta_h) \\
&\quad + c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) \\
&= (\mathbf{f}, \mathbf{v}_{ho}) + \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \mathbf{w}_h \in \mathbf{W}_h^0, q_h \in Q_h^0, \theta_h \in R_h^0. \quad (3.4)
\end{aligned}$$

The existence and uniqueness of the discrete solution to this scheme will be discussed in next section. Notice that the scheme (3.4) is equivalent to the following system: find $\mathbf{u}_h \in \mathbf{V}_h^0$, $\mathbf{B}_h \in \mathbf{W}_h^0$, $p_h \in Q_h^0$, $r_h \in R_h^0$, such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) + c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) = (\mathbf{f}, \mathbf{v}_{ho}), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \quad (3.5a)$$

$$b_h(\mathbf{u}_h, q_h) = 0, \quad \forall q_h \in Q_h^0, \quad (3.5b)$$

$$\tilde{a}_h(\mathbf{B}_h, \mathbf{w}_h) + \tilde{b}_h(\mathbf{w}_h, r_h) - \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) = \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}), \quad \forall \mathbf{w}_h \in \mathbf{W}_h^0, \quad (3.5c)$$

$$\tilde{b}_h(\mathbf{B}_h, \theta_h) = 0, \quad \forall \theta_h \in R_h^0. \quad (3.5d)$$

In what follows we shall show that the two relations (3.5b) and (3.5d) yield globally divergence-free approximations of the velocity and the magnetic field, i.e.

$$\mathbf{u}_{ho} \in \mathbf{H}(\text{div}, \Omega), \nabla \cdot \mathbf{u}_{ho} = 0, \quad (3.6)$$

$$\mathbf{B}_{ho} \in \mathbf{H}(\text{div}, \Omega), \nabla \cdot \mathbf{B}_{ho} = 0. \quad (3.7)$$

In fact, define a function $\varrho_{hb} \in L^2(\varepsilon_h)$ as follows: for any $e \in \varepsilon_h$,

$$\varrho_{hb}|_e = \begin{cases} -((\mathbf{u}_{ho} \cdot \mathbf{n}_e)|_{K_1})|_e - ((\mathbf{u}_{ho} \cdot \mathbf{n}_e)|_{K_2})|_e, & \text{if } e = K_1 \cap K_2, K_1, K_2 \in \mathcal{T}_h, \\ 0, & \text{if } e \subset \partial\Omega. \end{cases}$$

Setting $C_0 := \frac{1}{|\Omega|} \int_{\Omega} \nabla_h \cdot \mathbf{u}_{ho} d\mathbf{x}$ and taking $q_h = \{q_{ho}, q_{hb}\}$ in (3.5b) with $q_{ho} = \nabla_h \cdot \mathbf{u}_{ho} - C_0$, $q_{hb} = \varrho_{hb} - C_0$, we obtain

$$\begin{aligned} 0 &= -b_h(\mathbf{u}_h, q_h) = -(\nabla_{w,k} q_h, \mathbf{u}_{ho}) \\ &= (\nabla_h \cdot \mathbf{u}_{ho}, q_{ho}) - \sum_{T \in \mathcal{T}_h} \langle \mathbf{u}_{ho} \cdot \mathbf{n}_T, q_{hb} \rangle_{\partial T} \\ &= (\nabla_h \cdot \mathbf{u}_{ho}, \nabla_h \cdot \mathbf{u}_{ho} - C_0) - \sum_{T \in \mathcal{T}_h} \langle \mathbf{u}_{ho} \cdot \mathbf{n}_T, \varrho_{hb} - C_0 \rangle_{\partial T} \\ &= (\nabla_h \cdot \mathbf{u}_{ho}, \nabla_h \cdot \mathbf{u}_{ho}) - \sum_{T \in \mathcal{T}} \langle \mathbf{u}_{ho} \cdot \mathbf{n}_T, \varrho_{hb} \rangle_{\partial T} \\ &= \|\nabla_h \cdot \mathbf{u}_{ho}\|_0^2 + \sum_{e \in \varepsilon_h, e \not\subset \partial\Omega} \|(\mathbf{u}_{ho} \cdot \mathbf{n}_e)|_{K_1} + (\mathbf{u}_{ho} \cdot \mathbf{n}_e)|_{K_2}\|_{0,e}^2. \end{aligned}$$

This gives $\mathbf{u}_{ho} \in \mathbf{H}(\text{div}, \Omega)$ and $\nabla_h \cdot \mathbf{u}_{ho} = \nabla \cdot \mathbf{u}_{ho} = 0$, i.e. (3.6) holds.

Similarly, we can get (3.7).

As a result, we have the following conclusion.

Theorem 3.1. *The scheme (3.4) yields the globally divergence-free approximations of the velocity and the magnetic field in the sense that both (3.6) and (3.7) hold.*

We introduce spaces

$$\begin{aligned} \bar{\mathbf{V}}_h &:= \{\mathbf{v}_h \in \mathbf{V}_h^0 : b_h(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h^0\}, \\ \bar{\mathbf{W}}_h &:= \{\mathbf{w}_h \in \mathbf{W}_h^0 : \tilde{b}_h(\mathbf{w}_h, \theta_h) = 0, \forall \theta_h \in R_h^0\}. \end{aligned}$$

Thus, the solution $(\mathbf{u}_h, \mathbf{B}_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0$ of the scheme (3.4) also solves the following discretization problem: find $(\mathbf{u}_h, \mathbf{B}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + \tilde{a}_h(\mathbf{B}_h, \mathbf{w}_h) + c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) \\ = (\mathbf{f}, \mathbf{v}_{ho}) + \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}), \quad \forall (\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h, \end{aligned} \quad (3.8)$$

Remark 3.1. *It is easy to see that*

$$\bar{\mathbf{V}}_h \subset \{\mathbf{v}_h \in \mathbf{V}_h^0 : \mathbf{v}_{ho} \in \mathbf{H}(\text{div}, \Omega), \nabla \cdot \mathbf{v}_{ho} = 0\}, \quad (3.9)$$

$$\bar{\mathbf{W}}_h = \{\mathbf{w}_h \in \mathbf{W}_h^0 : \mathbf{w}_{ho} \in \mathbf{H}(\text{div}, \Omega), \nabla \cdot \mathbf{w}_{ho} = 0\}. \quad (3.10)$$

To discuss the existence and uniqueness of the discrete solution of the scheme (3.4) and derive error estimates, we will give some preliminary results in next subsection.

3.2 Preliminary results

In view of the definitions of weak gradient and curl operators, the Green's formula, the Cauchy-Schwarz inequality, the trace inequality and the inverse inequality, we can easily derive the following inequalities on \mathbf{V}_h .

Lemma 3.1. *Let $0 \leq k-1 \leq m \leq k$. For any $K \in \mathcal{T}_h$ and $\mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\}, \mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \mathbf{V}_h$, there hold*

$$\|\nabla \mathbf{v}_{ho}\|_{0,K} \lesssim \|\nabla_{w,m} \mathbf{v}_h\|_{0,K} + h_K^{-\frac{1}{2}} \|\mathbf{v}_{ho} - \mathbf{v}_{hb}\|_{0,\partial K}, \quad (3.11a)$$

$$\|\nabla_{w,m} \mathbf{v}_h\|_{0,K} \lesssim \|\nabla \mathbf{v}_{ho}\|_{0,K} + h_K^{-\frac{1}{2}} \|\mathbf{v}_{ho} - \mathbf{v}_{hb}\|_{0,\partial K}, \quad (3.11b)$$

$$\|\nabla \times \mathbf{w}_{ho}\|_{0,K} \lesssim \|\nabla_{w,m} \times \mathbf{w}_h\|_{0,K} + h_K^{-\frac{1}{2}} \|(\mathbf{w}_{ho} - \mathbf{w}_{hb})\|_{0,\partial K}, \quad (3.11c)$$

$$\|\nabla_{w,m} \times \mathbf{w}_h\|_{0,K} \lesssim \|\nabla \times \mathbf{w}_{ho}\|_{0,K} + h_K^{-\frac{1}{2}} \|(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}\|_{0,\partial K}. \quad (3.11d)$$

Introduce the following semi-norms respectively on \mathbf{V}_h^0 , \mathbf{W}_h^0 , Q_h^0 and R_h^0 :

$$\begin{aligned} |||\mathbf{v}_h|||_V &:= \left(\|\nabla_{w,k-1}\mathbf{v}_h\|_0^2 + \|\tau^{\frac{1}{2}}(\mathbf{v}_{ho} - \mathbf{v}_{hb})\|_{0,\partial\mathcal{T}_h}^2 \right)^{1/2}, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \\ |||\mathbf{w}_h|||_W &:= \left(\|\nabla_{w,k-1} \times \mathbf{w}_h\|_0^2 + \|\tau^{\frac{1}{2}}(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}\|_{0,\partial\mathcal{T}_h}^2 \right)^{1/2}, \quad \forall \mathbf{w}_h \in \mathbf{W}_h^0, \\ |||q_h|||_Q &:= \left(\|q_{ho}\|_0^2 + \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla_{w,k} q_h\|_{0,K}^2 \right)^{1/2}, \quad \forall q_h \in Q_h^0, \\ |||\theta_h|||_R &:= \left(\|\theta_{ho} - \bar{\theta}_{ho}\|_0^2 + \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla_{w,k} \theta_h\|_{0,K}^2 \right)^{1/2}, \quad \forall \theta_h \in R_h^0, \end{aligned}$$

where $\bar{\theta}_{ho} := \frac{1}{|\Omega|} \int_{\Omega} \theta_{ho} d\mathbf{x}$ denotes the mean value of θ_{ho} , and we recall that $\tau|_{\partial K} = h_K^{-1}$. It is easy to see that $|||\cdot|||_V$, $|||\cdot|||_Q$ and $|||\cdot|||_R$ are norms on \mathbf{V}_h^0 , Q_h^0 and R_h^0 , respectively (cf. [8]). As for the semi-norm $|||\cdot|||_W$, we have the following result.

Lemma 3.2. $|||\cdot|||_W$ is a norm on $\bar{\mathbf{W}}_h$.

Proof. For any $\mathbf{w}_h \in \bar{\mathbf{W}}_h$, it suffices to show that $|||\mathbf{w}_h|||_W = 0$ leads to $\mathbf{w}_h = 0$. From the definition of $|||\cdot|||_W$ and the estimate (3.11c), we immediately get

$$\nabla \times \mathbf{w}_{ho}|_K = 0, \quad (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}|_{\partial K} = 0, \quad \forall K \in \mathcal{T}_h.$$

Hence we have

$$\mathbf{w}_{ho} \in \mathbf{H}(\text{curl}; \Omega), \quad \nabla \times \mathbf{w}_{ho} = 0, \text{ and } \mathbf{w}_{ho} \times \mathbf{n}|_{\partial\Omega} = \mathbf{w}_{hb} \times \mathbf{n}|_{\partial\Omega} = 0.$$

Then there exists a potential function φ such that $\mathbf{w}_{ho} = \nabla\varphi$ in Ω .

On the other hand, from (3.10) we also have

$$\mathbf{w}_{ho} \in \mathbf{H}(\text{div}, \Omega), \quad \nabla \cdot \mathbf{w}_{ho}|_{\Omega} = 0.$$

As a result, we obtain $\Delta\varphi = 0$ in Ω . As the boundary condition $\nabla\varphi \times \mathbf{n}|_{\partial\Omega} = \mathbf{w}_{ho} \times \mathbf{n}|_{\partial\Omega} = 0$ implies that φ is a constant on $\partial\Omega$, we know that φ is a constant on Ω , which means that $\mathbf{w}_{ho} = \nabla\varphi = 0$. Finally, from the relation $(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}|_{\partial K} = 0$ for any $K \in \mathcal{T}_h$ it follows that $\mathbf{w}_{hb} = 0$. This finishes the proof. \blacksquare

Lemma 3.3. There hold

$$\|\nabla_h \mathbf{v}_{ho}\|_0 \lesssim |||\mathbf{v}_h|||_V, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \tag{3.12a}$$

$$\|\nabla_h \times \mathbf{w}_{ho}\|_0 \lesssim |||\mathbf{w}_h|||_W, \quad \forall \mathbf{w}_h \in \mathbf{W}_h^0, \tag{3.12b}$$

and

$$\|\mathbf{v}_{ho}\|_{0,q} \lesssim |||\mathbf{v}_h|||_V, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \tag{3.13a}$$

for $2 \leq q < \infty$ when $d = 2$, and for $2 \leq q \leq 6$ when $d = 3$.

Proof. The first two inequalities follow from Lemma 3.1 directly and the third one comes from [28, Lemma 3.5]. \blacksquare

We introduce the following mesh-dependent inner products and norms:

$$\begin{aligned} (u, v)_{\mathcal{T}_h} &:= \sum_{K \in \mathcal{T}_h} (u, v)_K, & \|u\|_{0,\mathcal{T}_h} &:= \left(\sum_{K \in \mathcal{T}_h} \|u\|_{0,K}^2 \right)^{1/2}, \\ \langle u, v \rangle_{\partial\mathcal{T}_h} &:= \sum_{K \in \mathcal{T}_h} \langle u, v \rangle_{\partial K}, & \|u\|_{0,\partial\mathcal{T}_h} &:= \left(\sum_{K \in \mathcal{T}_h} \|u\|_{0,\partial K}^2 \right)^{1/2}. \end{aligned}$$

Lemma 3.4. *There holds*

$$\|\mathbf{w}_{ho}\|_{0,3,\Omega} \lesssim \|\mathbf{w}_h\|_W, \quad \forall \mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \bar{\mathbf{W}}_h. \quad (3.14)$$

Proof. By (3.10) and the Green's formula we get

$$(\mathbf{w}_{ho}, \nabla \phi)_{\mathcal{T}_h} = 0, \quad \forall \phi \in H_0^1(\Omega).$$

Then, from [52, Theorem 3.1] we have

$$\begin{aligned} \|\mathbf{w}_{ho}\|_{0,3,\Omega} &\lesssim \|\nabla_h \times \mathbf{w}_{ho}\|_{0,\mathcal{T}_h} + \|\tau^{\frac{1}{2}}[\mathbf{w}_{ho}]\|_{0,\partial\mathcal{T}_h} \\ &= \|\nabla_h \times \mathbf{w}_{ho}\|_{0,\mathcal{T}_h} + \|\tau^{\frac{1}{2}}[\mathbf{w}_{ho} - \mathbf{w}_{hb}]\|_{0,\partial\mathcal{T}_h} \\ &\lesssim \|\nabla_h \times \mathbf{w}_{ho}\|_{0,\mathcal{T}_h} + \|\tau^{\frac{1}{2}}(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}\|_{0,\partial\mathcal{T}_h}. \end{aligned}$$

This estimate plus (3.12b) yields (3.14). \blacksquare

In light of the trace theorem, the inverse inequality and scaling arguments, we can get the following lemma (cf. [28, 58]).

Lemma 3.5. *For all $K \in \mathcal{T}_h$, $\varphi \in H^1(K)$, and $1 \leq q \leq \infty$, there holds*

$$\|\varphi\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}} \|\varphi\|_{0,q,K} + h_K^{1-\frac{1}{q}} |\varphi|_{1,q,K}.$$

In particular, for all $\varphi \in \mathcal{P}_k(K)$,

$$\|\varphi\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}} \|\varphi\|_{0,q,K}.$$

For any integer $s \geq 0$, $K \in \mathcal{T}_h$, $e \in \varepsilon_h$, let $Q_s^o : L^2(K) \rightarrow \mathcal{P}_s(K)$ and $Q_s^b : L^2(e) \rightarrow \mathcal{P}_s(e)$ be the standard L^2 projection operators. There vector/tensor analogues are denoted by \mathbf{Q}_s^o and \mathbf{Q}_s^b , respectively.

Lemma 3.6. [58] *For any $K \in \mathcal{T}_h$, $e \in \varepsilon_h$, and $1 \leq j \leq s+1$, there hold*

$$\begin{aligned} \|\mathbf{v} - Q_s^o \mathbf{v}\|_{0,K} + h_K |\mathbf{v} - Q_s^o \mathbf{v}|_{1,K} &\lesssim h_K^j |\mathbf{v}|_{j,K}, \quad \forall \mathbf{v} \in H^j(K), \\ \|\mathbf{v} - Q_s^o \mathbf{v}\|_{0,\partial K} + \|\mathbf{v} - Q_s^b \mathbf{v}\|_{0,\partial K} &\lesssim h_K^{j-1/2} |\mathbf{v}|_{j,K}, \quad \forall \mathbf{v} \in H^j(K), \\ \|Q_s^o \mathbf{v}\|_{0,K} &\leq \|\mathbf{v}\|_{0,K}, \quad \forall \mathbf{v} \in L^2(K), \\ \|Q_s^b \mathbf{v}\|_{0,e} &\leq \|\mathbf{v}\|_{0,e}, \quad \forall \mathbf{v} \in L^2(e). \end{aligned}$$

For any $K \in \mathcal{T}_h$, we introduce the local Raviart-Thomas (\mathcal{RT}) element space

$$\mathbb{RT}_s(K) = [\mathcal{P}_s(K)]^d + \mathbf{x} \mathcal{P}_s(K)$$

and the \mathcal{RT} projection operator $\mathbf{P}_s^{\mathcal{RT}} : [H^1(K)]^d \rightarrow \mathbb{RT}_s(K)$ (cf. [7]) defined by

$$\langle \mathbf{P}_s^{\mathcal{RT}} \mathbf{v} \cdot \mathbf{n}_e, w \rangle_e = \langle \mathbf{v} \cdot \mathbf{n}_e, w \rangle_e, \quad \forall w \in \mathcal{P}_s(e), e \in \partial K, \quad \text{for } s \geq 0, \quad (3.15a)$$

$$(\mathbf{P}_s^{\mathcal{RT}} \mathbf{v}, \mathbf{w})_K = (\mathbf{v}, \mathbf{w})_K, \quad \forall \mathbf{w} \in [\mathcal{P}_s(K)]^d, \quad \text{for } s \geq 1. \quad (3.15b)$$

Lemmas 3.7-3.9 give some properties of the \mathcal{RT} element space and the \mathcal{RT} projection.

Lemma 3.7. [7] *For any $\mathbf{v}_{ho} \in \mathbb{RT}_s(K)$, the relation $\nabla \cdot \mathbf{v}_{ho}|_K = 0$ implies that $\mathbf{v}_{ho} \in [\mathcal{P}_s(K)]^d$.*

Lemma 3.8. [7] *For any $K \in \mathcal{T}_h$ and $\mathbf{v} \in [H^1(K)]^d$, the following properties hold:*

$$(\nabla \cdot \mathbf{P}_s^{\mathcal{RT}} \mathbf{v}, \phi_h)_K = (\nabla \cdot \mathbf{v}, \phi_h)_K, \quad \forall \mathbf{v} \in [H^1(K)]^d, \phi_h \in \mathcal{P}_s(K),$$

$$\|\mathbf{v} - \mathbf{P}_s^{\mathcal{RT}} \mathbf{v}\|_{0,K} \lesssim h_K^j |\mathbf{v}|_{j,K}, \quad \forall 1 \leq j \leq s+1, \quad \forall \mathbf{v} \in [H^j(K)]^d.$$

By using the triangle inequality, the inverse inequality, Lemma 3.6 and Lemma 3.8 we can get more estimates for the \mathcal{RT} projection (cf. [28]):

Lemma 3.9. For any $K \in \mathcal{T}_h$, $\mathbf{v} \in [H^j(K)]^d$ and $1 \leq j \leq s+1$, the following estimates hold:

$$\begin{aligned} |\mathbf{v} - \mathbf{P}_s^{\mathcal{RT}} \mathbf{v}|_{1,K} &\lesssim h_K^{j-1} |\mathbf{v}|_{j,K}, \\ |\mathbf{v} - \mathbf{P}_s^{\mathcal{RT}} \mathbf{v}|_{0,\partial K} &\lesssim h_K^{j-\frac{1}{2}} |\mathbf{v}|_{j,K}, \\ |\mathbf{v} - \mathbf{P}_s^{\mathcal{RT}} \mathbf{v}|_{0,3,K} &\lesssim h_K^{j-\frac{d}{6}} |\mathbf{v}|_{j,K}, \\ |\mathbf{v} - \mathbf{P}_s^{\mathcal{RT}} \mathbf{v}|_{0,3,\partial K} &\lesssim h_K^{j-\frac{1}{3}-\frac{d}{6}} |\mathbf{v}|_{j,K}. \end{aligned}$$

We also have the following commutativity properties for the \mathcal{RT} projection, the L^2 projections and the discrete weak operators:

Lemma 3.10. [8, 46] For $k \geq 1$, there hold

$$\begin{aligned} \nabla_{w,k} \{Q_{k-1}^o q, Q_k^b q\} &= \mathbf{Q}_k^o (\nabla q), \quad \forall q \in H^1(\Omega), \\ \nabla_{w,k-1} \{\mathbf{P}_k^{\mathcal{RT}} \mathbf{v}, \mathbf{Q}_k^b \mathbf{v}\} &= \mathbf{Q}_{k-1}^o (\nabla \mathbf{v}), \quad \forall \mathbf{v} \in [H^1(\Omega)]^d, \\ \nabla_{w,k-1} \times \{\mathbf{Q}_k^o \mathbf{w}, \mathbf{Q}_k^b \mathbf{w}\} &= \mathbf{Q}_{k-1}^o (\nabla \times \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{H}(\text{curl}; \Omega). \end{aligned}$$

4 Existence and uniqueness of discrete solution

4.1 Stability conditions

Lemma 4.1. For any $\mathbf{u}_h, \mathbf{v}_h, \Phi_h \in \mathbf{V}_h^0$, and $\mathbf{B}_h, \mathbf{w}_h \in \mathbf{W}_h^0$, there hold the following stability conditions:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) \leq \frac{1}{H_a^2} |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V, \quad (4.1a)$$

$$a_h(\mathbf{v}_h, \mathbf{v}_h) = \frac{1}{H_a^2} |||\mathbf{v}_h|||_V^2, \quad (4.1b)$$

$$\tilde{a}_h(\mathbf{B}_h, \mathbf{w}_h) \leq \frac{1}{R_m^2} |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W, \quad (4.1c)$$

$$\tilde{a}_h(\mathbf{w}_h, \mathbf{w}_h) = \frac{1}{R_m^2} |||\mathbf{w}_h|||_W^2, \quad (4.1d)$$

$$c_h(\Phi_h; \mathbf{v}_h, \mathbf{v}_h) = 0, \quad (4.1e)$$

$$c_h(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) \leq M_h |||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V \lesssim |||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V, \quad (4.1f)$$

$$\tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) \leq \tilde{M}_h |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W |||\mathbf{v}_h|||_V \lesssim |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W |||\mathbf{v}_h|||_V, \quad (4.1g)$$

where

$$M_h := \sup_{\mathbf{o} \neq \Phi_h, \mathbf{u}_h, \mathbf{v}_h \in \bar{\mathbf{V}}_h} \frac{c_h(\Phi_h; \mathbf{u}_h, \mathbf{v}_h)}{|||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V}, \quad (4.2)$$

$$\tilde{M}_h := \sup_{\mathbf{o} \neq \mathbf{B}_h, \Psi_h \in \bar{\mathbf{W}}_h, \mathbf{o} \neq \mathbf{v}_h \in \bar{\mathbf{V}}_h} \frac{\tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \Psi_h)}{|||\Psi_h|||_W |||\mathbf{v}_h|||_V |||\mathbf{B}_h|||_W}. \quad (4.3)$$

Proof. From the definitions of $a_h(\cdot, \cdot)$, $\tilde{a}_h(\cdot, \cdot)$, the Cauchy-Schwarz inequality and Lemmas 3.1 we can easily get (4.1a)-(4.1d). The relation (4.1e) follows from the definition of $c_h(\cdot, \cdot)$, and the inequalities

$$c_h(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) \leq M_h |||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V$$

and

$$\tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) \leq \tilde{M}_h |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W |||\mathbf{v}_h|||_V$$

follow from the definitions of M_h and \tilde{M}_h , respectively. The thing left is to prove $c_h(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) \lesssim |||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V$ and $\tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) \lesssim |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W |||\mathbf{v}_h|||_V$.

For all $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h^0$, using the definitions of $c_h(\cdot, \cdot)$ and $\nabla_{w,k}$ we have

$$2c_h(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) = \frac{1}{N} \{ (\mathbf{v}_{ho} \otimes \Phi_{ho}, \nabla_h \mathbf{u}_{ho}) - (\mathbf{u}_{ho} \otimes \Phi_{ho}, \nabla_h \mathbf{v}_{ho}) \}$$

$$\begin{aligned}
& - \langle \mathbf{v}_{hb} \otimes \Phi_{hb} \mathbf{n}, \mathbf{u}_{ho} \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{u}_{hb} \otimes \Phi_{hb} \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} \} \\
& = \frac{1}{N} \{ (\mathbf{v}_{ho} \otimes \Phi_{ho}, \nabla_h \mathbf{u}_{ho}) - (\mathbf{u}_{ho} \otimes \Phi_{ho}, \nabla_h \mathbf{v}_{ho}) \\
& \quad + \langle (\mathbf{u}_{ho} - \mathbf{u}_{hb}) \otimes (\Phi_{ho} - \Phi_{hb}) \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} - \langle (\mathbf{u}_{ho} - \mathbf{u}_{hb}) \otimes \Phi_{ho} \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} \\
& \quad - \langle (\mathbf{v}_{ho} - \mathbf{v}_{hb}) \otimes (\Phi_{ho} - \Phi_{hb}) \mathbf{n}, \mathbf{u}_{ho} \rangle_{\partial \mathcal{T}_h} + \langle (\mathbf{v}_{ho} - \mathbf{v}_{hb}) \otimes \Phi_{ho} \mathbf{n}, \mathbf{u}_{ho} \rangle_{\partial \mathcal{T}_h} \} \\
& =: \sum_{i=1}^5 \mathcal{I}_i.
\end{aligned}$$

Combining the Hölder's inequality and Lemma 3.3 we obtain

$$\begin{aligned}
|\mathcal{I}_1| & = \frac{1}{N} |(\mathbf{v}_{ho} \otimes \Phi_{ho}, \nabla_h \mathbf{u}_{ho}) - (\mathbf{u}_{ho} \otimes \Phi_{ho}, \nabla_h \mathbf{v}_{ho})| \\
& \leq \frac{1}{N} (\|\mathbf{v}_{ho}\|_{0,4} \|\Phi_{ho}\|_{0,4} \|\nabla_h \mathbf{u}_{ho}\|_{0,2} + \|\mathbf{u}_{ho}\|_{0,4} \|\Phi_{ho}\|_{0,4} \|\nabla_h \mathbf{v}_{ho}\|_{0,2}) \\
& \lesssim \|\Phi_h\|_V \|\mathbf{u}_h\|_V \|\mathbf{v}_h\|_V.
\end{aligned}$$

Using the Hölder's inequality, the inverse inequality, Lemma 3.5 and Lemma 3.3, we have

$$\begin{aligned}
|\mathcal{I}_2| & = \frac{1}{N} |\langle (\mathbf{u}_{ho} - \mathbf{u}_{hb}) \otimes (\Phi_{ho} - \Phi_{hb}) \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h}| \\
& \leq \frac{1}{N} \sum_{K \in \mathcal{T}_h} \|\Phi_{ho} - \Phi_{hb}\|_{0,3,\partial K} \|\mathbf{u}_{ho} - \mathbf{u}_{hb}\|_{0,2,\partial K} \|\mathbf{v}_{ho}\|_{0,6,\partial K} \\
& \leq \frac{1}{N} \sum_{K \in \mathcal{T}_h} h_K^{-\frac{d-1}{6}} \|\Phi_{ho} - \Phi_{hb}\|_{0,2,\partial K} \|\mathbf{u}_{ho} - \mathbf{u}_{hb}\|_{0,2,\partial K} h_K^{-\frac{1}{6}} \|\mathbf{v}_{ho}\|_{0,6,K} \\
& \leq \frac{1}{N} \sum_{K \in \mathcal{T}_h} h_K^{-\frac{1}{2}} \|\Phi_{ho} - \Phi_{hb}\|_{0,2,\partial K} h_K^{-\frac{1}{2}} \|\mathbf{u}_{ho} - \mathbf{u}_{hb}\|_{0,2,\partial K} h_K^{1-\frac{d}{6}} \|\mathbf{v}_{ho}\|_{0,6,K} \\
& \lesssim \|\Phi_h\|_V \|\mathbf{u}_h\|_V \|\mathbf{v}_h\|_V, \\
|\mathcal{I}_3| & = \frac{1}{N} |\langle (\mathbf{u}_{ho} - \mathbf{u}_{hb}) \otimes \Phi_{ho} \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h}| \\
& \leq \frac{1}{N} \sum_{K \in \mathcal{T}_h} \|\Phi_{ho}\|_{0,4,\partial K} \|\mathbf{u}_{ho} - \mathbf{u}_{hb}\|_{0,2,\partial K} \|\mathbf{v}_{ho}\|_{0,4,\partial K} \\
& \leq \frac{1}{N} \sum_{K \in \mathcal{T}_h} h_K^{-\frac{1}{4}} \|\Phi_{ho}\|_{0,4,K} \|\mathbf{u}_{ho} - \mathbf{u}_{hb}\|_{0,2,\partial K} h_K^{-\frac{1}{4}} \|\mathbf{v}_{ho}\|_{0,4,K} \\
& \leq \frac{1}{N} \sum_{K \in \mathcal{T}_h} \|\Phi_{ho}\|_{0,4,K} h_K^{-\frac{1}{2}} \|\mathbf{u}_{ho} - \mathbf{u}_{hb}\|_{0,2,\partial K} \|\mathbf{v}_{ho}\|_{0,4,K} \\
& \lesssim \|\Phi_h\|_V \|\mathbf{u}_h\|_V \|\mathbf{v}_h\|_V.
\end{aligned}$$

Similarly, we can get

$$|\mathcal{I}_4| + |\mathcal{I}_5| \lesssim \|\Phi_h\|_V \|\mathbf{u}_h\|_V \|\mathbf{v}_h\|_V.$$

Combining the above estimates yields the inequalities (4.1f).

For any $\mathbf{B}_h, \mathbf{w}_h \in \mathbf{W}_h^0$, by the definition of $\nabla_{w,k} \times$ we have

$$\tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) = \frac{1}{R_m} (\nabla_h \times \mathbf{w}_{ho}, \mathbf{v}_{ho} \times \mathbf{B}_{ho}) - \frac{1}{R_m} \langle (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}, \mathbf{v}_{ho} \times \mathbf{B}_{ho} \rangle_{\partial \mathcal{T}_h}.$$

Using the Hölder's inequality, the inverse inequality, Lemmas 3.3, 3.4 and 3.5 again gives

$$\begin{aligned}
|(\nabla_h \times \mathbf{w}_{ho}, \mathbf{v}_{ho} \times \mathbf{B}_{ho})| & \leq |\nabla_h \times \mathbf{w}_{ho}|_{0,2} |\mathbf{B}_{ho}|_{0,3} |\mathbf{v}_{ho}|_{0,6} \\
& \lesssim \|\mathbf{w}_h\|_W \|\mathbf{B}_h\|_W \|\mathbf{v}_h\|_V, \\
|\langle (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}, \mathbf{v}_{ho} \times \mathbf{B}_{ho} \rangle_{\partial \mathcal{T}_h}| & \leq \sum_{K \in \mathcal{T}_h} |(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}|_{0,2,\partial K} |\mathbf{B}_{ho}|_{0,3,\partial K} |\mathbf{v}_{ho}|_{0,6,\partial K}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{K \in \mathcal{T}_h} |(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}|_{0,2,\partial K} h_K^{-\frac{1}{3}} |\mathbf{B}_{ho}|_{0,3,K} h_K^{-\frac{1}{6}} |\mathbf{v}_{ho}|_{0,6,K} \\
&\lesssim \sum_{K \in \mathcal{T}_h} h_K^{-\frac{1}{2}} |(\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}|_{0,2,\partial K} |\mathbf{B}_{ho}|_{0,3,K} |\mathbf{v}_{ho}|_{0,6,K} \\
&\lesssim |||\mathbf{w}_h|||_W |||\mathbf{B}_h|||_W |||\mathbf{v}_h|||_V.
\end{aligned}$$

As a result, the desired inequalities (4.1g) follows. \blacksquare

We have the following inf-sup inequalities.

Lemma 4.2. *There hold*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h^0} \frac{b_h(\mathbf{v}_h, q_h)}{|||\mathbf{v}_h|||_V} \gtrsim |||q_h|||_Q, \quad \forall q_h \in Q_h^0, \quad (4.4)$$

$$\sup_{\mathbf{w}_h \in \mathbf{W}_h^0} \frac{\tilde{b}_h(\mathbf{w}_h, \theta_h)}{|||\mathbf{w}_h|||_W} \gtrsim |||\theta_h|||_R, \quad \forall \theta_h \in R_h^0. \quad (4.5)$$

Proof. The first inequality follows from [8, Theorem 3.1].

For any $\theta_h = \{\theta_{ho}, \theta_{hb}\} \in R_h^0$, let $c_0 := \frac{1}{|\Omega|} \int_{\Omega} \theta_{ho} d\mathbf{x}$ and $\tilde{\theta}_h := \{\theta_{ho} - c_0, \theta_{hb} - c_0\} \in Q_h^0$. Then, according to the definition of $\tilde{b}(\cdot, \cdot)$, (3.11d), the relation $\nabla_{w,k}\{c_0, c_0\} = 0$, $\mathbf{W}_h^0 \supset \mathbf{V}_h^0$, the inequality $|||\mathbf{w}_h|||_W \leq |||\mathbf{w}_h|||_V$ and (4.4), we get

$$\begin{aligned}
\sup_{\mathbf{w}_h \in \mathbf{W}_h^0} \frac{\tilde{b}_h(\mathbf{w}_h, \theta_h)}{|||\mathbf{w}_h|||_W} &= \sup_{\mathbf{w}_h \in \mathbf{W}_h^0} \frac{(\mathbf{w}_{ho}, \nabla_{w,k}\theta_h)}{|||\mathbf{w}_h|||_W} \\
&= \sup_{\mathbf{w}_h \in \mathbf{W}_h^0} \frac{(\mathbf{w}_{ho}, \nabla_{w,k}\tilde{\theta}_h) + (\mathbf{w}_{ho}, \nabla_{w,k}\{c_0, c_0\})}{|||\mathbf{w}_h|||_W} \\
&\geq \sup_{\mathbf{w}_h \in \mathbf{V}_h^0} \frac{(\mathbf{w}_{ho}, \nabla_{w,k}\tilde{\theta}_h)}{|||\mathbf{w}_h|||_V} \\
&\gtrsim |||\tilde{\theta}_h|||_Q = \left(\|\theta_{ho} - c_0\|_0^2 + \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla_{w,k}\theta_h\|_{0,K}^2 \right)^{1/2}.
\end{aligned}$$

Taking $\mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} = \{\nabla_{w,k}\theta_h, \mathbf{0}\}$ in this relation and using Lemma 3.5, we further obtain

$$\begin{aligned}
\sup_{\mathbf{w}_h \in \mathbf{W}_h^0} \frac{\tilde{b}_h(\mathbf{w}_h, \theta_h)}{|||\mathbf{w}_h|||_W} &\gtrsim \frac{(\nabla_{w,k}\theta_h, \nabla_{w,k}\theta_h)}{|||\nabla_{w,k}\theta_h|||_0 + h^{-1/2} |||\nabla_{w,k}\theta_h \times \mathbf{n}|||_{0,\partial\mathcal{T}_h}} \\
&\gtrsim \frac{|||\nabla_{w,k}\theta_h|||_0^2}{h^{-1} |||\nabla_{w,k}\theta_h|||_0} \\
&\gtrsim |||\theta_h|||_R,
\end{aligned}$$

i.e. (4.5) holds. This completes the proof. \blacksquare

4.2 Existence and uniqueness results

For the discretization problems (3.4) and (3.8), we readily have the following equivalence result.

Lemma 4.3. *The problems (3.4) and (3.8) are equivalent in the sense that both (I) and (II) hold:*

- (I) *If $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$ is the solution to the problem (3.4), then \mathbf{u}_h and \mathbf{B}_h solve the problem (3.8);*
- (II) *If $\mathbf{u}_h \in \bar{\mathbf{V}}_h$ and $\mathbf{B}_h \in \bar{\mathbf{W}}_h$ solve the problem (3.8), then $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h)$ is a solution to the problem (3.4), where $p_h \in Q_h^0$ and $r_h \in R_h^0$ are determined by*

$$b_h(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_{ho}) - a_h(\mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \quad (4.6)$$

$$\tilde{b}_h(\mathbf{w}_h, r_h) = \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}) - \tilde{a}_h(\mathbf{B}_h, \mathbf{w}_h) + \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{W}_h^0. \quad (4.7)$$

From Lemma 4.1 it is easy to know that M_h and \tilde{M}_h are bounded and depend on the parameters N and R_m , respectively.

Lemma 4.4. *The problem (3.8) admits at least one solution $(\mathbf{u}_h, \mathbf{B}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$. In addition, there holds*

$$|||\mathbf{u}_h|||_V + |||\mathbf{B}_h|||_W \leq 2\zeta (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}), \quad (4.8)$$

where

$$\zeta := \max\{H_a, R_m\}, \quad (4.9)$$

$$\|\mathbf{f}\|_h := \sup_{\mathbf{0} \neq \mathbf{v}_h \in \bar{\mathbf{V}}_h} \frac{(\mathbf{f}, \mathbf{v}_{ho})}{|||\mathbf{v}_h|||_V}, \quad \|\mathbf{g}\|_{\tilde{h}} := \sup_{\mathbf{0} \neq \mathbf{w}_h \in \bar{\mathbf{W}}_h} \frac{(\mathbf{g}, \mathbf{w}_{ho})}{|||\mathbf{w}_h|||_W} \quad (4.10)$$

Proof. Taking $\mathbf{v}_h = \mathbf{u}_h$ and $\mathbf{w}_h = \mathbf{B}_h$ in (3.8), by Lemma 4.1 we have

$$\frac{1}{H_a^2} |||\mathbf{u}_h|||_V^2 + \frac{1}{R_m^2} |||\mathbf{B}_h|||_W^2 = (\mathbf{f}, \mathbf{u}_h) + \frac{1}{R_m} (\mathbf{g}, \mathbf{B}_h) \leq \|\mathbf{f}\|_h |||\mathbf{u}_h|||_V + \frac{1}{R_m} \|\mathbf{g}\|_{\tilde{h}} |||\mathbf{B}_h|||_W,$$

which yields

$$\begin{aligned} \frac{1}{2} \min\left\{\frac{1}{H_a^2}, \frac{1}{R_m^2}\right\} (|||\mathbf{u}_h|||_V + |||\mathbf{B}_h|||_W)^2 &\leq \min\left\{\frac{1}{H_a^2}, \frac{1}{R_m^2}\right\} (|||\mathbf{u}_h|||_V^2 + |||\mathbf{B}_h|||_W^2) \\ &\leq \frac{1}{H_a^2} |||\mathbf{u}_h|||_V^2 + \frac{1}{R_m^2} |||\mathbf{B}_h|||_W^2 \\ &\leq H_a^2 \|\mathbf{f}\|_h^2 + \|\mathbf{g}\|_{\tilde{h}}^2 \\ &\leq (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}})^2. \end{aligned}$$

Thus, we get the boundedness result (4.8).

Introduce a mapping $\mathbb{A} : \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h \rightarrow \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$, defined by $\mathbb{A}(\mathbf{u}_h, \mathbf{B}_h) = (\mathbf{w}_u, \mathbf{w}_B)$, where $(\mathbf{w}_u, \mathbf{w}_B) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ is given by

$$\begin{aligned} &a_h(\mathbf{w}_u, \mathbf{v}_h) + \tilde{a}_h(\mathbf{w}_B, \mathbf{w}_h) \\ &= (\mathbf{f}, \mathbf{v}_{ho}) + \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}) - c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) + \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h), \quad \forall (\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h. \end{aligned} \quad (4.11)$$

Clearly $(\mathbf{u}_h, \mathbf{B}_h)$ is a solution to (3.8) if it is a solution to

$$\mathbb{A}(\mathbf{u}_h, \mathbf{B}_h) = (\mathbf{u}_h, \mathbf{B}_h). \quad (4.12)$$

In order to show the system (4.12) has a solution, from the Leray-Schauder's principle it suffices to prove the following two assertions:

- (i) \mathbb{A} is a continuous and compact mapping;
- (ii) For any $0 \leq \lambda \leq 1$, the set $\bar{\mathbf{V}}_{\lambda, h} \times \bar{\mathbf{W}}_{\lambda, h} := \{(\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h : (\mathbf{v}_h, \mathbf{w}_h) = \lambda \mathbb{A}(\mathbf{v}_h, \mathbf{w}_h)\}$ is bounded.

In fact, let $\mathbf{u}_{1h}, \mathbf{u}_{2h} \in \bar{\mathbf{V}}_h$, $\mathbf{B}_{1h}, \mathbf{B}_{2h} \in \bar{\mathbf{W}}_h$, and set $\mathbb{A}(\mathbf{u}_{1h}, \mathbf{B}_{1h}) = (\mathbf{w}_{1u}, \mathbf{w}_{1B})$ and $\mathbb{A}(\mathbf{u}_{2h}, \mathbf{B}_{2h}) = (\mathbf{w}_{2u}, \mathbf{w}_{2B})$, then we have

$$\begin{aligned} &a_h(\mathbf{w}_{1u}, \mathbf{v}_h) + \tilde{a}_h(\mathbf{w}_{1B}, \mathbf{w}_h) + c_h(\mathbf{u}_{1h}; \mathbf{u}_{1h}, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_{1h}; \mathbf{B}_{1h}, \mathbf{B}_{1h}) - \tilde{c}_h(\mathbf{u}_{1h}; \mathbf{B}_{1h}, \mathbf{w}_h) \\ &= (\mathbf{f}, \mathbf{v}_{ho}) + \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}), \end{aligned} \quad (4.13)$$

$$\begin{aligned} &a_h(\mathbf{w}_{2u}, \mathbf{v}_h) + \tilde{a}_h(\mathbf{w}_{2B}, \mathbf{w}_h) + c_h(\mathbf{u}_{2h}; \mathbf{u}_{2h}, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_{2h}; \mathbf{B}_{2h}, \mathbf{B}_{2h}) - \tilde{c}_h(\mathbf{u}_{2h}; \mathbf{B}_{2h}, \mathbf{w}_h) \\ &= (\mathbf{f}, \mathbf{v}_{ho}) + \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{ho}), \end{aligned} \quad (4.14)$$

for all $(\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$. Subtracting (4.14) from (4.13), and taking $\mathbf{v}_h = \mathbf{w}_{1u} - \mathbf{w}_{2u}$, $\mathbf{w}_h = \mathbf{w}_{1B} - \mathbf{w}_{2B}$, we obtain

$$\begin{aligned} & a_h(\mathbf{w}_{1u} - \mathbf{w}_{2u}, \mathbf{w}_{1u} - \mathbf{w}_{2u}) + \tilde{a}_h(\mathbf{w}_{1B} - \mathbf{w}_{2B}, \mathbf{w}_{1B} - \mathbf{w}_{2B}) \\ &= -c_h(\mathbf{u}_{1h}; \mathbf{u}_{1h} - \mathbf{u}_{2h}, \mathbf{w}_{1u} - \mathbf{w}_{2u}) - c_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}; \mathbf{u}_{2h}, \mathbf{w}_{1u} - \mathbf{w}_{2u}) \\ &\quad - \tilde{c}_h(\mathbf{w}_{1u} - \mathbf{w}_{2u}; \mathbf{B}_{1h} - \mathbf{B}_{2h}, \mathbf{B}_{1h}) - \tilde{c}_h(\mathbf{w}_{1u} - \mathbf{w}_{2u}; \mathbf{B}_{2h}, \mathbf{B}_{1h} - \mathbf{B}_{2h}) \\ &\quad + \tilde{c}_h(\mathbf{u}_{1h}; \mathbf{B}_{1h} - \mathbf{B}_{2h}, \mathbf{w}_{1B} - \mathbf{w}_{2B}) + \tilde{c}_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}; \mathbf{B}_{2h}, \mathbf{w}_{1B} - \mathbf{w}_{2B}), \end{aligned}$$

which, together with Lemma 4.1, leads to

$$\begin{aligned} & \frac{1}{H_a^2} |||\mathbf{w}_{1u} - \mathbf{w}_{2u}|||_V^2 + \frac{1}{R_m^2} |||\mathbf{w}_{1B} - \mathbf{w}_{2B}|||_W^2 \\ & \leq M_h (|||\mathbf{u}_{1h}|||_V + |||\mathbf{u}_{2h}|||_V) |||\mathbf{u}_{1h} - \mathbf{u}_{2h}|||_V |||\mathbf{w}_{1u} - \mathbf{w}_{2u}|||_V \\ & \quad + \tilde{M}_h (|||\mathbf{B}_{1h}|||_W + |||\mathbf{B}_{2h}|||_W) |||\mathbf{B}_{1h} - \mathbf{B}_{2h}|||_W |||\mathbf{w}_{1u} - \mathbf{w}_{2u}|||_V \\ & \quad + \tilde{M}_h (|||\mathbf{u}_{1h}|||_V |||\mathbf{B}_{1h} - \mathbf{B}_{2h}|||_W + |||\mathbf{B}_{2h}|||_W |||\mathbf{u}_{1h} - \mathbf{u}_{2h}|||_V) |||\mathbf{w}_{1B} - \mathbf{w}_{2B}|||_W. \end{aligned}$$

This estimate plus (4.8) yields

$$\begin{aligned} & |||\mathbf{w}_{1u} - \mathbf{w}_{2u}|||_V + |||\mathbf{w}_{1B} - \mathbf{w}_{2B}|||_W \\ & \leq 2\zeta \left[H_a M_h (|||\mathbf{u}_{1h}|||_V + |||\mathbf{u}_{2h}|||_W) |||\mathbf{u}_{1h} - \mathbf{u}_{2h}|||_V + H_a \tilde{M}_h (|||\mathbf{B}_{1h}|||_W + |||\mathbf{B}_{2h}|||_W) |||\mathbf{B}_{1h} - \mathbf{B}_{2h}|||_W \right. \\ & \quad \left. + R_m \tilde{M}_h (|||\mathbf{u}_{1h}|||_V |||\mathbf{B}_{1h} - \mathbf{B}_{2h}|||_W + |||\mathbf{B}_{2h}|||_W |||\mathbf{u}_{1h} - \mathbf{u}_{2h}|||_V) \right] \\ & \leq 4\zeta^2 (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) \left[(2H_a M_h + R_m \tilde{M}_h) |||\mathbf{u}_{1h} - \mathbf{u}_{2h}|||_V + (2H_a \tilde{M}_h + R_m \tilde{M}_h) |||\mathbf{B}_{1h} - \mathbf{B}_{2h}|||_W \right] \\ & \leq 12\zeta^3 \max\{M_h, \tilde{M}_h\} (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (|||\mathbf{u}_{1h} - \mathbf{u}_{2h}|||_V + |||\mathbf{B}_{1h} - \mathbf{B}_{2h}|||_W). \end{aligned} \tag{4.15}$$

This result implies that \mathbb{A} is equicontinuous and uniformly bounded, since

$$\mathbb{A}(\mathbf{u}_{1h}, \mathbf{B}_{1h}) - \mathbb{A}(\mathbf{u}_{2h}, \mathbf{B}_{2h}) = (\mathbf{w}_{1u} - \mathbf{w}_{2u}, \mathbf{w}_{1B} - \mathbf{w}_{2B}).$$

Thus, \mathbb{A} is compact by the Arzelá-Ascoli theorem [6], and (i) holds.

The work left is to show (ii). If $\lambda = 0$, then $\bar{\mathbf{V}}_{\lambda,h} \times \bar{\mathbf{W}}_{\lambda,h} = \{(0, 0)\}$. For $\lambda \in (0, 1]$ and $(\mathbf{v}_u, \mathbf{v}_B) \in \bar{\mathbf{V}}_{\lambda,h} \times \bar{\mathbf{W}}_{\lambda,h}$, using (4.11) we have

$$\lambda^{-1}(a_h(\mathbf{v}_u, \mathbf{v}_h) + \tilde{a}_h(\mathbf{v}_B, \mathbf{w}_h)) + c_h(\mathbf{v}_u; \mathbf{v}_u, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{v}_B, \mathbf{v}_B) - \tilde{c}_h(\mathbf{v}_u; \mathbf{v}_B; \mathbf{w}_h) = (\mathbf{f}, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{g}, \mathbf{w}_h).$$

Similar to (4.8), there holds

$$|||\mathbf{v}_u|||_V + |||\mathbf{v}_B|||_W \leq 2\zeta \lambda (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}).$$

As a result, (ii) holds. This completes the proof. ■

Denote

$$\delta := 12\zeta^3 \max\{M_h, \tilde{M}_h\} = 12 \max\{H_a, R_m\}^3 \max\{M_h, \tilde{M}_h\},$$

and we have the following uniqueness result.

Lemma 4.5. *Under the smallness condition that*

$$\delta (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) < 1, \tag{4.16}$$

the problem (3.8) admits a unique solution $(\mathbf{u}_h, \mathbf{B}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$.

Proof. Let $(\mathbf{u}_{1h}, \mathbf{B}_{1h}), (\mathbf{u}_{2h}, \mathbf{B}_{2h}) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ be two solutions of problem (3.8). Then it suffices to show $\mathbf{u}_{1h} = \mathbf{u}_{2h}$ and $\mathbf{B}_{1h} = \mathbf{B}_{2h}$. In fact, from (3.8) it follows that, for any $(\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$,

$$a_h(\mathbf{u}_{1h}, \mathbf{v}_h) + \tilde{a}_h(\mathbf{B}_{1h}, \mathbf{w}_h) + c_h(\mathbf{u}_{1h}; \mathbf{u}_{1h}, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_{1h}, \mathbf{B}_{1h}) - \tilde{c}_h(\mathbf{u}_{1h}; \mathbf{B}_{1h}, \mathbf{w}_h) = (\mathbf{f}, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{g}, \mathbf{w}_h),$$

$$a_h(\mathbf{u}_{2h}, \mathbf{v}_h) + \tilde{a}_h(\mathbf{B}_{2h}, \mathbf{w}_h) + c_h(\mathbf{u}_{2h}; \mathbf{u}_{2h}, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_{2h}, \mathbf{B}_{2h}) - \tilde{c}_h(\mathbf{u}_{2h}; \mathbf{B}_{2h}, \mathbf{w}_h) = (\mathbf{f}, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{g}, \mathbf{w}_h).$$

Subtracting the above first equation from the second one and choosing $\mathbf{v}_h = \mathbf{u}_{1h} - \mathbf{u}_{2h}$, $\mathbf{w}_h = \mathbf{B}_{1h} - \mathbf{B}_{2h}$, we get

$$\begin{aligned} & a_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}, \mathbf{u}_{1h} - \mathbf{u}_{2h}) + \tilde{a}_h(\mathbf{B}_{1h} - \mathbf{B}_{2h}, \mathbf{B}_{1h} - \mathbf{B}_{2h}) \\ &= -c_h(\mathbf{u}_{1h}; \mathbf{u}_{1h} - \mathbf{u}_{2h}, \mathbf{u}_{1h} - \mathbf{u}_{2h}) - c_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}; \mathbf{u}_{2h}, \mathbf{u}_{1h} - \mathbf{u}_{2h}) \\ &\quad - \tilde{c}_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}; \mathbf{B}_{1h} - \mathbf{B}_{2h}, \mathbf{B}_{1h}) - \tilde{c}_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}; \mathbf{B}_{2h}, \mathbf{B}_{1h} - \mathbf{B}_{2h}) \\ &\quad + \tilde{c}_h(\mathbf{u}_{1h}; \mathbf{B}_{1h} - \mathbf{B}_{2h}, \mathbf{B}_{1h} - \mathbf{B}_{2h}) + \tilde{c}_h(\mathbf{u}_{1h} - \mathbf{u}_{2h}; \mathbf{B}_{2h}, \mathbf{B}_{1h} - \mathbf{B}_{2h}), \end{aligned}$$

which, together with Lemma 4.1, leads to

$$\begin{aligned} & \frac{1}{H_a^2} \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V^2 + \frac{1}{R_m^2} \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W^2 \\ & \leq M_h (\|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V^2 \|\mathbf{u}_{1h}\|_V + \tilde{M}_h (\|\mathbf{B}_{1h}\|_W + \|\mathbf{B}_{2h}\|_W) \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V \\ & \quad + \tilde{M}_h \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W^2 \|\mathbf{u}_{1h}\|_V + \tilde{M}_h \|\mathbf{B}_{2h}\|_W \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V). \end{aligned}$$

This estimate plus (4.8) yields

$$\begin{aligned} & \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V + \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W \\ & \leq 2\zeta \{ H_a M_h \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V \|\mathbf{u}_{1h}\|_V + H_a \tilde{M}_h (\|\mathbf{B}_{1h}\|_W + \|\mathbf{B}_{2h}\|_W) \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W \\ & \quad + R_m \tilde{M}_h \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W \|\mathbf{u}_{1h}\|_V + R_M \tilde{M}_h \|\mathbf{B}_{2h}\|_W \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_W \} \\ & \leq 4\zeta^2 (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) \left((H_a M_h + R_M \tilde{M}_h) \|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_W + (2H_a \tilde{M}_h + R_m \tilde{M}_h) \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W \right) \\ & \leq 12\zeta^3 \max\{M_h, \tilde{M}_h\} (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (\|\mathbf{u}_{1h} - \mathbf{u}_{2h}\|_V + \|\mathbf{B}_{1h} - \mathbf{B}_{2h}\|_W). \end{aligned}$$

In view of the assumption (4.16), the above inequality implies

$$\mathbf{u}_{1h} = \mathbf{u}_{2h}, \quad \mathbf{B}_{1h} = \mathbf{B}_{2h}.$$

This completes the proof. ■

Finally, we have the following existence and uniqueness results for the WG scheme (3.4).

Theorem 4.1. *The scheme (3.4) admits at least one solution $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$ and there holds the boundedness result (4.8). In addition, the scheme admits a unique solution under the smallness condition (4.16).*

Proof. The existence and uniqueness of the discrete solutions \mathbf{u}_h and \mathbf{B}_h follow from Lemmas 4.3, 4.4 and 4.5, and the existence and uniqueness of the discrete solution, p_h , to (4.6) and the discrete solution, r_h , to (4.7) follow from the two discrete inf-sup inequalities in Lemma 4.2. ■

5 Error estimates

This section is devoted to establish the error estimates of the WG scheme (3.4). To this end, we assume that the weak solution, $(\mathbf{u}, \mathbf{B}, p, r)$, to the problem (1.1)-(1.5) satisfies the following regularity conditions:

$$\mathbf{u} \in \mathbf{V} \cap [H^{k+1}(\Omega)]^d, \quad \mathbf{B} \in \mathbf{W} \cap [H^{k+1}(\Omega)]^d, \quad p \in L_0^2(\Omega) \cap H^k(\Omega), \quad r \in H_0^1(\Omega) \cap H^k(\Omega). \quad (5.1)$$

Here we recall that $k \geq 1$. We set

$$\begin{aligned} \Pi_1 \mathbf{u}|_K &:= \{\mathbf{P}_k^{\mathcal{RT}}(\mathbf{u}|_K), \mathbf{Q}_k^b(\mathbf{u}|_K)\}, \quad \Pi_2 \mathbf{B}|_K := \{\mathbf{P}_k^{\mathcal{RT}}(\mathbf{B}|_K), \mathbf{Q}_k^b(\mathbf{B}|_K)\}, \\ \Pi_3 p|_K &:= \{Q_{k-1}^o(p|_K), Q_k^b(p|_K)\}, \quad \Pi_4 r|_K := \{Q_{k-1}^o(r|_K), Q_k^b(r|_K)\}. \end{aligned}$$

for any $K \in \mathcal{T}_h$.

Lemma 5.1. For for any $(\mathbf{v}_h, \mathbf{w}_h, q_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$, there hold

$$\begin{aligned} & a_h(\Pi_1 \mathbf{u}, \mathbf{v}_h) + \tilde{a}_h(\Pi_2 \mathbf{B}, \mathbf{w}_h) + b_h(\mathbf{v}_h, \Pi_3 p) - b_h(\Pi_1 \mathbf{u}, q_h) + \tilde{b}_h(\mathbf{w}_h, \Pi_4 r) - \tilde{b}_h(\Pi_2 \mathbf{B}, \theta_h) \\ & + c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) - \tilde{c}_h(\Pi_1 \mathbf{u}; \Pi_2 \mathbf{B}, \mathbf{w}_h) \\ & = (\mathbf{f}, \mathbf{v}_{ho}) + \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_h) + E_u(\mathbf{u}, \mathbf{v}_h) + E_B(\mathbf{B}, \mathbf{w}_h) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) \\ & + E_{\tilde{B}1}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) + E_{\tilde{B}2}(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h), \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} E_u(\mathbf{u}, \mathbf{v}_h) &:= \frac{1}{H_a^2} \langle (\nabla \mathbf{u} - \mathbf{Q}_{k-1}^o \nabla \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{hb} - \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} + \frac{1}{H_a^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h}, \\ E_B(\mathbf{B}, \mathbf{w}_h) &:= -\frac{1}{R_m^2} \langle \nabla \times \mathbf{B} - \mathbf{Q}_{k-1}^o (\nabla \times \mathbf{B}), (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &+ \frac{1}{R_m^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial \mathcal{T}_h}, \\ E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) &:= \frac{1}{2N} \langle \mathbf{u} \otimes \mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \otimes \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}, \nabla_h \mathbf{v}_{ho} \rangle - \frac{1}{2N} \langle (\mathbf{u} \otimes \mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \otimes \mathbf{Q}_k^b \mathbf{u}) \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} \\ &- \frac{1}{2N} \langle \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \cdot \nabla_h \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}, \mathbf{v}_{ho} \rangle - \frac{1}{2N} \langle \mathbf{v}_{hb} \otimes \mathbf{Q}_k^b \mathbf{u} \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \rangle_{\partial \mathcal{T}_h}, \\ E_{\tilde{B}1}(\mathbf{B}, \mathbf{v}_h) &:= -\frac{1}{R_m} \langle \nabla_h \times (\mathbf{B} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B}), \mathbf{v}_{ho} \times \mathbf{B} \rangle + \frac{1}{R_m} \langle \nabla_h \times \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B}, \mathbf{v}_{ho} \times (\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B} - \mathbf{B}) \rangle \\ &- \frac{1}{R_m} \langle (\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, \mathbf{v}_{ho} \times \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B} \rangle_{\partial \mathcal{T}_h}, \\ E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \mathbf{w}_h) &:= -\frac{1}{R_m} \langle \nabla_h \times \mathbf{w}_{ho}, (\mathbf{u} \times \mathbf{B} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \times \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B}) \rangle - \frac{1}{R_m} \langle \mathbf{w}_{ho} \times \mathbf{n}, \mathbf{u} \times \mathbf{B} \rangle_{\partial \mathcal{T}_h} \\ &- \frac{1}{R_m} \langle (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \times \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

In addition, we have

$$\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_K \in [\mathcal{P}_k(K)]^d \quad \text{and} \quad \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B}|_K \in [\mathcal{P}_k(K)]^d, \quad \forall K \in \mathcal{T}_h. \quad (5.3)$$

Proof. We first show (5.3). For any $K \in \mathcal{T}_h$, using Lemma 3.8 we get

$$\begin{aligned} (\nabla \cdot \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}, \phi_h)_K &= (\nabla \cdot \mathbf{u}, \phi_h)_K = 0, \quad \forall \phi_h \in \mathcal{P}_k(K), \\ (\nabla \cdot \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B}, \theta_h)_K &= (\nabla \cdot \mathbf{B}, \theta_h)_K = 0, \quad \forall \theta_h \in \mathcal{P}_k(K), \end{aligned}$$

which give

$$\nabla \cdot \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} = 0, \quad \nabla \cdot \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{B} = 0. \quad (5.4)$$

Then the result (5.3) follows from Lemma 3.7.

From the definitions of the bilinear forms $a_h(\cdot, \cdot)$ and the weak gradient, the second commutativity property in Lemma 3.10, the properties of the projection \mathbf{Q}_m^o ($m = k, k-1$), the Green's formula, the relation $\langle \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} = 0$ and the definition of $E_u(\mathbf{u}, \mathbf{v}_h)$, we immediately get, for any $\mathbf{v}_h \in \mathbf{V}_h^0$,

$$\begin{aligned} a_h(\Pi_1 \mathbf{u}, \mathbf{v}_h) &= \frac{1}{H_a^2} \langle \nabla_{w,k-1} \Pi_1 \mathbf{u}, \nabla_{w,k-1} \mathbf{v}_h \rangle + \frac{1}{H_a^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} \\ &= \frac{1}{H_a^2} \langle \mathbf{Q}_{k-1}^o \nabla \mathbf{u}, \nabla_{w,k-1} \mathbf{v}_h \rangle + \frac{1}{H_a^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} \\ &= -\frac{1}{H_a^2} \langle \nabla_h \cdot \mathbf{Q}_{k-1}^o \nabla \mathbf{u}, \mathbf{v}_{ho} \rangle + \frac{1}{H_a^2} \langle \mathbf{Q}_{k-1}^o \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} + \frac{1}{H_a^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} \\ &= \frac{1}{H_a^2} \langle \mathbf{Q}_{k-1}^o \nabla \mathbf{u}, \nabla_h \mathbf{v}_{ho} \rangle + \frac{1}{H_a^2} \langle \mathbf{Q}_{k-1}^o \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_{hb} - \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} + \frac{1}{H_a^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} \\ &= -\frac{1}{H_a^2} \langle \Delta \mathbf{u}, \mathbf{v}_{ho} \rangle + \frac{1}{H_a^2} \langle (\nabla \mathbf{u} - \mathbf{Q}_{k-1}^o \nabla \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{hb} - \mathbf{v}_{ho} \rangle_{\partial \mathcal{T}_h} + \frac{1}{H_a^2} \langle \tau(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}), \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial \mathcal{T}_h} \end{aligned}$$

$$= -\frac{1}{H_a^2}(\Delta \mathbf{u}, \mathbf{v}_{ho}) + E_u(\mathbf{u}, \mathbf{v}_h).$$

Similarly, in light of the definitions of the bilinear forms $\tilde{a}_h(\cdot, \cdot)$ and the weak curl, the third commutativity property in Lemma 3.10, the properties of the projection \mathbf{Q}_m^o , the Green's formula, the relation $\langle \nabla \times \mathbf{B}, \mathbf{w}_{hb} \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} = 0$, and the definition of $E_B(\mathbf{B}, \mathbf{w}_h)$, we obtain, for any $\mathbf{w}_h \in \mathbf{W}_h^0$,

$$\begin{aligned} \tilde{a}_h(\Pi_2 \mathbf{B}, \mathbf{w}_h) &= \frac{1}{R_m^2}(\nabla_{w,k-1} \times \Pi_2 \mathbf{B}, \nabla_{w,k-1} \times \mathbf{w}_h) + \frac{1}{R_m^2} \langle \tau(\mathbf{P}_k^{\mathcal{RT}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m^2}(\mathbf{Q}_{k-1}^o(\nabla \times \mathbf{B}), \nabla_{w,k-1} \times \mathbf{w}_h) + \frac{1}{R_m^2} \langle \tau(\mathbf{P}_k^{\mathcal{RT}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m^2}(\nabla_h \times (\mathbf{Q}_{k-1}^o(\nabla \times \mathbf{B})), \mathbf{w}_{ho}) + \frac{1}{R_m^2} \langle \mathbf{Q}_{k-1}^o(\nabla \times \mathbf{B}), \mathbf{w}_{hb} \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &\quad + \frac{1}{R_m^2} \langle \tau(\mathbf{P}_k^{\mathcal{RT}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m^2}(\mathbf{Q}_{k-1}^o(\nabla \times \mathbf{B}), \nabla_h \times \mathbf{w}_{ho}) - \frac{1}{R_m^2} \langle \mathbf{Q}_{k-1}^o(\nabla \times \mathbf{B}), (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &\quad + \frac{1}{R_m^2} \langle \tau(\mathbf{P}_k^{\mathcal{RT}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m^2}(\nabla \times \nabla \times \mathbf{B}, \mathbf{w}_{ho}) - \frac{1}{R_m^2} \langle \nabla \times \mathbf{B} - \mathbf{Q}_{k-1}^o(\nabla \times \mathbf{B}), (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &\quad + \frac{1}{R_m^2} \langle \tau(\mathbf{P}_k^{\mathcal{RT}} \mathbf{B} - \mathbf{Q}_k^b \mathbf{B}) \times \mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb}) \times \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m^2}(\nabla \times \nabla \times \mathbf{B}, \mathbf{w}_{ho}) + E_B(\mathbf{B}, \mathbf{w}_h). \end{aligned}$$

In view of the definitions of $b_h(\cdot, \cdot)$ and the weak gradient, the first commutativity property in Lemma 3.10, the projection property, and the relations (3.15a), (5.4) and $\langle \mathbf{u} \cdot \mathbf{n}, q_{hb} \rangle_{\partial\mathcal{T}_h} = 0$, we get

$$\begin{aligned} b_h(\mathbf{v}_h, \Pi_3 p) - b_h(\Pi_1 \mathbf{u}, q_h) &= (\nabla_{w,k} \{Q_{k-1}^o p, Q_k^b p\}, \mathbf{v}_{ho}) - (\nabla_{w,k} q_h, \mathbf{P}_k^{\mathcal{RT}} \mathbf{u}) \\ &= (Q_k^o \nabla p, \mathbf{v}_{ho}) + (\nabla \cdot \mathbf{P}_k^{\mathcal{RT}} \mathbf{u}, q_{ho}) - \langle \mathbf{P}_k^{\mathcal{RT}} \mathbf{u} \cdot \mathbf{n}, q_{hb} \rangle_{\partial\mathcal{T}_h} \\ &= (\nabla p, \mathbf{v}_{ho}) - \langle \mathbf{u} \cdot \mathbf{n}, q_{hb} \rangle_{\partial\mathcal{T}_h} \\ &= (\nabla p, \mathbf{v}_{ho}), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \tilde{b}_h(\mathbf{w}_h, \Pi_4 r) - \tilde{b}_h(\Pi_2 \mathbf{B}, \theta_h) &= \frac{1}{R_m}(\nabla_{w,k} \{Q_{k-1}^o r, Q_k^b r\}, \mathbf{w}_{ho}) - \frac{1}{R_m}(\nabla_{w,k} \theta_h, \mathbf{P}_k^{\mathcal{RT}} \mathbf{B}) \\ &= \frac{1}{R_m}(Q_k^o \nabla r, \mathbf{w}_{ho}) + \frac{1}{R_m}(\nabla \cdot \mathbf{P}_k^{\mathcal{RT}} \mathbf{B}, \theta_{ho}) - \frac{1}{R_m} \langle \mathbf{P}_k^{\mathcal{RT}} \mathbf{B} \cdot \mathbf{n}, \theta_{hb} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m}(\nabla r, \mathbf{w}_{ho}) - \frac{1}{R_m} \langle \mathbf{B} \cdot \mathbf{n}, \theta_{hb} \rangle_{\partial\mathcal{T}_h} \\ &= \frac{1}{R_m}(\nabla r, \mathbf{w}_{ho}), \quad \forall \mathbf{w}_h \in \mathbf{W}_h^0. \end{aligned}$$

By the Green's formula and the definitions of $c_h(\cdot, \cdot, \cdot)$, the weak divergence and $E_{\bar{u}}(\cdot, \cdot)$ we get

$$\begin{aligned} c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \mathbf{v}_h) &= \frac{1}{2N}(\nabla_{w,k} \cdot \{\mathbf{P}_k^{\mathcal{RT}} \mathbf{u} \otimes \mathbf{P}_k^{\mathcal{RT}} \mathbf{u}, \mathbf{Q}_k^b \mathbf{u} \otimes \mathbf{Q}_k^b \mathbf{u}\}, \mathbf{v}_{ho}) \\ &\quad - \frac{1}{2N}(\nabla_{w,k} \cdot \{\mathbf{v}_{ho} \otimes \mathbf{P}_k^{\mathcal{RT}} \Phi, \mathbf{v}_{hb} \otimes \mathbf{Q}_k^b \mathbf{u}\}, \mathbf{P}_k^{\mathcal{RT}} \mathbf{u}) \\ &= \frac{1}{2N}(\nabla \cdot (\mathbf{u} \otimes \mathbf{u}), \mathbf{v}_{ho}) + \frac{1}{2N}(\mathbf{u} \otimes \mathbf{u} - \mathbf{P}_k^{\mathcal{RT}} \mathbf{u} \otimes \mathbf{P}_k^{\mathcal{RT}} \mathbf{u}, \nabla_h \mathbf{v}_{ho}) \\ &\quad - \frac{1}{2N} \langle (\mathbf{u} \otimes \mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \otimes \mathbf{Q}_k^b \mathbf{u}) \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial\mathcal{T}_h} \\ &\quad + \frac{1}{2N}(\nabla \cdot (\mathbf{u} \otimes \mathbf{u}), \mathbf{v}_{ho}) + \frac{1}{2N}(\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{P}_k^{\mathcal{RT}} \mathbf{u} \cdot \nabla_h \mathbf{P}_k^{\mathcal{RT}} \mathbf{u}, \mathbf{v}_{ho}) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2N} \langle (\mathbf{v}_{hb} \otimes \mathbf{Q}_k^b \mathbf{u} \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u})_{\partial\mathcal{T}_h} \\
& = \frac{1}{N} (\nabla \cdot (\mathbf{u} \otimes \mathbf{u}), \mathbf{v}_{ho}) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h).
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) & = -\frac{1}{R_m} (\nabla \times \mathbf{B} \times \mathbf{B}, \mathbf{v}_{ho}) + E_{\tilde{B}1}(\mathbf{B}_h, \mathbf{v}_h), \\
-\tilde{c}_h(\Pi_1 \mathbf{u}; \Pi_2 \mathbf{B}, \mathbf{w}_h) & = -\frac{1}{R_m} (\nabla \times (\mathbf{u} \times \mathbf{B}), \mathbf{w}_{ho}) - E_{\tilde{B}2}(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h).
\end{aligned}$$

Combining the above relations and (1.1), we finally arrive at the desired conclusion (5.2). \blacksquare

Lemma 5.2. For any $\mathbf{v}_h \in \mathbf{V}_h^0$ and $\mathbf{w}_h \in \mathbf{W}_h^0$, there hold

$$|E_u(\mathbf{u}, \mathbf{v}_h)| \lesssim h^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V, \quad (5.5)$$

$$|E_B(\mathbf{B}, \mathbf{w}_h)| \lesssim h^k \|\mathbf{B}\|_{k+1} \|\mathbf{w}_h\|_W, \quad (5.6)$$

$$|E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h)| \lesssim h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V, \quad (5.7)$$

$$|E_{\tilde{B}1}(\mathbf{B}, \mathbf{v}_h)| \lesssim h^k \|\mathbf{B}\|_2 \|\mathbf{B}\|_{k+1} \|\mathbf{v}_h\|_V, \quad (5.8)$$

$$|E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \mathbf{w}_h)| \lesssim h^k (\|\mathbf{u}\|_2 \|\mathbf{B}\|_{k+1} + \|\mathbf{B}\|_2 \|\mathbf{u}\|_{k+1}) \|\mathbf{w}_h\|_W. \quad (5.9)$$

Proof. We only show (5.7), since the other results can be derived similarly.

We shall estimate the four terms of $E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h)$ one by one. Using the Cauchy-Schwarz inequality, the Hölder's inequality, the Sobolev embedding theorem, and Lemmas 3.8, 3.9 and 3.3, we have

$$\begin{aligned}
& |(\mathbf{u} \otimes \mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \otimes \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}, \nabla_h \mathbf{v}_{ho})| \\
& \leq |((\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}) \otimes \mathbf{u}, \nabla_h \mathbf{v}_{ho})| + |(\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \otimes (\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}), \nabla_h \mathbf{v}_{ho})| \\
& \leq \|\mathbf{u}\|_{0,\infty,\Omega} \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} + \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,3,K} |\mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,6,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} \\
& \leq \|\mathbf{u}\|_{0,\infty,\Omega} \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} + \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,3,K} (|\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,6,K} + |\mathbf{u}|_{0,6,K}) \|\nabla_h \mathbf{v}_{ho}\|_{0,K} \\
& \leq \|\mathbf{u}\|_{0,\infty,\Omega} \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} + (|\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,6,\Omega} + |\mathbf{u}|_{0,6,\Omega}) \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,3,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} \\
& \lesssim h^{k+1} \|\mathbf{u}\|_{0,\infty,\Omega} \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V + \|\mathbf{u}\|_1 \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}|_{0,3,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} \\
& \lesssim h^{k+1} \|\mathbf{u}\|_{0,\infty} \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V + h^{k+1-d/6} \|\mathbf{u}\|_1 \|\mathbf{u}\|_{k+1} \|\nabla_h \mathbf{v}_{ho}\|_0 \\
& \lesssim h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
& |\langle (\mathbf{u} \otimes \mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \otimes \mathbf{Q}_k^b \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{ho} \rangle_{\partial\mathcal{T}_h}| \\
& = |\langle (\mathbf{u} \otimes \mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \otimes \mathbf{Q}_k^b \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial\mathcal{T}_h}| \\
& \leq |\langle (\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}) \otimes (\mathbf{u} - \mathbf{Q}_k^o \mathbf{u}) \mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial\mathcal{T}_h}| + |\langle (\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}) \otimes \mathbf{Q}_k^o \mathbf{u} \mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial\mathcal{T}_h}| \\
& \quad + |\langle (\mathbf{Q}_k^o \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}) \otimes (\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}) \mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial\mathcal{T}_h}| + |\langle \mathbf{Q}_k^o \mathbf{u} \otimes (\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}) \mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb} \rangle_{\partial\mathcal{T}_h}| \\
& \leq \sum_{K \in \mathcal{T}_h} (|\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}|_{0,\partial K} \|\mathbf{u} - \mathbf{Q}_k^o \mathbf{u}\|_{0,\partial K} \|\mathbf{v}_{ho} - \mathbf{v}_{hb}\|_{0,\infty,\partial K} + |\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}|_{0,\partial K} |\mathbf{Q}_k^o \mathbf{u}|_{0,6,\partial K} \|\mathbf{v}_{ho} - \mathbf{v}_{hb}\|_{0,3,\partial K}) \\
& \quad + \sum_{K \in \mathcal{T}_h} (|\mathbf{Q}_k^o \mathbf{u} - \mathbf{Q}_k^b \mathbf{u}|_{0,\partial K} \|\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}\|_{0,\partial K} \|\mathbf{v}_{ho} - \mathbf{v}_{hb}\|_{0,\infty,\partial K} + |\mathbf{u} - \mathbf{Q}_k^b \mathbf{u}|_{0,\partial K} |\mathbf{Q}_k^o \mathbf{u}|_{0,6,\partial K} \|\mathbf{v}_{ho} - \mathbf{v}_{hb}\|_{0,3,\partial K}) \\
& \lesssim h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V,
\end{aligned}$$

$$|(\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u} \cdot \nabla_h \mathbf{P}_k^{\mathcal{R}\mathcal{T}} \mathbf{u}, \mathbf{v}_{ho})|$$

$$\begin{aligned}
&\leq |((\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u}) \cdot \nabla \mathbf{u}, \mathbf{v}_{ho})| + |(\mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} \cdot (\nabla \mathbf{u} - \nabla_h \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u}), \mathbf{v}_{ho})| \\
&\leq \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u}|_{0,3,K} |\nabla \mathbf{u}|_{0,K} \|\mathbf{v}_{ho}\|_{0,6,K} + \sum_{K \in \mathcal{T}_h} |\nabla \mathbf{u} - \nabla_h \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u}|_{0,K} |\mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u}|_{0,6,K} \|\mathbf{v}_{ho}\|_{0,3,K} \\
&\lesssim h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V,
\end{aligned}$$

and

$$\begin{aligned}
&|\langle \mathbf{v}_{hb} \otimes \mathbf{Q}_k^b \mathbf{u} \cdot \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} \rangle_{\partial \mathcal{T}_h}| = |\langle \mathbf{v}_{hb} \otimes \mathbf{Q}_k^b \mathbf{u} \cdot \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \rangle_{\partial \mathcal{T}_h}| \\
&\leq |\langle (\mathbf{v}_{ho} - \mathbf{v}_{hb}) \otimes (\mathbf{Q}_k^b \mathbf{u} - \mathbf{Q}_k^o \mathbf{u}) \cdot \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \rangle_{\partial \mathcal{T}_h}| + |\langle \mathbf{v}_{ho} \otimes (\mathbf{Q}_k^b \mathbf{u} - \mathbf{Q}_k^o \mathbf{u}) \cdot \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \rangle_{\partial \mathcal{T}_h}| \\
&\quad + |\langle (\mathbf{v}_{ho} - \mathbf{v}_{hb}) \otimes \mathbf{Q}_k^o \mathbf{u} \cdot \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \rangle_{\partial \mathcal{T}_h}| + |\langle \mathbf{v}_{ho} \otimes \mathbf{Q}_k^o \mathbf{u} \cdot \mathbf{n}, \mathbf{P}_k^{\mathcal{R}\mathcal{T}}\mathbf{u} - \mathbf{Q}_k^b \mathbf{u} \rangle_{\partial \mathcal{T}_h}| \\
&\lesssim h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_V.
\end{aligned}$$

As a result, the desired estimate (5.7) follows. \blacksquare

Theorem 5.1. Let $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$ be the solution to the WG scheme (3.4). Under the regularity assumption (5.1) and the smallness condition (4.16) there hold the following estimates:

$$|||\Pi_1 \mathbf{u} - \mathbf{u}_h|||_V + |||\Pi_2 \mathbf{B} - \mathbf{B}_h|||_W \lesssim h^k C_1(\mathbf{u}, \mathbf{B}), \quad (5.10)$$

$$|||\Pi_3 p - p_h|||_Q + |||\Pi_4 r - r_h|||_R \lesssim h^k C_1(\mathbf{u}, \mathbf{B}) + h^{2k} C_2(\mathbf{u}, \mathbf{B}), \quad (5.11)$$

where

$$C_1(\mathbf{u}, \mathbf{B}) := (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1}) (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2),$$

$$C_2(\mathbf{u}, \mathbf{B}) := (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1})^2 (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2)^2$$

Proof. From (3.4) and Lemma 5.1, we can get the error equation

$$\begin{aligned}
&a_h(\Pi_1 \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \tilde{a}_h(\Pi_2 \mathbf{B} - \mathbf{B}_h, \mathbf{w}_h) + b_h(\mathbf{v}_h, \Pi_3 p - p_h) \\
&\quad - b_h(\Pi_1 \mathbf{u} - \mathbf{u}_h, q_h) + \tilde{b}_h(\mathbf{w}_h, \Pi_4 r - r_h) - \tilde{b}(\Pi_2 \mathbf{B} - \mathbf{B}_h, \theta_h) \\
&\quad + c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \mathbf{v}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B}; \Pi_2 \mathbf{B}) \\
&\quad - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - \tilde{c}_h(\Pi_1 \mathbf{u}; \Pi_2 \mathbf{B}, \mathbf{w}_h) + \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) \\
&= E_u(\mathbf{u}, \mathbf{v}_h) + E_B(\mathbf{B}, \mathbf{w}_h) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) + E_{\tilde{B}1}(\mathbf{B}_h, \mathbf{v}_h) + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \mathbf{w}_h),
\end{aligned}$$

for any $(\mathbf{v}_h, \mathbf{w}_h, q_h, \theta_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$. Taking $(\mathbf{v}_h, \mathbf{w}_h, q_h, \theta_h) = (\Pi_1 \mathbf{u} - \mathbf{u}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_3 p - p_h, \Pi_4 r - r_h)$ in this relation and using Lemma 4.1 we get

$$\begin{aligned}
&\frac{1}{H_a^2} |||\Pi_1 \mathbf{u} - \mathbf{u}_h|||_V^2 + \frac{1}{R_m^2} |||\Pi_2 \mathbf{B} - \mathbf{B}_h|||_W^2 \\
&= a_h(\Pi_1 \mathbf{u} - \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) + \tilde{a}_h(\Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) \\
&= E_u(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_B(\mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) + E_{\tilde{u}}(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\
&\quad + E_{\tilde{B}1}(\mathbf{B}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) \\
&\quad - c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) + c_h(\mathbf{u}_h; \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) - \tilde{c}_h(\Pi_1 \mathbf{u} - \mathbf{u}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) \\
&\quad + \tilde{c}_h(\Pi_1 \mathbf{u} - \mathbf{u}_h; \mathbf{B}_h, \mathbf{B}_h) + \tilde{c}_h(\Pi_1 \mathbf{u}; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) - \tilde{c}_h(\mathbf{u}_h; \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) \\
&= E_u(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_B(\mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) + E_{\tilde{u}}(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\
&\quad + E_{\tilde{B}1}(\mathbf{B}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) \\
&\quad - c_h(\Pi_1 \mathbf{u} - \mathbf{u}_h; \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\
&\quad + \tilde{c}_h(\mathbf{u}_h; \Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) - \tilde{c}_h(\Pi_1 \mathbf{u} - \mathbf{u}_h; \Pi_2 \mathbf{B} - \mathbf{B}_h, \mathbf{B}_h),
\end{aligned}$$

where in the last '=' we have used the relation $c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u} - \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) = 0$. In view of Lemma 5.2 and the definitions of M_h and \tilde{M}_h in (4.2) and (4.3), we further have

$$\begin{aligned}
&\frac{1}{H_a^2} |||\Pi_1 \mathbf{u} - \mathbf{u}_h|||_V^2 + \frac{1}{R_m^2} |||\Pi_2 \mathbf{B} - \mathbf{B}_h|||_W^2 \\
&\leq C(h^k \|\mathbf{u}\|_{k+1} |||\Pi_1 \mathbf{u} - \mathbf{u}_h|||_V + h^k \|\mathbf{B}\|_{k+1} |||\Pi_2 \mathbf{B} - \mathbf{B}_h|||_W + h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} |||\Pi_1 \mathbf{u} - \mathbf{u}_h|||_V)
\end{aligned}$$

$$+ h^k \|\mathbf{B}\|_2 \|\mathbf{B}\|_{k+1} \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + h^k (\|\mathbf{u}\|_2 \|\mathbf{B}\|_{k+1} + \|\mathbf{B}\|_2 \|\mathbf{u}\|_{k+1}) \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W) \\ + M_h \|\mathbf{u}_h\|_V \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V^2 + \tilde{M}_h \|\mathbf{u}_h\|_V \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W^2 + \tilde{M}_h \|\mathbf{B}_h\|_W \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W,$$

which, together with (4.8), yields

$$\begin{aligned} & \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W \\ \leq & 2\zeta C (h^k \|\mathbf{u}\|_{k+1} + h^k \|\mathbf{B}\|_{k+1} + h^k \|\mathbf{B}\|_2 \|\mathbf{B}\|_{k+1} + h^k \|\mathbf{u}\|_2 \|\mathbf{u}\|_{k+1} + h^k \|\mathbf{u}\|_2 \|\mathbf{B}\|_{k+1} + h^k \|\mathbf{B}\|_2 \|\mathbf{u}\|_{k+1}) \\ & + 2\zeta \left(H_a M_h \|\mathbf{u}_h\|_V \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + R_m \tilde{M}_h \|\mathbf{u}_h\|_V \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W \right. \\ & \quad \left. + \frac{1}{2} R_m \tilde{M}_h \|\mathbf{B}_h\|_W \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + \frac{1}{2} R_m \tilde{M}_h \|\mathbf{B}_h\|_W \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W \right) \\ \leq & 2\zeta C h^k (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1}) (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2) \\ & + 4\zeta^2 (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) \left(H_a M_h \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + R_m \tilde{M}_h \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W \right. \\ & \quad \left. + \frac{1}{2} R_m \tilde{M}_h \|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + \frac{1}{2} R_m \tilde{M}_h \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W \right) \\ \leq & 2\zeta C h^k (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1}) (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2) \\ & + 4\zeta^2 (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (H_a M_h + \frac{1}{2} R_m \tilde{M}_h + R_m \tilde{M}_h + \frac{1}{2} R_m \tilde{M}_h) (\|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W) \\ \leq & 2\zeta C h^k (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1}) (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2) \\ & + 12\zeta^3 \max\{M_h, \tilde{M}_h\} (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (\|\Pi_1 \mathbf{u} - \mathbf{u}_h\|_V + \|\Pi_2 \mathbf{B} - \mathbf{B}_h\|_W), \end{aligned}$$

Since the smallness condition (4.16) implies

$$12\zeta^3 \max\{M_h, \tilde{M}_h\} (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) = \delta (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) < 1,$$

we immediately obtain the desired estimate (5.10).

Next let us estimate the pressure error. Taking $(\mathbf{w}_h, q_h, r_h) = (0, 0, 0)$ in the equation (5.2), we have

$$a_h(\Pi_1 \mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \Pi_3 p) + c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) = (\mathbf{f}, \mathbf{v}_{ho}) + E_u(\mathbf{u}, \mathbf{v}_h) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h),$$

which, together with (3.5a), gives

$$\begin{aligned} b_h(\mathbf{v}_h, \Pi_3 p - p_h) = & E_u(\mathbf{u}, \mathbf{v}_h) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) - a_h(\Pi_1 \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - c_h(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \mathbf{v}_h) + c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ & - \tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) \\ = & E_u(\mathbf{u}, \mathbf{v}_h) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) - a_h(\Pi_1 \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) \\ & - c_h(\Pi_1 \mathbf{u} - \mathbf{u}_h; \Pi_1 \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_h; \Pi_1 \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - c_h(\Pi_1 \mathbf{u} - \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ & - \tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) - \tilde{c}_h(\mathbf{v}_h; \Pi_2 \mathbf{B} - \mathbf{B}_h, \mathbf{B}_h). \end{aligned}$$

Thus, using the inf-sup condition (4.4), Lemmas 4.1 and 5.2, and the estimate (5.10), we get

$$\begin{aligned} \|\Pi_3 p - p_h\|_Q & \lesssim \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h^0} \frac{b_h(\mathbf{v}_h, \Pi_3 p - p_h)}{\|\mathbf{v}_h\|_V} \\ & \lesssim h^k (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1}) (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2) + h^{2k} (\|\mathbf{u}\|_{k+1} + \|\mathbf{B}\|_{k+1})^2 (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2)^2 \\ & \lesssim h^k C_1(\mathbf{u}, \mathbf{B}) + h^{2k} C_2(\mathbf{u}, \mathbf{B}). \end{aligned}$$

Similarly, by using the inf-sup condition (4.5), Lemmas 4.1 and 5.2, and (5.10), we can obtain

$$\|\Pi_4 r - r_h\|_R \lesssim h^k C_1(\mathbf{u}, \mathbf{B}) + h^{2k} C_2(\mathbf{u}, \mathbf{B}).$$

Combining the above two inequalities leads to the desired result (5.11). \blacksquare

In light of Theorem 5.1, Lemmas 3.1, 3.6, 3.8 and 3.9, and the triangle inequality, we can finally get the following main error estimates.

Theorem 5.2. Under the same conditions as in Theorem 5.1, there hold

$$\|\nabla \mathbf{u} - \nabla_h \mathbf{u}_{ho}\|_0 + \|\nabla \mathbf{u} - \nabla_{w,k-1} \mathbf{u}_h\|_0 \lesssim h^k C_1(\mathbf{u}, \mathbf{B}), \quad (5.12)$$

$$\|\nabla \times \mathbf{B} - \nabla_h \times \mathbf{B}_{ho}\|_0 + \|\nabla \times \mathbf{B} - \nabla_{w,k-1} \times \mathbf{B}_h\|_0 \lesssim h^k C_1(\mathbf{u}, \mathbf{B}), \quad (5.13)$$

$$\|p - p_{ho}\|_0 + h\|\nabla p - \nabla_{w,k} p\|_0 \lesssim h^k C_1(\mathbf{u}, \mathbf{B}) + h^k \|p\|_k + h^{2k} C_2(\mathbf{u}, \mathbf{B}), \quad (5.14)$$

$$\|r - r_{ho} - (\bar{r} - \bar{r}_{ho})\|_0 + h\|\nabla r - \nabla_{w,k} r\|_0 \lesssim h^k C_1(\mathbf{u}, \mathbf{B}) + h^k \|r\|_k + h^{2k} C_2(\mathbf{u}, \mathbf{B}), \quad (5.15)$$

where \bar{r} and \bar{r}_{ho} denote the mean values of r and r_{ho} on Ω , respectively.

Remark 5.1. From the estimates (5.12) and (5.13) we see that the errors of the velocity and the magnetic field are independent of the pressure and the magnetic pseudo-pressure. This means that our WG scheme is pressure-robust.

6 Local elimination property and iteration scheme

6.1 Local elimination

In this subsection, we shall show that in the WG scheme (3.4) the approximations $(\mathbf{u}_{ho}, \mathbf{B}_{ho}, p_{ho}, r_{ho})$ of the velocity, the magnetic field, the pressure and the magnetic pseudo-pressure defined in the interior of elements can be locally eliminated by the using the numerical traces $(\mathbf{u}_{hb}, \mathbf{B}_{hb}, p_{hb}, r_{hb})$ defined on the boundaries of the elements. After the local elimination the resulting system only contains the degrees of freedom of $(\mathbf{u}_{hb}, \mathbf{B}_{hb}, r_{hb}, p_{hb})$ as unknowns.

For any $K \in \mathcal{T}_h$, we take $\mathbf{v}_{ho}|_{\mathcal{T}_h/K} = 0$, $\mathbf{v}_{hb} = 0$, $\mathbf{w}_{ho}|_{\mathcal{T}_h/K} = 0$, $\mathbf{w}_{hb} = 0$, $q_{ho}|_{\mathcal{T}_h/K} = 0$, $q_{hb} = 0$, $\theta_{ho}|_{\mathcal{T}_h/K} = 0$, $\theta_{hb} = 0$ in the scheme (3.4), and obtain the following local problem:

Find $(\mathbf{u}_{ho}, \mathbf{B}_{ho}, r_{ho}, p_{ho}) \in [\mathcal{P}_k(K)]^d \times [\mathcal{P}_k(K)]^d \times \mathcal{P}_{k-1}(K) \times \mathcal{P}_{k-1}(K)$ such that

$$\begin{aligned} & a_{h,K}(\mathbf{u}_{ho}, \mathbf{v}_{ho}) + \tilde{a}_{h,K}(\mathbf{B}_{ho}, \mathbf{w}_{ho}) \\ & + b_{h,K}(\mathbf{v}_{ho}, p_{ho}) - b_{h,K}(\mathbf{u}_{ho}, q_{ho}) + \tilde{b}_{h,K}(\mathbf{w}_{ho}, r_{ho}) - \tilde{b}_{h,K}(\mathbf{B}_{ho}, \theta_{ho}) \\ & + c_{h,K}(\mathbf{u}_{ho}; \mathbf{u}_{ho}, \mathbf{v}_{ho}) + \tilde{c}_{h,K}(\mathbf{v}_{ho}; \mathbf{B}_{ho}, \mathbf{B}_{ho}) - \tilde{c}_{h,K}(\mathbf{u}_{ho}; \mathbf{B}_{ho}, \mathbf{w}_{ho}) \\ & = \mathbf{F}_K(\mathbf{v}_{ho}) + \mathbf{G}_K(\mathbf{w}_{ho}), \quad \forall (\mathbf{v}_{ho}, \mathbf{w}_{ho}, \theta_{ho}, q_{ho}) \in [\mathcal{P}_k(K)]^d \times [\mathcal{P}_k(K)]^d \times \mathcal{P}_{k-1}(K) \times \mathcal{P}_{k-1}(K), \end{aligned} \quad (6.1)$$

where

$$\begin{aligned} a_{h,K}(\mathbf{u}_{ho}, \mathbf{v}_{ho}) &:= \frac{1}{H_a^2} (\nabla_{w,k-1} \{\mathbf{u}_{ho}, \mathbf{0}\}, \nabla_{w,k-1} \{\mathbf{v}_{ho}, \mathbf{0}\})_K + s_{h,K}(\mathbf{u}_{ho}, \mathbf{v}_{ho}), \\ s_{h,K}(\mathbf{u}_{ho}, \mathbf{v}_{ho}) &:= \frac{1}{H_a^2} \langle \tau \mathbf{u}_{ho}, \mathbf{v}_{ho} \rangle_{\partial K}, \\ \tilde{a}_{h,K}(\mathbf{B}_{ho}, \mathbf{w}_{ho}) &:= \frac{1}{R_m^2} (\nabla_{w,k-1} \times \{\mathbf{B}_{ho}, \mathbf{0}\}, \nabla_{w,k-1} \times \{\mathbf{w}_{ho}, \mathbf{0}\})_K + \tilde{s}_{h,K}(\mathbf{B}_{ho}, \mathbf{w}_{ho}), \\ \tilde{s}_{h,K}(\mathbf{B}_{ho}, \mathbf{w}_{ho}) &:= \frac{1}{R_m^2} \langle \tau \mathbf{B}_{ho} \times \mathbf{n}, \mathbf{w}_{ho} \times \mathbf{n} \rangle_{\partial K}; \\ b_{h,K}(\mathbf{v}_{ho}, q_{ho}) &:= (\nabla_{w,k} \{p_{ho}, 0\}, \mathbf{v}_{ho})_K, \quad \tilde{b}_{h,K}(\mathbf{w}_{ho}, r_{ho}) := \frac{1}{R_m} (\nabla_{w,k} \{r_{ho}, 0\}, \mathbf{w}_{ho})_K, \\ c_{h,K}(\mathbf{u}_{ho}; \mathbf{u}_{ho}, \mathbf{v}_{ho}) &:= \frac{1}{2N} (\nabla_{w,k} \cdot \{\mathbf{u}_{ho} \otimes \mathbf{u}_{ho}, \mathbf{0} \otimes \mathbf{0}\}, \mathbf{v}_{ho})_K - \frac{1}{2N} (\nabla_{w,k} \cdot \{\mathbf{v}_{ho} \otimes \mathbf{u}_{ho}, \mathbf{0} \otimes \mathbf{0}\}, \mathbf{u}_{ho})_K, \\ \tilde{c}_{h,K}(\mathbf{v}_{ho}; \mathbf{B}_{ho}, \mathbf{B}_{ho}) &:= \frac{1}{R_m} (\nabla_{w,k} \times \{\mathbf{B}_{ho}, \mathbf{0}\}, \mathbf{v}_{ho} \times \mathbf{B}_{ho})_K, \\ \mathbf{F}_K(\mathbf{v}_{ho}) &:= (\mathbf{f}, \mathbf{v}_{h0})_K - \frac{1}{H_a^2} (\nabla_{w,k-1} \{\mathbf{0}, \mathbf{u}_{hb}\}, \nabla_{w,k-1} \{\mathbf{v}_{ho}, \mathbf{0}\})_K + \frac{1}{H_a^2} \langle \tau \mathbf{u}_{hb}, \mathbf{v}_{ho} \rangle_{\partial K} \\ & - (\nabla_{w,k} \{0, p_{hb}\}, \mathbf{v}_{ho})_K - \frac{1}{2N} (\nabla_{w,k} \cdot \{\mathbf{0} \otimes \mathbf{0}, \mathbf{u}_{hb} \otimes \mathbf{u}_{hb}\}, \mathbf{v}_{ho})_K, \\ \mathbf{G}_K(\mathbf{w}_{ho}) &:= \frac{1}{R_m} (\mathbf{g}, \mathbf{w}_{h0})_K - \frac{1}{R_m^2} (\nabla_{w,k-1} \times \{\mathbf{0}, \mathbf{B}_{hb}\}, \nabla_{w,k-1} \times \{\mathbf{w}_{ho}, \mathbf{0}\})_K \\ & + \frac{1}{R_m^2} \langle \tau \mathbf{B}_{hb} \times \mathbf{n}, \mathbf{w}_{ho} \times \mathbf{n} \rangle_{\partial K} - \frac{1}{R_m} (\nabla_{w,k} \{0, r_{hb}\}, \mathbf{w}_{ho})_K. \end{aligned}$$

For any $K \in \mathcal{T}_h$, we define the semi-norms as follows:

$$\begin{aligned} |||\mathbf{v}_{ho}|||_{D,K} &= (\|\nabla_{w,k-1}\{\mathbf{v}_{ho}, \mathbf{0}\}\|_{0,K}^2 + \|\tau^{\frac{1}{2}}\mathbf{v}_{ho}\|_{0,\partial K}^2)^{\frac{1}{2}}, \\ |||\mathbf{w}_{ho}|||_{C,K} &:= (\|\nabla_{w,k-1} \times \{\mathbf{w}_{ho}, \mathbf{0}\}\|_{0,K}^2 + \|\tau^{\frac{1}{2}}\mathbf{w}_{ho} \times \mathbf{n}\|_{0,\partial K}^2)^{\frac{1}{2}}. \end{aligned}$$

By following the same routine as in subsection 4.2, we can derive the existence and uniqueness results for the local problem (6.1).

Theorem 6.1. *For any $K \in \mathcal{T}_h$ and given numerical traces $\mathbf{u}_{hb}|_{\partial K}, \mathbf{B}_{hb}|_{\partial K}, p_{hb}|_{\partial K}$ and $r_{hb}|_{\partial K}$, the local problem (6.1) admits at least one solution. Moreover, under the smallness condition*

$$\tilde{\delta} (H_a \|\mathbf{F}_K\|_h + \|\mathbf{G}_K\|_{\tilde{h}}) < 1,$$

the problem (6.1) admits a unique solution, where

$$\begin{aligned} \tilde{\delta} &:= 12 \max\{H_a, R_m\}^3 \max\{M_{h,K}, \tilde{M}_{h,K}\}, \\ M_{h,K} &:= \sup_{\mathbf{0} \neq \Phi_{ho}, \mathbf{u}_{ho}, \mathbf{v}_{ho} \in \bar{\mathbf{V}}_{h,K}} \frac{c_{h,K}(\Phi_{ho}; \mathbf{u}_{ho}, \mathbf{v}_{ho})}{|||\Phi_{ho}|||_{D,K} |||\mathbf{u}_{ho}|||_{D,K} |||\mathbf{v}_{ho}|||_{D,K}}, \\ \tilde{M}_{h,K} &:= \sup_{\substack{\mathbf{0} \neq \mathbf{w}_{ho}, \mathbf{B}_{ho} \in \bar{\mathbf{W}}_{h,K}, \\ \mathbf{0} \neq \mathbf{v}_{ho} \in \bar{\mathbf{V}}_{h,K}}} \frac{\tilde{c}_{h,K}(\mathbf{v}_{ho}; \mathbf{B}_{ho}, \mathbf{w}_{ho})}{|||\mathbf{w}_{ho}|||_{C,K} |||\mathbf{v}_{ho}|||_{D,K} |||\mathbf{B}_{ho}|||_{C,K}}, \\ \|\mathbf{F}_K\|_h &:= \sup_{\mathbf{0} \neq \mathbf{v}_{ho} \in \bar{\mathbf{V}}_{h,K}} \frac{\mathbf{F}_K(\mathbf{v}_{ho})}{|||\mathbf{v}_{ho}|||_{D,K}}, \quad \|\mathbf{G}_K\|_{\tilde{h}} := \sup_{\mathbf{0} \neq \mathbf{w}_{ho} \in \bar{\mathbf{W}}_{h,K}} \frac{\mathbf{G}_K(\mathbf{w}_{ho})}{|||\mathbf{w}_{ho}|||_{C,K}}, \\ \bar{\mathbf{V}}_{h,K} &:= \{\mathbf{v}_{ho} \in [\mathcal{P}_k(K)]^d; b_{h,K}(\mathbf{v}_{ho}, q_{ho}) = 0, \forall q_{ho} \in [\mathcal{P}_{k-1}(K)]\}, \\ \bar{\mathbf{W}}_{h,K} &:= \{\mathbf{w}_{ho} \in [\mathcal{P}_k(K)]^d; \tilde{b}_{h,K}(\mathbf{w}_{ho}, \theta_{ho}) = 0, \forall \theta_{ho} \in [\mathcal{P}_{k-1}(K)]\}, \\ |||\mathbf{v}_{ho}|||_{D,K} &:= (\|\nabla_{w,k-1}\{\mathbf{v}_{ho}, \mathbf{0}\}\|_{0,K}^2 + \|\tau^{\frac{1}{2}}\mathbf{v}_{ho}\|_{0,\partial K}^2)^{\frac{1}{2}}, \\ |||\mathbf{w}_{ho}|||_{C,K} &:= (\|\nabla_{w,k-1} \times \{\mathbf{w}_{ho}, \mathbf{0}\}\|_{0,K}^2 + \|\tau^{\frac{1}{2}}\mathbf{w}_{ho} \times \mathbf{n}\|_{0,\partial K}^2)^{\frac{1}{2}}. \end{aligned}$$

6.2 Oseen's iteration scheme

The WG scheme (3.4) is nonlinear, and we shall adopt the following Oseen's iterative algorithm for it: given \mathbf{u}_h^0 and \mathbf{B}_h^0 , find $(\mathbf{u}_h^n, \mathbf{B}_h^n, p_h^n, r_h^n)$ with $n = 1, 2, \dots$, such that

$$a_h(\mathbf{u}_h^n, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h^n) - b_h(\mathbf{u}_h^n, q_h) + c_h(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n) = (\mathbf{f}, \mathbf{v}_{ho}), \quad (6.2a)$$

$$\tilde{a}_h(\mathbf{B}_h^n, \mathbf{w}_h) + \tilde{b}_h(\mathbf{w}_h, r_h^n) - \tilde{b}_h(\mathbf{B}_h^n, \theta_h) - \tilde{c}_h(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, \mathbf{w}_h) = \frac{1}{R_m}(\mathbf{g}, \mathbf{w}_{ho}), \quad (6.2b)$$

for all $(\mathbf{v}_h, \mathbf{w}_h, q_h, \theta_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$.

Remark 6.1. Notice that the above Oseen's iterative scheme can be rewritten as follows: given \mathbf{u}_h^0 and \mathbf{B}_h^0 , for $n = 1, 2, \dots$,

Step 1: find (\mathbf{B}_h^n, r_h^n) such that

$$\begin{aligned} &\tilde{a}_h(\mathbf{B}_h^n, \mathbf{w}_h) + \tilde{b}_h(\mathbf{w}_h, r_h^n) + \tilde{b}_h(\mathbf{B}_h^n, \theta_h) \\ &= \frac{1}{R_m}(\mathbf{g}, \mathbf{w}_{ho}) + \tilde{c}_h(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, \mathbf{w}_h), \quad \forall (\mathbf{w}_h, \theta_h) \in \mathbf{W}_h^0 \times R_h^0; \end{aligned}$$

Step 2: find (\mathbf{u}_h^n, p_h^n) such that

$$\begin{aligned} &a_h(\mathbf{u}_h^n, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h^n) + b_h(\mathbf{u}_h^n, q_h) + c_h(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h) \\ &= (\mathbf{f}, \mathbf{v}_{ho}) - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h^0 \times Q_h^0. \end{aligned}$$

We have the following convergence result.

Theorem 6.2. Let $(\mathbf{u}_h, \mathbf{B}_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$ be the solution of the WG scheme (3.4). Under the condition (4.16) the Oseen's iteration scheme (6.2) is convergent in the following sense:

$$\lim_{n \rightarrow \infty} |||\mathbf{u}_h^n - \mathbf{u}_h|||_V = 0, \quad \lim_{n \rightarrow \infty} |||\mathbf{B}_h^n - \mathbf{B}_h|||_W = 0, \quad \lim_{n \rightarrow \infty} |||p_h^n - p_h|||_Q = 0, \quad \lim_{n \rightarrow \infty} |||r_h^n - r_h|||_R = 0.$$

Proof. Denote $e_u^n := \mathbf{u}_h^n - \mathbf{u}_h, e_B^n := \mathbf{B}_h^n - \mathbf{B}_h, e_p^n := p_h^n - p_h, e_r^n := r_h^n - r_h$. Subtracting (6.2) from (3.4), for all $(\mathbf{v}_h, \mathbf{w}_h, q_h, \theta_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0$, we have

$$\begin{aligned} & a_h(e_u^n, \mathbf{v}_h) + \tilde{a}_h(e_B^n, \mathbf{w}_h) \\ &= -b_h(\mathbf{v}_h, e_p^n) + b_h(e_u^n, q_h) + c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) \\ &\quad - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n) - \tilde{b}_h(\mathbf{w}_h, e_r^n) + \tilde{b}_h(e_B^n, \theta_h) - \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) + \tilde{c}_h(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, \mathbf{w}_h). \end{aligned} \quad (6.3)$$

Taking $\mathbf{v}_h = e_u^n, \mathbf{w}_h = e_B^n, q_h = e_p^n, \theta_h = e_r^n$ in (6.3) and using Lemma 4.1, we get

$$\begin{aligned} & \frac{1}{H_a^2} |||e_u^n|||_V^2 + \frac{1}{R_m^2} |||e_B^n|||_W^2 \\ &= c_h(\mathbf{u}_h; \mathbf{u}_h, e_u^n) - c_h(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, e_u^n) + \tilde{c}_h(e_u^n; \mathbf{B}_h, \mathbf{B}_h) - \tilde{c}_h(e_u^n; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n) - \tilde{c}_h(\mathbf{u}_h; \mathbf{B}_h, e_B^n) + \tilde{c}_h(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, e_B^n) \\ &= -c_h(e_u^{n-1}; \mathbf{u}_h, e_u^n) - \tilde{c}_h(e_u^n; e_B^{n-1}, \mathbf{B}_h) + \tilde{c}_h(\mathbf{u}_h; e_B^{n-1}, e_B^n) \\ &\leq M_h |||\mathbf{u}_h|||_V |||e_u^{n-1}|||_V |||e_u^n|||_V + \tilde{M}_h |||e_B^{n-1}|||_W |||e_u^n|||_V |||\mathbf{B}_h|||_W + \tilde{M}_h |||e_B^{n-1}|||_W |||\mathbf{u}_h|||_V |||e_B^n|||_W, \end{aligned}$$

where in the second '=' we have used the relation $c_h(\mathbf{u}_h^{n-1}; e_u^n, e_u^n) = 0$. Similar to the proof of (4.15), the above estimate plus (4.8) yields

$$\begin{aligned} & |||e_u^n|||_V + |||e_B^n|||_W \\ &\leq 2\zeta \left(H_a M_h |||\mathbf{u}_h|||_V |||e_u^{n-1}|||_V + H_a \tilde{M}_h |||e_B^{n-1}|||_W |||\mathbf{B}_h|||_W + R_m \tilde{M}_h |||e_B^{n-1}|||_W |||\mathbf{u}_h|||_V \right) \\ &\leq 4\zeta^2 (H_a M_h + H_a \tilde{M}_h + R_m \tilde{M}_h) (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (|||e_u^{n-1}|||_V + |||e_B^{n-1}|||_W) \\ &\leq 12\zeta^3 \max\{M_h, \tilde{M}_h\} (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (|||e_u^{n-1}|||_V + |||e_B^{n-1}|||_W) \\ &\leq \delta (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) (|||e_u^{n-1}|||_V + |||e_B^{n-1}|||_W) \\ &\leq \dots \\ &\leq (\delta (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}))^n (|||e_u^0|||_V + |||e_B^0|||_W), \end{aligned}$$

which, together with the smallness condition (4.16), i.e. $0 \leq \delta (H_a \|\mathbf{f}\|_h + \|\mathbf{g}\|_{\tilde{h}}) < 1$, gives

$$\lim_{n \rightarrow \infty} (|||\mathbf{u}_h^n - \mathbf{u}_h|||_V + |||\mathbf{B}_h^n - \mathbf{B}_h|||_W) = 0. \quad (6.4)$$

From (6.3) we can get for any $\mathbf{v}_h \in \mathbf{V}_h^0$,

$$\begin{aligned} b_h(\mathbf{v}_h, e_p^n) &= -a_h(e_u^n, \mathbf{v}_h) + c_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h) + \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - \tilde{c}_h(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n) \\ &= -a_h(e_u^n, \mathbf{v}_h) - c_h(e_u^{n-1}; \mathbf{u}_h, \mathbf{v}_h) - c_h(e_u^{n-1}; e_u^n, \mathbf{v}_h) - \tilde{c}_h(\mathbf{v}_h; e_B^{n-1}, \mathbf{B}_h) - \tilde{c}_h(\mathbf{v}_h; e_B^{n-1}, e_B^n). \end{aligned}$$

Using the inf-sup condition (4.4) and Lemma 4.1, we have

$$\begin{aligned} |||e_p^n|||_Q &\leq \sup_{\mathbf{v}_h \in \mathbf{V}_h^0} \frac{b_h(\mathbf{v}_h, e_p^n)}{|||\mathbf{v}_h|||_V} \\ &\leq \frac{1}{H_a^2} |||e_u^n|||_V + M_h (|||e_u^{n-1}|||_V |||\mathbf{u}_h|||_V + |||e_u^{n-1}|||_V |||e_u^n|||_V) + \tilde{M}_h (|||e_B^{n-1}|||_W |||\mathbf{B}_h|||_W + |||e_B^{n-1}|||_W |||e_B^n|||_W), \end{aligned}$$

which together with (6.4), yields

$$\lim_{n \rightarrow \infty} |||p_h^n - p_h|||_Q = \lim_{n \rightarrow \infty} |||e_p^n|||_Q = 0.$$

Similarly, by using the inf-sup condition (4.4), Lemma 4.1 and (6.4), we can obtain

$$\lim_{n \rightarrow \infty} |||r_h^n - r_h|||_R = 0.$$

This completes the proof. ■

7 Numerical examples

In this section, we give two 2D numerical examples to verify the performance of the WG scheme (3.4) for the steady incompressible MHD flow (1.1). We apply the Oseen's iterative scheme with an initial guess $(\mathbf{u}_{ho}^0, \mathbf{B}_{ho}^0) = (0, 0)$ and the stop criterion

$$\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_0 < 1e-8$$

in all numerical experiments.

In the examples of the model (1.1), we set

$$\Omega = [0, 1]^2, H_a = N = R_m = 1,$$

and we use regular triangular meshes for the computation (cf. Figure 1).

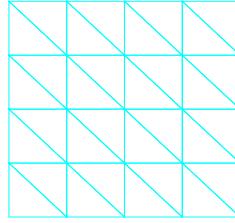


Figure 1: The meshes: 4×4 mesh .

Example 7.1. *The exact solution $(\mathbf{u}, \mathbf{B}, p, r)$ is of the form*

$$\begin{cases} u_1 = -x^2(x-1)^2y(y-1)(2y-1), & u_2 = y^2(y-1)^2x(x-1)(2x-1), \\ B_1 = -x^2(x-1)^2y(y-1)(2y-1), & B_2 = y^2(y-1)^2x(x-1)(2x-1), \\ p = x(x-1)(x-1/2)y(y-1)(y-1/2), \\ r = x(x-1)(x-1/2)y(y-1)(y-1/2). \end{cases}$$

We compute the WG scheme (3.4) with $k = 1, 2$. Numerical results are listed in Tables 1 and 2.

Example 7.2. *The exact solution $(\mathbf{u}, \mathbf{B}, p, r)$ is of the form*

$$\begin{cases} u_1 = \sin(\pi x) \cos(\pi y), & u_2 = -\sin(\pi y) \cos(\pi x), \\ B_1 = -x^2(x-1)^2y(y-1)(2y-1), & B_2 = y^2(y-1)^2x(x-1)(2x-1), \\ p = x^6 - y^6, & r = x(x-1)(x-1/2)y(y-1)(y-1/2). \end{cases}$$

We compute the WG scheme (3.4) with $k = 1, 2$. Numerical results are listed in Tables 3 and 4.

Table 1 - Table 4 show the histories of convergence for the velocity \mathbf{u}_{ho} , the magnetic field \mathbf{B}_{ho} , the pressure p_{ho} , and the magnetic pseudo-pressure r_{ho} . Results of

$$\text{div}U_h := \text{Max}_{K \in \mathcal{T}_h} h_K^{-1} \|\nabla \cdot \mathbf{u}_{ho}\|_{0,K}$$

and

$$\text{div}B_h := \text{Max}_{K \in \mathcal{T}_h} h_K^{-1} \|\nabla \cdot \mathbf{B}_{ho}\|_{0,K}$$

are also listed to verify the divergence-free property. From the numerical results of the two examples, we have the following conclusions:

- The convergence rates of $\|\nabla \mathbf{u} - \nabla_h \mathbf{u}_{ho}\|_0$, $\|\nabla \mathbf{u} - \nabla_{w,k-1} \mathbf{u}_h\|_0$, $\|\nabla \times \mathbf{B} - \nabla_h \times \mathbf{B}_{ho}\|_0$, $\|\nabla \mathbf{B} - \nabla_{w,k-1} \mathbf{B}_h\|_0$, $\|p - p_{ho}\|_0$, $h\|\nabla p - \nabla_{w,k} p_h\|_0$ and $h\|\nabla r - \nabla_{w,k} r_h\|_0$ for the WG scheme with $k = 1, 2$ are of k^{th} orders, which are consistent with the established theoretical results in Theorem 5.2.
- The convergence rate of $\|r - r_{ho}\|_0$ is also of k^{th} order.
- The convergence rates of $\|\mathbf{u} - \mathbf{u}_{ho}\|_0$ and $\|\mathbf{B} - \mathbf{B}_{ho}\|_0$ are of $(k + 1)^{th}$ orders.

- Based on the facts that

$$\|\nabla_h \cdot \mathbf{u}_{ho}\|_{0,\infty} \lesssim \max_{K \in \mathcal{T}_h} h_K^{-1} \|\nabla \cdot \mathbf{u}_{ho}\|_{0,K} = \text{div}U_h,$$

and

$$\|\nabla_h \cdot \mathbf{B}_{ho}\|_{0,\infty} \lesssim \max_{K \in \mathcal{T}_h} h_K^{-1} \|\nabla \cdot \mathbf{B}_{ho}\|_{0,K} = \text{div}B_h,$$

we can see that the the discrete velocity and the discrete magnetic field are globally divergence-free.

Table 1: History of convergence results with $k = 1$ for Example 7.1

mesh	$\frac{\ \mathbf{u}-\mathbf{u}_{ho}\ _0}{\ \mathbf{u}\ _0}$		$\frac{\ \nabla \mathbf{u}-\nabla_{w,k-1}\mathbf{u}_h\ _0}{\ \nabla \mathbf{u}\ _0}$		$\frac{\ \nabla \mathbf{u}-\nabla_h \mathbf{u}_{ho}\ _0}{\ \nabla \mathbf{u}\ _0}$		$\text{div}U_h$	
	error	order	error	order	error	order		
2×2	2.3094e+00	-	9.0044e-01	-	1.2207e+00	-	3.4001e-16	
4×4	5.6715e-01	1.87	5.1165e-01	0.77	6.4277e-01	0.92	3.0878e-16	
8×8	1.5224e-01	1.90	2.7237e-01	0.91	3.2258e-01	0.99	3.4348e-16	
16×16	3.9918e-02	1.93	1.3841e-01	1.00	1.6098e-01	1.00	1.3878e-17	
32×32	1.0236e-02	1.96	6.9404e-02	1.00	8.0401e-02	1.00	3.8511e-16	
64×64	2.5908e-03	1.98	3.4720e-02	1.00	4.0183e-02	1.00	3.4153e-14	
128×128	6.5160e-04	1.99	1.7363e-02	1.00	2.0089e-02	1.00	2.9676e-13	
mesh	$\frac{\ \mathbf{B}-\mathbf{B}_{ho}\ _0}{\ \mathbf{B}\ _0}$		$\frac{\ \nabla \times \mathbf{B}-\nabla_{w,k-1} \times \mathbf{B}_h\ _0}{\ \nabla \mathbf{B}\ _0}$		$\frac{\ \nabla \times \mathbf{B}-\nabla_h \times \mathbf{B}_{ho}\ _0}{\ \nabla \mathbf{B}\ _0}$		$\text{div}B_h$	
	error	order	error	order	error	order		
2×2	4.7569e+00	-	9.8373e-01	-	1.5806e+00	-	4.8572e-17	
4×4	1.1606e+00	2.04	5.6441e-01	0.82	8.2061e-01	0.94	7.9797e-16	
8×8	2.9482e-01	1.98	2.7675e-01	1.03	3.7929e-01	1.11	4.0939e-16	
16×16	7.4064e-02	1.99	1.3198e-01	1.07	1.6468e-01	1.20	4.1633e-16	
32×32	1.8525e-02	2.00	6.4624e-02	1.03	7.5900e-02	1.11	5.6899e-15	
64×64	4.6309e-03	2.00	3.2115e-02	1.01	3.6869e-02	1.04	2.0886e-14	
128×128	1.1577e-03	2.00	1.6032e-02	1.00	1.8283e-02	1.01	2.9830e-13	
mesh	$\frac{\ p-p_{ho}\ _0}{\ p\ _0}$		$\frac{h\ \nabla p-\nabla_{w,k}p_h\ _0}{\ \nabla r\ _0}$		$\frac{\ r-r_{ho}\ _0}{\ r\ _0}$		$\frac{h\ \nabla r-\nabla_{w,k}r_h\ _0}{\ \nabla r\ _0}$	
	error	order	error	order	error	order		
2×2	7.0956e+00	-	6.2931e-01	-	4.7146e+00	-	7.1008e-01	-
4×4	4.4806e-01	0.73	2.5901e-01	1.20	2.8776e+00	-0.05	4.4508e-01	0.70
8×8	2.3643e-01	0.92	1.2598e-01	1.03	1.4532e-01	0.71	2.0693e-01	1.10
16×16	1.1985e-01	0.98	6.2587e-02	1.00	7.2538e-01	0.98	9.2089e-02	1.16
32×32	6.0129e-02	1.00	3.1264e-02	1.00	3.6249e-01	1.00	4.5880e-02	1.00
64×64	3.0089e-02	1.00	1.5628e-02	1.00	1.8122e-02	1.00	2.5771e-02	0.89
128×128	1.5048e-02	1.00	7.8132e-03	1.00	5.4339e-03	1.00	1.5907e-02	0.90

Table 2: History of convergence results with $k = 2$ for Example 7.1

mesh	$\ \mathbf{u} - \mathbf{u}_{ho}\ _0$		$\ \nabla \mathbf{u} - \nabla_{w,k-1} \mathbf{u}_h\ _0$		$\ \nabla \mathbf{u} - \nabla_h \mathbf{u}_{ho}\ _0$		$divUh$
	error	order	error	order	error	order	
2×2	4.0948e-01	-	4.5399e-01	-	8.7670e-01	-	1.9385e-17
4×4	5.6888e-02	2.85	1.3014e-01	1.80	2.4354e-01	1.84	8.4007e-15
8×8	7.3768e-03	2.95	3.4894e-02	1.89	6.2798e-02	1.95	4.6346e-15
16×16	9.3177e-04	2.99	8.9496e-03	1.96	1.5637e-02	2.00	1.7873e-15
32×32	1.1713e-04	2.99	2.2591e-03	1.99	3.8806e-03	2.01	9.7438e-15
64×64	1.4693e-05	3.00	5.6704e-04	2.00	9.6538e-04	2.00	5.2180e-14
128×128	1.8403e-06	2.99	1.4201e-04	2.00	2.4068e-04	2.00	6.9921e-13
mesh	$\ \mathbf{B} - \mathbf{B}_{ho}\ _0$		$\ \nabla \times \mathbf{B} - \nabla_{w,k-1} \times \mathbf{B}_h\ _0$		$\ \nabla \times \mathbf{B} - \nabla_h \times \mathbf{B}_{ho}\ _0$		$divBh$
	error	order	error	order	error	order	
2×2	7.3927e-01	-	5.9051e-01	-	6.4000e+00	-	3.8289e-16
4×4	1.9191e-01	1.95	1.2959e-01	2.19	1.3377e+00	2.26	1.8881e-16
8×8	3.2161e-02	2.58	2.8450e-02	2.19	3.3459e-01	2.00	2.0293e-15
16×16	4.5933e-03	2.81	7.0130e-03	2.02	8.5809e-02	1.96	2.9827e-15
32×32	6.0891e-04	2.92	1.7591e-03	2.00	2.1581e-02	1.99	5.4936e-15
64×64	7.8185e-05	2.96	4.4098e-04	2.00	5.3937e-03	2.00	6.0488e-14
128×128	9.8991e-06	2.98	1.1041e-04	2.00	1.3468e-03	2.00	6.7897e-13
mesh	$\ p - p_{ho}\ _0$		$h \ \nabla p - \nabla_{w,k} p_h\ _0$		$\ r - r_{ho}\ _0$		$h \ \nabla r - \nabla_{w,k} r_h\ _0$
	error	order	error	order	error	order	
2×2	6.2098e+00	-	1.6225e+01	-	6.4000e+00	-	1.6227e+01
4×4	6.6197e-02	1.69	3.0672e+00	1.99	1.3377e+00	2.26	4.0583e+00
8×8	1.7485e-02	1.92	1.8913e+00	1.93	3.3459e-01	2.00	1.0203e+00
16×16	4.4316e-03	1.98	3.8976e-01	1.96	5.3937e-02	2.00	2.6077e-01
32×32	1.1117e-03	1.99	6.2237e-02	1.90	8.5809e-02	1.96	7.0588e-02
64×64	2.7815e-04	1.99	3.0583e-02	1.91	2.1581e-02	1.99	2.2239e-02
128×128	6.9553e-05	1.99	8.7568e-03	1.89	5.3937e-03	2.00	8.7574e-03

 Table 3: History of convergence results with $k = 1$ for Example 7.2

mesh	$\ \mathbf{u} - \mathbf{u}_{ho}\ _0$		$\ \nabla \mathbf{u} - \nabla_{w,k-1} \mathbf{u}_h\ _0$		$\ \nabla \mathbf{u} - \nabla_h \mathbf{u}_{ho}\ _0$		$divUh$
	error	order	error	order	error	order	
2×2	5.8720e-01	-	5.0386e-01	-	7.5274e-01	-	4.4409e-16
4×4	1.5498e-01	1.92	2.6999e-01	0.90	3.5470e-01	1.08	1.7764e-15
8×8	3.9245e-02	1.98	1.3795e-01	0.96	1.7228e-01	1.04	2.1316e-14
16×16	9.8383e-03	1.99	6.9412e-02	0.99	8.5323e-02	1.01	2.8422e-14
32×32	2.4610e-03	1.99	3.4774e-02	0.99	4.2550e-02	1.00	1.1369e-13
64×64	6.1533e-04	1.99	1.7399e-02	0.99	2.1260e-02	1.00	9.5497e-12
128×128	1.5384e-04	2.00	8.7017e-03	0.99	1.0628e-02	1.00	1.0368e-10
mesh	$\ \mathbf{B} - \mathbf{B}_{ho}\ _0$		$\ \nabla \times \mathbf{B} - \nabla_{w,k-1} \times \mathbf{B}_h\ _0$		$\ \nabla \times \mathbf{B} - \nabla_h \times \mathbf{B}_{ho}\ _0$		$divBh$
	error	order	error	order	error	order	
2×2	4.7546e+00	-	9.9565e-01	-	1.5813e+00	-	1.7347e-17
4×4	1.1650e+00	2.02	5.6617e-01	0.81	8.2285e-01	0.94	3.4694e-18
8×8	2.9537e-01	1.97	2.7683e-01	1.03	3.7935e-01	1.11	3.1919e-16
16×16	7.4159e-02	1.99	1.3199e-01	1.06	1.6468e-01	1.20	9.7145e-17
32×32	1.8545e-02	1.99	6.4625e-02	1.03	7.5899e-02	1.11	1.4218e-14
64×64	4.6359e-03	2.00	3.2115e-02	1.00	3.6869e-02	1.04	5.5865e-14
128×128	1.1589e-03	2.00	1.6032e-02	1.00	1.8283e-02	1.01	5.3192e-13
mesh	$\ p - p_{ho}\ _0$		$h \ \nabla p - \nabla_{w,k} p_h\ _0$		$\ r - r_{ho}\ _0$		$h \ \nabla r - \nabla_{w,k} r_h\ _0$
	error	order	error	order	error	order	
2×2	1.5904e+00	-	5.0377e-01	-	4.8444e+00	-	7.4476e-01
4×4	8.2249e-01	0.95	2.4127e-01	1.06	3.5434e-01	-0.84	4.8998e-01
8×8	4.0154e-01	1.03	1.2071e-01	0.99	5.1369e-01	0.82	2.1646e-01
16×16	1.9969e-01	1.00	6.0868e-02	0.98	2.9593e-01	0.80	9.4338e-02
32×32	9.9854e-02	0.99	3.0661e-02	0.98	1.8125e-01	1.01	4.6403e-02
64×64	4.9969e-02	0.99	1.5415e-02	0.99	7.2736e-02	1.01	2.5885e-02
128×128	2.5000e-02	0.99	7.7379e-03	0.99	3.6275e-02	1.00	1.5930e-02

Table 4: History of convergence results with $k = 2$ for Example 7.2

mesh	$\ \mathbf{u} - \mathbf{u}_{ho}\ _0$		$\ \nabla \mathbf{u} - \nabla_{w,k-1} \mathbf{u}_h\ _0$		$\ \nabla \mathbf{u} - \nabla_h \mathbf{u}_{ho}\ _0$		$divUh$	
	error	order	error	order	error	order		
2×2	1.1135e-01	-	1.6761e-01	-	4.3095e-01	-	4.2931e-15	
4×4	1.4860e-02	2.91	4.3804e-02	1.94	1.0261e-01	2.07	1.3417e-14	
8×8	1.8798e-03	2.99	1.1123e-02	1.98	2.4719e-02	2.05	1.1394e-13	
16×16	2.3542e-04	3.00	2.7981e-03	1.99	6.1043e-03	2.01	1.8455e-13	
32×32	2.9435e-05	3.00	7.0135e-04	2.00	1.5210e-03	2.00	1.1427e-12	
64×64	3.6793e-06	3.00	1.7554e-04	2.00	3.7993e-04	2.00	6.1542e-12	
128×128	4.5990e-07	3.00	4.3909e-05	2.00	9.4962e-05	2.00	1.8475e-10	
mesh	$\ \mathbf{B} - \mathbf{B}_{ho}\ _0$		$\ \nabla \times \mathbf{B} - \nabla_{w,k-1} \times \mathbf{B}_h\ _0$		$\ \nabla \times \mathbf{B} - \nabla_h \times \mathbf{B}_{ho}\ _0$		$divBh$	
	error	order	error	order	error	order		
2×2	7.3926e-01	-	5.9072e-01	-	1.1218e+00	-	5.0090e-17	
4×4	1.9145e-01	1.95	1.3030e-01	2.18	4.3639e-01	1.36	2.8295e-16	
8×8	3.2158e-02	2.57	2.8597e-02	2.18	1.3865e-01	1.65	4.8643e-16	
16×16	4.6052e-03	2.80	7.0452e-03	2.02	3.8347e-02	1.85	1.3153e-15	
32×32	6.1557e-04	2.90	1.7668e-03	2.00	9.9908e-03	1.94	2.0630e-14	
64×64	8.1469e-05	2.92	4.4291e-04	2.00	2.5434e-03	1.97	1.1610e-13	
128×128	1.1444e-05	2.83	1.1089e-04	2.00	6.4124e-04	1.98	7.3211e-13	
mesh	$\ \mathbf{p} - \mathbf{p}_{ho}\ _0$		$\frac{h}{\ \mathbf{p}\ _0} \ \nabla \mathbf{p} - \nabla_{w,k} \mathbf{p}_h\ _0$		$\ \mathbf{r} - \mathbf{r}_{ho}\ _0$		$\frac{h}{\ \mathbf{r}\ _0} \ \nabla \mathbf{r} - \nabla_{w,k} \mathbf{r}_h\ _0$	
	error	order	error	order	error	order		
2×2	7.9002e-01	-	4.8151e-01	-	6.4267e+00	-	3.5362e-01	-
4×4	1.4304e-01	2.47	2.4399e-01	1.98	1.6916e+00	1.93	8.8431e-02	1.99
8×8	2.9207e-02	2.30	1.2339e-01	1.98	4.8829e-01	1.79	2.2139e-02	1.99
16×16	6.6988e-03	2.12	6.2086e-02	1.99	1.2901e-01	1.92	5.5663e-03	1.97
32×32	1.6187e-03	2.05	3.1145e-02	1.99	3.2709e-02	1.98	7.0588e-02	1.90
64×64	3.9905e-04	2.02	1.5599e-02	1.99	8.1999e-03	2.00	2.2239e-02	1.89
128×128	9.9147e-05	2.01	7.8059e-03	1.99	2.0504e-03	2.00	8.7574e-03	1.89

8 Conclusions

In this paper, we have developed a weak Galerkin method of arbitrary order for the steady incompressible Magnetohydrodynamics flow. The well-posedness of the discrete scheme has been established. The method yields globally divergence-free approximations of velocity and magnetic field, and is of optimal order convergence for the velocity, the magnetic field, the pressure, and the magnetic pseudo-pressure approximations. The proposed Oseen's iteration algorithm is unconditionally convergent. Numerical experiments have verified the theoretical results.

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