# Empty Rainbow Triangles in $k$-colored Point Sets 

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$\square$


#### Abstract

Let $S$ be a set of $n$ points in general position in the plane. Suppose that each point of $S$ has been assigned one of $k \geq 3$ possible colors and that there is the same number, $m$, of points of each color class. A polygon with vertices on $S$ is empty if it does not contain points of $S$ in its interior; and it is rainbow if all its vertices have different colors. Let $f(k, m)$ be the minimum number of empty rainbow triangles determined by $S$. In this paper we give tight asymptotic bounds for this function. Furthermore, we show that $S$ may not determine an empty rainbow quadrilateral for some arbitrarily large values of $k$ and $m$.


## 1 Introduction

A set of points in the plane is in general position if no three of its points are collinear. In this paper all sets of points are in general position. The well known Erdős-Szekeres theorem [13] states that for every positive integer $r \geq 3$ there exists a positive integer $n(r)$ such that every set of $n(r)$ (or more points) in the plane contains the vertices of a convex polygon of $r$ vertices.

Let $S$ be a set of $n$ points in the plane. A polygon with vertices on $S$ is said to be empty if it does not contain a point of $S$ in its interior. An r-hole of $S$

[^0]is an empty convex polygon of $r$ sides with vertices on $S$. In 1978, Erdős 12 asked if for every $r$, every sufficiently large set of points in the plane contains an $r$-hole. Klein [13] had already noted that every set of 5 points contains a 4-hole. Harboth 17 showed that every set of 10 points contains a 5 -hole. Horton 18 constructed arbitrarily large sets of points without 7 -holes. The case for 6 -holes remained open until Nicolás [22] and Gerken [16], independently showed that every sufficiently large point set contains a 6 -hole.

Once the existence of $r$-holes for some given $r$ in every sufficiently large point set is established, it is natural to ask what is the minimum number of $r$-holes in every set of $n$ points in the plane. Katchalski and Meir 20 first considered this question for triangles. They showed that every set of $n$ points determines $\Omega\left(n^{2}\right)$ empty triangles and provided an example of a point set determining $O\left(n^{2}\right)$ empty triangles. The lower and upper bounds on this number have been improved throughout the years [5, 11, 24, 15, 2, 2, The problem of determining the minimum number of $r$-holes in every set of $n$ points in the plane has also been considered in these papers.

Colored variants of these problems where first studied by Devillers, Hurtado, Károlyi and Seara 10. A point set is $k$-colored if every one of its points is assigned one of $k$ available colors. We say that an $r$-hole on $S$ is monochromatic if all its vertices are of the same color, and that it is rainbou ${ }^{1}$ if all its vertices are of different colors. Many chromatic variants on problems regarding $r$-holes in colored points sets have been studied since; see [6, 23, 1, 3, 7, 21, 4, 19, 8, 14, 25, In particular, Aichholzer, Fabila-Monroy, Flores-Peñaloza, Hackl, Huemer, and Urrutia showed that every 2 -colored set of $n$ points in the plane determines $\Omega\left(n^{5 / 4}\right)$ empty monochromatic triangles [1]. This was later improved to $\Omega\left(n^{4 / 3}\right)$ by Pach and Tóth [23]. The current best upper bound on this number is $O\left(n^{2}\right)$ and this is conjectured to be the right asymptotic value.

In this paper we consider the problem of counting the number of empty rainbow triangles in $k$-colored point sets in which there are the same number, $m$, of points of each color class. Let $f(k, m)$ be the minimum number of empty rainbow triangles in such a point set. We give the following tight asymptotic bound for $f(k, m)$.

## Theorem 1.1.

$$
f(k, m)= \begin{cases}\Theta\left(k^{2} m\right) & \text { if } m<k \\ \Theta\left(k^{3}\right) & \text { if } m \geq k\end{cases}
$$

Note that in contrast to the number of empty monochromatic triangles, the number of empty rainbow triangles does not necessarily grow with the number of points.

[^1]
## 2 Lower Bound

Proof of the lower bound in Theorem 1.1. Let $S$ be a $k$-colored set of points with $m$ points of each color class. Without loss of generality assume that no two points of $S$ have the same $x$-coordinate. Assume that the set of colors is $\{1, \ldots, k\}$. For each $1 \leq i \leq k$, let $p_{i}$ be the leftmost point of color $i$. Without loss of generality assume that when sorted by $x$-coordinate these points are $p_{1}, \ldots, p_{k}$.

Let $1 \leq i \leq k$ and let $r_{i}:=\min \{i, m\}$. We show that there are at least $\left(r_{i}^{2}-3 r_{i}+2\right) / 2$ empty rainbow triangles having a point of color $i$ as its rightmost point. Let $q_{1}:=p_{i}, q_{2}, \ldots, q_{r_{i}-2}$ be the first $r_{i}-2$ points of color $i$ when sorted by $x$-coordinate. For each $1 \leq j \leq r_{i}-2$ do the following. Sort the points of $S$ to the left of $q_{j}$ counterclockwise by angle around $q_{j}$. Note that any two consecutive points in this order define an empty triangle with $q_{j}$ as its rightmost point. Since the points $p_{1}, \ldots, p_{i-1}$ are to the left of $q_{j}$, there are at least $i-2$ of these empty triangles such that the first point is of a color $l$ distinct from $i$, and the next point is of a color distinct from $l$. Furthermore, for at least $(i-2)-(j-1)=i-j-1$ of these triangles the next point is not of color $i$; thus, they are rainbow. We have at least

$$
\begin{equation*}
\sum_{j=1}^{r_{i}-1} i-j-1=\frac{\left(r_{i}-1\right)\left(2 i-r_{i}-2\right)}{2} \tag{1}
\end{equation*}
$$

empty rainbow triangles with a point of color $i$ as its rightmost point. If $i \leq m$ then the right hand side of (1) is equal to

$$
\frac{i^{2}-3 i+2}{2}
$$

Thus, if $m \geq k$ then $S$ determines at least

$$
\sum_{i=3}^{k} \frac{i^{2}-3 i+2}{2}=\frac{1}{6} k^{3}-\frac{1}{2} k^{2}+\frac{1}{3} k=\Omega\left(k^{3}\right)
$$

empty rainbow triangles; and if $m<k$ then $S$ determines at least

$$
\begin{aligned}
& \sum_{i=3}^{k} \frac{\left(r_{i}-1\right)\left(2 i-r_{i}-2\right)}{2} \\
& =\sum_{i=3}^{m} \frac{i^{2}-3 i+2}{2}+\sum_{i=m+1}^{k} \frac{(m-1)(2 i-m-2)}{2} \\
& =\frac{1}{2} k^{2} m-\frac{1}{2} k m^{2}+\frac{1}{6} m^{3}-\frac{1}{2} k^{2}+\frac{1}{2} k-\frac{1}{6} m \\
& =\Omega\left(k^{2} m\right)+\Omega\left(k m^{2}+m^{3}\right) \\
& =\Omega\left(k^{2} m\right)
\end{aligned}
$$

empty rainbow triangles

## 3 Upper Bound

In this section we construct a $k$-colored point set which provides our upper bounds for $f(k, m)$.

### 3.1 The Empty Triangles of the Horton Set

As a building block for our construction we use Horton sets [18]; in this section we characterize the empty triangles of the Horton set. Let $H$ be a set of $n$ points in the plane with no two points having the same $x$-coordinate; sort its points by their $x$-coordinate so that $H=\left\{p_{0}, p_{1}, \ldots, p_{n-1}\right\}$. Let $H_{0}$ be the subset of the even-indexed points of $H$, and $H_{1}$ be the subset of the odd-indexed points of $H$. That is, $H_{0}=\left\{p_{0}, p_{2}, \ldots\right\}$ and $H_{1}=\left\{p_{1}, p_{3}, \ldots\right\}$. Let $X$ and $Y$ be two finite sets of points in the plane. We say that $X$ is high above $Y$ if: every line determined by two points in $X$ is above every point in $Y$; and every line determined by two points in $Y$ is below every point in $X$.

Definition 1. H is a Horton set if

1. $|H|=1$; or
2. $|H| \geq 2 ; H_{0}$ and $H_{1}$ are Horton sets; and $H_{1}$ is high above $H_{0}$.

Assume that $H$ is a Horton set. We say that an edge $e:=\left(p_{i}, p_{j}\right)$ is a visible edge of $H$ if one of the following two conditions are met.

- Both $i$ and $j$ are even and for every even $i<l<j$, the point $p_{l}$ is below the line passing through $e$. In this case we say that $e$ is visible from above.
- Both $i$ and $j$ are odd and for every odd $i<l<j$, the point $p_{l}$ is above the line passing through $e$. In this case we say that $e$ is visible from below.

Lemma 3.1. The number of visible edges of $H$ is less than $2 n$.
Proof. Let $s:=100 \cdots 0$ be a binary string starting with a 1 and followed by a trail of 0 's of length at most $\left\lceil\log _{2}(n)\right\rceil$. Every consecutive pair of points of $H_{s}$ defines a visible edge from below of $H$. Moreover, all visible edges from below of $H$ are of this form, for some $s$. Note that $\left|H_{s}\right| \leq n / 2^{|s|}+1$. A similar analysis holds for the edges visible from above of $H$, using the binary strings starting with a 0 and followed by a trail of 1 's of length at most $\left\lceil\log _{2}(n)\right\rceil$. The number of visible edges of $H$ is at most

$$
2 \sum_{i=1}^{\left\lceil\log _{2}(n)\right\rceil} \frac{n}{2^{i}}<2 n
$$

The visible edges of $H$ allows to characterize its empty triangles recursively as follows.

Lemma 3.2. Let $p_{i}, p_{j}$ and $p_{l}$ be the vertices of a triangle $\tau$ of $H$ such that either

- $\left(p_{i}, p_{j}\right)$ is an edge visible from below and $p_{l} \in H_{0}$; or
- $\left(p_{i}, p_{j}\right)$ is an edge visible from above and $p_{l} \in H_{1}$.

Then $\tau$ is empty. Moreover, every empty triangle of $H$ with at least one vertex in each of $H_{0}$ and $H_{1}$ is of one these forms.

Proof. If $\tau$ is such a triangle then its emptiness follows from the definition of the Horton set. Suppose now that $\tau:=p_{i} p_{j} p_{l}$ is an empty triangle of $H$ with $p_{i}, p_{j} \in H_{0}$ and $p_{l} \in H_{1}$, or $p_{i}, p_{j} \in H_{1}$ and $p_{l} \in H_{0}$. Then, for $\tau$ to be empty, $p_{i} p_{j}$ must be an edge visible from above (resp. below).

We can now get a good upper bound on the number of empty triangles of $H$.

Corollary 3.3. The number of empty triangles of $H$ is at most $2 n^{2}$.
Proof. Let $T(n)$ be the number of empty triangles in a Horton set of $n$ points. Then $T(n)$ is equal to the number of empty triangles with at least one vertex in each of $H_{0}$ and $H_{1}$, plus the number of empty triangles with all their vertices in $H_{0}$ or all their vertices in $H_{1}$. By the definition of Horton sets and Lemma 3.2 we have that

$$
T(n)<T\left(\left\lceil\frac{n}{2}\right\rceil\right)+T\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+n^{2} \leq 2 n^{2}
$$

### 3.2 Blockers

Our strategy is to start with a Horton set $H$ of $k$ points and replace each point $p_{i}$ of $H$ with a cluster $C_{i}$ of $m$ points. All of the points of $C_{i}$ are of the same color and are at a distance of at most some $\varepsilon$ from $p_{i}$. We choose $\varepsilon$ to be arbitrarily small. Let $S$ be the resulting set. Note that every rainbow triangle of $S$ must have all its vertices in different clusters. Moreover, since each $C_{i}$ is arbitrarily close to $p_{i}$ we have the following. If $\tau$ is an empty triangle of $S$ with vertices in different clusters $C_{i}, C_{j}$ and $C_{l}$ then $p_{i}, p_{j}$ and $p_{l}$ are the vertices of an empty triangle in $H$. In principle, this gives $m^{3}$ empty rainbow triangles in $S$ per empty triangle of $H$. However, we can place the points within each cluster in such a way so that only very few of these triangles are actually empty.

Let $p_{i} \in H$, and $r:=\min \left\{\left\lceil\log _{2}(k)\right\rceil+2,\lceil m / 2\rceil\right\}$. In what follows we iteratively define real numbers

$$
\varepsilon=\varepsilon_{1}>\varepsilon_{2}>\cdots>\varepsilon_{r+1}>0
$$

in the process we also place a subset $B_{i}$ of points of $C_{i}$ at some of these distances; we refer to the points in $B_{i}$ as blockers. For $t=1 \ldots, r$ suppose that $\varepsilon_{t}$ has been defined and possibly some points of $B_{i}$ have been placed. Consider every pair
of points $p_{j}, p_{l} \in H$ distinct from $p_{i}$. Let $q \in B_{i}$ be at distance $\varepsilon_{t}$ or more from $p_{i}$ and such that $q$ is in the interior of every triangle with vertices $p_{i}, p_{j}^{\prime}$ and $p_{l}^{\prime}$, where $p_{j}^{\prime}$ and $p_{l}^{\prime}$ are at a distance of at most $\varepsilon$ of $p_{j}$ and $p_{l}$, respectively. Let $\tau$ be the triangle with vertices $p_{i}^{\prime}, p_{j}^{\prime}$ and $p_{l}^{\prime}$, where $p_{i}^{\prime}$ is any point at a distance of at most $\varepsilon_{t+1}$ from $p_{i}$. We define $\varepsilon_{t+1}<\varepsilon_{t}$ small enough so that every such $q$ is in the interior of every such $\tau$. We say that $q$ blocks the triangle with vertices $p_{i}^{\prime}, p_{j}^{\prime}$ and $p_{l}^{\prime}$.

We construct $B_{i}$ iteratively as follows. We say that a blocker point at distance $\varepsilon_{t}$ from $p_{i}$ is at layer $t$. Let $s_{0}, \ldots, s_{r^{\prime}}$ be the binary strings such that:

1) $s_{0}=\emptyset$;
2) for every $0 \leq t<r^{\prime}, s_{t+1}=s_{t} 0$ or $s_{t+1}=s_{t} 1$; and
3) $H_{s_{r^{\prime}}}=\left\{p_{i}\right\}$.

By 2) and 3) we have that $p_{i} \in H_{s_{t}}$ for every $0 \leq t \leq r^{\prime}$. Note that $r^{\prime} \leq$ $\left\lceil\log _{2}(k)\right\rceil$.

Sort the points of $H \backslash\left\{p_{i}\right\}$ counterclockwise by angle around $p_{i}$. For every $t=0, \ldots, r^{\prime}-2$ and as long as we have placed at most $m-2$ blocker points, we place two blocker points at a distance from $\varepsilon_{t+1}$ from $p_{i}$ as follows.

- Suppose that $s_{t+1}=s_{t} 0$. Place one blocker point just after the leftmost point of $H_{s_{t} 1}$ in order by angle around $p_{i}$; place another blocker point just before the rightmost point of $H_{s_{t} 1}$ in order by angle around $p_{i}$, as depicted in Figure 1(a).
- Suppose that $s_{t+1}=s_{t} 1$. Place one blocker point just after the leftmost point of $H_{s_{t} 0}$ in order by angle around $p_{i}$; place another blocker point just before the rightmost point of $H_{s_{t} 0}$ in order by angle around $p_{i}$.

Let $B_{i}^{\prime} \subset B_{i}$ the set of these blocker points. If $\left|B_{i}^{\prime}\right|<m$ then we proceed to place the remaining points of $C_{i}$. If $m-\left|B_{i}^{\prime}\right|<k$ then place the remaining points of $C_{i}$ in any way at a distance of at most $\varepsilon_{r}$ of $p_{i}$; in this case we have that $B_{i}=B_{i}^{\prime}$. Suppose that $m-\left|B_{i}^{\prime}\right| \geq k$. For every $t=0, \ldots, r^{\prime}-1$ we place additional blocker points as follows.

- If $s_{t+1}=s_{t} 0$ then place a blocker point, at a distance of $\varepsilon_{r}$ from $p_{i}$, between any two consecutive points of $H_{s_{t} 1}$ in order by angle around $p_{i}$; see Figure 1(b).
- If $s_{t+1}=s_{t} 1$ then place a blocker point, at a distance of $\varepsilon_{r}$ from $p_{i}$, between any two consecutive points of $H_{s_{t} 0}$ in order by angle around $p_{i}$.

Let $B_{i}^{\prime \prime} \subset B_{i}$ the set of these blocker points. No more blocker points are added and $B_{i}=B_{i}^{\prime} \cup B_{i}^{\prime \prime}$. If $m>\left|B_{i}\right|=\left|B_{i}^{\prime}\right|+\left|B_{i}^{\prime \prime}\right|$ then place the remaining points of $C_{i}$ in any way at a distance of at most $\varepsilon_{r+1}$ from $p_{i}$.

We are now ready to prove our upper bounds on $f(k, m)$.


Figure 1: (a) Blocking triangles with one point in $H_{s 0}$ and one point in $H_{s 1}$ (b) Blocking triangles with two vertices in $H_{s 1}$.

Proof of the upper bound in Theorem 1.1. We count the number of empty rainbow triangles determined by $S$ as constructed above. To every empty triangle $\tau^{\prime}$ of $S$ with vertices $p_{i}^{\prime} \in C_{i}, p_{j}^{\prime} \in C_{j}$ and $p_{l}^{\prime} \in C_{l}$, we assign the empty triangle $\tau$ of $H$ with vertices $p_{i}, p_{j}$ and $p_{l}$. Let $s$ be the binary string such that the vertices of $\tau$ are contained in $H_{s}$ but not in $H_{s 0}$ and $H_{s 1}$. We say that $\tau$ is in layer $|s|+1$. Without loss of generality suppose that $p_{j}$ and $p_{l}$ are both contained in $H_{s 0}$ or are both contained in $H_{s 1}$.

If $\tau^{\prime}$ contains a blocker point of each of $B_{j}^{\prime}$ and $B_{l}^{\prime}$ then these blocker points are at layer $|s|+1$. In this case there at most $2(|s|+1)$ possible choices for each of $p_{i}^{\prime}$ and $p_{j}^{\prime}$. Otherwise, $m<2(|s|+1)$ and there are at most $m$ possible choices for each of $p_{j}^{\prime}$ and $p_{l}^{\prime}$. If $m \geq k+2\left\lceil\log _{2}(k)\right\rceil$ then $\tau^{\prime}$ contains a point from $B_{i}^{\prime \prime}$; and there at most $k+2\left\lceil\log _{2}(k)\right\rceil$ possible choices for $p_{i}^{\prime}$. Otherwise, $m<k+2\left\lceil\log _{2}(k)\right\rceil$ and there at most $m$ possible choices for $p_{i}^{\prime}$. Summarizing, $\tau$ is assigned to at most the following number of empty rainbow triangles of $S$ :

$$
\begin{array}{ll}
m^{3} & \text { if } m \leq 2|s|+1 ; \\
4(|s|+1)^{2} m & \text { if } 2|s|+1<m<k+2\left\lceil\log _{2}(k)\right\rceil ; \text { and } \\
4(|s|+1)^{2}\left(k+2\left\lceil\log _{2}(k)\right\rceil\right) & \text { if } m \geq k+2\left\lceil\log _{2}(k)\right\rceil .
\end{array}
$$

By Lemma 3.2, $\left(p_{j}, p_{l}\right)$ is a visible edge of $H_{s}$. Since $\left|H_{s}\right| \leq\left\lceil k / 2^{|s|}\right\rceil$, by Lemma 3.1 there are at most $2\left\lceil k / 2^{|s|}\right\rceil\left\lceil k / 2^{|s|+1}\right\rceil \leq 8\left(k^{2} / 2^{2|s|}\right)$ empty triangles in $H_{s}$. Thus, for every $1 \leq t \leq\left\lceil\log _{2}(k)\right\rceil$ there at most $2^{t-1} 8\left(k^{2} / 2^{2(t-1)}\right)=$ $8 k^{2} / 2^{t-1}$ empty triangles in $H$ at layer $t$. Let $m^{\prime}:=\min \left\{m, k+2\left\lceil\log _{2}(k)\right\rceil\right\}$. Therefore, the number of empty rainbow triangles determined by $S$ is at most

$$
\begin{equation*}
\sum_{t=1}^{\left\lfloor m^{\prime} / 2\right\rfloor} 4 t^{2} m^{\prime}\left(\frac{8 k^{2}}{2^{t-1}}\right)+\sum_{t=\lfloor m / 2\rfloor+1}^{\left\lceil\log _{2}(k)\right\rceil} m^{3}\left(\frac{8 k^{2}}{2^{t-1}}\right) \tag{2}
\end{equation*}
$$

where the second term is set to 0 if $\lfloor m / 2\rfloor>\left\lceil\log _{2}(k)\right\rceil$.
If $m^{\prime}=m$ then 2 is at most

$$
\begin{aligned}
& \sum_{t=1}^{\lfloor m / 2\rfloor} 4 t^{2} m\left(\frac{8 k^{2}}{2^{t-1}}\right)+\sum_{t=\lfloor m / 2\rfloor+1}^{\left\lceil\log _{2}(k)\right\rceil} 4 t^{2} m\left(\frac{8 k^{2}}{2^{t-1}}\right) \\
& =\sum_{t=1}^{\left\lceil\log _{2}(k)\right\rceil} 4 t^{2} m\left(\frac{8 k^{2}}{2^{t-1}}\right) \\
& =32 k^{2} m \sum_{t=1}^{\left\lceil\log _{2}(k)\right\rceil}\left(\frac{t^{2}}{2^{t-1}}\right) \\
& \leq 384 k^{2} m \\
& =O\left(k^{2} m\right)
\end{aligned}
$$

If $m^{\prime}=k+2\left\lceil\log _{2}(k)\right\rceil$ then $\sqrt{2}$ is at most

$$
\begin{aligned}
& \sum_{t=1}^{\left\lceil\log _{2}(k)\right\rceil} 4 t^{2}\left(k+2\left\lceil\log _{2}(k)\right\rceil\right)\left(\frac{8 k^{2}}{2^{t-1}}\right) \\
& =32 k^{2}\left(k+2\left\lceil\log _{2}(k)\right\rceil\right) \sum_{t=1}^{\left\lceil\log _{2}(k)\right\rceil}\left(\frac{t^{2}}{2^{t-1}}\right) \\
& \leq 384 k^{2}\left(k+2\left\lceil\log _{2}(k)\right\rceil\right) \\
& =O\left(k^{3}\right)
\end{aligned}
$$

Therefore,

$$
f(k, m)= \begin{cases}O\left(k^{2} m\right) & \text { if } m \leq k \\ O\left(k^{3}\right) & \text { if } m>k\end{cases}
$$

## 4 Empty Rainbow Quadrilaterals

A natural generalization is to consider empty rainbow polygons; we construct a $k$-colored point set with the same number of points in each color class and that does not determine an empty rainbow quadrilateral. First, we observe the following.

Lemma 4.1. The point set depicted in Figure 2 does not determine an empty rainbow quadrilateral.

Proof. Let $\tau$ be a rainbow quadrilateral of the point set depicted in Figure 2. Note that $\tau$ must have $A, B$ and $C$ as vertices. Thus, at least two of $A B, A C$
and $B C$ are sides of $\tau$. Assume without loss of generality that $A B$ and $A C$ are sides of $\tau$. If the fourth vertex of $\tau$ is not one of the red points near $A$ then these points are inside $\tau$; and $\tau$ is not empty. If the fourth vertex of $\tau$ is one of the red points near $A$ then by construction the other red point near $A$ is inside $\tau$; and again $\tau$ is not empty.


Figure 2: A colored point set without an empty rainbow quadrilateral
We use Lemma 4.1 to construct our point set. First we take a regular $(k-1)$ gon, $P$, with vertices $p_{1}, \ldots, p_{k-1}$; we replace every point $p_{i}$ with a cluster $C_{i}$ of $m$ points of color $i$. Let $P^{\prime}$ be a copy of $P$ with vertices $p_{1}^{\prime}, \ldots, p_{k-1}^{\prime}$, which is rotated by $\frac{2 \pi}{2(k-1)}=\frac{\pi}{k-1}$. So $p_{1}, p_{1}^{\prime}, p_{2}, \ldots, p_{k-1}, p_{k-1}^{\prime}$ form a regular $2(k-1)$ gon. Let $\varepsilon$ be sufficiently small. For every $1 \leq i \leq k-1$, we place the points of $C_{i}$ at a distance of at most $\varepsilon$ from $p_{i}$. For $1 \leq i \leq k-1\left(p_{0}^{\prime}=p_{k-1}^{\prime}\right)$ we place at least $2(k-3)$ points of color $k$ arbitrarily close to the line segment $p_{i-1}^{\prime} p_{i}^{\prime}$ and so that the following holds. Let $q_{1}$ and $q_{2}$ be any two consecutive points of $P$ distinct from $p_{i}$. In the triangle with vertices $p_{i}, q_{1}$ and $q_{2}$ there are at least two points on $p_{i-1}^{\prime} p_{i}^{\prime}$ of color $k$. Furthermore, these points are at a distance of at least $\varepsilon$ to the lines $\overline{p_{i} q_{1}}$ and $\overline{p_{i} q_{2}}$. Note that $m \geq 2 k^{2}-8 k+6$. Let $C_{k}$ be the set of points of color $k$ in this construction. The construction for $k=6$ is depicted in Figure 3


Figure 3: A construction of a 6-colored point set without empty rainbow 4-gons; the clusters $C_{i}$ are drawn enlarged.

Theorem 4.2. The set $\bigcup_{i=1}^{k} C_{i}$ is a $k$-colored point set, with the same number of points of each color class, that does not determine an empty rainbow quadrilateral.

Proof. Let $\tau$ be a rainbow triangle with vertices $q_{i} \in C_{i}, q_{j} \in C_{j}$ and $q_{l} \in C_{l}$. We show that $\tau$ has a structure like the point set depicted in Figure 2, Consider the points with color $k$ near the line segment $p_{i-1}^{\prime} p_{i}^{\prime}$ and that are between the line segments $q_{i} q_{j}$ and $q_{i} q_{l}$. By construction, there are at least two points of color $k$ near $p_{i-1}^{\prime} p_{i}^{\prime}$ between $p_{i} p_{j}$ and $p_{i} p_{l}$. Since these points are at a distance of at most $\varepsilon$ to either of these lines, they are also between $q_{i} q_{j}$ and $q_{i} q_{l}$. By the same argument there are points of color $k$ on $p_{j-1}^{\prime} p_{j}^{\prime}$ and $p_{l-1}^{\prime} p_{l}^{\prime}$ inside $\tau$. Therefore, by Lemma 4.1, the point set $\bigcup_{i=1}^{k} C_{i}$ does not determine an empty rainbow quadrilateral.

The construction described above determines many monochromatic quadrilaterals. This leads us to the following question.

Problem 1. Does every sufficiently large $k$-colored ( $k \geq 4$ ) point set with the same number of points in each color class determines an empty rainbow quadrilateral or an empty monochromatic quadrilateral?

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[^1]:    ${ }^{1}$ In 10 rainbow $r$-holes are called heterochromatic. We prefer to use "rainbow", because this term is used, with this meaning, in the more general setting of anti-Ramsey problems.

