On the Number of Order Types in Integer Grids of Small Size

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Abstract

Let $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ be two sets of n labeled points in general position in the plane. We say that these two point sets have the same order type if for every triple of indices (i, j, k), p_k is above the directed line from p_i to p_j if and only if q_k is above the directed line from q_i to q_j . In this paper we give the first non-trivial lower bounds on the number of different order types of n points that can be realized in integer grids of polynomial size.

1 Introduction

Let A and B be two arrays of n distinct numbers. We say that A and B have the same order type if for every pair i, j of different indices we have that A[i] < A[j] if and only if B[i] < B[j]. Goodman and Pollack [5] introduced a higher dimensional analogue of this idea. Let $S := \{p_1, \ldots, p_n\}$ be a set of n labeled points in general position in the plane. The relationship that A[i] < A[j] is equivalent to A[i] being the left of A[j] in the real line. This left-right relationship can be generalized to point sets as follows. For a given triple (i, j, k) of distinct indices, p_k may be above or below the directed line from p_i to p_j . Two sets of n labeled points in the plane have the same order type if they have these same above-below relationships.¹ In dimension d > 2 this is generalized by considering all

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¹In the literature, it is more common to consider two points sets as having the same order type if there is a bijection between them that preserves these above-below relationships. In this paper we only consider *labeled* order types; thus, a relabeling of S usually produces a different order type.

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(d+1) tuples of points of a given labeled point set in \mathbb{R}^d . The first *d* points of the tuple define an oriented hyperplane and the relationship is whether the last point is above or below this hyperplane. The order type is defined also for point sets not in general position. In this case the last point can be below, above or on the corresponding hyperplane.

Suppose that for each A[i] we are given the number of elements of A that are to the left of A[i], from this information alone we can sort A and recover the left-right relationships mentioned above. Remarkably, this also holds for higher dimensions. For every pair of indices i and j, let $\lambda(i, j)$ be the number of elements of S above the directed line from i to j. The λ -matrix of S is the $n \times n$ matrix whose (i, j) entry is equal to $\lambda(i, j)$. Goodman and Pollack [5] showed that from the λ -matrix of a point set one can recover the above-below relationships of its triples. This also holds in dimension d > 2: if for a given n-point set in \mathbb{R}^d , one is given the number of points above which oriented hyperplanes.

The λ -matrix of set of n points in the plane can be codified with $O(n^2 \log n)$ bits. This implies that if f(n) is the number of different possible order types of a set of n points in general position in the plane then $f(n) \leq \exp(O(n^2 \log n))$. Goodman and Pollack [6] showed that this bound is far away from the real value of f(n). They showed that

$$f(n) \le \exp(4(1 + O(1/\log n))n\log n).$$

To lower bound f(n), consider the following procedure (see [6]). Suppose that we want to extend S to an (n+1) point configuration by adding a point p_{n+1} to S. Consider the line arrangement spanned by all the straight lines passing through a pair of points in S. It was proved by Zaslavsky [10] that this line arrangement has

$$\binom{\binom{n}{2}}{2} + \binom{\binom{n}{2}}{1} + 1 - n\binom{n-2}{2} \ge \frac{1}{8}n^4,$$

cells. Adding p_{n+1} in different cells of the arrangement produces point sets with different order types. We may use this argument by starting from $\{p_1, p_2, p_3\}$ and iteratively adding the remaining points; at each step we consider the number of different options that produce different order types. This yields

$$f(n) \ge \prod_{i=1}^{n} \frac{1}{8}i^4 = \frac{n!^4}{8^n} = \exp(4(1 + O(1/\log n))n\log n),$$

were the last term is obtained by using Stirling's formula.

The order type of a point configuration abstracts the convexity relationships between its subsets. As a result, for various questions regarding point sets, two point sets having the same order type are equivalent. However, an arbitrary assignment of "above" or "below" relationships to triples of indices in $\{1, \ldots, n\}$ might not be realizable as the order type of a labeled set of n points.

Aichholzer, Aurenhammer and Krasser [1] have produced a database with a point set for each realizable order type of at most 10 points. Although it is a relatively small value of n, this database has proven to be very useful. Chazelle asked in 1987 (see [7]): what is the number of bits needed to store a representative of any given realizable order type of n points? Equivalently, what is the minimum size of an integer grid, so that it contains a representative of every realizable order type of n points? Goodman, Pollack and Sturmfels [7, 8] showed that there are order types of n points whose every realization with positive integer coordinates has a coordinate of size greater than $2^{2^{c_1n}}$,



Figure 1: Q_{13}

for some positive constant c_1 ; they also showed that every order type of n points can be realized with positive integer coordinates of size at most $2^{2^{c_2n}}$, for some positive constant c_2 . In the book "Research Problems in Discrete Geometry" [3] by Brass, Moser and Pach, we find the following problem.

Problem 1. For a given constant $\alpha > 0$, what is the number of order types of n points that can be represented by integer coordinates smaller than n^{α} ?

In this paper we show the first non trivial lower bounds for Problem 1. Let $g(n, \alpha)$ be the number of different order types realizable in an integer grid of size n^{α} .

For starters one may ask what is the smallest integer grid in which at least one order type is realizable. This is equivalent to ask what is the size of the minimum integer grid so that it contains a set of n points such that no three of them are collinear. This is known as the *no-three-in-a-line* problem and was introduced by Dudeny [4] in 1917.

Erdős showed (see [9]) that if p is a prime then the set

$$Q_p := \{(i, i^2 \mod p) : 0 \le i < p\}$$

is in general position. This point set is shown in Figure 1 for p = 13. Therefore, at least one order type can be realized in integer grids of linear size.

Suppose that n^{α} is such that at least one order type of n points can be realized in an $n^{\alpha} \times n^{\alpha}$ integer grid. Any permutation of the labels of a point set that preserves the order type must preserve the clockwise cyclic order of the points in the convex hull. Moreover, for every point $p \in S$, the clockwise cyclic order by angle of the points of $S \setminus \{p\}$ around p, must be also be preserved. These two observations together imply that at least (n-1)! other different order types are realizable in this grid. By Stirling's approximation, this at least

$$\exp(n\log n - n + O(\log n))$$

As a result we consider a meaningful lower bound for $g(n, \alpha)$ to be of

$$\exp(c \cdot n \log n)$$

for some c > 1. In this paper, in Section 2, we prove the following lower bounds.

Theorem 1. If $\alpha > 2$ then

$$g(n, \alpha) \ge \exp\left(2n\log n - O(n\log\log n)\right).$$

Theorem 2. If $\alpha \geq 2.5$ then

$$g(n, \alpha) \ge \exp(3n \log n - O(n \log \log n)).$$

We have the following upper bounds. Note that there are at most $n^{2n} = \exp(2n \log n)$ different sets of n points in an $n \times n$ integer grid. Thus

$$g(n,1) \le \exp(2n\log n).$$

By using the point sets found in [7], one can produce many point sets whose order types cannot be realized in an integer grid of size n^{α} . Let P be a point set of $\log(\alpha \log n)$ points whose order type cannot be realized with integer coordinates smaller than n^{α} . Consider $P \cup Q$, where Q is any point set of $n - \log(\alpha \log n)$ points such that $P \cup Q$ is in general position; note that $P \cup Q$ cannot be realized with integer coordinates smaller than n^{α} . Therefore, for every $\alpha > 0$, there are at least

$$f\left(n - \frac{\log\left(\alpha \log n\right)}{c_1}\right)$$

realizable order types of n points but not realizable in integer grids of size n^{α} .

2 Lower Bound Constructions

In this section we prove Theorems 1 and 2; we present two constructions that produce many point sets with different order types in integer grids of size n^{α} for $\alpha > 2$ and $\alpha \ge 2.5$, respectively. Our approach is similar to the one used to lower bound f(n): we iteratively place points and lower bound the number of different available choices that produce different order types. With the caveat that if we now consider the line arrangement spanned by the straight lines passing through pairs of already placed points, a given cell might not contain a grid point.

To work around this problem, we do the following. We place a portion of our points in a special configuration C; and choose a set of straight lines passing through pairs of points in C. Then, we define a set T of isothethic squares of side length equal to ℓ such that any two squares are separated by one of our chosen lines. Afterwards, we place the remaining points. This is done as follows. At each step we first choose a square from T that

(1) has not been chosen before; and

(2) contains a point p of integer coordinates that does not produce a triple of collinear points with the previously placed points.

We then choose p as our next point.

Our strategy is to lower bound, at each step, the number of squares in T that satisfy (1) and (2). We say that these squares are *alive*; otherwise, we say that they are *dead*. Suppose that a square of T that has not been chosen yet. If less than ℓ lines passing through a pair of previously placed points intersect this square, then it is still alive. In what follows, we use this observation extensively.

2.1 Cross Configuration

Let n be an arbitrarily large positive integer and let p be the smallest prime greater than $n/4 \log n$. In this case the configuration \mathcal{C} consists of four sets \mathcal{U} , \mathcal{L} , \mathcal{R} and \mathcal{D} ; each set is an affine copy of Q_p . \mathcal{L} and \mathcal{R} are rotated by 90° and stretched vertically. \mathcal{U} and \mathcal{D} are stretched horizontally. \mathcal{L} and \mathcal{R} are placed at the same height, with \mathcal{L} to the left of \mathcal{R} ; \mathcal{U} and \mathcal{D} are placed at the same x-coordinate and between \mathcal{L} and \mathcal{R} ; \mathcal{U} is above $\mathcal{L} \cup \mathcal{R} \cup \mathcal{D}$ and \mathcal{D} is below $\mathcal{L} \cup \mathcal{R} \cup \mathcal{U}$. Every point in \mathcal{U} is joined with a straight line with the point in \mathcal{D} with the same x-coordinate; every point in \mathcal{L} is joined with a straight line with the point in \mathcal{R} with the same y-coordinate. These are our chosen set of lines. See Figure 2. Let $k := \lceil \log n \rceil$. The precise definitions are

$$\begin{aligned} \mathcal{U} &:= \{ (i \cdot 34p \cdot k^2, (34 \cdot (i^2 \mod p)) \cdot k^2) : 0 \le i$$

Simple (but tedious) arithmetic shows that C is in general position. The set of chosen straight lines form a rectangular grid. In the interior of each of these rectangles place an isothethic square with $32pk^2 \times 32pk^2$ integer grid points. Let T be the set of these squares. Baker, Harman and Pintz [2] showed that the interval $[x, x + x^{21/40}]$ contains a prime number, for x sufficiently large. Thus, $p = n/4 \log n + O(n^{21/40})$. Therefore $|C| = n/\log n + O(n^{21/40})$, $|T| = (p-1)^2$ and $\ell = 32pk^2$ for this construction.

We now iteratively place the remaining n - 4p points. At each stage the number of lines passing through a pair of the so far placed points is less than $n^2/2$; each of these lines intersects less than 2psquares of T; each square must touched by at least $32pk^2$ straight lines before being dead. Therefore, the number of alive squares at every stage is at least

$$(p-1)^2 - \frac{n^2 p}{32pk^2} = \frac{1}{2}p^2 - O(p) \ge \frac{n^2}{32\log^2 n}.$$

where the last inequality holds for sufficiently large n.

Therefore, we obtain at least

$$\prod_{i=1}^{n-4p} \frac{n^2}{32\log^2 n} = \frac{n^{2(n-4p)}}{(32\log^2 n)^{n-4p}} = \exp\left(2n\log n - O(n\log\log n)\right)$$

different order types with this procedure. Since C is contained in an integer grid of side length equal to $\Theta(p^2k^2) = \Theta(n^2)$, this proves Theorem 1.



Figure 2: Cross configuration for p = 5

2.2 Regular Polygon Configuration

Let n be an arbitrarily large positive integer; let m be the smallest multiple of 16 larger than $n/\log n$ and let $L := \lceil 64n^2/m^{1/2} \rceil$. Let $\mathcal{C} := \{v_0, \ldots, v_{m-1}\}$ be the vertices, in clockwise order, of a regular polygon P of side length equal to L. These points may not have integer coordinates; their coordinates will be rounded up to the nearest integer later on. Let q^* be the center of this polygon. For $1 \leq i \leq m$, let Δ_i be the triangle with vertices v_{i-1}, v_i , and q^* . In what follows we define a set T_i of squares inside Δ_i .



Figure 3: The regular polygon construction with m = 32

Starting at the line segment joining v_{i-1} and v_i , let e_1, \ldots, e_{m-1} be the line segments joining v_{i-1} and every other vertex of P, sorted clockwise by angle around v_{i-1} . Starting at the line segment joining v_i and v_{i-1} , let f_1, \ldots, f_{m-1} be the line segments joining v_i and every other vertex of P, sorted counterclockwise by angle around v_i . Let C_1 be the the circumcircle of P. Since every pair of consecutive vertices of P defines a chord of C_1 , and these chords have the same length, the angle between any two consecutive e_i and e_{i+1} is the same. Let γ be this angle. Moreover, the angle between any two consecutive f_j and f_{j+1} is also equal to γ ; note that

$$\gamma = \frac{1}{m}\pi.$$

For indices $2 \le j \le m/2$ and $2 \le k \le m/2$, let $p_{j,k}$ be the intersection of e_j and f_k ; note that $p_{j,k}$ is contained in Δ_i . Let

$$Q := \left\{ p_{j,k} : j, k \text{ even and } \frac{m}{8} \le j, k < \frac{m}{4} \right\}$$

Note that

$$|Q| = \frac{m^2}{256}.$$

See Figure 3. For each $p_{j,k}$ in Q, place an isothethic square of side length equal to

$$\ell := \frac{L}{m}$$

centered at $p_{j,k}$. Let T_i be the set of these squares. The next lemma shows that the squares in T_i are well separated by the e_j 's and f_k 's



Figure 4: The proof of Lemma 3

Lemma 3. Let $p_{j,k}$ be a point in Q. Then the distances from $p_{j,k}$ to e_{j-1} , e_{j+1} , f_{k-1} and f_{k+1} are greater than

$$\left(\sqrt{2}-1\right)\pi\ell+O\left(\frac{\ell}{m^2}\right).$$

Proof. We show that the distances from $p_{j,k}$ to e_{j-1} and e_{j+1} are at least the required value. The proof for f_{k-1} and f_{k+1} is similar. Note that among the $p_{j,k}$'s in Q,

$$p := p_{m/8,m/4-1}$$

is the point closest to v_i . Consider the triangle with vertices v_i, v_{i-1} and p (see Figure 4). By the law of sines the distance from p to v_i is equal to

$$\frac{\sin(\pi/8)}{\sin(5\pi/8 + \pi/m)} L > \frac{\sin(\pi/8)}{\sin(5\pi/8)} L = \left(\sqrt{2} - 1\right) L.$$

Therefore, the distances from $p_{j,k}$ to e_{j-1} and e_{j+1} are at least $\tan(\gamma) \cdot (\sqrt{2}-1)L$. The result follows from the facts that Maclaurin series of $\tan(x)$ is equal to $x + O(x^3)$ and that $\gamma = \pi/m$.

We are ready to define the set of squares, let

$$T := \bigcup_{i=1}^m T_i.$$

Let C_2 be the circle with center q^* and passing through $p_{m/4,m/4}$. Note that T is contained in the annulus A bounded by C_1 and C_2 . The following lemma upper bounds the number of squares in T that a given straight line can intersect.



Figure 5: The proof of Lemma 4



$$\frac{m^{3/2}}{4}$$

squares of T.

Proof. Let φ be a straight line. Note that φ intersects A in at most two straight line segments. We upper bound the number of squares in T that a straight line segment s can intersect. Each time s intersects a square in T_i , it must intersect and edge e_j or f_k ; moreover, only half of these edges define a square in T_i . Therefore, s intersects at most $\frac{m}{8}$ squares in T_i . We upper bound the number of the triangles Δ_i 's that s can intersect. For this we upper bound the length of s.

Let R and r be the radius of the circles C_1 and C_2 , respectively. Note that s has maximum length when it is tangent to C_2 and its endpoints are in C_1 . Therefore,

$$||s|| \le 2\sqrt{R^2 - r^2}.$$

Since C_2 passes through $p_{m/4,m/4}$, the distance from C_2 to the edge v_i, v_{i-1} is equal to L/2. Since P is a regular polygon $R = \frac{1}{2}L \csc(\pi/m)$ and its apotheme is equal to $R \cos(\pi/m)$ (see Figure 5). This implies that $r = \frac{1}{2}L(\cot(\pi/m) - 1)$. Therefore,

$$||s|| \le 2\sqrt{R^2 - r^2} \le \sqrt{2} \cdot L\sqrt{\cot\left(\frac{\pi}{m}\right)} \le \sqrt{\frac{2m}{\pi}}L - O\left(\frac{L}{m^{3/2}}\right) = \frac{1}{2}$$

the last term comes from the fact that the Maclaurin series of $\sqrt{\cot(x)}$ is equal to $\sqrt{\frac{1}{x}} - O(x^{3/2})$.

Now we lower bound the length of $s \cap \triangle_i$. Note that $s \cap \triangle_i$ has minimum length when s is tangent to C_2 and parallel to the edge v_{i-1}, v_i . Thus,

$$||s \cap \triangle_i|| \ge 2 \tan\left(\frac{\pi}{m}\right) r = \left(1 - \tan\left(\frac{\pi}{m}\right)\right) L > \sqrt{\frac{2}{\pi}L}$$

where the last term holds for sufficiently large n. Therefore, s intersects a most \sqrt{m} of the triangles \triangle_i . The result follows.

To end the construction we round the coordinates of the v_i 's to their nearest integer. Redefine the e_j 's and f_k 's accordingly. By Lemma 3, a square in T_i centered at $p_{j,k}$ is separated from edges $e_{j'}$ and $f_{k'}$ different e_j and f_k by a distance of at least $(\sqrt{2}-1) \pi \ell$. The endpoints of the new e_j 's and f_k 's are at a distance of at most one of their original positions. Since $(\sqrt{2}-1) \pi > 1$, the squares in T_i are still separated by the straight lines containing the e_i 's and f_k 's.

We now iteratively place the remaining n-m points. At every stage the number of lines passing through every pair of the so far placed points is less than $n^2/2$; each of these lines intersects at most $m^{3/2}/4$ squares of T; each square must be touched by at least $\ell = L/m$ straight lines before being dead. Thus, the number of squares alive at every stage is at least

$$\frac{m^3}{256} - \frac{n^2 m^{5/2}}{8L} \ge \frac{m^3}{512} \ge \frac{n^3}{512 \log^3 n}$$

Therefore, we obtain at least

$$\prod_{i=1}^{n-m} \frac{n^3}{512 \log^3 n} = \frac{n^{3(n-m)}}{(512 \log n)^{3(n-m)}} = \exp(3n \log n - O(n \log \log n))$$

different order types with this procedure. Recall that $m \leq n/\log n + 16$ and $L \leq 64n^2/m^{1/2} + 1$. Therefore, these point sets lie in an integer grid of side length equal to $L \cdot m = \Theta \left(n^{2.5}/\sqrt{\log n}\right)$. This proves Theorem 2.

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