

Drawing the Almost Convex Set in an Integer Grid of Minimum Size

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Abstract

In 2001, Károlyi, Pach and Tóth introduced a family of point sets to solve an Erdős-Szekeres type problem; which have been used to solve several other Erdős-Szekeres type problems. In this paper we refer to these sets as nested almost convex sets. A nested almost convex set \mathcal{X} has the property that the interior of every triangle determined by three points in the same convex layer of \mathcal{X} , contains exactly one point of \mathcal{X} . In this paper, we introduce a characterization of nested almost convex sets. Our characterization implies that there exists at most one (up to order type) nested almost convex set of n points. We use our characterization to obtain a linear time algorithm to construct nested almost convex sets of n points, with integer coordinates of absolute values at most $O(n^{\log_2 5})$. Finally, we use our characterization to obtain an $O(n \log n)$ -time algorithm to determine whether a set of points is a nested almost convex set.

1 Introduction.

We say that a set of points in the plane is in *general position* if no three of them are collinear. Throughout this paper all points sets are in general position. In [9], Erdős asked for the minimum integer $E(s, l)$ that satisfies the following. Every set of at least $E(s, l)$ points, contains s points in convex position and at most l points in its interior. A k -hole of \mathcal{X} is a polygon with k vertices, all of which belong to \mathcal{X} and has no points of \mathcal{X} in its interior; the polygon may be convex or non-convex. In 1983, Horton surprised the community with a simple proof that $E(s, l)$ does not exist for $l = 0$ and $s \geq 7$ [12]; Horton constructed arbitrarily large point set with no convex 7-holes. Note that for $l = 0$, $E(s, l)$ is the minimum integer such that every set of at least $E(s, 0)$ points contains at least one s -hole.

In 2001 [14] Károlyi, Pach and Tóth introduce a family of sets that, although was not given a name, it was used in other works related to the original question of Erdős. In this paper we refer the elements of this family as *nested almost convex sets*. They have been used in the following problems.

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A modular version of the Erdős problem. In 2001 [14] Károlyi, Pach and Tóth use the nested almost convex sets to prove that, for any $s \geq 5l/6 + O(1)$, there is an integer $B(s, l)$ with the following property. Every set of at least $B(s, l)$ points in general position contains s points in convex position such that the number of points in the interior of their convex hull is 0, modulo (l) . This "modular" version of the Erdős problem was proposed by Bialostocki, Dierker, and Voxman [5]. This was proved for $s \geq l + 2$ by Bialostocki et al. The original upper bound on $B(s, l)$ was later improved by Caro in [7].

A version of the Erdős problem in almost convex sets. We say that \mathcal{X} is an *almost convex* set if every triangle with vertices in \mathcal{X} contains at most one point of \mathcal{X} in its interior. Let $N(s)$ be the smallest integer such that every almost convex set of at least $N(s)$ points contains an s -hole. In 2007 [17] Valtr Lippner and Károlyi use the nested almost convex sets to prove that:

$$N(s) = \begin{cases} 2^{(s+1)/2} - 1 & \text{if } s \geq 3 \text{ is odd} \\ \frac{3}{2}2^{s/2} - 1 & \text{if } s \geq 4 \text{ is even.} \end{cases} \quad (1)$$

The authors use the nested almost convex sets to attain the equality in (1). The existence of $N(s)$ was first proved by Károlyi, Pach and Tóth in [14]. The upper bound for $N(s)$ was improved by Kun and Lippner in [15], and it was improved again by Valtr in [16].

Maximizing the number of non-convex 4-holes. In 2014 [1] Aichholzer, Fabila-Monroy, González-Aguilar, Hackl, Heredia, Huemer, Urrutia and Vogtenhuber prove that the maximum number of non-convex 4-holes in a set of n points is at most $n^3/2 - \Theta(n^2)$. The authors use the nested almost convex sets to prove that some sets have $n^3/2 - \Theta(n^2 \log(n))$ non-convex 4-holes.

Blocking 5-holes. A set B blocks the convex k -holes in \mathcal{X} , if any k -hole of \mathcal{X} contains at least one element of B in the interior of its convex hull. In 2015 [6] Cano, Garcia, Hurtado, Sakai, Tejel and Urrutia use the nested almost convex sets to prove that: $n/2 - 2$ points are always necessary and sometimes sufficient to block the 5-holes of a point set with n elements in convex position and $n = 4k$. The authors use the nested almost convex sets as an example of a set for which $n/2 - 2$ points are sufficient to block its 5-holes.

We now define formally the nested almost convex sets.

Definition 1.1. Let \mathcal{X} be a point set; let k be the number of convex layers of \mathcal{X} ; and for $1 \leq j \leq k$, let R_j be the set of points in the j -th convex layer of \mathcal{X} . We say that \mathcal{X} is a nested almost convex set if:

1. $\mathcal{X}_j := R_1 \cup R_2 \cup \dots \cup R_j$ is in general position,
2. the vertices in the convex hull of \mathcal{X}_j are the elements of R_j , and
3. any triangle determined by three points of R_j contains precisely one point of \mathcal{X}_{j-1} in its interior.

In this paper, we give a characterization of when a set of points is a nested almost convex set. This is done by first defining a family of trees. If there exists a map, that satisfies certain properties, from the point set to the nodes of a tree

in the family, then the point set is a nested almost convex set. This map encodes a lot of information about the point set. For example, it determines the location of any given point with respect to the convex hull; we use this information to obtain an $O(n \log n)$ -time algorithm to decide whether a set of points is a nested almost convex set. This map also determines the orientation of any given triplet of points. This implies that for every n there exists essentially at most one nested almost convex set. We further apply this information to obtain a linear-time algorithm that produces a representation of a nested almost convex set of n points on a small integer grid of size $O(n^{\log_2 5})$.

The *order type* of a point set $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ is a mapping that assigns to each ordered triplet (x_i, x_j, x_k) an orientation. If x_k is to the left of the directed line from x_i to x_j , the orientation of (x_i, x_j, x_k) is counterclockwise. If x_k is to the right of the directed line from x_i to x_j , the orientation of (x_i, x_j, x_k) is clockwise. We say that two sets of points have the same order type, if there exist a bijection between these sets that preserves the orientation of all triplets.

The order type was introduced by Goodman and Pollack in [10], and it has been widely used in Combinatorial Geometry to classify point sets; two sets of points are essentially the same if they have the same order type. As a consequence of the characterization of nested almost convex sets presented in Section 2, we have the following.

Theorem 1.2. *If $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$ there is exactly one order type that correspond to a nested almost convex set with n points; for other values of n , nested almost convex sets with n points do not exist.*

In previous papers, two constructions of nested almost convex sets have been presented. The first construction was introduced by Károlyi, Pach and Tóth in [14]. The second construction was introduced by Valtr, Lippner and Károlyi in [17] six years later.

Construction 1: Let X_1 be a set of two points. Assume that $j > 0$ and that X_j has been constructed. Let z_1, \dots, z_r denote the vertices of R_j in clockwise order. Let P_j be the polygon with vertices in R_j . Let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$, let ℓ_i denote the line through z_i orthogonal to the bisector of the angle of P_j at z_i . Let z'_i and z''_i be two points in ℓ_i at distance ε_j of z_i . Finally, move z'_i and z''_i away from P_j at distance δ_j , in the direction orthogonal to ℓ_i , and denote the resulting points by u'_i and u''_i , respectively. Let $R_{j+1} = \{u'_i, u''_i : i = 1 \dots r\}$ and $X_{j+1} = X_j \cup R_{j+1}$. It is easy to see that if ε_j and $\frac{\varepsilon_j}{\delta_j}$ are sufficiently small, then X_{j+1} is an almost convex set. See Figure 1a.

Construction 2: Let X_1 be a set of one point. Let R_2 be a set of three points such that, the point in X_1 is in the interior of the triangle determined by R_2 . Let $X_2 = X_1 \cup R_2$. Now recursively, suppose that X_j and R_j have been constructed and construct the next convex layer R_{j+1} as in Construction 1. See Figure 1b.

Computers are frequently used to decide whether particular sets satisfy some properties. Thus, a representation of large nested almost convex sets could be necessary. Construction 1 and Construction 2 provide such representations; however, the coordinates of the points in those constructions are not integers or are too large with respect to the value of n . This is prone to rounding errors

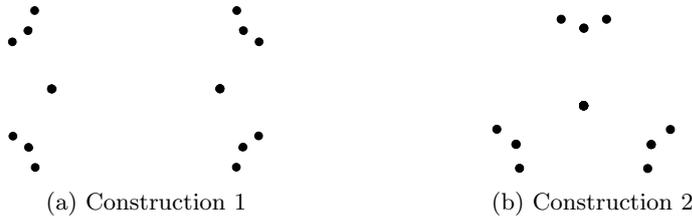


Figure 1: Examples of Almost Convex Sets

or increases the cost of computation. Thus, it is better if the coordinates of the points in the representations are small integers.

A *drawing* of \mathcal{X} is a set of points with integer coordinates and with the same order type than \mathcal{X} . The *size* of a drawing is the maximum of the absolute values of its coordinates. Other works on point sets drawings are [3, 4, 11, 13].

In Section 3, we prove that a nested almost convex set of n points (if it exists), can be drawn in an integer grid of size $O(n^{\log_2 5}) \simeq O(n^{2.322})$. Furthermore, we provide a linear time algorithm to find this drawing. A lower bound of $\Omega(n^{1.5})$ on the size of any drawing of a nested almost convex set of n points can be derived from the following observations. Any drawing of an n -point set in convex position has size $\Omega(n^{1.5})$ [13]; and every nested almost convex set of n points has a $\Theta(n)$ points in convex position. This is presented in detail in Section 2.

In Section 4, we are interested in finding an algorithm to decide whether a given point set is a nested almost convex set. A straightforward $O(n^4)$ -time algorithm for this problem can be given using Definition 1.1. This can be improved to $O(n^2)$ as follows. Using the algorithm presented in Section 3 an instance of nested almost convex set can be constructed. Recently in [2], Aloupis, Iacono, Langerman, Öskan and Wührer gave an $O(n^2)$ -time algorithm to decide whether two given sets of n points have the same order type. Thus, using their algorithm and our instance solves the decision problem in $O(n^2)$ time. We further improve on this by presenting $O(n \log n)$ time algorithm.

2 Characterization of Nested Almost Convex Sets.

In this section we prove Theorem 2.1, in which the nested almost convex sets are characterized. First we introduce some definitions.

Throughout this section: \mathcal{X} will denote a set of n points in general position; k will denote the number of convex layers of \mathcal{X} ; R_j will denote the set of points in the j -th convex layer of \mathcal{X} , R_1 being the most internal; and \mathcal{X}_j will denote the set of points in \mathcal{X} , that are in R_j or in the interior of its convex hull.

$T_1(\mathbf{k})$: We define $T_1(k)$ as the complete binary tree with $2^{k+1} - 1$ nodes. The j -level of $T_1(k)$ is defined as the set of the nodes at distance j from the root.

Type 1: We say that \mathcal{X} is of *type 1* if $|R_j| = 2^j$ for $1 \leq j \leq k - 1$. Note that if \mathcal{X} is of Type 1, then for every $1 \leq j \leq k$, the number of points in R_j is

equal to the number of nodes in the j -level of $T_1(k)$.

Type 1 labeling: An injective function $\psi : \mathcal{X} \rightarrow T_1(k)$ is a *type 1 labeling*, if \mathcal{X} is Type 1 and ψ labels the nodes (different to the root) of $T_1(k)$ with different points of \mathcal{X} .

$T_2(k)$: We define $T_2(k)$ as the tree that, its root has three children, and each child is the root of a complete binary tree with $2^{k-1} - 1$ nodes. The j -level of $T_2(k)$ is defined as the set of the nodes at distance $j - 1$ from the root.

Type 2: We say that \mathcal{X} is of *type 2* if $|R_1| = 1$ and $|R_j| = 3 \cdot 2^{j-2}$ for $2 \leq j \leq k$. Note that if \mathcal{X} is of Type 2, then for every $1 \leq j \leq k$, the number of points in R_j is equal to the number of nodes in the j -level of $T_2(k)$.

Type 2 labeling: An injective function $\psi : \mathcal{X} \rightarrow T_2(k)$ is a *Type 2 labeling*, if \mathcal{X} is Type 2 and ψ labels the nodes (also the root) of $T_2(k)$ with different points of \mathcal{X} .

Labeling: Let T be equal to $T_1(k)$ or $T_2(k)$. We say that a map $\psi : \mathcal{X} \rightarrow T$ is a *labeling*, if ψ is a Type 1 labeling or a Type 2 labeling. Note that, if \mathcal{X} admits a labeling then $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$.

In the following, when the map $\psi : \mathcal{X} \rightarrow T$ is clear from the context, we say that a point is the *label* of a node of T if the point is mapped to the node by ψ . This way, given a node u of T , we denote by x_u its label. We denote by $u(l)$ and $u(r)$ the left and right children of u in T , respectively.

Nested: We say that a labeling is *nested* if, for $1 \leq j \leq k$, the left to right order of labels of the nodes in the j -level of T , corresponds to the counterclockwise order of the points in R_j .

Adoptable: Given a point p in R_j and two points q_1, q_2 in R_{j+1} , we say that q_1, q_2 are *adoptable from p* if, for every other point q_3 in R_{j+1} , p is in the interior of the triangle determined by q_1, q_2, q_3 . We say that a nested labeling is *adoptable* if, for every node u in T , $x_{u(l)}$ and $x_{u(r)}$ are adoptable from x_u .

We denote by $R_j(u)$ the set of points in R_j that label a descendant of u . With respect to the counterclockwise order, we denote by: $\text{first}[R_j(u)]$, the first point in $R_j(u)$; $\text{last}[R_j(u)]$, the last point in $R_j(u)$; $\text{previous}[R_j(u)]$, the point in R_j previous to $\text{first}[R_j(u)]$; and $\text{next}[R_j(u)]$, the point in R_j next to $\text{last}[R_j(u)]$. See figure 2.

Well laid: We say that a nested labeling is *well laid* if, for every u in T , x_u is in the intersection of the triangle determined by $\text{previous}[R_k(u)]$, $\text{first}[R_k(u)]$, $\text{last}[R_k(u)]$ and the triangle determined by $\text{first}[R_k(u)]$, $\text{last}[R_k(u)]$, $\text{next}[R_k(u)]$.

Let u be a node of T . We denote by \mathcal{X}_u the set of points x_v such that v is descendant of u in T . We denote by $\overline{\mathcal{X}_u}$ the set $\mathcal{X}_u \cup \{x_u\}$. Given two sets of points A and B , we call any directed line from a point in A to a point in B , an (A, B) -line.

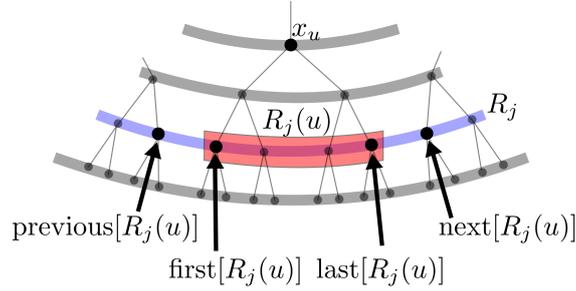


Figure 2: Illustration of $R_j(u)$, $\text{first}[R_j(u)$, $\text{last}[R_j(u)$, $\text{previous}[R_j(u)$, and $\text{next}[R_j(u)$.

Internal separation: We say that a nested labeling is an internal separation if for every node u of T , every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_{u(l)}, \mathcal{X}_{u(r)})$ -line ℓ .

External separation: We say that a nested labeling is an external separation if for every node u of T , every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_{u(l)}, \{x_u\})$ -line and to the left of every $(\{x_u\}, \mathcal{X}_{u(r)})$ -line.

Theorem 2.1. *Let \mathcal{X} be a point set in general position. Then the following statements are equivalent:*

1. \mathcal{X} is a nested almost convex set.
2. \mathcal{X} admits a labeling that is nested, adoptable and well laid.
3. \mathcal{X} admits a labeling that is an internal separation and an external separation.

Proof of Theorem 2.1

The proof of Theorem 2.1 is divided into three parts: first we prove that $1 \implies 2$; afterwards we prove that $2 \implies 3$; and finally we prove that $3 \implies 1$.

$1 \implies 2$

In this part we assume that \mathcal{X} is a nested almost convex set, and we introduce a labeling ψ' that is nested, adoptable and well laid.

It is clear from the definition of labeling that a necessary condition for \mathcal{X} to admit a labeling is that \mathcal{X} must be type 1 or type 2. In the following lemma we prove that, if \mathcal{X} is a nested almost convex, then \mathcal{X} is type 1 or type 2.

Lemma 2.2. *If \mathcal{X} is a nested almost convex set then we have one of the following cases:*

1. $|R_j| = 2^j$ for $1 \leq j \leq k-1$.
2. $|R_1| = 1$ and $|R_j| = 3 \cdot 2^{j-2}$ for $2 \leq j \leq k$.

Proof. Suppose that R_1 has three or more points. In this case, the interior of the convex hull of R_1 has at least one point of \mathcal{X} ; this contradicts that R_1 is the first convex layer of \mathcal{X} . Thus $R_1 = \mathcal{X}_1$, and \mathcal{X}_1 has one or two points. This proves the lemma for $j = 1$.

Any triangulation of R_{j+1} , has $|R_{j+1}| - 2$ triangles and each triangle has exactly one point of \mathcal{X}_j in its interior; thus $|\mathcal{X}_j| = |R_{j+1}| - 2$. In particular, if $|\mathcal{X}_1| = 2$ or $|\mathcal{X}_1| = 1$ then $|\mathcal{X}_2| = 4$ or $|\mathcal{X}_2| = 3$, respectively. This proves the lemma for $j = 2$.

For the other cases, note that

$$|R_{j+1}| = |\mathcal{X}_j| + 2 = |R_j| + |\mathcal{X}_{j-1}| + 2 = 2|R_j|.$$

□

Now we define ψ' on a subset of nodes of T depending on whether \mathcal{X} is of type 1 or type 2.

- If \mathcal{X} is of type 1: ψ' labels the two nodes in the 1-level of $T_1(k)$, with the two points in R_1 .
- If \mathcal{X} is of type 2: ψ' labels the node in the 1-level of $T_2(k)$, with the point in R_1 ; ψ' labels the three nodes in the 1-level of $T_2(k)$, with the three points in R_2 (such that, the left to right order of labels of the nodes in the 2-level of T , coincides to the counterclockwise order of the points in R_2).

To define ψ' on the other nodes of T , we use the following Lemma.

Lemma 2.3. *Let p_0, \dots, p_t be the set of points in R_j in counterclockwise order. Then, the points in R_{j+1} can be listed in counterclockwise order as $q_0, q_1, \dots, q_{2t+1}$, where the points q_{2i}, q_{2i+1} are adoptable from p_i for $0 \leq i \leq t$.*

Proof. Let \mathcal{T} be the set of triangles determined by three consecutive points of R_{j+1} in counterclockwise order. We first show that:

Claim 2.3.1. *Each point of R_j is in exactly two consecutive triangles of \mathcal{T} .*

Assume that $j \geq 2$ (and note that Claim 2.3.1 holds for $j = 1$). Let Δ be the interior of a triangle of \mathcal{T} . By the almost convex set definition, there is one point of \mathcal{X}_j in Δ . This point must be in R_j , since the convex hull of R_{j+1} without Δ (and its boundary) is convex. Thus, there is one point of R_j in the interior of each triangle of \mathcal{T} . As the triangles of \mathcal{T} are defined by consecutive points of R_{j+1} , each point of R_j is in at most two triangles of \mathcal{T} . Thereby Claim 2.3.1 follows from $|\mathcal{T}| = |R_{j+1}| = 2|R_j|$.

The two triangles of \mathcal{T} that contain p_0 , are defined by four consecutive points of R_{j+1} ; let q_0 be the second of these points. Let $q_0, q_1, \dots, q_{2t+1}$ be the points of R_{j+1} in counterclockwise order. Note that, for each p_i , the middle two points of the four points that define the two triangles that contain p_i , are q_{2i} and q_{2i+1} . Thus q_{2i} and q_{2i+1} are adoptable from p_i .

□

Now we define ψ' on the other nodes of T recursively. For each labeled node u , ψ' labels $u(l)$ and $u(r)$ with the two points adoptable from the label of u . We do this so that, the left to right order of the labels of the nodes in the $(j+1)$ -level of T , correspond to the counterclockwise order of the points in R_{j+1} . Note that

ψ' is nested and adoptable. It remains to prove that ψ' is well laid. We prove this in Lemma 2.5.

Lemma 2.4. *If u is a node of T , the label of every descendant of u is contained in the convex hull of $R_k(u)$.*

Proof. We claim that every set $R_{j-1}(u)$, with at least two points, is contained in the convex hull of $R_j(u)$. Let p be a point in $R_{j-1}(u)$ and let q and q' be the labels of the children of the node labeled by p . By construction of ψ' , q and q' are adoptable from p . As $R_{j-1}(u)$ has at least two points, $R_j(u)$ has at least four points. Let Δ be a triangle determined by q , q' and another point of $R_j(u)$. By definition of adoptable, p is in the interior of Δ and in consequence in the interior of the convex hull of R_j . An inductive application of the previous claim proves this lemma. \square

Lemma 2.5. *Let u be a node of T . Then x_u is in the intersection of the triangle determined by $\text{previous}[R_k(u)]$, $\text{first}[R_k(u)]$ and $\text{last}[R_k(u)]$ and the triangle determined by $\text{first}[R_k(u)]$, $\text{last}[R_k(u)]$ and $\text{next}[R_k(u)]$.*

Proof. Let j be the index such that the j -level of T contains u . Let R'_k be the set that contains $\text{first}(R_k(v))$ and $\text{last}(R_k(v))$ for all nodes v in the j -level of T . Let \mathcal{T} be the set of triangles determined by three consecutive points of R'_k in counterclockwise order. We first show the following claim.

Claim 2.5.1. *Each point of R_j is in exactly two consecutive triangles of \mathcal{T} .*

Note that every point of $\mathcal{X} \setminus \mathcal{X}_j$, is the label of some descendant of a node v in the j -level of T . Thus, by Lemma 2.4, every point of $\mathcal{X} \setminus \mathcal{X}_j$ is in the convex hull of $R_k(v)$ for some node v in the j -level of T . Let \mathcal{A} be the region obtained from the convex hull of \mathcal{X} , by removing the convex hull of $R_k(v)$ for each v in the j -level of T . Note that the set of points of \mathcal{X} that are in \mathcal{A} is \mathcal{X}_j .

Let Δ be the interior of a triangle of \mathcal{T} . By the nested almost convex set definition, there is one point of \mathcal{X} in Δ . As Δ is contained in \mathcal{A} , this point must be in \mathcal{X}_j . This point must also be in R_j , since \mathcal{A} without Δ (and its boundary) is convex. Thus, there is one point of R_j in the interior of each triangle of \mathcal{T} . As the triangles of \mathcal{T} are defined by consecutive points of R'_k , each point of R_j is in at most two triangles of \mathcal{T} . Thereby Claim 2.5.1 follows from $|\mathcal{T}| = |R'_k| = 2|R_j|$.

Let Δ' be the intersection of the triangle determined by $\text{previous}[R_{j+1}(u)]$, $\text{first}[R_{j+1}(u)]$ and $\text{last}[R_{j+1}(u)]$, with the triangle determined by $\text{first}[R_{j+1}(u)]$, $\text{last}[R_{j+1}(u)]$ and $\text{next}[R_{j+1}(u)]$. Note that $\text{first}[R_{j+1}(u)]$ and $\text{last}[R_{j+1}(u)]$ are the labels of the children of u . By definition of ψ' , x_u is in the interior of every triangle determined by $\text{first}[R_{j+1}(u)]$, $\text{last}[R_{j+1}(u)]$ and every other point of R_{j+1} ; thus x_u is in Δ' . By Claim 2.5.1, x_u is in the interior of two triangles of \mathcal{T} , but there are only two triangles of \mathcal{T} that intersect Δ' ; these are the triangles determined by $\text{previous}[R_k(u)]$, $\text{first}[R_k(u)]$ and $\text{last}[R_k(u)]$, and the triangle determined by $\text{first}[R_k(u)]$, $\text{last}[R_k(u)]$, $\text{next}[R_k(u)]$. \square

2 \implies 3

In this part we assume that there is a labeling ψ' of \mathcal{X} that is nested, adoptable and well laid; and we prove that ψ' is an internal separation and an external separation.

Lemma 2.6. ψ' is an internal separation.

Proof. Let u be a node of T and recall that $u(l)$, $u(r)$ are the left and right children of u , respectively. We need to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line.

Let ℓ be the directed segment from $\text{first}[R_k(u(l))]$ to $\text{last}[R_k(u(r))]$. By Lemma 2.5, each point in $\mathcal{X}/\mathcal{X}_u$ is in the interior of a triangle whose vertices are to the left of, or on ℓ ; thus every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of ℓ . By Lemma 2.4, every point in $\overline{\mathcal{X}_{u(l)}} \cup \overline{\mathcal{X}_{u(r)}}$ is to the right of ℓ . We claim that:

Claim 2.6.1. No $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line intersects ℓ .

As the end points of ℓ , $\text{first}[R_k(u(l))]$ and $\text{last}[R_k(u(r))]$, are in the boundary of the convex hull of \mathcal{X} ; to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line, it is enough to show Claim 2.6.1.

Let P_1 be the polygonal chain that starts at $q_1 := \text{first}[R_k(u(l))]$, follows the points of $R_k(u(l))$ in counterclockwise order, and ends at $q_2 := \text{last}[R_k(u(l))]$. Similarly, let P_2 be the polygonal chain that starts at $q_3 := \text{first}[R_k(u(r))]$, follows the points of $R_k(u(r))$ in counterclockwise order, and ends at $q_4 := \text{last}[R_k(u(r))]$. To prove Claim 2.6.1 it is enough to show that every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line intersects both P_1 and P_2 .

Let q be the intersection point of the diagonals of the quadrilateral defined by q_1 , q_2 , q_3 and q_4 . By Lemma 2.4 and Lemma 2.5, $\overline{\mathcal{X}_{u(l)}}$ is contained in the convex hull of $P_1 \cup \{q\}$, and $\overline{\mathcal{X}_{u(r)}}$ is contained in the convex hull of $P_2 \cup \{q\}$. Let ℓ' be an $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line. Note that the slope of ℓ' , is in the range from the slope of the line define by q_1 and q_3 , to the slope of the line define by q_2 and q_4 , in counterclockwise order. Thus ℓ' intersects both P_1 and P_2 . □

Lemma 2.7. ψ' is an external separation.

Proof. Let u be a node of T . We need to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$ -line and to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$ -line. We prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$ -line. That every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$ -line can be proven in a similar way.

Let P be the polygonal chain that starts at $\text{next}[R_k(u)]$, follows the points of R_k in counterclockwise order, and ends at $\text{previous}[R_k(u)]$. Note that, by Lemma 2.5, $\mathcal{X}/\overline{\mathcal{X}_u}$ is contained in the convex hull of P . Thus, to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$ -line, it is enough to show that x_u is to the right of the directed line from $\text{last}[R_k(u(l))]$ to $\text{next}[R_k(u)]$. See Figure 3.

Let j be the index such that the j -level of T contains u . For $j < i \leq k$, let ℓ_i be the directed line from $\text{last}[R_i(u(l))]$ to $\text{next}[R_i(u)]$. We show that x_u is to the right of ℓ_i by induction. As $x_{u(l)}$ and $x_{u(r)}$ are adoptable from x_u , and $x_{u(l)} = \text{last}[R_{j+1}(u(l))]$; x_u is in the interior of the triangle determined by $\text{last}[R_{j+1}(u(l))]$, $x_{u(r)}$ and $\text{next}[R_{j+1}(u)]$. Thus the induction holds for $i = j + 1$. Suppose that x_u is to the right of ℓ_i . Let $\text{last}[R_{i+1}(u(l))]$ and p be the two children of $\text{last}[R_i(u(l))]$. Let $\text{next}[R_{i+1}(u)]$ and q be the two children of $\text{next}[R_i(u)]$. Let \square be the quadrilateral determined by $\text{last}[R_{i+1}(u(l))]$, p , q and $\text{next}[R_{i+1}(u)]$. As $\text{last}[R_{i+1}(u(l))]$, p , q and $\text{next}[R_{i+1}(u)]$ are in R_{i+1} , and any

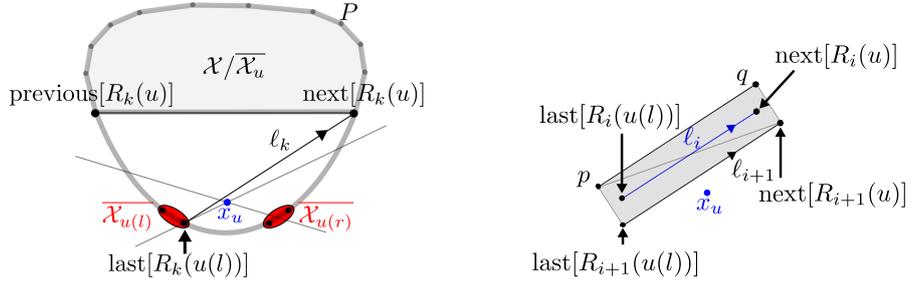


Figure 3: Illustration of the proof of Lemma 2.7

triangulation of \square has two triangles; there are two points of \mathcal{X}_i in \square . As those points are $\text{last}[R_i(u(l))]$ and $\text{next}[R_i(u)]$, x_u is not in the interior of \square . Thus x_u is not between ℓ_i and ℓ_{i+1} , and therefore x_u is to the right of ℓ_{i+1} . \square

3 \implies 1

In this part we finish the proof of Theorem 2.1. We assume that there is a labeling ψ' of \mathcal{X} that is an internal separation and an external separation, and we prove that \mathcal{X} is a nested almost convex set. For this it is enough to prove Lemma 2.8. As consequence of Lemma 2.8 and Theorem 2.1, Theorem 1.2 holds.

Lemma 2.8. *Let \mathcal{X} be an n -point set that admits a labeling $\psi : \mathcal{X} \rightarrow T$ that is an internal separation and an external separation. Then the order type of \mathcal{X} is determined by T and:*

- If $n = 2^{k-1} - 2$, then \mathcal{X} has the same order type than any n -point set obtained from Construction 1.
- If $n = 3 \cdot 2^{k-1} - 2$, then \mathcal{X} has the same order type than any n -point set obtained from Construction 2.

Proof. The labeling that \mathcal{X} admits can be a type 1 labeling or a type 2 labeling. If \mathcal{X} admits a type 1 labeling, $|\mathcal{X}| = 2^{k+1} - 2$ for some integer k ; in this case, an almost convex set with the same cardinality than \mathcal{X} can be obtained using Construction 1. If \mathcal{X} admits a type 2 labeling, $|\mathcal{X}| = 3 \cdot 2^{k-1} - 2$ for some integer k ; in this case, an almost convex set with the same cardinality than \mathcal{X} can be obtained using Construction 2. Let \mathcal{Y} be an almost convex set with $|\mathcal{X}|$ points obtained from Construction 1 or Construction 2. We prove that \mathcal{X} and \mathcal{Y} have the same order type, and that this order type is determined by T .

Assume that \mathcal{X} admits a type 1 labeling. The case when \mathcal{X} admits a type 2 labeling can be proven in a similar way. As \mathcal{Y} is an almost convex set, \mathcal{Y} admits a labeling that is an internal separation and an external separation. Let $\psi_Y : \mathcal{Y} \rightarrow T$ be such type 1 labeling.

Let $f := \psi_Y^{-1}(\psi')$. We prove that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a bijection that preserves the orientation of all triplets. Let x_1, x_2, x_3 be different points in \mathcal{X} , let u_1, u_2, u_3 be the nodes of T that x_1, x_2, x_3 label in ψ' , and let y_1, y_2, y_3 be the labels of u_1, u_2, u_3 in ψ_Y . Note that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(x_3) = y_3$. To prove

that (x_1, x_2, x_3) and (y_1, y_2, y_3) have the same orientation, we show that the position of u_1, u_2 and u_3 in T determines the orientation of any labeling of u_1, u_2 and u_3 .

Given a node w of T , denote by T_w the subtree of T that contains every descendant of w . Let w be the farthest node from the root of T , such that at least two of u_1, u_2, u_3 are in T_w . If two of u_1, u_2, u_3 are in the left subtree of T_w or, two of u_1, u_2, u_3 are in the right subtree of T_w ; the orientation of the labels of u_1, u_2, u_3 is determined by an external separation. If there are not two of u_1, u_2, u_3 in the left subtree of T_w or in the right subtree of T_w ; there is one of u_1, u_2, u_3 in the left subtree of T_w , one u_1, u_2, u_3 in the right subtree of T_w , and the other one is not in the left or right subtree of T_w . In this case, the orientation of the labels of u_1, u_2, u_3 is determined by an internal separation. \square

3 Drawings of Nested Almost Convex Sets with Small Size.

Let \mathcal{X}' be a nested almost convex set with n points, and let k be the number of convex layers of \mathcal{X}' . In this section we construct a drawing of \mathcal{X}' of size $O(n^{\log_2 5})$. This section is divided into three parts. First, we construct a $2^{k+1} - 2$ point set \mathcal{X} with integer coordinates and size $2 \cdot 5^{k+1}$. Afterwards, we prove that \mathcal{X} is a nested almost convex set. Finally, we obtain a subset of \mathcal{X} that is a drawing of \mathcal{X}' .

Construction of \mathcal{X} .

Recall that $T_1(k)$ is the complete binary tree with $2^{k+1} - 1$ nodes, and the j -level of $T_1(k)$ is the set of nodes at distance j from the root of $T_1(k)$. Before defining \mathcal{X} , we will construct a point set \mathcal{Y} in convex position, and for each node u in $T_1(k)$, we will define a set $\mathcal{Y}_u \subset \mathcal{Y}$ of consecutive points of \mathcal{Y} in counterclockwise order. The point x_u will denote the midpoint between the first and last points of \mathcal{Y}_u in counterclockwise order. The set \mathcal{X} will be the set of points x_u such that u is a node of $T_1(k)$ different from the root.

Let p, o and q be points in the plane and let $c \in [0, 1]$. We denote by \overline{op} and \overline{oq} the segments from o to p and from o to q , respectively. We say that $\alpha = (q, o, p)$ is a *corner*, if the angle from \overline{op} to \overline{oq} counterclockwise is less than π . Let $\alpha := (q, o, p)$ be a corner. We denote by $\text{LeftPoint}(\alpha, c)$ the point in the segment \overline{oq} at distance $c|\overline{oq}|$ from o . We denote by $\text{RightPoint}(\alpha, c)$ the point in the segment \overline{op} at distance $c|\overline{op}|$ from o . See Figure 4.

Recursively, we define a corner α_u for each node u of $T_1(k)$. The corner of the root of $T_1(k)$ is defined as $((0, 2 \cdot 5^{k+1}), (0, 0), (2 \cdot 5^{k+1}, 0))$. Let u be a node for which its corner α_u has been defined; the corners of its left and right children, $u(l)$ and $u(r)$, are defined as follows (See Figure 4):

$$\begin{aligned} \alpha_{u(l)} &= (\text{LeftPoint}(\alpha_u, 2/5), \text{LeftPoint}(\alpha_u, 1/5), \text{RightPoint}(\alpha_u, 1/5)) \\ \alpha_{u(r)} &= (\text{LeftPoint}(\alpha_u, 1/5), \text{RightPoint}(\alpha_u, 1/5), \text{RightPoint}(\alpha_u, 2/5)) \end{aligned}$$

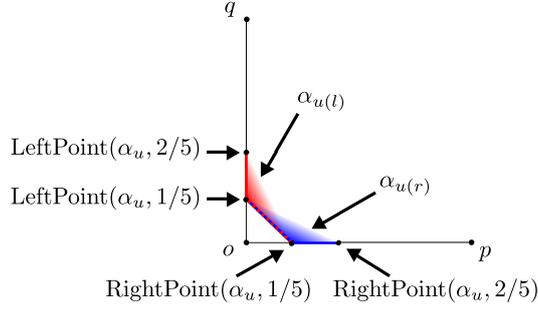


Figure 4: Illustration of corners α_u , $\alpha_{u(l)}$ and $\alpha_{u(r)}$, where $\alpha_u = (q, o, p)$.

Let v be a leaf of $T_1(k+1)$. Note that v is a child of a leaf u of $T_1(k)$. If v is the left child of u , let $y_v := \text{LeftPoint}(\alpha_u, 1/5)$. If v is the right child of u , let $y_v := \text{RightPoint}(\alpha_u, 1/5)$. We define \mathcal{Y} as the set of points y_v such that v is a leaf of $T_1(k+1)$. Given a node u of $T_1(k)$, we define \mathcal{Y}_u as the set of points y_v such that v is a descendant of u , and v is a leaf of $T_1(k+1)$. With respect to the counterclockwise order, we denote by: $\text{first}[\mathcal{Y}_u]$, the first point in \mathcal{Y}_u ; $\text{last}[\mathcal{Y}_u]$, the last point in \mathcal{Y}_u ; $\text{previous}[\mathcal{Y}_u]$, the point in \mathcal{Y}_u previous to $\text{first}[\mathcal{Y}_u]$; and $\text{next}[\mathcal{Y}_u]$, the point in \mathcal{Y}_u next to $\text{last}[\mathcal{Y}_u]$.

Lemma 3.1. *Let u be a node of $T_1(k)$. Let v_1, v_2, \dots, v_t be the leaves of $T_1(k+1)$, that are descendant of u , ordered from left to right. Then $y_{v_1}, y_{v_2}, \dots, y_{v_t}$ are in convex position, and are the points in \mathcal{Y}_u in counterclockwise order.*

Proof. Let $(q, o, p) := \alpha_u$; $q' := \text{LeftPoint}(\alpha_u, 2/5)$; and $p' := \text{RightPoint}(\alpha_u, 2/5)$. Let $\Delta(u)$ be the triangle determined by q' , o and p' . inductively from the leaves to the root of $T_1(k)$, it can be proven that:

1. The set of points of \mathcal{Y} in $\Delta(u)$ is \mathcal{Y}_u ; from which: $\text{first}[\mathcal{Y}_u]$ is on the segment from o to q' , $\text{last}[\mathcal{Y}_u]$ is on the segment from o to p' , and the other points are in the interior of $\Delta(u)$.
2. The points $q', y_{v_1}, y_{v_2}, \dots, y_{v_t}, p'$ are in convex position, and appear in this order counterclockwise.

This proof follows from 2. □

By Lemma 3.1, \mathcal{Y} is in convex position, and for each node u in $T_1(k)$, \mathcal{Y}_u is a subset of consecutive points of \mathcal{Y} in counterclockwise order. We denote by x_u the midpoint between $\text{first}[\mathcal{Y}_u]$ and $\text{last}[\mathcal{Y}_u]$. Let \mathcal{X} be the set of points x_u such that u is a node of $T_1(k)$ different from the root.

Let u be a node of $T_1(k)$ at distance j from the root, let $(q, o, p) := \alpha_u$ and let v be a leaf of $T_1(k+1)$. Recursively note that, the coordinates of q , o and p are divisible by $2 \cdot 5^{k+1-j}$. Thus, the coordinates of y_v are divisible by 2, x_u has integer coordinates, and \mathcal{X} has size $2 \cdot 5^{k+1}$.

\mathcal{X} is a nested Almost Convex Set.

In this subsection we prove that \mathcal{X} is a nested almost convex set. By Theorem 2.1, it is enough to prove that \mathcal{X} admits a labeling that is an internal

separation and an external separation. Let $\psi : \mathcal{X} \rightarrow T_1(k)$ be the type 1 labeling that labels each node u of $T_1(k)$ different from the root, with x_u . We prove that ψ is both an internal separation and an external separation.

Lemma 3.2. *If u is a node of $T_1(k)$ at distance j from the root, then $\text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u, c_j)$ and $\text{last}[\mathcal{Y}_u] = \text{RightPoint}(\alpha_u, c_j)$, where*

$$c_j = \frac{1}{4} \left(1 - 5^{(j-k-1)} \right).$$

Proof. Note that

$$c_j = \sum_{i=k}^j \left(\frac{1}{5} \right)^{k+1-j}.$$

If $j = k$, then: u is a leaf of $T_1(k)$; $c_j = 1/5$; and $\text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u, c_j)$ and $\text{last}[\mathcal{Y}_u] = \text{RightPoint}(\alpha_u, c_j)$. Suppose that $j < k$, and that this lemma holds for larger values of j . Let $u(l)$ and $u(r)$ be the left and right children of u . Note that by induction,

$$\text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_{u(l)}, c_{j+1}) = \text{LeftPoint}(\alpha_u, c^*)$$

where $c^* = (1/5)c_{j+1} + 1/5 = c_j$; thus $\text{first}[\mathcal{Y}_u] := \text{LeftPoint}(\alpha_u, c_j)$. In a similar way $\text{last}[\mathcal{Y}_u] := \text{RightPoint}(\alpha_u, c_j)$. \square

Lemma 3.3. *ψ is an internal separation.*

Proof. Let u be a node of $T_1(k)$ different from the root, and let $u(l)$, $u(r)$ be the left and right children of u , respectively. We need to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line.

Let ℓ be the directed segment from $\text{first}[\mathcal{Y}_{u(l)}]$ to $\text{last}[\mathcal{Y}_{u(r)}]$. As each point in $\mathcal{X}/\mathcal{X}_u$ is the midpoint between two points that are not to the right of ℓ , every point in $\mathcal{X}/\mathcal{X}_u$ is not to the right of ℓ . As every point in $\overline{\mathcal{X}_{u(l)}} \cup \overline{\mathcal{X}_{u(r)}}$ is the midpoint between a point to the right of ℓ and a point that is not to the left of ℓ , every point in $\overline{\mathcal{X}_{u(l)}} \cup \overline{\mathcal{X}_{u(r)}}$ is to the right of ℓ . We claim that:

Claim 3.3.1. *No $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line intersects ℓ .*

As the endpoints of ℓ , $\text{first}[\mathcal{Y}_{u(l)}]$ and $\text{last}[\mathcal{Y}_{u(r)}]$, are in the boundary of the convex hull of \mathcal{Y} ; to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line, it is enough to show Claim 3.3.1.

Let P_1 be the polygonal chain that starts at $\text{first}[\mathcal{Y}_{u(l)}]$, follows the points of $\mathcal{Y}_{u(l)}$ in counterclockwise order, and ends at $\text{last}[\mathcal{Y}_{u(l)}]$. Similarly, let P_2 be the polygonal chain that starts at $\text{first}[\mathcal{Y}_{u(r)}]$, follows the points of $\mathcal{Y}_{u(r)}$ in counterclockwise order, and ends at $\text{last}[\mathcal{Y}_{u(r)}]$. To prove Claim 3.3.1 it is enough to show that every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$ -line intersects P_1 and P_2 . This follows from the fact that $\overline{\mathcal{X}_{u(l)}}$ is contained in the convex hull of P_1 , and $\overline{\mathcal{X}_{u(r)}}$ is contained in the convex hull of P_2 . \square

Lemma 3.4. *Let u be a node of $T_1(k)$ at distance j from the root, and let $(q, o, p) := \alpha_u$. Suppose that the nodes in the j -level of $T_1(k)$, are ordered from left to right.*

1. *If u is not the first node, then the points o , $\text{first}[\mathcal{Y}_u]$, $\text{previous}[\mathcal{Y}_u]$ and q are collinear, and $\text{previous}[\mathcal{Y}_u] = \text{LeftPoint}(u, c)$ for some $c > 3/5$.*

2. If u is not the last node, then the points o , $\text{last}[\mathcal{Y}_u]$, $\text{next}[\mathcal{Y}_u]$ and p are collinear, and $\text{next}[\mathcal{Y}_u] = \text{RightPoint}(u, c)$ for some $c > 3/5$.

Proof. To prove 1 and 2, note that, for any two consecutive nodes in the j -level of $T_1(k)$, there is a segment that contains one side of each the corners corresponding to these nodes; then apply Lemma 3.2. \square

Lemma 3.5. ψ is an external separation.

Proof. Let u be a node of $T_1(k)$ and $u(l)$, $u(r)$ be the left and right children of u , respectively. We need to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$ -line and to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$ -line. We prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$ -line. That every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$ -line can be proven in a similar way.

Let P be the polygonal chain that starts at $\text{next}[\mathcal{Y}_u]$, follows the points of \mathcal{Y} in counterclockwise order, and ends at $\text{previous}[\mathcal{Y}_u]$. Note that $\mathcal{X}/\overline{\mathcal{X}_u}$ is contained in the convex hull of P . Thus, to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$ -line, it is enough to show that $\text{next}[\mathcal{Y}_u]$ is to the left of the directed line from $\text{last}[\mathcal{Y}_{u(l)}]$ to x_u .

Let ℓ be the directed line from $\text{last}[\mathcal{Y}_{u(l)}]$ to x_u and let $(q, o, p) := \alpha_u$. Note that x_u and $\text{last}[\mathcal{Y}_{u(l)}]$ are in the interior of the wedge determined by α_u , from \overline{op} to \overline{oq} in counterclockwise order. By Lemma 3.4-2, $\text{next}[\mathcal{Y}_u]$ is on \overline{op} and $\text{next}[\mathcal{Y}_u] = \text{RightPoint}(u, c)$ for some $c > 3/5$. To finish this proof we show that ℓ intersects \overline{op} at a point $\text{RightPoint}(u, c')$ for some $c' < 3/5$.

Consider the following coordinate system, o is the origin, p has coordinates $(1, 0)$ and q has coordinates $(0, 1)$. Assume that this is the new coordinate system. Let t be such that the intersection point between ℓ and the abscissa is the point $(t, 0)$; thereby, we need to prove that $t < 3/5$.

By Lemma 3.2, $\text{first}[\mathcal{Y}_u]$ and $\text{last}[\mathcal{Y}_u]$ have coordinates $(0, c_j)$ and $(c_j, 0)$; thus, x_u has coordinates $(c_j/2, c_j/2)$. By construction of $\alpha_{u(l)}$ and Lemma 3.2, $\text{last}[\mathcal{Y}_{u(l)}]$ is in the segment from $(0, 1/5)$ to $(1/5, 0)$ in $\text{RightPoint}(u(l), c_{j+1})$. Thus $\text{last}[\mathcal{Y}_{u(l)}]$ has coordinates $(\frac{1}{5}c_{j+1}, \frac{1}{5}(1 - c_{j+1}))$ and the equation of ℓ is

$$x = \frac{c_{j+1}/5 - c_j/2}{(1 - c_{j+1})/5 - c_j/2}(y - c_j/2) + c_j/2$$

taking $y = 0$, $s = k - j$, and replacing c_j and c_{j+1} , we have that

$$t = -\frac{1}{40 \cdot 5^s} - \frac{1}{40(1 + 3/5^s)} - \frac{1}{40(3 \cdot 5^s + 5^{2s})} + \frac{3}{8(3/5^s + 1)} + \frac{1}{8(3 + 5^s)} + \frac{1}{8}$$

finally, as $5^s \geq 1$

$$t < \frac{3}{8} + \frac{1}{8(4)} + \frac{1}{8} = \frac{17}{32} < \frac{3}{5}.$$

\square

Construction of a Drawing of \mathcal{X} .

In this subsection we find a subset of \mathcal{X} that is a drawing of \mathcal{X}' . By Theorem 1.2, there are two cases: \mathcal{X}' is of type 1 and has $n = 2^{k+1} - 2$ points; or \mathcal{X}' is of type 2 and has $n = 3 \cdot 2^{k-1} - 2$ points. By Theorem 1.2, if \mathcal{X}' is type 1, \mathcal{X}' and \mathcal{X} have the same order type and \mathcal{X} is a drawing of \mathcal{X}' . Assume that \mathcal{X}' is type 2.

Let w be the root of $T_1(k)$; u and u' be the children of w ; $u(l)$ and $u(r)$ be the children of u ; and $u'(l)$ and $u'(r)$ be the children of u' . We define T as the tree obtained from $T_1(k)$, by making $u'(l)$ the third child of u and removing w , u' , $u'(r)$ and every descendant of $u'(r)$. Recall that $T_2(k)$ is a tree such that, its root has three children, and each child is the root of a complete binary tree with $2^{k-1} - 1$ points. Note that T and $T_2(k)$ are isomorphic.

Let \mathcal{X}_2 be the set of points x_u such that u is in T . Let $\psi' : \mathcal{X}_2 \rightarrow T$ be such that $\psi'(x_u) = u$. Note that: as ψ is an internal separation, ψ' is an internal separation; and as ψ is an external separation, ψ' is external separation. Thus by Theorem 2.1, \mathcal{X}_2 is a nested almost convex set.

By Theorem 1.2, as \mathcal{X}_2 has $3 \cdot 2^{k-1} - 2$ points, \mathcal{X}_2 and \mathcal{X}' have the same order type and \mathcal{X}_2 is a drawing of \mathcal{X}' .

4 Decision Algorithm for Nested Almost Convexity.

Let \mathcal{X} be a set of n points. In this section, we present an $O(n \log n)$ time algorithm, to decide whether \mathcal{X} is a nested almost convex set. This algorithm is based in Theorem 2.1-2 and consists of four steps. At each step, it is verified if \mathcal{X} satisfies a certain property; \mathcal{X} is a nested almost convex set if and only if \mathcal{X} satisfies each of these properties.

By Theorem 1.2, if \mathcal{X} is a nested almost convex set, then $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$ for some integer k . The first step is to verify whether \mathcal{X} has one of those cardinalities. If $n = 2^{k-1} - 2$ let $T := T_1(k)$. If $n = 3 \cdot 2^{k-1} - 2$ let $T := T_2(k)$. Recall that: the j -level of $T_1(k)$ is defined as the set of the nodes at distance j from the root; and the j -level of $T_2(k)$ is defined as the set of the nodes at distance $j - 1$ from the root.

By Lemma 2.2, if \mathcal{X} is a nested almost convex set then: for $1 \leq j \leq k$, the number of nodes in the j -level of T is equal to the number of nodes in the j -th convex layer of \mathcal{X} . The second step is to verify whether \mathcal{X} satisfies Lemma 2.2. Chazelle [8] showed that, the convex layers of a given an n -point set can be found in $O(n \log n)$ time; thus the second step can be done in $O(n \log n)$ time. We denote by R_j the set of points in the j -th convex layer of \mathcal{X} .

The third step is to verify whether \mathcal{X} satisfies Lemma 2.3. For $1 \leq j \leq k - 1$, we do the following. Let p_0, \dots, p_t be the points in R_j in counterclockwise order. We search for two consecutive points in R_{j+1} that are adoptable by p_0 . If those points exist, they are the only pair of consecutive points in R_{j+1} that are adoptable by p_0 . Let $q_0, q_1, \dots, q_{2t+1}$ be the points in R_{j+1} in counterclockwise order, such that q_0 and q_1 are adoptable by p_0 . Then we verify whether q_{2i}, q_{2i+1} are adoptable by p_i for $0 \leq i \leq t$.

Let p be in R_j , and let $q_r, q_{r+1}, q_{r+2}, q_{r+3}$ be four consecutive points in R_{j+1} . Note that q_{r+1} and q_{r+2} are adoptable by p , if and only if, p is in the intersection of the triangle determined by q_r, q_{r+1} and q_{r+2} , and the triangle determined by q_{r+1}, q_{r+2} and q_{r+3} . Thus, we can verify whether q_{2i}, q_{2i+1} are adoptable by p_i in constant time; the third step hence requires linear time.

If \mathcal{X} satisfies Lemma 2.3, we can define a labeling $\psi : \mathcal{X} \rightarrow T$ like the one defined in Section 2-2. The fourth step is to verify if ψ is well laid, this requires linear time.

According to the proof of Theorem 2.1, \mathcal{X} is a nested almost convex set if and only if \mathcal{X} verifies the properties in previous four steps. This can be done in $O(n \log n)$ time.

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