# Piercing Pairwise Intersecting Geodesic Disks by Five Points* 

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#### Abstract

Given a simple polygon $P$ on $n$ vertices, and a set $\mathcal{D}$ of $m$ pairwise intersecting geodesic disks in $P$, we show that five points in $P$ are always sufficient to pierce all the disks in $\mathcal{D}$. This improves the previous bound of 14 , obtained by Bose, Carmi, and Shermer [1.


## 1 Introduction

The problem of piercing geometric objects with as few points as possible has attracted the attention of researchers for the past century. The research so far has been focused on convex objects and disks in the plane. The most known result for piercing geometric objects with set of minimum cardinality, is known as Helly's theorem [5:6], and works for convex sets in the plane. This theorem states the following: Given a set of $m$ convex objects in $\mathbb{R}^{d}$ such that $m>d+1$, if every $d+1$ of these objects have a point in common, then all of them have a point in common. This means that one point is sufficient to pierce all the objects. This claim does not hold when the convex objects are only pairwise intersecting. However, for a set of disks in the plane, where every pair of disks intersects, it has proven by Danzer [3] and by Stacho [8, 9] that four points are sufficient to pierce all the disks. These proofs are not amenable to design efficient (subquadratic-time) algorithms for computing the piercing points. Recently, linear-time algorithms have been presented by Har-Peled et al. (4) for computing five points that pierce $m$ pairwise intersecting disks, and by Carmi et al. 2 for computing four points.

Let $P$ be a simple polygon. A geodesic disk $D$ with radius $r$ centered at a point $c \in P$ is the set of all points $x \in P$, such that the length of the shortest path from $x$ to $c$ is at most $r$. Bose et al. [1] showed that for any set $\mathcal{D}$ of pairwise intersecting geodesic disks in $P, 14$ points are sufficient to pierce all the disks in $\mathcal{D}$ and these points can be computed in linear time. In this paper, we prove that five points are sufficient to pierce all the disks in $\mathcal{D}$, which improve the result of Bose et al. [1]. More precisely, we prove the following theorem.

Theorem 1. Given a simple polygon $P$ on $n$ vertices, and a set $\mathcal{D}$ of $m$ pairwise intersecting geodesic disks in $P$, five points in $P$ are sufficient to pierce all the disks in $\mathcal{D}$.

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## 2 The Setup and Preliminaries

For simplicity of presentation, we adapt some notation that appeared in [1]. Moreover, we use the convention that all indices are taken modulo the size of the set involved. Let $P$ be a simple $n$-vertex polygon in the plane and let $v_{1}, v_{2}, \ldots, v_{n}$ be its vertices sorted in clockwise order. For two points $x, y \in P$, the geodesic (shortest) path from $x$ to $y$ is denoted as $\Pi(x, y)$ and its length is the sum of the lengths of its edges, and is denoted as $|\Pi(x, y)|$. A geodesic disk with radius $r \geq 0$ centered at a point $c \in P$ is the set $\{y \in P:|\Pi(c, y)| \leq r\}$. A geodesic triangle on three points $a, b, c \in P$, denoted by $\triangle(a, b, c)$, is a weekly-simple polygon whose boundary consists of the paths $\Pi(a, b), \Pi(b, c)$, and $\Pi(a, c)$; see Figure 1. A pseudo triangle is a simple polygon with three convex vertices.

A set $X=\left\{x_{1}, \ldots, x_{k}\right\}$ of at least three points in $P$ is geodesically collinear if there exist two points $x_{i}, x_{j} \in X$, such that $X \subset \Pi\left(x_{i}, x_{j}\right)$. Given three points $a, b, c \in P$ that are not geodesically collinear, the paths $\Pi(a, b)$ and $\Pi(a, c)$ have a common subpath until they diverge at a point $a^{\prime}$. Similarly, let $b^{\prime}$ (resp., $c^{\prime}$ ) be the point where the paths $\Pi(b, a)$ and $\Pi(b, c)$ (resp., the paths $\Pi(c, a)$ and $\Pi(c, b))$ diverge; see Figure 1]. Pollack et al. [7] observed that $\triangle\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$ is a pseudo triangle. We refer to $\triangle\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$ as the geodesic core of $\triangle(a, b, c)$ and denote it by $\nabla(a, b, c)$. Pollack et al. 7 observed the following observation.
Observation 1. Let $a, b$ and $c$ be three points in $P$. Then the geodesic core $\nabla(a, b, c)$ has only reflex angles along its boundary and the interior of this triangle is fully contained in $P$.


Figure 1: $\triangle(a, b, c)$ is a geodesic triangle. $\triangle\left(a^{\prime}, b^{\prime}, c\right)$ is a pseudo triangle. $\nabla(a, b, c)=\triangle\left(a^{\prime}, b^{\prime}, c\right)$ is the geodesic core of $\triangle(a, b, c)$

Moreover, Pollack et al. [7] proved the following lemma about distances between a point and a geodesic path.
Lemma 1 ( $[7)$. Let $a, b$ and $c$ be three points in $P$. Let $g$ be the function defined on $\Pi(b, c)$, such that $g(x)=|\Pi(a, x)|$, for every point $x$ on $\Pi(b, c)$. Then, $g$ is a convex function with its maximum occurring either at $b$ or $c$. That is, $g(x) \leq \max \{g(b), g(c)\}$, for every point $x$ on $\Pi(b, c)$.

The following observations follow from Lemma 1 .
Observation 2. Let $a$ and $b$ be two points, such that the segment $\overline{a b}$ is entirely contained in $P$. Then, any disk $D \in \mathcal{D}$ that contains both $a$ and $b$ must contain the segment $\overline{a b}$.

Observation 3. Let $D$ be a geodesic disk in $\mathcal{D}$ with center $c \in P$, and let a and $b$ be two points in $D$. Then, the pseudo-triangle $\triangle(c, a, b)$ is contained in $D$.

Observation 4. Let $D$ be geodesic disk with center $c \in P$ and radius $r$. Let $q$ and $b$ be two points, such that $|\Pi(c, q)|+1 \leq r,|q b| \leq 1$, and the segment $\overline{q b}$ is entirely contained in $P$. Then, $b$ is contained in $D$.

Let $\mathcal{D}=\left\{D_{1}, D_{2}, \ldots, D_{m}\right\}$ be a set of $m$ pairwise intersecting geodesic disks in $P$. For each $1 \leq i \leq m$, let $c_{i}$ and $r_{i}$ denote the center and the radius of $D_{i}$, respectively. The set $\mathcal{D}$ is called Helly if there is a point that pierces all the disks in $\mathcal{D}$. For a point $x \in P$, we define a function $f(x)=y$ to be the smallest radius of a geodesic disk centered at $x$ that intersects all the disks in $\mathcal{D}$. A disk $D$ with radius $r$ centered at $c$ is called minimal with respect to $\mathcal{D}$ if every point $x$ in the close neighborhood of $c$ in $P$ has $f(x)>r$. Let $D^{*}$ be the disk with center $c^{*}$ that minimizes $f\left(c^{*}\right)$, and let $r^{*}=f\left(c^{*}\right)$ be its radius. Bose et al. [1] proved the following lemma regarding the properties of $D^{*}$.

Lemma 2 ( 11 ). If $\mathcal{D}$ is not Helly, then $D^{*}$ satisfies the following properties:

- $r^{*}>0$;
- $D^{*}$ does not intersect the boundary of $P$;
- $D^{*}$ is tangent to at least 3 geodesic disks $D_{1}, D_{2}, D_{3}$ in $\mathcal{D}$ at 3 distinct points $t_{1}, t_{2}, t_{3}$, respectively;
- $c^{*}$ is contained in the interior of $\triangle\left(t_{1}, t_{2}, t_{3}\right)$; and
- $D^{*}$ does not intersect the boundary of the geodesic core $\nabla\left(c_{1}, c_{2}, c_{3}\right)$, where $c_{1}, c_{2}, c_{3}$ are the centers of $D_{1}, D_{2}, D_{3}$, respectively.
Assume, w.l.o.g., that $r^{*}=1$ and that $c^{*}$ is located at the origin $(0,0)$. Let $D_{1}, D_{2}, D_{3}$ be the three geodesic disks from Lemma 2 that are tangent to $D^{*}$ at the points $t_{1}, t_{2}, t_{3}$, respectively. For each $i \in\{1,2,3\}$, let $\ell_{i}$ be the line that is tangent to $D_{i}$ and passes through $t_{i}$; see Figure 2 . Let $m_{i, j}$ be the intersection point between the lines $\ell_{i}$ and $\ell_{j}$, for every distinct $i, j \in\{1,2,3\}$. Assume, w.l.o.g., that $\ell_{1}$ is horizontal and the angle $\angle\left(m_{1,2}, m_{2,3}, m_{3,1}\right)$ is the largest in the triangle $\triangle\left(m_{1,2}, m_{2,3}, m_{3,1}\right)$; see Figure 2 .

For two points $p$ and $q$, let $\overline{p q}$ denote the line segment connecting them. For every distinct $i, j \in\{1,2,3\}$, let $\ell_{i, j}$ be the line passing through $m_{i, j}$ perpendicular to $\overline{c^{*} m_{i, j}}$; see Figure 2. Let $t_{i(j)}$ be the intersection point between $\ell_{i, j}$ and the line passing through $\overline{c^{*} t_{i}}$. The following lemma was proven in [1].

Lemma 3 ( 1$]$ ). The path $\Pi\left(c_{i}, c_{j}\right)$ does not intersect $\triangle\left(t_{i(j)}, c^{*}, t_{j(i)}\right)$, for any distinct $i, j \in$ $\{1,2,3\}$.

Let $g_{1}, g_{2}, g_{3}$ and $g_{4}$ be the points located at the coordinates $(2,0),(0,2),(-2,0)$, and $(0,-2)$, respectively. The following corollary follows from Lemma 3 and the assumption that $\ell_{1}$ is horizontal and the angle $\angle\left(m_{1,2}, m_{2,3}, m_{3,1}\right)$ is the largest in the triangle $\triangle\left(m_{1,2}, m_{2,3}, m_{3,1}\right)$.


Figure 2: The smallest disk $D^{*}$ is located at the origin. $\ell_{1}, \ell_{2}$, and $\ell_{3}$ are the tangent lines between $D^{*}$ and the disks $D_{1}, D_{2}$, and $D_{3}$, respectively. The path $\Pi\left(c_{i}, c_{j}\right)$ does not intersect $\triangle\left(t_{i(j)}, c^{*}, t_{j(i)}\right)$, for any distinct $i, j \in\{1,2,3\}$.

Corollary 1. The polygon $P$ does not intersect the triangles $\triangle\left(g_{1}, c^{*}, g_{4}\right)$ and $\triangle\left(g_{3}, c^{*}, g_{4}\right)$.
For a point $p \in P$, let $x(p)$ and $y(p)$ denote the $x$-coordinate and the $y$-coordinate of $p$, respectively. We divide the plane into 4 quadrants $Q_{1}, Q_{2}, Q_{3}$, and $Q_{4}$ as follows; see Figure 3 .

- $Q_{1}=\left\{p \in \mathbb{R}^{2}: x(p) \geq 0\right.$ and $\left.y(p) \geq 0\right\} ;$
- $Q_{2}=\left\{p \in \mathbb{R}^{2}: x(p) \leq 0\right.$ and $\left.y(p) \geq 0\right\} ;$
- $Q_{3}=\left\{p \in \mathbb{R}^{2}: x(p) \leq 0\right.$ and $\left.y(p) \leq 0\right\} ;$ and
- $Q_{4}=\left\{p \in \mathbb{R}^{2}: x(p) \geq 0\right.$ and $\left.y(p) \leq 0\right\}$.

For each $i \in\{1,2,3,4\}$, let $z_{i} \in Q_{i}$ be the point whose distance from $g_{i}, g_{i+1}$ and the boundary of $D^{*}$ is equal; see Figure 3. The computation of the points $z_{i}$ is not involved. For example, we compute $z_{1}$ by solving the following equations system:

$$
\begin{aligned}
& \left|z_{1} g_{1}\right|=\sqrt{\left(x\left(z_{1}\right)-2\right)^{2}+y\left(z_{1}\right)^{2}}=d, \\
& \left|z_{1} g_{2}\right|=\sqrt{x\left(z_{1}\right)^{2}+\left(y\left(z_{1}\right)-2\right)^{2}}=d, \\
& \left|z_{1} c^{*}\right|=\sqrt{x\left(z_{1}\right)^{2}+y\left(z_{1}\right)^{2}}=d+1 .
\end{aligned}
$$

This implies that $z_{1}=(a, a), z_{2}=(-a, a), z_{3}=(-a,-a)$, and $z_{4}=(a,-a)$, where $a=\frac{3}{4-2 \sqrt{2}} \approx$ 2.56. For $1 \leq i \leq 4$, let $p r_{i}$ be the region bounded by the parabola that contains all the points that are closer to $g_{i}$ than to $D^{*}$; see Figure 3. Moreover, let $t_{1}^{+}=(2,1.5), t_{1}^{-}=(2,-1.5), t_{2}^{+}=(-1.5,2)$, $t_{2}^{-}=(1.5,2), t_{3}^{+}=(-2,-1.5), t_{3}^{-}=(-2,1.5), t_{4}^{+}=(1.5,-2)$, and $t_{4}^{-}=(-1.5,-2)$. The following observations follow from the definition of $p r_{i}$.


Figure 3: The quadrants $Q_{i}$ and the points $z_{i}$, for each $1 \leq i \leq 4$. The segments $\overline{z_{i} g_{i}}$ and $\overline{z_{i-1} g_{i}}$ are entirely contained in the region bounded by the parabola $p r_{i}$ (depicted in purple).

Observation 5. Let $p$ be a point on $\overline{z_{i} g_{i}}$ or on $\overline{z_{i-1} g_{i}}$, where $i \in\{1,2,3,4\}$. Then, for any point $q$ on the boundary of $D^{*}$, we have $\left|p g_{i}\right| \leq|p q|$.

Observation 6. Let $p$ be a point on $\overline{t_{i}^{+} t_{i}^{-}}$, where $i \in\{1,2,3,4\}$. Then, for any point $q$ on the boundary of $D^{*}$, we have $\left|p g_{i}\right| \leq|p q|$.

Let $D \in \mathcal{D}$ be a disk with center $c$ and radius $r$. Throughout the rest of the paper, we use the following notations. For each $i \in\{1,2,3\}$, let $q_{i}$ be the intersection point of the path $\Pi\left(c, c_{i}\right)$ with the boundary of $D_{i}$. Let $q^{*}$ be the intersection point of the path $\Pi\left(c, c^{*}\right)$ with the boundary of $D^{*}$. Thus, $\left|\Pi\left(c, q^{*}\right)\right| \leq r$ and $\left|\Pi\left(c, q_{i}\right)\right| \leq r$, for each $i \in\{1,2,3\}$. Let $c^{\prime}$ be the point on $\Pi\left(c, c^{*}\right)$, such that the edge $\left(c^{\prime}, c^{*}\right)$ is the last edge in $\Pi\left(c, c^{*}\right)$. That is, $c^{\prime}$ is the first point on $\Pi\left(c, c^{*}\right)$ that is visible from $c^{*}$. Finally, let $\alpha_{2}$ (resp., $\alpha_{3}$ ) be the acute angle between $\ell_{2}$ (resp., $\ell_{3}$ ) and the $x$-axis; see Figure 4.

Observation 7. If the polygon $P$ intersects the segment $\overline{z_{4} g_{1}}$ or $\overline{z_{4} g_{4}}$, then $\alpha_{2}>\frac{\pi}{5}$; see Figure 4 . Similarly, if the polygon intersects the segment $\overline{z_{3} g_{3}}$ or $\overline{z_{3} g_{4}}$, then $\alpha_{3}>\frac{\pi}{5}$.

Proof. By Lemma 3, the polygon does not intersect the triangle $\triangle\left(t_{2(1)}, c^{*}, t_{1(2)}\right)$; see Figure 4 . Using a simple geometric calculation, for $\alpha_{2}=\frac{\pi}{5}$, the acute angle between $\ell_{1,2}$ and the $x$-axis is $\beta=\frac{\pi-\alpha_{2}}{2}=\frac{2 \pi}{5}$ and the coordinates of $m_{1,2}$ are $\left(\frac{\cos \alpha_{2}+1}{\sin \alpha_{2}},-1\right)$. Thus, for $\alpha_{2}=\frac{\pi}{5}, \ell_{1,2}$ passes through $z_{4}$. Therefore, for $0<\alpha_{2} \leq \frac{\pi}{5}$, the point $z_{4}$ is contained in the triangle $\triangle\left(t_{2(1)}, c^{*}, t_{1(2)}\right)$, and, the polygon cannot intersect the segment $\overline{z_{4} g_{1}}$.

In the following lemma, we show that, for each $i \in\{1,2,3,4\}$, if the polygon does not intersect the segments $\overline{z_{i} g_{i}}$ and $\overline{z_{i} g_{i+1}}$, then every disk $D \in \mathcal{D}$ with $c^{\prime}$ in $Q_{i}$ is pierced by at least one of the points $c^{*}, g_{i}$ or $g_{i+1}$.

Lemma 4. Let $D \in \mathcal{D}$ be a disk with $c^{\prime}$ in $Q_{i}$, where $i \in\{1,2,3,4\}$. If the polygon does not intersect the segments $\overline{z_{i} g_{i}}$ nor $\overline{z_{i} g_{i+1}}$, then $D$ contains at least one of the points $c^{*}, g_{i}$ or $g_{i+1}$.


Figure 4: For $0<\alpha_{2} \leq \frac{\pi}{5}$, the polygon cannot intersect the segment $\overline{z_{4} g_{1}}$.

Proof. Let $c$ and $r$ be the center and the radius of $D$, respectively. We distinguish between two cases:
Case 1: The path $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{i} g_{i}}$ or $\overline{z_{i} g_{i+1}}$ at a point $p$. Assume w.l.o.g., $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{i} g_{i}}$; see Figure 5 (for $i=1$ ). By Observation 5, we have $\left|p g_{i}\right| \leq\left|p q^{*}\right|$. Moreover, since the polygon does not intersect $\overline{z_{i} g_{i}}$, we have $\left|\Pi\left(c, g_{i}\right)\right| \leq|\Pi(c, p)|+\left|p g_{i}\right| \leq|\Pi(c, p)|+\left|p q^{*}\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$. Therefore, $D$ contains $g_{i}$.


Figure 5: $c^{\prime} \in Q_{1}$ and the path $\Pi\left(c^{\prime}, c^{*}\right)$ intersects $\overline{z_{1} g_{1}}$.
Case 2: The path $\Pi\left(c, c^{*}\right)$ does not intersect $\overline{z_{i} g_{i}}$ nor $\overline{z_{i} g_{i+1}}$. We prove this case for $i=1$; the proof of the other cases are symmetric. Consider the path $\Pi\left(c, c_{1}\right)$ and notice that it intersects the $x$-axis at a point $q$. Since $\left|\Pi\left(q, q_{1}\right)\right| \geq 1$, we have $|\Pi(c, q)|+1 \leq|\Pi(c, q)|+\left|\Pi\left(q, q_{1}\right)\right|=\left|\Pi\left(c, q_{1}\right)\right| \leq r$. By the case assumption, and by the fact that the polygon does not intersect $\overline{z_{1} g_{1}}$ nor $\overline{z_{1} g_{2}}, \Pi\left(c, c_{1}\right)$ has a vertex $p$ inside the quadrilateral defined by $c^{*}, g_{1}, z_{1}, g_{2}$, such that the polygon does not intersect the segment $\overline{p q}$; see Figure 6. Hence, $x\left(c^{*}\right) \leq x(q) \leq x\left(z_{1}\right)$ and $|\Pi(c, q)|=|\Pi(c, p)|+|p q|$. Moreover, $\left|c^{*} g_{1}\right|=2$, and, by Corollary 1 , the polygon does not intersect $\overline{c^{*} g_{1}}$.

- If $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}\right)$, then, since $\left|c^{*} g_{1}\right|=2$, we have $\left|c^{*} q\right| \leq 1$ or $\left|q g_{1}\right| \leq 1$, and by Observation 4, $D$ contains at least one of the points $c^{*}$ or $g_{1}$; see Figure 6(a).
- If $x\left(g_{1}\right)<x(q) \leq x\left(z_{1}\right)$, then, since $x\left(g_{1}\right)=2$ and $x\left(z_{1}\right)<3$, we have $\left|q g_{1}\right| \leq 1$, and thus $\left|\Pi\left(c, g_{1}\right)\right| \leq|\Pi(c, p)|+|p q|+\left|q g_{1}\right|<|\Pi(c, p)|+|p q|+1=|\Pi(c, q)|+1 \leq r$; see Figure 6(b). Therefore, $D$ contains $g_{1}$.

Notice that, by Corollary 1 , the polygon does not intersect $\overline{c^{*} g_{1}}, \overline{c^{*} g_{3}}$, nor $\overline{c^{*} g_{4}}$. Thus, for $i=3$ and $i=4$, in Case 2, we have $p=c=c^{\prime}$ is $D$ 's center.

(a)

(b)

Figure 6: The path $\Pi\left(c^{\prime}, c^{*}\right)$ does not intersect $\overline{\bar{z}_{1} g_{1}}$ nor $\overline{z_{1} g_{2}}$, (a) $p=c$. (b) $p \neq c$.
In the following, we define eight points $g_{i}^{+} \in Q_{i}$ and $g_{i}^{-} \in Q_{i-1}$, for each $i \in\{1,2,3,4\}$, and we prove some lemmas regarding these points. For each $i \in\{1,2,3,4\}$, let $m_{i}^{+}$(resp., $m_{i}^{-}$) be the tangent line to $D^{*}$ that passes through $g_{i}$ and has a positive (resp., negative) slope; see Figure 7 (for an illustration of $m_{1}^{+}$and $m_{1}^{-}$).

The point $g_{1}^{+}$is defined as follows. Let $D^{\prime}$ be the disk of radius 1 centered at the point $(1,0)$. We sweep with a line $\ell$ that is tangent to $D^{*}$ in counterclockwise order starting with $\ell=m_{1}^{+}$and we stop when $\ell$ intersects either $D_{3}$ or the polygon inside the quadrilateral defined by $c^{*}, g_{1}, z_{1}, g_{2}$; see Figure 7 (a). Let $u$ be the intersection point of $\ell$ with $D^{\prime}$ in $Q_{1}$ when we stop the sweeping. We also sweep upwards with a horizontal line $\ell_{h}$ that passes through the point $c^{*}$, and stop when $\ell_{h}$ intersects the polygon inside $D^{\prime}$, or when $\ell_{h}$ 's $y$-coordinate is 1 ; see Figure 7 (b). Let $w$ be the intersection point of $\ell_{h}$ with $D^{\prime}$ in $Q_{1}$ when we stop the sweeping. We set $g_{1}^{+}$as the lowest point among $u$ and $w$.

The point $g_{1}^{-}$is defined as follows. We sweep with a line $\ell$ that is tangent to $D^{*}$ in clockwise order starting with $\ell=m_{1}^{-}$and we stop when $\ell$ intersects either $D_{1}$ or the polygon inside the quadrilateral defined by $c^{*}, g_{1}, z_{4}, g_{4}$; see Figure 7 . Let $u$ be the intersection point of $m_{1}^{-}$with $D^{\prime}$ in $Q_{4}$ when we stop the sweeping. We also sweep downwards with a horizontal line $\ell_{h}$ that passes through the point $c^{*}$, and stop either when $\ell_{h}$ intersects the polygon inside $D^{\prime}$, or when $\ell_{h}$ 's $y$-coordinate is -1 . Let $w$ be the intersection point of $\ell_{h}$ with $D^{\prime}$ in $Q_{4}$ when we stop the sweeping. We set $g_{1}^{-}$as the highest point among $u$ and $w$.


Figure 7: Defining $g_{1}^{+}$and $g_{1}^{-}$. (a) $g_{1}^{+}$is defined by the intersection of $m_{1}^{+}$with $D_{3}$ and $g_{1}^{-}$is defined by the intersection of $m_{1}^{-}$with $D_{1}$. (b) $g_{1}^{+}$is defined by the intersection of $\ell_{h}$ with the polygon inside $D^{\prime}$ and $g_{1}^{-}$is defined by the intersection of $m_{1}^{-}$with the polygon outside $D^{\prime}$.

We define $g_{2}^{+}=(-1,1)$ and $g_{2}^{-}=(1,1)$, and we define $g_{3}^{+}, g_{3}^{-}, g_{4}^{+}$, and $g_{4}^{-}$similarly to $g_{1}^{+}$and $g_{1}^{-}$, where the sweeping line $\ell$ starts with $m_{3}^{+}, m_{3}^{-}, m_{4}^{+}$, and $m_{4}^{-}$, respectively. For $g_{3}^{+}$and $g_{3}^{-}, D^{\prime}$ is centered at $(-1,0)$ and, for $g_{4}^{+}$and $g_{4}^{-}, D^{\prime}$ is centered at $(0,-1)$.

Lemma 5. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$ and let $g_{i}^{\prime} \in\left\{g_{i}, g_{i}^{+}, g_{i}^{-}\right\}$, for each $i \in\{1,2,3,4\}$.
(i) If $c^{\prime} \in Q_{1}$, and $\Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}^{\prime}\right)$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{\prime}\right)\right| \leq r$; see Figure $8(a)$.
(ii) If $c^{\prime} \in Q_{2}$ and $\Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x\left(g_{3}^{\prime}\right) \leq x(q) \leq x\left(c^{*}\right)$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{3}^{\prime}\right)\right| \leq r$; see Figure $8(b)$.
(iii) If $c^{\prime} \in Q_{3}$ and $\Pi\left(c, c_{2}\right)$ intersects the $x$-axis at a point $q$ with $x\left(g_{3}^{\prime}\right) \leq x(q) \leq x\left(c^{*}\right)$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{3}^{\prime}\right)\right| \leq r$; see Figure $8(c)$.
(iv) If $c^{\prime} \in Q_{3}$ and $\Pi\left(c, c_{2}\right)$ intersects the $y$-axis at a point $q$ with $y\left(g_{4}^{\prime}\right) \leq y(q) \leq y\left(c^{*}\right)$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{\prime}\right)\right| \leq r$; see Figure $8(d)$.
(v) If $c^{\prime} \in Q_{4}$ and $\Pi\left(c, c_{3}\right)$ intersects the $x$-axis at a point $q$ with $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}^{\prime}\right)$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{\prime}\right)\right| \leq r$; see Figure $8(e)$.
(vi) If $c^{\prime} \in Q_{4}$ and $\Pi\left(c, c_{3}\right)$ intersects the $y$-axis at a point $q$ with $y\left(g_{4}^{\prime}\right) \leq y(q) \leq y\left(c^{*}\right)$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{\prime}\right)\right| \leq r$; see Figure $\delta(f)$.


Figure 8: Illustration of Lemma 5 (a) Item (i), (b) Item (ii), (c) Item (iii), (d) Item (iv), (e) Item (v), and (f) Item (vi).

Proof. We prove Item (i), the proofs of the other five items are symmetric.

Notice that $|\Pi(c, q)|+1 \leq r$. Moreover, by Corollary 1 , the polygon does not intersect $\overline{c^{*} g_{1}}$; see Figure 9.

- If $x\left(c^{*}\right) \leq x(q) \leq 1$, then, $\left|c^{*} q\right| \leq 1$; see Figure 9 (a). Thus, $\left|\Pi\left(c, c^{*}\right)\right| \leq|\Pi(c, q)|+\left|q c^{*}\right| \leq$ $|\Pi(c, q)|+1 \leq r$.
- If $1<x(q) \leq x\left(g_{1}^{\prime}\right)$, then $\left|\underline{q g_{1}^{\prime}}\right| \leq 1$; see Figure 9 (b). Moreover, by the definition of $g_{1}^{\prime}$, the polygon does not intersect $\overline{q g_{1}^{\prime}}$. Thus, $\left|\Pi\left(c, g_{1}^{\prime}\right)\right| \leq|\Pi(c, q)|+\left|q g_{1}^{\prime}\right| \leq|\Pi(c, q)|+1 \leq r$.

(a)

(b)

Figure 9: Illustration of the proof of Lemma 5. Item (i): (a) $x\left(c^{*}\right) \leq x(q) \leq 1$, and (b) $1 \leq x(q) \leq$ $x\left(g_{1}^{\prime}\right)$.

Lemma 6. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$.
(i) If $c^{\prime} \in Q_{1}, \Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(g_{1}^{+}\right), \Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{1} g_{2}}$ and the polygon intersects the segment $\overline{z_{2} g_{2}}$, then $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$; see Figure 10 (a).
(ii) If $c^{\prime} \in Q_{1}, \Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(g_{1}^{+}\right), \Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{1} g_{2}}$ and the polygon does not intersect the segment $\overline{z_{2} g_{2}}$, then $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{2}\right)\right| \leq r$; see Figure $10(b)$.
(iii) If $c^{\prime} \in Q_{3}, \Pi\left(c, c_{2}\right)$ intersects the $y$-axis at a point $q$ with $y(q)<y\left(g_{4}^{-}\right)$, and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{3} g_{3}}$, then, $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{3}\right)\right| \leq r$; see Figure $10(c)$.
(iv) If $c^{\prime} \in Q_{4}, \Pi\left(c, c_{3}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(g_{\perp}^{-}\right)$, and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{\bar{z}_{4} g_{4}}$, then, $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}\right)\right| \leq r$; see Figure $10(d)$.

Proof. We prove Items (i) and (ii), the proofs of the other two items are symmetric to the proof of Item (ii).

Let $\ell_{v}$ be the vertical line passing through $g_{1}^{+}$. Since $\Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(g_{1}^{+}\right), \Pi\left(c, c^{*}\right)$ intersects $\ell_{v}$ at a point $p$ and the polygon cannot intersect the segment


Figure 10: Illustration of Lemma 66 (a) Item (i), (b) Item (ii), (c) Item (iii), and (d) Item (iv).
$\overline{p g_{1}^{+}}$. Let $b=\left(b_{x}, 2\right)$ be the point on $\Pi\left(c, c^{*}\right)$; see Figure 12. We distinguish between two cases.
Case 1: $\quad b_{x} \leq \frac{3}{2}$.
Proof of item (i): Since the polygon intersects the segment $\overline{z_{2} g_{2}}$, we have $y\left(q^{*}\right) \leq y\left(g_{1}^{+}\right)$. Thus, the angle $\angle\left(p, g_{1}^{+}, q^{*}\right)$ is the largest in the triangle $\triangle\left(p, g_{1}^{+}, q^{*}\right)$; see Figure 11. Thus, $\left|p g_{1}^{+}\right| \leq\left|\Pi\left(p, q^{*}\right)\right|$. Therefore, $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq|\Pi(c, p)|+\left|p g_{1}^{+}\right| \leq|\Pi(c, p)|+\left|\Pi\left(, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$.

## Proof of item (ii):

- If the polygon does not intersect the segment $\overline{g_{2} b}$, then, by Observation $6,\left|b g_{2}\right| \leq\left|b q^{*}\right|$, and thus $\left|\Pi\left(c, g_{2}\right)\right|=|\Pi(c, b)|+\left|b g_{2}\right| \leq|\Pi(c, b)|+\left|b q^{*}\right| \leq\left|\Pi\left(c, q^{*}\right)\right| \leq r$; see Figure 12(a).
- Otherwise, the polygon intersects the segment $\overline{g_{2} b}$.
- If the polygon intersects the disk $D^{\prime}$, then $g_{1}^{+}$is defined as the intersection of the sweeping horizontal line $\ell_{h}$ with $D^{\prime}$, and thus $y\left(q^{*}\right) \leq y\left(g_{1}^{+}\right)$; see Figure $12(\mathrm{~b})$. Thus, $\left|p g_{1}^{+}\right| \leq\left|p q^{*}\right|$. Therefore, since the polygon does not intersect the segment $p g_{1}^{+}$, we have $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq$ $|\Pi(c, p)|+\left|p g_{1}^{+}\right| \leq|\Pi(c, p)|+\left|p q^{*}\right| \leq|\Pi(c, p)|+\left|\Pi\left(p, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$.


Figure 11: $b_{x} \leq \frac{3}{2}$ and the polygon intersects the segment $\overline{z_{2} g_{2}}$.

(a)

(b)

Figure 12: $b_{x} \leq \frac{3}{2}$ : (a) The polygon does not intersect the segment $\overline{g_{2} b}$. (b) The polygon intersects the disk $D^{\prime}$.

- Otherwise, let $\ell_{3}^{\prime}$ be the horizontal line that is tangent to $D^{*}$ at the point $(0,1)$ and let $D_{b}$ be the disk centered at $b$ and is tangent to $\ell_{3}^{\prime}$. Let $\ell_{b}$ be a tangent line of $D_{b}$ and $D^{*}$ as depicted in Figure 13(b). Let $g$ be the intersection point of $\ell_{b}$ with $D^{\prime}$ in $Q_{1}$. Since $g$ is below $g_{1}^{+}$on the boundary of $D^{\prime}$, we have $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq|\Pi(c, g)|$. Therefore, to prove that $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, it is sufficient to prove that $|\Pi(c, g)| \leq r$.
Let $\ell_{v}^{\prime}$ be the vertical line passing through $g$. Since $\Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(z_{1}\right), \Pi\left(c, c^{*}\right)$ intersects $\ell_{v}^{\prime}$ at a point $p^{\prime}$ and the polygon cannot intersect the segment $\overline{p^{\prime} g}$. Let $b_{x}$ denote the $x$-coordinate of $b$, and notice that the coordinates of $g=\left(g_{x}, g_{y}\right)$ depend on $b_{x}$. To compute the coordinates of $g$, we compute the intersection point between the tangent line $\ell_{b}$ with $D^{\prime}$ in $Q_{1}$. The equation of $\ell_{b}$ is


Figure 13: $b_{x} \leq \frac{3}{2}$ and the polygon intersects the segment $\overline{g_{2} b}$ but not $D^{\prime}$.
$y=\frac{2}{b_{x}} x-\frac{\sqrt{4+b_{x}^{2}}}{b_{x}}$, and the equation of $D^{\prime}$ is $(x-1)^{2}+y^{2}=1$. Hence, we have
$g_{x}=\frac{b_{x}^{2}+2 \sqrt{4+b_{x}^{2}}+2 b_{x} \sqrt{\sqrt{4+b_{x}^{2}}-1}}{4+b_{x}^{2}}$ and $g_{y}=\frac{2 b_{x}-b_{x} \sqrt{4+b_{x}^{2}}+4 \sqrt{\sqrt{4+b_{x}^{2}}-1}}{4+b_{x}^{2}}$.
Since $\frac{b_{x}^{2}+2 \sqrt{4+b_{x}^{2}}+2 b_{x} \sqrt{\sqrt{4+b_{x}^{2}}-1}}{4+b_{x}^{2}}-b_{x}>0$, for every $0<b_{x} \leq \frac{3}{2}$, we have $b_{x} \leq g_{x}$, for every $0<b_{x} \leq \frac{3}{2}$.
Let $g^{\prime}=\left(g_{x}, 2\right)$ and notice that the polygon does not intersect the segment $\overline{g^{\prime} g}$; see Figure 13(b). Moreover, since the polygon does not intersect the segment $\overline{p^{\prime} g}$, we have $\left|\Pi\left(c, g^{\prime}\right)\right| \leq|\Pi(c, b)|$. We now claim that

$$
\left|g^{\prime} g\right|=2-g_{y}<\sqrt{4+b_{x}^{2}}-1=\left|b q^{*}\right|, \text { for every } 0<b_{x} \leq g_{x} .
$$

That is, $3-\sqrt{4+b_{x}^{2}}<g_{y}=\frac{2 b_{x}-b_{x} \sqrt{4+b_{x}^{2}}+4 \sqrt{\sqrt{4+b_{x}^{2}}-1}}{4+b_{x}^{2}}$. To see the correctness of this inequality, we need to show that $\left(3-\sqrt{4+b_{x}^{2}}\right)\left(4+b_{x}^{2}\right)<2 b_{x}-b_{x} \sqrt{4+b_{x}^{2}}+4 \sqrt{\sqrt{4+b_{x}^{2}}-1}$. This is true since the left side of this inequality has maximum value equals 4 , when $b_{x}=0$, and the right side of this inequality has minimum value equals 4 , when $b_{x}=0$, for each $0<b_{x} \leq \frac{3}{2}$. Therefore, we have $|\Pi(c, g)| \leq\left|\Pi\left(c, g^{\prime}\right)\right|+\left|g^{\prime} g\right| \leq|\Pi(c, b)|+\left|b q^{*}\right| \leq$ $\left|\Pi\left(c, q^{*}\right)\right| \leq r$.

Case 2: $b_{x}>\frac{3}{2}$. We show that in this case we have $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, which proves both Items (i) and (ii). Let $a=\left(a_{x}, a_{y}\right)$ be the intersection point of $\Pi\left(c, c^{*}\right)$ with the segment $\overline{z_{1} g_{2}}$. Since $\Pi\left(p, c^{*}\right)$ intersects $\ell_{v}$, we have $a_{x} \geq b_{x}>\frac{3}{2}$. Since the polygon does not intersect the segment $\overline{a g_{1}^{+}}$, we have $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq|\Pi(c, a)|+\left|a g_{1}^{+}\right|$. Thus, it is sufficient to prove that $\left|a g_{1}^{+}\right| \leq\left|a q^{*}\right|$, for each $\frac{3}{2} \leq a_{x} \leq x\left(z_{1}\right)$.

Let $\ell_{3}^{\prime}$ be the horizontal line passing through point $(0,1)$ and let $D_{a}$ be the disk centered at $a$ and is tangent to $\ell_{3}^{\prime}$. Let $\ell_{a}$ be a tangent line to $D_{a}$ and $D^{*}$ as depicted in Figure 14 (b). Let $g$ be the intersection point of $\ell_{a}$ with $D^{\prime}$ in $Q_{1}$. We distinguish between two cases.

(a)

(b)

Figure 14: $b_{x}>\frac{3}{2}$ and $y(g) \leq y\left(g_{1}^{+}\right)$.
Case 2.1: $y(g) \leq y\left(g_{1}^{+}\right)$(i.e., $g$ is below $g_{1}^{+}$on the boundary of $D^{\prime}$ ). Since $1 \leq x\left(g_{1}^{+}\right) \leq 2$, we have $\left|a g_{1}^{+}\right| \leq|a g|$. Therefore, to prove the lemma, it is sufficient to prove that $|a g| \leq\left|a q^{*}\right|$, for each $\frac{3}{2} \leq a_{x} \leq x\left(z_{1}\right)$.

Since $a$ is on the segment $\overline{z_{1} g_{2}}$ and the equation of the line passing through $z_{1}$ and $g_{2}$ is $y=\frac{4 \sqrt{2}-5}{3} x+2$, we have $a_{y}=\frac{(4 \sqrt{2}-5) a_{x}}{3}+2$. Notice that $x(g)$ and $y(g)$ (the coordinates of $g)$ depend on $a_{x}$. To compute these coordinates, we compute the intersection point between the tangent line $\ell_{a}$ with $D^{\prime}$ in $Q_{1}$. The equation of $\ell_{a}$ is $y=\left(x+\frac{3}{4 \sqrt{2}-5}\right) m-\frac{6}{(4 \sqrt{2}-5) a_{x}}-1$, where

$$
\begin{aligned}
m=\frac{1}{8(5 \sqrt{2}-6) a_{x}} & \left((12 \sqrt{2}-15) a_{x}+18-\right. \\
& \left.\sqrt{-3\left((120 \sqrt{2}-171) a_{x}^{2}-(1712 \sqrt{2}-2420) a_{x}+480 \sqrt{2}-684\right)}\right) .
\end{aligned}
$$

Let $t$ be the point on the segment $\overline{z_{1} g_{2}}$, where $x(t)=\frac{3}{2}$. We prove that $|a g| \leq\left|a q^{*}\right|$, for each $\frac{3}{2} \leq a_{x} \leq x\left(z_{1}\right)$ by dividing the segment $\overline{t z_{1}}$ into 7 intervals defined by the points $t_{0}, t_{1}, \ldots, t_{7}$, where $x\left(t_{0}\right)=x(t)=\frac{3}{2}, x\left(t_{1}\right)=1.52, x\left(t_{2}\right)=1.56, x\left(t_{3}\right)=1.63, x\left(t_{4}\right)=1.74, x\left(t_{5}\right)=1.9$, $x\left(t_{6}\right)=2.15$, and $x\left(t_{7}\right)=x\left(z_{1}\right)=\frac{3(2+\sqrt{2})}{4}$. For each $1 \leq i \leq 7$, we compute the intersection point $g_{i}^{\prime}$ of $l_{a}$ with the disk $D^{\prime}$, where $a=t_{i}$, and we show that $\left|a g_{i}^{\prime}\right|=\sqrt{\left(a_{x}-x\left(g_{i}^{\prime}\right)\right)^{2}+\left(a_{y}-y\left(g_{i}^{\prime}\right)\right)^{2}} \leq$ $\sqrt{a_{x}^{2}+a_{y}^{2}}-1=\left|a q^{*}\right|$, for each $x\left(t_{i-1}\right) \leq a_{x} \leq x\left(t_{i}\right)$.

- For $i=1$, we have $g_{1}^{\prime}=(1.8033,0.5955)$, and thus

$$
\sqrt{\left(a_{x}-1.8033\right)^{2}+\left(a_{y}-0.5955\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 1.5 \leq a_{x} \leq 1.52 .
$$

- For $i=2$, we have $g_{2}^{\prime}=(1.8152,0.5792)$, and thus

$$
\sqrt{\left(a_{x}-1.8152\right)^{2}+\left(a_{y}-0.5792\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 1.52 \leq a_{x} \leq 1.56
$$

- For $i=3$, we have $g_{3}^{\prime}=(1.8347,0.5507)$, and thus

$$
\sqrt{\left(a_{x}-1.8347\right)^{2}+\left(a_{y}-0.5507\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 1.56 \leq a_{x} \leq 1.63
$$

- For $i=4$, we have $g_{4}^{\prime}=(1.8623,0.5063)$, and thus

$$
\sqrt{\left(a_{x}-1.8623\right)^{2}+\left(a_{y}-0.5063\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 1.63 \leq a_{x} \leq 1.74
$$

- For $i=5$, we have $g_{5}^{\prime}=(1.8966,0.4429)$, and thus

$$
\sqrt{\left(a_{x}-1.8966\right)^{2}+\left(a_{y}-0.4429\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 1.174 \leq a_{x} \leq 1.9 .
$$

- For $i=6$, we have $g_{6}^{\prime}=(1.9376,0.3478)$, and thus

$$
\sqrt{\left(a_{x}-1.9376\right)^{2}+\left(a_{y}-0.3478\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 1.9 \leq a_{x} \leq 2.15 .
$$

- For $i=7$, we have $g_{7}^{\prime}=(1.9787,0.2053)$, and thus

$$
\sqrt{\left(a_{x}-1.9787\right)^{2}+\left(a_{y}-0.2053\right)^{2}} \leq \sqrt{a_{x}^{2}+a_{y}^{2}}-1, \text { for each } 2.15 \leq a_{x} \leq \frac{3(2+\sqrt{2})}{4}
$$

These inequalities hold since the function $\sqrt{a_{x}^{2}+a_{y}^{2}}-1-\sqrt{\left(a_{x}-x\left(g_{i}^{\prime}\right)\right)^{2}+\left(a_{y}-y\left(g_{i}^{\prime}\right)\right)^{2}}$ is monotonic in the interval $x\left(t_{i-1}\right) \leq a_{x} \leq x\left(t_{i}\right)$, and has minimum value when $a_{x}=x\left(t_{i-1}\right)$, for each $1 \leq i \leq 7$. Thus, for each $a_{x} \leq x\left(t_{i}\right)$, where $1 \leq i \leq 7$, we have $|a g| \leq\left|a g_{i}^{\prime}\right|$. This proves that $|a g| \leq\left|a q^{*}\right|$.

(a)

(b)

Figure 15: $b_{x}>\frac{3}{2}$ and $y\left(g^{\prime}\right)>y\left(g_{1}^{+}\right)$.
Case 2.2: $y(g)>y\left(g_{1}^{+}\right)$(i.e., $g$ is above $g_{1}^{+}$on the boundary of $D^{\prime}$ ). Observe that this case can happen only if the polygon intersects the line $\ell_{3}^{\prime}$. Recall that $\ell_{v}$ is the vertical line passing
through $g_{1}^{+}, p$ is the intersection point of $\Pi\left(c, c^{*}\right)$ with $\ell_{v}$, and the polygon does not intersect the segment $\overline{p g_{1}^{+}}$. Let $\ell_{h}$ be the horizontal line passing through $g_{1}^{+}$. Since the polygon intersects $\ell_{3}^{\prime}$, we have $y\left(q^{*}\right) \leq y\left(g_{1}^{+}\right)$, i.e., $q^{*}$ is below $\ell_{h}$; see Figure 15 . Thus, the angle $\angle\left(p, g_{1}^{+}, q^{*}\right) \geq \frac{\pi}{2}$, and, since the polygon does not intersect $p g_{1}^{+}$, we have $\left|p g_{1}^{+}\right| \leq\left|\Pi\left(p, q^{*}\right)\right|$. Therefore, $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq$ $|\Pi(c, p)|+\left|p g_{1}^{+}\right| \leq|\Pi(c, p)|+\left|\Pi\left(p, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$.

Lemma 7. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$.
(i) If $c^{\prime} \in Q_{1}, \Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(g_{1}^{+}\right)$, and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{1} g_{1}}$, then $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$; see Figure $16(a)$.
(ii) if $c^{\prime} \in Q_{3}, \Pi\left(c, c_{2}\right)$ intersects the $y$-axis at a point $q$ with $y(q)<x\left(g_{4}^{-}\right)$, and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{3} g_{4}}$, then $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$; see Figure 16 (b).
(iii) If $c^{\prime} \in Q_{4}, \Pi\left(c, c_{3}\right)$ intersects the $x$-axis at a point $q$ with $x(q)>x\left(g_{1}^{-}\right)$, and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{4} g_{1}}$, then $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$; see Figure $16(c)$.


Figure 16: Illustration of Lemma 7 7 Item (i), (b) Item (ii), and (c) Item (iii).

Proof. We prove Item (i), the proofs of the other items are symmetric.
Let $\ell_{v}$ be the vertical line passing through $g_{1}^{+}$, and let $p$ be the intersection point of $\Pi\left(c, c^{*}\right)$ with $\ell_{v}$; see Figure 17. We distinguish between two cases.
Case 1: $y(p) \geq y\left(g_{1}^{+}\right)$; see Figure 17(a). Let $a$ be the intersection point of $\Pi\left(c, c^{*}\right)$ with $\overline{z_{1} g_{1}}$. By the definition of $z_{1}$, we have $\left|a g_{1}\right| \leq\left|a q^{*}\right|$, and thus $\left|\Pi\left(c, g_{1}\right)\right| \leq r$. Moreover, since $p$ is above $g_{1}^{+}$, $g_{1}^{+}$is inside the pseudo-triangle $\triangle\left(c, q^{*}, g_{1}\right)$; see Figure 17 (a). Thus, by Observation 3, $D$ contains $g_{1}^{+}$, and therefore $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$.

(a)

(b)

(c)

Figure 17: Illustration of the proof of Lemma 7. Item (i): (a) $y(p) \geq y\left(g_{1}^{+}\right)$, (b) $y(p)<y\left(g_{1}^{+}\right)$and $\Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$ above $g_{1}^{+}$, and (c) $y(p)<y\left(g_{1}^{+}\right)$and $\Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$ below $g_{1}^{+}$.

Case 2: $y(p)<y\left(g_{1}^{+}\right)$; see Figure 17(b). Let $\ell_{t}$ be the line that is tangent to $D^{*}$ and passes through $g_{1}^{+}$, and observe that $\Pi\left(c, c_{3}\right)$ intersects this line.

- If $\Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$ above $g_{1}^{+}$, then $g_{1}^{+}$is inside the pseudo-triangle $\triangle\left(c, q^{*}, q_{3}\right)$; see Figure 17 (b). Thus, by Observation $3, D$ contains $j g_{1}^{+}$, and therefore $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$.
- If $\Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$ below $g_{1}^{+}$, then let $\ell_{h}$ be the horizontal line passing through $g_{1}^{+}$; see Figure 17 (c). Let $a$ be the intersection point of $\Pi\left(c, c_{3}\right)$ with the boundary of $D^{\prime}$, and let $b$ be the intersection point of $\Pi\left(c, c_{3}\right)$ with $\ell_{h}$. Observe that $x(b) \leq x\left(g_{1}^{+}\right) \leq x(a)$ and $y(b)=$ $y\left(g_{1}^{+}\right) \geq y(a)$. Hence, the angle $\angle\left(a, g_{1}^{+}, b\right)$ is the largest in the triangle $\triangle\left(a, g_{1}^{+}, b\right)$. Thus, $\left|a g_{1}^{+}\right| \leq|a b| \leq\left|\Pi\left(a, q_{3}\right)\right|$. Therefore, $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq|\Pi(c, a)|+\left|a g_{1}^{+}\right| \leq|\Pi(c, a)|+\left|\Pi\left(a, q_{3}\right)\right|=$ $\left|\Pi\left(c, q_{3}\right)\right| \leq r$.

Lemma 8. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$.
(i) If $c^{\prime} \in Q_{1}$ and $\Pi\left(c, c^{*}\right)$ does not intersect the segments $\overline{z_{1} g_{1}}$ nor $\overline{z_{1} g_{2}}$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$; see Figure $18(a)$.
(ii) If $c^{\prime} \in Q_{3}, \Pi\left(c, c^{*}\right)$ does not intersect the segments $\overline{z_{3} g_{4}}$ nor $\overline{z_{3} g_{3}}$, and $\Pi\left(c, c_{2}\right)$ intersects the negative $y$-axis, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$; see Figure $18(b)$.
(iii) If $c^{\prime} \in Q_{4}, \Pi\left(c, c^{*}\right)$ does not intersect the segments $\overline{\bar{z}_{4} g_{4}}$ nor $\overline{\bar{z}_{4} g_{1}}$, and $\Pi\left(c, c_{3}\right)$ intersects the positive $x$-axis, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$; see Figure $18(c)$.

(a)

(b)

(c)

Figure 18: Illustration of Lemma 8: (a) Item (i), (b) Item (ii), and (c) Item (iii).

Proof. We prove Item (i), the proofs of the other two items are symmetric.
Since $c^{\prime} \in Q_{1}$, the path $\Pi\left(c, c_{1}\right)$ intersects the positive $x$-axis at a point $q$. If $x\left(c^{*}\right) \leq x(q) \leq$ $x\left(g_{1}^{+}\right)$, then by Lemma $5,\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$. Otherwise, $x\left(g_{1}^{+}\right)<x(q) \leq x\left(z_{1}\right)$. Let $\ell_{h}$ be the horizontal line passing through $g_{1}^{+}$.

- If $\Pi\left(c, c_{1}\right)$ intersects $\ell_{h}$, then let $p$ be this intersection point; see Figure 19(a). Thus, $x\left(g_{1}^{+}\right) \leq$ $x(p) \leq x\left(z_{1}\right)$ and the polygon does not intersect the segment $\overline{g_{1}^{+} p}$. Let $q_{1}$ be the intersection point of $\Pi\left(c, c_{1}\right)$ with $\ell_{1}$. Since, $\left|p g_{1}^{+}\right|=|p q|+(x(p)-2), x(p)<3$, and $\left|p q_{1}\right|=|p q|+1$, we have $\left|p g_{1}^{+}\right|<\left|p q_{1}\right| \leq\left|\Pi\left(p, q_{1}\right)\right|$. Therefore, since $\left|\Pi\left(c, q_{1}\right)\right| \leq r$, we have $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq$ $|\Pi(c, p)|+\left|p g_{1}^{+}\right| \leq|\Pi(c, p)|+\left|\Pi\left(p, q_{1}\right)\right|=\left|\Pi\left(c, q_{1}\right)\right| \leq r$.
- If $\Pi\left(c, c_{1}\right)$ does not intersect $\ell_{h}$, then let $\ell_{t}$ be the tangent to $D^{*}$ with positive slope that passes through $g_{1}^{+}$; see Figure 19(b).
- If $\Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$ above $g_{1}^{+}$, then $g_{1}^{+}$is inside the pseudo-triangle $\triangle\left(c, q^{*}, q_{3}\right)$. Thus, by Observation 3, $D$ contains $g_{1}^{+}$, and therefore $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$.
- If $\Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$ below $g_{1}^{+}$, then let $\ell_{h}$ be the horizontal line passing through $g_{1}^{+}$. Let $a$ be the intersection point of $\Pi\left(c, c_{3}\right)$ with the boundary of $D^{\prime}$, and let $b$ be the intersection point of $\Pi\left(c, c_{3}\right)$ with $\ell_{h}$. Observe that $x(b) \leq x\left(g_{1}^{+}\right) \leq x(a)$ and $y(b)=$ $y\left(g_{1}^{+}\right) \geq y(a)$. Hence, the angle $\angle\left(a, g_{1}^{+}, b\right)$ is the largest in the triangle $\triangle\left(a, g_{1}^{+}, b\right)$. Thus, $\left|a g_{1}^{+}\right| \leq|a b| \leq\left|\Pi\left(a, q_{3}\right)\right|$. Therefore, $\Pi\left(c, g_{1}^{+}\right) \leq|\Pi(c, a)|+\left|a g_{1}^{+}\right| \leq|\Pi(c, a)|+\left|\Pi\left(a, q_{3}\right)\right|=$ $\left|\Pi\left(c, q_{3}\right)\right| \leq r$.

(a)

(b)

Figure 19: Illustration of the proof of Lemma 8 . Item (i): (a) $\Pi\left(c, c_{1}\right)$ intersects $\ell_{h}$, and (b) $\Pi\left(c, c_{1}\right)$ does not intersect $\ell_{h}$.

Lemma 9. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$.
(i) If $c^{\prime} \in Q_{1}$ and $g_{1}^{-} \neq g_{1}$, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$; see Figure 20 (a).
(ii) If $c^{\prime} \in Q_{4}, g_{1}^{+} \neq g_{1}$ and $\Pi\left(c, c_{3}\right)$ intersects the positive $x$-axis, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$; see Figure 20 (b)
(iii) If $c^{\prime} \in Q_{4}, g_{4}^{-} \neq g_{4}$ and $\Pi\left(c, c_{3}\right)$ intersects the negative $y$-axis, then $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$; see Figure $20(c)$.


Figure 20: Illustration of Lemma 9, (a) Item (i), (b) Item (ii), and (c) Item (iii).

Proof. We prove Item (i), the proofs of the other two items are symmetric.
Since $c^{\prime} \in Q_{1}, \Pi\left(c, c_{1}\right)$ intersects the positive $x$-axis at a point $q$. If $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}^{-}\right)$, then, by Lemma 5, $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$. Otherwise, $x(q)>x\left(g_{1}^{-}\right)$. Let $\ell_{t}$ be the line of negative slope that is tangent to $D^{*}$ and passes through $g_{1}^{-}$, and observe that if $g_{1}^{-} \neq g_{1}$, then $\Pi\left(c, c_{1}\right)$ intersects this line; see Figure 21 .

- If $\Pi\left(c, c_{1}\right)$ intersects $\ell_{t}$ below $g_{1}^{-}$, then $g_{1}^{-}$is inside the pseudo-triangle $\triangle\left(c, q^{*}, q_{1}\right)$; see Figure 21 (a). Thus, by Observation 3, $D$ contains $g_{1}^{-}$, and therefore $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$.
- If $\Pi\left(c, c_{1}\right)$ intersects $\ell_{t}$ above $g_{1}^{-}$, then let $\ell_{h}$ be the horizontal line passing through $g_{1}^{-}$; see Figure 21(b). Let $a$ be the intersection point of $\Pi\left(c, c_{1}\right)$ with the boundary of $D^{\prime}$, and let $b$ be the intersection point of $\Pi\left(c, c_{1}\right)$ with $\ell_{h}$. Observe that $x(b) \leq x\left(g_{1}^{-}\right) \leq x(a)$ and $y(b)=$
$y\left(g_{1}^{-}\right) \leq y(a)$. Hence, the angle $\angle\left(a, g_{1}^{-}, b\right)$ is the largest in the triangle $\triangle\left(a, g_{1}^{-}, b\right)$. Thus, $\left|a g_{1}^{-}\right| \leq|a b| \leq\left|\Pi\left(a, q_{1}\right)\right|$. Therefore, $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq|\Pi(c, a)|+\left|a g_{1}^{-}\right| \leq|\Pi(c, a)|+\left|\Pi\left(a, q_{1}\right)\right|=$ $\left|\Pi\left(c, q_{1}\right)\right| \leq r$.


Figure 21: Illustration of the proof of Lemma 9. Item (i): (a) $\Pi\left(c, c_{1}\right)$ intersects $\ell_{t}$ below $g_{1}^{-}$, and (b) $\Pi\left(c, c_{1}\right)$ intersects $\ell_{t}$ above $g_{1}^{-}$.

Lemma 10. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$ and $c^{\prime} \in Q_{2}$, such that $\Pi\left(c, c_{1}\right)$ intersects the $x$-axis at a point $q$ with $x(q)<x\left(z_{2}\right)$, and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{2} g_{2}}$. If $\alpha_{2}>\frac{\pi}{3}$, then $\left|\Pi\left(c, g_{2}^{+}\right)\right| \leq r$.

Proof. Let $\ell_{v}$ be the vertical line passing through $g_{2}^{+}$and let $p$ be the intersection point of $\Pi\left(c, c^{*}\right)$ with $\ell_{v}$; see Figure 22 .

- If $y(p) \geq y\left(g_{2}^{+}\right)=1$, then, since $y\left(q^{*}\right) \leq 1$, the angle $\angle\left(p, g_{2}^{+}, q^{*}\right)$ is the largest in the triangle $\triangle\left(p, g_{2}^{+}, q^{*}\right)$; see Figure 22(a). Since the polygon does not intersect $\overline{p g_{2}^{+}}$, we have $\left|p g_{2}^{+}\right| \leq$ $\left|\Pi\left(p, q^{*}\right)\right|$. Therefore, $\left|\Pi\left(c, g_{2}^{+}\right)\right| \leq|\Pi(c, p)|+\left|p g_{2}^{+}\right| \leq|\Pi(c, p)|+\left|\Pi\left(p, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$.
- If $y(p)<y\left(g_{2}^{+}\right)$, then consider the path $\Pi\left(c, c_{2}\right)$ and notice that, since $\alpha_{2}>\frac{\pi}{3}$, this path intersects $\ell_{v}$ at a point $a$; see Figure 22(b). If $y(a) \geq y\left(g_{2}^{+}\right)$, then $g_{2}^{+}$is inside the pseudo triangle $\triangle\left(c, q^{*}, q_{2}\right)$, and by Observation $3, D$ contains $g_{2}^{+}$. Otherwise, $0 \leq y(a)<y\left(g_{2}^{+}\right)$. In this case, $\left|a g_{2}^{+}\right| \leq \underline{1}$, and, since $\alpha_{2}>\frac{\pi}{3}$, we have $\left|\Pi\left(a, q_{2}\right)\right|>1$. Moreover, since the polygon does not intersect $\overline{a g_{2}^{+}}$, we have $\left|a g_{2}^{+}\right|<\left|\Pi\left(a, q_{2}\right)\right|$. Therefore, $\left|\Pi\left(c, g_{2}^{+}\right)\right| \leq|\Pi(c, a)|+\left|a g_{2}^{+}\right|<$ $|\Pi(c, a)|+\left|\Pi\left(a, q_{2}\right)\right|=\left|\Pi\left(c, q_{2}\right)\right| \leq r$.

Lemma 11. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$ and $c^{\prime} \in Q_{4}$, such that $\Pi\left(c, c_{3}\right)$ intersects the $x$-axis at a point $q$ where $x(q)>x\left(g_{1}\right)$. If $\alpha_{3}>\frac{\pi}{6}$, then

- if $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{1}}$, then $\left|\Pi\left(c, g_{1}\right)\right| \leq r$; and


Figure 22: Illustration of the proof of Lemma 10, (a) $\Pi\left(c, c^{*}\right)$ intersects $\ell_{v}$ above $g_{2}^{+}$, and (b) $\Pi\left(c, c^{*}\right)$ intersects $\ell_{v}$ below $g_{2}^{+}$.

- if $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$, then $\left|\Pi\left(c, g_{1}\right)\right| \leq r$ and $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$.

Proof. Let $\ell_{t}$ be the line tangent to $D^{*}$ with a positive slope that passes through $g_{1}$, and notice that the acute angle between $\ell_{t}$ and the $x$-axis is $\frac{\pi}{3}$. Since $\alpha_{3}>\frac{\pi}{6}, \Pi\left(c, c_{3}\right)$ intersects $\ell_{t}$. Moreover, since $x(q)>x\left(g_{1}\right), g_{1}$ is inside the pseudo-triangle $\triangle\left(c, q^{*}, q_{3}\right)$; see Figure 23(a). Thus, by Observation 3 . $D$ contains $g_{1}$, and therefore $\left|\Pi\left(c, g_{1}\right)\right| \leq r$. Let $g=(-1,-1)$, and notice that $g_{1}^{-}$is on the small arc $\overparen{g_{1} g}$ of $D^{\prime}$ between $g_{1}$ and $g$; see Figure 23 (b). If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$, then $\overparen{g_{1} g}$ is contained in the pseudo-triangle $\triangle\left(c, q^{*}, q_{3}\right)$. Thus, by Observation $3, D$ contains both $g_{1}$ and $g_{1}^{-}$. Therefore, $\left|\Pi\left(c, g_{1}\right)\right| \leq r$ and $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$.

(a)

(b)

Figure 23: Illustration of the proof of Lemma 11; (a) $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{1}}$, and (b) $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$.

The following lemma and its proof is symmetric to Lemma 11 .
Lemma 12. Let $D \in \mathcal{D}$ be a disk centered at $c$ with radius $r$ and $c^{\prime} \in Q_{4}$, such that $\Pi\left(c, c_{3}\right)$ intersects the $y$-axis at a point $q$ where $y(q)<y\left(g_{4}\right)$. If $\alpha_{3} \leq \frac{\pi}{3}$, then

- if $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$, then $\left|\Pi\left(c, g_{4}\right)\right| \leq r$; see Figure 24(a); and
- if $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{1}}$, then $\left|\Pi\left(c, g_{4}\right)\right| \leq r$ and $\left|\Pi\left(c, g_{4}^{+}\right)\right| \leq r$; see Figure 24(b).

(a)

(b)

Figure 24: Illustration of Lemma 12, (a) $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$, and (b) $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{1}}$.

## 3 The Algorithm

In this section, we show how to compute a set $S$ of five points that pierce all the disks of $\mathcal{D}$. The algorithm, in a high-level description, works as follows. It first initializes $S$ by $\left\{c^{*}\right\}$. Then, it goes over the segments $\overline{z_{i} g_{i}}, \overline{z_{i} g_{i+1}}$, for each $i=1,2,3,4$ (in a fixed order), and, for each segment, it checks whether the polygon intersects the segment, and adds to $S$ a point $g_{i}^{\prime} \in\left\{g_{i}, g_{i}^{+}, g_{i}^{-}\right\}$.

Recall that $\alpha_{2}$ (resp., $\alpha_{3}$ ) is the acute angle between $\ell_{2}$ (resp., $\ell_{3}$ ) and the $x$-axis, and notice that at most one of them is greater than $\frac{\pi}{3}$. We distinguish between three cases:
(i) $\alpha_{2}>\frac{\pi}{3}$;
(ii) $\alpha_{3}>\frac{\pi}{3}$;
(iii) $\alpha_{2} \leq \frac{\pi}{3}$ and $\alpha_{3} \leq \frac{\pi}{3}$.

Notice that Case (i) and Case (ii) are symmetric. In Algorithm 1, we describe how to compute $S$ in Case (i), and, in Algorithm 2, we describe how to compute $S$ in Case (iii).

```
Algorithm 1 Compute \(S\) when \(\alpha_{2}>\frac{\pi}{3}\)
    \(g_{1}^{\prime} \leftarrow g_{1}, g_{2}^{\prime} \leftarrow g_{2}, g_{3}^{\prime} \leftarrow g_{3}, g_{4}^{\prime} \leftarrow g_{4}\)
    if \(P\) does not intersect \(\overline{z_{1} g_{1}}\) then
        if \(P\) intersects \(\overline{z_{1} g_{2}}\) or \(\overline{z_{2} g_{2}}\) then
            \(g_{1}^{\prime} \leftarrow g_{1}^{+}\)
            if \(P\) intersects \(\overline{z_{2} g_{2}}\) then
                \(g_{2}^{\prime} \leftarrow g_{2}^{+}\)
        else
            if \(P\) intersects \(\overline{z_{4} g_{4}}\) then
                \(g_{1}^{\prime} \leftarrow g_{1}^{-}\)
    if \(P\) does not intersect \(\overline{z_{2} g_{3}}\) then
        if \(P\) intersects \(\overline{z_{2} g_{2}}\) then
            \(g_{2}^{\prime} \leftarrow g_{2}^{+}\)
    if \(P\) does not intersect \(\overline{z_{3} g_{4}}\) then
        if \(P\) intersects \(\overline{z_{3} g_{3}}\) then
            \(g_{4}^{\prime} \leftarrow g_{4}^{-}\)
    return \(S=\left\{c^{*}, g_{1}^{\prime}, g_{2}^{\prime}, g_{3}^{\prime}, g_{4}^{\prime}\right\}\)
```

```
Algorithm 2 Compute \(S\) when \(\alpha_{2} \leq \frac{\pi}{3}\) and \(\alpha_{3} \leq \frac{\pi}{3}\)
    \(g_{1}^{\prime} \leftarrow g_{1}, g_{2}^{\prime} \leftarrow g_{2}, g_{3}^{\prime} \leftarrow g_{3}, g_{4}^{\prime} \leftarrow g_{4}\)
    if \(P\) does not intersect \(\overline{z_{1} g_{1}}\) then
        if \(P\) intersects \(\overline{z_{1} g_{2}}\) then
            \(g_{1}^{\prime} \leftarrow g_{1}^{+}\)
    if \(P\) does not intersect \(\overline{z_{2} g_{3}}\) then
        if \(P\) intersects \(\overline{z_{2} g_{2}}\) then
            \(g_{3}^{\prime} \leftarrow g_{3}^{-}\)
    if \(P\) does not intersect \(\overline{z_{1} g_{1}}, \overline{z_{1} g_{2}}, \overline{z_{2} g_{2}}\) nor \(\overline{z_{2} g_{3}}\) then
        if \(P\) intersects \(\overline{z_{3} g_{4}}\) then
            \(g_{3}^{\prime} \leftarrow g_{3}^{+}\)
        if \(P\) intersects \(\overline{z_{4} g_{4}}\) then
            \(g_{1}^{\prime} \leftarrow g_{1}^{-}\)
    return \(S=\left\{c^{*}, g_{1}^{\prime}, g_{2}^{\prime}, g_{3}^{\prime}, g_{4}^{\prime}\right\}\)
```


## 4 Correctness

Let $D \in \mathcal{D}$ be a disk with center $c$ and radius $r$. For each $i \in\{1,2,3\}$, let $q_{i}$ be the intersection point of the path $\Pi\left(c, c_{i}\right)$ with the line $\ell_{i}$. Let $q^{*}$ be the intersection point of the path $\Pi\left(c, c^{*}\right)$ with the boundary of $D^{*}$, and let $c^{\prime}$ be the point on $\Pi\left(c, c^{*}\right)$, such that the edge ( $c^{\prime}, c^{*}$ ) is the last edge in $\Pi\left(c, c^{*}\right)$. That is, $c^{\prime}$ is the first point on $\Pi\left(c, c^{*}\right)$ that is visible from $c^{*}$. We prove that the set $S=\left\{c^{*}, g_{1}^{\prime}, g_{2}^{\prime}, g_{3}^{\prime}, g_{4}^{\prime}\right\}$ (that is computed by the algorithm) pierces all the disks of $\mathcal{D}$. In the proof, we distinguish between three cases: (i) $\alpha_{2}>\frac{\pi}{3}$; (ii) $\alpha_{3}>\frac{\pi}{3}$; and (iii) $\alpha_{2} \leq \frac{\pi}{3}$ and $\alpha_{3} \leq \frac{\pi}{3}$. Following the algorithm, we show in Section 4.1 the proof for Case (i) and in Section 4.2 the proof for Case (iii) (since Case (i) and Case (ii) are symmetric).

### 4.1 Case (i): $\alpha_{2}>\frac{\pi}{3}$

Let $D \in \mathcal{D}$ be a disk with center $c$ and radius $r$. We show that $D$ is pierced by at least one of the points of $S$. We distinguish between four cases according to which quadrant $c^{\prime}$ belongs to.

### 4.1.1 $\quad c^{\prime} \in Q_{1}$

We prove that $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{2}^{\prime}$, or $c^{*}$. We distinguish between four cases.
Case 1: The polygon does not intersect $\overline{z_{1} g_{1}}, \overline{z_{1} g_{2}}, \overline{z_{2} g_{2}}$, nor $\overline{z_{4} g_{4}}$. In this case, $g_{1}^{\prime}=g_{1}$ and $g_{2}^{\prime}=g_{2}$, and by Lemma 4, $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{2}^{\prime}$, and $c^{*}$.
Case 2: The polygon intersects $\overline{z_{1} g_{1}}$; see Figure 25. In this case, $g_{1}^{\prime}=g_{1}$. Consider the path $\Pi\left(c, c_{1}\right)$ and notice that this path intersects the positive $x$-axis. Let $q$ be this intersection point. Thus, $|\Pi(c, q)|+1 \leq r$.
(i) If $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}\right)$, then by Lemma 5. Item (i), $D$ contains at least one of the points $c^{*}$ or $g_{1}$; see Figure 25(a).
(ii) If $x\left(g_{1}\right)<x(q) \leq x\left(z_{1}\right)$, then, since $x\left(g_{1}\right)=2$ and $x\left(z_{1}\right)<3$, we have $\left|q g_{1}\right|<1$. Since $q$ is the intersection point of $\Pi\left(c, c_{1}\right)$ with the $x$-axis, the polygon does not intersect $\overline{q g_{1}}$. Thus, $\left|\Pi\left(c, g_{1}\right)\right| \leq|\Pi(c, q)|+\left|q g_{1}\right| \leq|\Pi(c, q)|+1 \leq r$. Therefore, $D$ contains $g_{1}$.
(iii) If $x(q)>x\left(z_{1}\right)$, then consider the path $\Pi\left(c, c^{*}\right)$ and let $p$ be the intersection point of $\Pi\left(c, c^{*}\right)$ with $\overline{z_{1} g_{1}}$; see Figure 25 (b). Thus, the polygon does not intersect $\overline{p g_{1}}$, and, by Observation 5 we have $\left|p g_{1}\right| \leq\left|p q^{*}\right| \leq\left|\Pi\left(p, q^{*}\right)\right|$. Thus, $\left|\Pi\left(c, g_{1}\right)\right| \leq|\Pi(c, p)|+\left|p g_{1}\right| \leq|\Pi(c, p)|+\left|p q^{*}\right| \leq$ $|\Pi(c, p)|+\left|\Pi\left(p, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$. Therefore, $D$ contains $g_{1}$.

(a)

(b)

Figure 25: Illustration of the proof of Case 2. (a) $x\left(c^{*}\right) \leq x(q) \leq x\left(z_{1}\right)$, and (b) $x(q)>x\left(z_{1}\right)$.
Case 3: The polygon does not intersect $\overline{z_{1} g_{1}}$ but intersects $\overline{z_{1} g_{2}}$ or $\overline{z_{2} g_{2}}$. In this case, $g_{1}^{\prime}=g_{1}^{+}$. Consider the path $\Pi\left(c, c_{1}\right)$ and notice that this path intersects the positive $x$-axis at a point $q$.
(i) If $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}^{+}\right)$, then, by Lemma 5. Item (i), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{+}$.
(ii) If $x(q)>x\left(g_{1}^{+}\right)$and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{1} g_{2}}$, then,

- if the polygon intersects the segment $\overline{z_{2} g_{2}}$, then, by Lemma 6. Item (i), $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{+}$; and
- if the polygon does not intersect the segment $\overline{z_{2} g_{2}}$, then, in this case, $g_{2}^{\prime}=g_{2}$, and, by Lemma 6. Item (ii), $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{2}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{+}$or $g_{2}$.
(iii) If $x(q)>x\left(g_{1}^{+}\right)$and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{1} g_{1}}$, then, by Lemma 7. Item (i), $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{+}$.
(iv) If $x(q)>x\left(g_{1}^{+}\right)$and $\Pi\left(c, c^{*}\right)$ does not intersect the segments $\overline{z_{1} g_{1}}$ nor $\overline{z_{1} g_{2}}$, then, by Lemma 8 , Item (i), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{+}$.

Case 4: The polygon does not intersect $\overline{z_{1} g_{1}}, \overline{z_{1} g_{2}}$ nor $\overline{z_{2} g_{2}}$ but intersects $\overline{z_{4} g_{4}}$. In this case, $g_{1}^{\prime}=g_{1}^{-}$, and thus, by Lemma 9. Item (i), $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$ or $\left|\Pi\left(c, c^{*}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{-}$or $c^{*}$.

### 4.1.2 $\quad c^{\prime} \in Q_{2}$

We prove that $D$ is pierced by at least one of the points $g_{2}^{\prime}, g_{3}^{\prime}$, or $c^{*}$. We distinguish between three cases.
Case 1: The polygon does not intersect $\overline{z_{2} g_{2}}$ nor $\overline{z_{2} g_{3}}$. In this case, $g_{2}^{\prime}=g_{2}$ and $g_{3}^{\prime}=g_{3}$, and by Lemma 4. $D$ is pierced by at least one of the points $g_{2}^{\prime}, g_{3}^{\prime}$, or $c^{*}$.
Case 2: The polygon intersects $\overline{z_{2} g_{3}}$. In this case, $g_{3}^{\prime}=g_{3}$ and $D$ contains at least one of the points $c^{*}$ or $g_{3}$ (the proof is symmetric to Case 2 in Section 4.1.1); see Figure 26 .

(a)

(b)

Figure 26: Case 2: (a) $x\left(g_{3}\right) \leq x(q) \leq x\left(c^{*}\right)$, and (b) $x(q)<x\left(z_{2}\right)$.
Case 3: The polygon does not intersect $\overline{z_{2} g_{3}}$ but intersects $\overline{z_{2} g_{2}}$. In this case, $g_{2}^{\prime}=g_{2}^{+}=(-1,1)$ and $g_{3}^{\prime}=g_{3}$. Consider the path $\Pi\left(c, c_{1}\right)$ and notice that it intersects the negative $x$-axis at a point $q$. Thus, $|\Pi(c, q)|+1 \leq r$.
(i) If $x\left(g_{3}\right) \leq x(q) \leq x\left(c^{*}\right)$, then, by Lemma 5. Item (ii), we have $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{3}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{3}$.
(ii) If $x\left(z_{2}\right) \leq x(q) \leq x\left(g_{3}\right)$, then, since $x\left(z_{2}\right)>-3$ and $x\left(g_{3}\right)=-2$, we have $\left|q g_{3}\right|<1$. Since the polygon does intersect $\overline{q g_{3}}$, we have $\left|\Pi\left(c, g_{3}\right)\right| \leq|\Pi(c, q)|+\left|q g_{3}\right|<|\Pi(c, q)|+1 \leq r$. Therefore, $D$ contains $g_{3}$.
(iii) If $x(q)<x\left(z_{2}\right)$, then consider the path $\Pi\left(c, c^{*}\right)$, and notice that this path intersects either $\overline{z_{2} g_{2}}$ or $\overline{z_{2} g_{3}}$.

- If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{2} g_{2}}$, then, since $\alpha_{2}>\frac{\pi}{3}$, by Lemma 10 , we have $\left|\Pi\left(c, g_{2}^{+}\right)\right| \leq r$, and therefore $D$ contains $g_{2}^{+}$.
- If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{2} g_{3}}$, then let $p$ be this intersection point. Since the polygon does not intersect $\overline{p g_{3}}$, by Observation5, we have $\left|p g_{3}\right| \leq\left|p q^{*}\right| \leq\left|\Pi\left(p, q^{*}\right)\right|$. Thus, $\left|\Pi\left(c, g_{3}\right)\right| \leq$ $|\Pi(c, p)|+\left|p g_{3}\right| \leq|\Pi(c, p)|+\left|\Pi\left(p, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$. Therefore, $D$ contains $g_{3}$.


### 4.1.3 $\quad c^{\prime} \in Q_{3}$

We prove that $D$ is pierced by at least one of the points $g_{3}^{\prime}, g_{4}^{\prime}$, or $c^{*}$.
Case 1: The polygon does not intersect $\overline{z_{3} g_{3}}$ nor $\overline{z_{3} g_{4}}$. In this case, $g_{3}^{\prime}=g_{3}$ and $g_{4}^{\prime}=g_{4}$, and by Lemma 4, $D$ is pierced by at least one of the points $g_{3}^{\prime}, g_{4}^{\prime}$, and $c^{*}$.
Case 2: The polygon intersects $\overline{z_{3} g_{4}}$. In this case $g_{3}^{\prime}=g_{3}$ and $g_{4}^{\prime}=g_{4}$. Consider the path $\Pi\left(c, c_{2}\right)$, and notice that it intersects either the negative $y$-axis or the negative $x$-axis. Let $q$ be this intersection point. Thus, $|\Pi(c, q)|+1 \leq r$.
(i) If $\Pi\left(c, c_{2}\right)$ intersects the negative $y$-axis, then $D$ contains at least one of the points $c^{*}$ or $g_{4}$ (the proof is symmetric to Case 2 in Section 4.1.1); see Figure 27.

(a)

(b)

Figure 27: $\Pi\left(c, c_{2}\right)$ intersects the $y$-axis (a) $y\left(g_{4}\right) \leq y(q) \leq y\left(c^{*}\right)$, and (b) $y(q)<y\left(g_{4}\right)$.
(ii) If $\Pi\left(c, c_{2}\right)$ intersects the negative $x$-axis and $x\left(g_{3}\right) \leq x(q) \leq x\left(c^{*}\right)$, then, by Lemma 5, Item (iii), $D$ contains at least one of the points $c^{*}$ or $g_{3}$; see Figure 28(a).
(iii) If $\Pi\left(c, c_{2}\right)$ intersects the negative $x$-axis and $x(q)<x\left(g_{3}\right)$, then consider the path $\Pi\left(c, c^{*}\right)$ and notice that, since $\alpha_{2}>\frac{\pi}{3}$ and $x(q)<x\left(g_{3}\right)$, this path intersects $\overline{z_{3} g_{3}}$ at a point $p$ and the
polygon does not intersect $\overline{p g_{3}}$; see Figure $28(\mathrm{~b})$. Hence, by Observation 5, we have $\left|p g_{3}\right| \leq$ $\left|p q^{*}\right| \leq\left|\Pi\left(p, q^{*}\right)\right|$. Thus, $\left|\Pi\left(c, g_{3}\right)\right| \leq|\Pi(c, p)|+\left|p g_{3}\right| \leq|\Pi(c, p)|+\left|\Pi\left(p, q^{*}\right)\right|=\left|\Pi\left(c, q^{*}\right)\right| \leq r$. Therefore, $D$ contains $g_{3}$.


Figure 28: Case 2: (a) $\Pi\left(c, c_{2}\right)$ intersects the $x$-axis and $x\left(g_{3}\right) \leq x(q) \leq x\left(c^{*}\right)$, and (b) $\Pi\left(c, c_{2}\right)$ intersects the $x$-axis and $x(q)<x\left(g_{3}\right)$.

Case 3: The polygon does not intersect $\overline{z_{3} g_{4}}$ but intersects $\overline{z_{3} g_{3}}$. In this case $g_{3}^{\prime}=g_{3}$ and $g_{4}^{\prime}=g_{4}^{-}$. Consider the path $\Pi\left(c, c_{2}\right)$ and notice that it intersects either the negative $y$-axis or the negative $x$-axis. Let $q$ be this intersection point. Thus, $|\Pi(c, q)|+1 \leq r$.
(i) $\Pi\left(c, c_{2}\right)$ intersects the negative $x$-axis, then $D$ contains at least one of the points $c^{*}$ and $g_{3}$ (the proof is symmetric to the proof of Items (ii) and (iii) in the previous case); see Figure 29 .

(a)

(b)

Figure 29: $\Pi\left(c, c_{2}\right)$ intersects the $x$-axis. (a) $x\left(g_{3}\right) \leq x(q) \leq x\left(c^{*}\right)$, and (b) $x(q)<x\left(g_{3}\right)$.
(ii) $\Pi\left(c, c_{2}\right)$ intersects the negative $y$-axis and $y\left(g_{4}^{-}\right) \leq y(q) \leq y\left(c^{*}\right)$, then, by Lemma 5. Item (iv), we have $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{4}^{-}$.
(iii) If $y(q)<y\left(g_{4}^{-}\right)$and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{3} g_{3}}$, then, by Lemma 6, Item (iii), $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{3}\right)\right| \leq r$, and therefore $D$ contains $g_{4}^{-}$or $g_{3}$.
(iv) If $y(q)<y\left(g_{4}^{-}\right)$and $\Pi\left(c, c^{*}\right)$ intersects the segment $\overline{z_{3} g_{4}}$, then, by Lemma 7 . Item (ii), $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$, and therefore $D$ contains $g_{4}^{-}$.
(v) If $y(q)<y\left(g_{4}^{-}\right)$and $\Pi\left(c, c^{*}\right)$ does not intersect the segments $\overline{z_{3} g_{4}}$ nor $\overline{z_{3} g_{3}}$, then, by Lemma 8 , Item (ii), $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$, and therefore $D$ contains $g_{4}^{-}$.

### 4.1.4 $\quad c^{\prime} \in Q_{4}$

We prove that $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{4}^{\prime}$, or $c^{*}$. Consider the path $\Pi\left(c, c_{3}\right)$, and notice that it intersects either the positive $x$-axis or the negative $y$-axis at a point $q$.
The point $q$ is on the positive $x$-axis.
Case 1: $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}^{\prime}\right)$. By Lemma 5. Item (v), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{\prime}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{\prime}$.
Case 2: $x(q)>x\left(g_{1}^{\prime}\right)$. We distinguish between three cases.
Case 2.1: $g_{1}^{\prime}=g_{1}$.
(i) If the polygon intersects $\overline{z_{1} g_{1}}$, then $\Pi\left(c, c_{3}\right)$ intersects $\overline{z_{1} g_{1}}$ at a point $p$. Thus, $g_{1}$ is inside the pseudo-triangle $\triangle\left(c, q^{*}, p\right)$, and, by Observation 3, $D$ contains $g_{1}$.
(ii) If the polygon intersects $\overline{z_{4} g_{1}}$, then $D$ contains at least one of the points $c^{*}$ or $g_{1}$ (the proof is symmetric to the proof of Case 2 in Section 4.1.1).
(iii) If the polygon does not intersect $\overline{z_{1} g_{1}}$ nor $\overline{z_{4} g_{1}}$, then, since $g_{1}^{\prime}=g_{1}$, the polygon does not intersect $\overline{z_{4} g_{4}}$. If $g_{4}^{\prime}=g_{4}$, then, by Lemma 4, $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{4}^{\prime}$, and $c^{*}$. Otherwise, $g_{4}^{\prime}=g_{4}^{-}$. In this case, the polygon intersects $\overline{z_{3} g_{3}}$, and, by Observation 7 , we have $\alpha_{3}>\frac{\pi}{5}$. Thus, by Lemma 11, $\left|\Pi\left(c, g_{1}\right)\right| \leq r$, and therefore $D$ contains $g_{1}$.

Case 2.2: $g_{1}^{\prime}=g_{1}^{+} \neq g_{1}$. By Lemma 9. Item (ii), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{+}$.
Case 2.3: $g_{1}^{\prime}=g_{1}^{-}$.
(i) If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$, then, if $g_{4}^{\prime}=g_{4}$, then, by Lemma 6, Item (iv), $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{-}$or $g_{4}$. Otherwise, $g_{4}^{\prime}=g_{4}^{-}$. In this case, the polygon intersects $\overline{z_{3} g_{3}}$, and, by Observation 7, we have $\alpha_{3}>\frac{\pi}{5}$. Thus, by Lemma 11, $\left|\Pi\left(c, g_{1}\right)\right| \leq r$, and therefore $D$ contains $g_{1}$.
(ii) If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{1}}$, then by Lemma 7 , Item (iii), $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{-}$.
(iii) If $\Pi\left(c, c^{*}\right)$ does not intersect $\overline{z_{4} g_{1}}$ nor $\overline{z_{4} g_{4}}$, then, by Lemma 8, Item (iii), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{-}$.

The point $q$ is on the negative $y$-axis.
Case 1: $y\left(g_{4}^{\prime}\right) \leq y(q) \leq y\left(c^{*}\right)$. By Lemma 5. Item (vi), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{\prime}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{4}^{\prime}$.
Case 2: $y(q)<y\left(g_{4}^{\prime}\right)$. We distinguish between two cases.
Case 2.1: $g_{4}^{\prime}=g_{4}^{-} \neq g_{4}$. By Lemma 9. Item (iii), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{-}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{4}^{-}$.
Case 2.2: $g_{4}^{\prime}=g_{4}$. Then, since $\alpha_{3} \leq \frac{\pi}{3}$, by Lemma $12,\left|\Pi\left(c, g_{4}\right)\right| \leq r$, and therefore $D$ contains $g_{4}$.

### 4.2 Case (iii): $\alpha_{2} \leq \frac{\pi}{3}$ and $\alpha_{3} \leq \frac{\pi}{3}$

Let $D \in \mathcal{D}$ be a disk with center $c$ and radius $r$. We show that $D$ is pierced by at least one of the points of $S$. Notice that in Algorithm 2, $Q_{1}$ is symmetric to $Q_{2}$ and $Q_{3}$ is symmetric to $Q_{4}$. Therefore, we show the correctness for the cases where $c^{\prime} \in Q_{1}$ and $c^{\prime} \in Q_{4}$.

### 4.2.1 $\quad c^{\prime} \in Q_{1}$

We prove that $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{2}^{\prime}$, or $c^{*}$. We distinguish between four cases.

Case 1: The polygon does not intersect $\overline{z_{1} g_{1}}, \overline{z_{1} g_{2}}$, nor $\overline{z_{4} g_{4}}$. In this case, $g_{1}^{\prime}=g_{1}$ and $g_{2}^{\prime}=g_{2}$, and by Lemma 4, $D$ contains at least one of the points $g_{1}^{\prime}, g_{2}^{\prime}$, and $c^{*}$.
Case 2: The polygon intersects $\overline{z_{1} g_{1}}$; see Figure 25. In this case, $g_{1}^{\prime}=g_{1}$, and $D$ contains at least one of the points $c^{*}$ or $g_{1}$. The proof is the same as in Case 2 of Section 4.1.1.
Case 3: The polygon does not intersect $\overline{z_{1} g_{1}}$, but intersects $\overline{z_{1} g_{2}}$. In this case, $g_{1}^{\prime}=g_{1}^{+}$and $g_{2}^{\prime}=g_{2}$, and $D$ contains at least one of the points $g_{2}, g_{1}^{+}$and $c^{*}$. The proof is the same as in Case 3 of Section 4.1.1.
Case 4: The polygon does not intersect $\overline{z_{1} g_{1}}$ nor $\overline{z_{1} g_{2}}$ but intersects $\overline{z_{4} g_{4}}$. In this case $g_{1}^{\prime}=g_{1}^{-}$and $g_{2}^{\prime}=g_{2}$, and $D$ contains at least one of the points $c^{*}$ or $g_{1}^{-}$. The proof is the same as in Case 4 of Section 4.1.1.

### 4.2.2 $\quad c^{\prime} \in Q_{4}$

We prove that $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{4}^{\prime}$, and $c^{*}$. Notice that in this case where both $\alpha_{2}$ and $\alpha_{3}$ are less or equal to $\frac{\pi}{3}$, Algorithm 2 does not change $g_{4}$, thus $g_{4}^{\prime}=g_{4}$. Consider the path $\Pi\left(c, c_{3}\right)$, and notice that it intersects either the positive $x$-axis or the negative $y$-axis at a point $q$.
The point $q$ is on the positive $x$-axis.
Case 1: $x\left(c^{*}\right) \leq x(q) \leq x\left(g_{1}^{\prime}\right)$. By Lemma 5 . Item (v), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{\prime}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{\prime}$.
Case 2: $x(q)>x\left(g_{1}^{\prime}\right)$. We distinguish between three cases.
Case 2.1: $g_{1}^{\prime}=g_{1}$.
(i) If the polygon intersects $\overline{\bar{z}_{1} g_{1}}$, then $\Pi\left(c, c_{3}\right)$ intersects $\overline{z_{1} g_{1}}$ at a point $p$. Thus, $g_{1}$ is inside the pseudo-triangle $\triangle\left(c, q^{*}, p\right)$, and, by Observation 3, $D$ contains $g_{1}$.
(ii) If the polygon intersects $\overline{z_{4} g_{1}}$, then $D$ contains at least one of the points $c^{*}$ or $g_{1}$ (the proof is symmetric to the proof of Case 2 in Section 4.1.1).
(iii) If the polygon does not intersect $\overline{z_{1} g_{1}}$ nor $\overline{z_{4} g_{1}}$, then, since $g_{1}^{\prime}=g_{1}$, the polygon does not intersect $\overline{\bar{z}_{4} g_{4}}$. Since $g_{4}^{\prime}=g_{4}$, by Lemma 4. $D$ is pierced by at least one of the points $g_{1}^{\prime}, g_{4}^{\prime}$, and $c^{*}$.

Case 2.2: $g_{1}^{\prime}=g_{1}^{+} \neq g_{1}$. By Lemma 9, Item (ii), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{+}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{+}$.
Case 2.3: $g_{1}^{\prime}=g_{1}^{-}$.
(i) If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{4}}$, then, since $g_{4}^{\prime}=g_{4}$, by Lemma 6 . Item (iv), we have $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{-}$or $g_{4}$.
(ii) If $\Pi\left(c, c^{*}\right)$ intersects $\overline{z_{4} g_{1}}$, then by Lemma 7 , Item (iii), $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$, and therefore $D$ contains $g_{1}^{-}$.
(iii) If $\Pi\left(c, c^{*}\right)$ does not intersect $\overline{z_{4} g_{1}}$ nor $\overline{z_{4} g_{4}}$, then, by Lemma 8, Item (iii), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{1}^{-}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{1}^{-}$.

The point $q$ is on the negative $y$-axis.
Case 1: $y\left(g_{4}^{\prime}\right) \leq y(q) \leq y\left(c^{*}\right)$. By Lemma 5. Item (vi), $\left|\Pi\left(c, c^{*}\right)\right| \leq r$ or $\left|\Pi\left(c, g_{4}^{\prime}\right)\right| \leq r$, and therefore $D$ contains $c^{*}$ or $g_{4}^{\prime}$.
Case 2: $y(q)<y\left(g_{4}^{\prime}\right)$. Since, $g_{4}^{\prime}=g_{4}$ and $\alpha_{3} \leq \frac{\pi}{3}$, by Lemma 12 , we have $\left|\Pi\left(c, g_{4}\right)\right| \leq r$, and therefore $D$ contains $g_{4}$.

## 5 Conclusion

We have shown that five points are sufficient to pierce a set of pairwise intersecting geodesic disks inside a polygon $P$. This improves the upper bound of 14 , which was provided by Bose et al. [1]. This upper bound is very close to the lower bound for stabbing pairwise intersecting disks in the plane, which was proven to be four.

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