Piercing Pairwise Intersecting Geodesic Disks by Five Points^{*}

A. Karim Abu-Affash[†] Paz Carmi[‡] Meytal Maman[§]

December 14, 2021

Abstract

Given a simple polygon P on n vertices, and a set \mathcal{D} of m pairwise intersecting geodesic disks in P, we show that five points in P are always sufficient to pierce all the disks in \mathcal{D} . This improves the previous bound of 14, obtained by Bose, Carmi, and Shermer [1].

1 Introduction

The problem of piercing geometric objects with as few points as possible has attracted the attention of researchers for the past century. The research so far has been focused on convex objects and disks in the plane. The most known result for piercing geometric objects with set of minimum cardinality, is known as Helly's theorem [5,6], and works for convex sets in the plane. This theorem states the following: Given a set of m convex objects in \mathbb{R}^d such that m > d + 1, if every d + 1 of these objects have a point in common, then all of them have a point in common. This means that one point is sufficient to pierce all the objects. This claim does not hold when the convex objects are only pairwise intersecting. However, for a set of disks in the plane, where every pair of disks intersects, it has proven by Danzer [3] and by Stacho [8,9] that four points are sufficient to pierce all the disks. These proofs are not amenable to design efficient (subquadratic-time) algorithms for computing the piercing points. Recently, linear-time algorithms have been presented by Har-Peled et al. [4] for computing five points that pierce m pairwise intersecting disks, and by Carmi et al. [2] for computing four points.

Let P be a simple polygon. A geodesic disk D with radius r centered at a point $c \in P$ is the set of all points $x \in P$, such that the length of the shortest path from x to c is at most r. Bose et al. [1] showed that for any set \mathcal{D} of pairwise intersecting geodesic disks in P, 14 points are sufficient to pierce all the disks in \mathcal{D} and these points can be computed in linear time. In this paper, we prove that five points are sufficient to pierce all the disks in \mathcal{D} , which improve the result of Bose et al. [1]. More precisely, we prove the following theorem.

Theorem 1. Given a simple polygon P on n vertices, and a set \mathcal{D} of m pairwise intersecting geodesic disks in P, five points in P are sufficient to pierce all the disks in \mathcal{D} .

 $^{^*}$ This work was partially supported by Grant 2016116 from the United States – Israel Binational Science Foundation.

[†]Software Engineering Department, Shamoon College of Engineering, Beer-Sheva 84100, Israel, abuaa1@sce.ac.il.

[‡]Department of Computer Science, Ben-Gurion University, Beer-Sheva 84105, Israel, carmip@cs.bgu.ac.il.

[§]Department of Computer Science, Ben-Gurion University, Beer-Sheva 84105, Israel, meytal.maman@gmail.com.

2 The Setup and Preliminaries

For simplicity of presentation, we adapt some notation that appeared in [1]. Moreover, we use the convention that all indices are taken modulo the size of the set involved. Let P be a simple *n*-vertex polygon in the plane and let $v_1, v_2, ..., v_n$ be its vertices sorted in clockwise order. For two points $x, y \in P$, the geodesic (shortest) path from x to y is denoted as $\Pi(x, y)$ and its length is the sum of the lengths of its edges, and is denoted as $|\Pi(x, y)|$. A geodesic disk with radius $r \ge 0$ centered at a point $c \in P$ is the set $\{y \in P : |\Pi(c, y)| \le r\}$. A geodesic triangle on three points $a, b, c \in P$, denoted by $\Delta(a, b, c)$, is a weekly-simple polygon whose boundary consists of the paths $\Pi(a, b), \Pi(b, c),$ and $\Pi(a, c)$; see Figure 1. A pseudo triangle is a simple polygon with three convex vertices.

A set $X = \{x_1, ..., x_k\}$ of at least three points in P is geodesically collinear if there exist two points $x_i, x_j \in X$, such that $X \subset \Pi(x_i, x_j)$. Given three points $a, b, c \in P$ that are not geodesically collinear, the paths $\Pi(a, b)$ and $\Pi(a, c)$ have a common subpath until they diverge at a point a'. Similarly, let b' (resp., c') be the point where the paths $\Pi(b, a)$ and $\Pi(b, c)$ (resp., the paths $\Pi(c, a)$ and $\Pi(c, b)$) diverge; see Figure 1. Pollack et al. [7] observed that $\Delta(a', b', c')$ is a pseudo triangle. We refer to $\Delta(a', b', c')$ as the geodesic core of $\Delta(a, b, c)$ and denote it by $\nabla(a, b, c)$. Pollack et al. [7] observed the following observation.

Observation 1. Let a, b and c be three points in P. Then the geodesic core $\nabla(a, b, c)$ has only reflex angles along its boundary and the interior of this triangle is fully contained in P.



Figure 1: $\triangle(a, b, c)$ is a geodesic triangle. $\triangle(a', b', c)$ is a pseudo triangle. $\bigtriangledown(a, b, c) = \triangle(a', b', c)$ is the geodesic core of $\triangle(a, b, c)$

Moreover, Pollack et al. [7] proved the following lemma about distances between a point and a geodesic path.

Lemma 1 ([7]). Let a, b and c be three points in P. Let g be the function defined on $\Pi(b, c)$, such that $g(x) = |\Pi(a, x)|$, for every point x on $\Pi(b, c)$. Then, g is a convex function with its maximum occurring either at b or c. That is, $g(x) \leq \max\{g(b), g(c)\}$, for every point x on $\Pi(b, c)$.

The following observations follow from Lemma 1.

Observation 2. Let a and b be two points, such that the segment \overline{ab} is entirely contained in P. Then, any disk $D \in \mathcal{D}$ that contains both a and b must contain the segment \overline{ab} .

Observation 3. Let D be a geodesic disk in \mathcal{D} with center $c \in P$, and let a and b be two points in D. Then, the pseudo-triangle $\Delta(c, a, b)$ is contained in D.

Observation 4. Let D be geodesic disk with center $c \in P$ and radius r. Let q and b be two points, such that $|\Pi(c,q)| + 1 \leq r$, $|qb| \leq 1$, and the segment \overline{qb} is entirely contained in P. Then, b is contained in D.

Let $\mathcal{D} = \{D_1, D_2, ..., D_m\}$ be a set of m pairwise intersecting geodesic disks in P. For each $1 \leq i \leq m$, let c_i and r_i denote the center and the radius of D_i , respectively. The set \mathcal{D} is called *Helly* if there is a point that pierces all the disks in \mathcal{D} . For a point $x \in P$, we define a function f(x) = y to be the smallest radius of a geodesic disk centered at x that intersects all the disks in \mathcal{D} . A disk D with radius r centered at c is called *minimal* with respect to \mathcal{D} if every point x in the close neighborhood of c in P has f(x) > r. Let D^* be the disk with center c^* that minimizes $f(c^*)$, and let $r^* = f(c^*)$ be its radius. Bose et al. [1] proved the following lemma regarding the properties of D^* .

Lemma 2 ([1]). If \mathcal{D} is not Helly, then D^* satisfies the following properties:

- $r^* > 0;$
- D^* does not intersect the boundary of P;
- D^{*} is tangent to at least 3 geodesic disks D₁, D₂, D₃ in D at 3 distinct points t₁, t₂, t₃, respectively;
- c^* is contained in the interior of $\triangle(t_1, t_2, t_3)$; and
- D^* does not intersect the boundary of the geodesic core $\nabla(c_1, c_2, c_3)$, where c_1, c_2, c_3 are the centers of D_1, D_2, D_3 , respectively.

Assume, w.l.o.g., that $r^* = 1$ and that c^* is located at the origin (0,0). Let D_1, D_2, D_3 be the three geodesic disks from Lemma 2 that are tangent to D^* at the points t_1, t_2, t_3 , respectively. For each $i \in \{1, 2, 3\}$, let ℓ_i be the line that is tangent to D_i and passes through t_i ; see Figure 2. Let $m_{i,j}$ be the intersection point between the lines ℓ_i and ℓ_j , for every distinct $i, j \in \{1, 2, 3\}$. Assume, w.l.o.g., that ℓ_1 is horizontal and the angle $\angle(m_{1,2}, m_{2,3}, m_{3,1})$ is the largest in the triangle $\triangle(m_{1,2}, m_{2,3}, m_{3,1})$; see Figure 2.

For two points p and q, let \overline{pq} denote the line segment connecting them. For every distinct $i, j \in \{1, 2, 3\}$, let $\ell_{i,j}$ be the line passing through $m_{i,j}$ perpendicular to $\overline{c^*m_{i,j}}$; see Figure 2. Let $t_{i(j)}$ be the intersection point between $\ell_{i,j}$ and the line passing through $\overline{c^*t_i}$. The following lemma was proven in [1].

Lemma 3 ([1]). The path $\Pi(c_i, c_j)$ does not intersect $\triangle(t_{i(j)}, c^*, t_{j(i)})$, for any distinct $i, j \in \{1, 2, 3\}$.

Let g_1, g_2, g_3 and g_4 be the points located at the coordinates (2,0), (0,2), (-2,0), and (0,-2), respectively. The following corollary follows from Lemma 3 and the assumption that ℓ_1 is horizontal and the angle $\angle(m_{1,2}, m_{2,3}, m_{3,1})$ is the largest in the triangle $\triangle(m_{1,2}, m_{2,3}, m_{3,1})$.



Figure 2: The smallest disk D^* is located at the origin. ℓ_1, ℓ_2 , and ℓ_3 are the tangent lines between D^* and the disks D_1, D_2 , and D_3 , respectively. The path $\Pi(c_i, c_j)$ does not intersect $\Delta(t_{i(j)}, c^*, t_{j(i)})$, for any distinct $i, j \in \{1, 2, 3\}$.

Corollary 1. The polygon P does not intersect the triangles $\triangle(g_1, c^*, g_4)$ and $\triangle(g_3, c^*, g_4)$.

For a point $p \in P$, let x(p) and y(p) denote the x-coordinate and the y-coordinate of p, respectively. We divide the plane into 4 quadrants Q_1 , Q_2 , Q_3 , and Q_4 as follows; see Figure 3.

• $Q_1 = \{ p \in \mathbb{R}^2 : x(p) \ge 0 \text{ and } y(p) \ge 0 \};$

•
$$Q_2 = \{ p \in \mathbb{R}^2 : x(p) \le 0 \text{ and } y(p) \ge 0 \};$$

- $Q_3 = \{ p \in \mathbb{R}^2 : x(p) \le 0 \text{ and } y(p) \le 0 \}; \text{ and }$
- $Q_4 = \{ p \in \mathbb{R}^2 : x(p) \ge 0 \text{ and } y(p) \le 0 \}.$

For each $i \in \{1, 2, 3, 4\}$, let $z_i \in Q_i$ be the point whose distance from g_i, g_{i+1} and the boundary of D^* is equal; see Figure 3. The computation of the points z_i is not involved. For example, we compute z_1 by solving the following equations system:

$$\begin{aligned} |z_1g_1| &= \sqrt{(x(z_1) - 2)^2 + y(z_1)^2} = d, \\ |z_1g_2| &= \sqrt{x(z_1)^2 + (y(z_1) - 2)^2} = d, \\ |z_1c^*| &= \sqrt{x(z_1)^2 + y(z_1)^2} = d + 1. \end{aligned}$$

This implies that $z_1 = (a, a)$, $z_2 = (-a, a)$, $z_3 = (-a, -a)$, and $z_4 = (a, -a)$, where $a = \frac{3}{4-2\sqrt{2}} \approx 2.56$. For $1 \le i \le 4$, let pr_i be the region bounded by the parabola that contains all the points that are closer to g_i than to D^* ; see Figure 3. Moreover, let $t_1^+ = (2, 1.5)$, $t_1^- = (2, -1.5)$, $t_2^+ = (-1.5, 2)$, $t_2^- = (1.5, 2)$, $t_3^+ = (-2, -1.5)$, $t_3^- = (-2, 1.5)$, $t_4^+ = (1.5, -2)$, and $t_4^- = (-1.5, -2)$. The following observations follow from the definition of pr_i .



Figure 3: The quadrants Q_i and the points z_i , for each $1 \le i \le 4$. The segments $\overline{z_i g_i}$ and $\overline{z_{i-1} g_i}$ are entirely contained in the region bounded by the parabola pr_i (depicted in purple).

Observation 5. Let p be a point on $\overline{z_ig_i}$ or on $\overline{z_{i-1}g_i}$, where $i \in \{1, 2, 3, 4\}$. Then, for any point q on the boundary of D^* , we have $|pg_i| \leq |pq|$.

Observation 6. Let p be a point on $\overline{t_i^+ t_i^-}$, where $i \in \{1, 2, 3, 4\}$. Then, for any point q on the boundary of D^* , we have $|pg_i| \leq |pq|$.

Let $D \in \mathcal{D}$ be a disk with center c and radius r. Throughout the rest of the paper, we use the following notations. For each $i \in \{1, 2, 3\}$, let q_i be the intersection point of the path $\Pi(c, c_i)$ with the boundary of D_i . Let q^* be the intersection point of the path $\Pi(c, c^*)$ with the boundary of D^* . Thus, $|\Pi(c, q^*)| \leq r$ and $|\Pi(c, q_i)| \leq r$, for each $i \in \{1, 2, 3\}$. Let c' be the point on $\Pi(c, c^*)$, such that the edge (c', c^*) is the last edge in $\Pi(c, c^*)$. That is, c' is the first point on $\Pi(c, c^*)$ that is visible from c^* . Finally, let α_2 (resp., α_3) be the acute angle between ℓ_2 (resp., ℓ_3) and the x-axis; see Figure 4.

Observation 7. If the polygon P intersects the segment $\overline{z_4g_1}$ or $\overline{z_4g_4}$, then $\alpha_2 > \frac{\pi}{5}$; see Figure 4. Similarly, if the polygon intersects the segment $\overline{z_3g_3}$ or $\overline{z_3g_4}$, then $\alpha_3 > \frac{\pi}{5}$.

Proof. By Lemma 3, the polygon does not intersect the triangle $\triangle(t_{2(1)}, c^*, t_{1(2)})$; see Figure 4. Using a simple geometric calculation, for $\alpha_2 = \frac{\pi}{5}$, the acute angle between $\ell_{1,2}$ and the *x*-axis is $\beta = \frac{\pi - \alpha_2}{2} = \frac{2\pi}{5}$ and the coordinates of $m_{1,2}$ are $(\frac{\cos \alpha_2 + 1}{\sin \alpha_2}, -1)$. Thus, for $\alpha_2 = \frac{\pi}{5}, \ell_{1,2}$ passes through z_4 . Therefore, for $0 < \alpha_2 \leq \frac{\pi}{5}$, the point z_4 is contained in the triangle $\triangle(t_{2(1)}, c^*, t_{1(2)})$, and, the polygon cannot intersect the segment $\overline{z_4g_1}$.

In the following lemma, we show that, for each $i \in \{1, 2, 3, 4\}$, if the polygon does not intersect the segments $\overline{z_i g_i}$ and $\overline{z_i g_{i+1}}$, then every disk $D \in \mathcal{D}$ with c' in Q_i is pierced by at least one of the points c^* , g_i or g_{i+1} .

Lemma 4. Let $D \in \mathcal{D}$ be a disk with c' in Q_i , where $i \in \{1, 2, 3, 4\}$. If the polygon does not intersect the segments $\overline{z_i g_i}$ nor $\overline{z_i g_{i+1}}$, then D contains at least one of the points c^* , g_i or g_{i+1} .



Figure 4: For $0 < \alpha_2 \leq \frac{\pi}{5}$, the polygon cannot intersect the segment $\overline{z_4g_1}$.

Proof. Let c and r be the center and the radius of D, respectively. We distinguish between two cases:

Case 1: The path $\Pi(c, c^*)$ intersects $\overline{z_i g_i}$ or $\overline{z_i g_{i+1}}$ at a point p. Assume w.l.o.g., $\Pi(c, c^*)$ intersects $\overline{z_i g_i}$; see Figure 5 (for i = 1). By Observation 5, we have $|pg_i| \leq |pq^*|$. Moreover, since the polygon does not intersect $\overline{z_i g_i}$, we have $|\Pi(c, g_i)| \leq |\Pi(c, p)| + |pg_i| \leq |\Pi(c, p)| + |pq^*| = |\Pi(c, q^*)| \leq r$. Therefore, D contains g_i .



Figure 5: $c' \in Q_1$ and the path $\Pi(c', c^*)$ intersects $\overline{z_1g_1}$.

Case 2: The path $\Pi(c, c^*)$ does not intersect $\overline{z_i g_i}$ nor $\overline{z_i g_{i+1}}$. We prove this case for i = 1; the proof of the other cases are symmetric. Consider the path $\Pi(c, c_1)$ and notice that it intersects the *x*-axis at a point *q*. Since $|\Pi(q, q_1)| \ge 1$, we have $|\Pi(c, q)| + 1 \le |\Pi(c, q)| + |\Pi(q, q_1)| = |\Pi(c, q_1)| \le r$. By the case assumption, and by the fact that the polygon does not intersect $\overline{z_1 g_1}$ nor $\overline{z_1 g_2}$, $\Pi(c, c_1)$ has a vertex *p* inside the quadrilateral defined by c^*, g_1, z_1, g_2 , such that the polygon does not intersect the segment \overline{pq} ; see Figure 6. Hence, $x(c^*) \le x(q) \le x(z_1)$ and $|\Pi(c, q)| = |\Pi(c, p)| + |pq|$. Moreover, $|c^*g_1| = 2$, and, by Corollary 1, the polygon does not intersect $\overline{c^*g_1}$.

- If $x(c^*) \leq x(q) \leq x(g_1)$, then, since $|c^*g_1| = 2$, we have $|c^*q| \leq 1$ or $|qg_1| \leq 1$, and by Observation 4, D contains at least one of the points c^* or g_1 ; see Figure 6(a).
- If $x(g_1) < x(q) \le x(z_1)$, then, since $x(g_1) = 2$ and $x(z_1) < 3$, we have $|qg_1| \le 1$, and thus $|\Pi(c,g_1)| \le |\Pi(c,p)| + |pq| + |qg_1| < |\Pi(c,p)| + |pq| + 1 = |\Pi(c,q)| + 1 \le r$; see Figure 6(b). Therefore, D contains g_1 .

Notice that, by Corollary 1, the polygon does not intersect $\overline{c^*g_1}$, $\overline{c^*g_3}$, nor $\overline{c^*g_4}$. Thus, for i = 3 and i = 4, in Case 2, we have p = c = c' is D's center.



Figure 6: The path $\Pi(c', c^*)$ does not intersect $\overline{z_1g_1}$ nor $\overline{z_1g_2}$, (a) p = c. (b) $p \neq c$.

In the following, we define eight points $g_i^+ \in Q_i$ and $g_i^- \in Q_{i-1}$, for each $i \in \{1, 2, 3, 4\}$, and we prove some lemmas regarding these points. For each $i \in \{1, 2, 3, 4\}$, let m_i^+ (resp., m_i^-) be the tangent line to D^* that passes through g_i and has a positive (resp., negative) slope; see Figure 7 (for an illustration of m_1^+ and m_1^-).

The point g_1^+ is defined as follows. Let D' be the disk of radius 1 centered at the point (1, 0). We sweep with a line ℓ that is tangent to D^* in counterclockwise order starting with $\ell = m_1^+$ and we stop when ℓ intersects either D_3 or the polygon inside the quadrilateral defined by c^*, g_1, z_1, g_2 ; see Figure 7(a). Let u be the intersection point of ℓ with D' in Q_1 when we stop the sweeping. We also sweep upwards with a horizontal line ℓ_h that passes through the point c^* , and stop when ℓ_h intersects the polygon inside D', or when ℓ_h 's y-coordinate is 1; see Figure 7(b). Let w be the intersection point of ℓ_h with D' in Q_1 when we stop the sweeping. We set g_1^+ as the lowest point among u and w.

The point g_1^- is defined as follows. We sweep with a line ℓ that is tangent to D^* in clockwise order starting with $\ell = m_1^-$ and we stop when ℓ intersects either D_1 or the polygon inside the quadrilateral defined by c^*, g_1, z_4, g_4 ; see Figure 7. Let u be the intersection point of m_1^- with D' in Q_4 when we stop the sweeping. We also sweep downwards with a horizontal line ℓ_h that passes through the point c^* , and stop either when ℓ_h intersects the polygon inside D', or when ℓ_h 's y-coordinate is -1. Let w be the intersection point of ℓ_h with D' in Q_4 when we stop the sweeping. We set g_1^- as the highest point among u and w.



Figure 7: Defining g_1^+ and g_1^- . (a) g_1^+ is defined by the intersection of m_1^+ with D_3 and g_1^- is defined by the intersection of m_1^- with D_1 . (b) g_1^+ is defined by the intersection of ℓ_h with the polygon inside D' and g_1^- is defined by the intersection of m_1^- with the polygon outside D'.

We define $g_2^+ = (-1,1)$ and $g_2^- = (1,1)$, and we define g_3^+ , g_3^- , g_4^+ , and g_4^- similarly to g_1^+ and g_1^- , where the sweeping line ℓ starts with m_3^+ , m_3^- , m_4^+ , and m_4^- , respectively. For g_3^+ and g_3^- , D' is centered at (-1,0) and, for g_4^+ and g_4^- , D' is centered at (0,-1).

Lemma 5. Let $D \in \mathcal{D}$ be a disk centered at c with radius r and let $g'_i \in \{g_i, g_i^+, g_i^-\}$, for each $i \in \{1, 2, 3, 4\}$.

- (i) If $c' \in Q_1$, and $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(c^*) \leq x(q) \leq x(g'_1)$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_1)| \leq r$; see Figure 8(a).
- (ii) If $c' \in Q_2$ and $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(g'_3) \leq x(q) \leq x(c^*)$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_3)| \leq r$; see Figure 8(b).
- (iii) If $c' \in Q_3$ and $\Pi(c, c_2)$ intersects the x-axis at a point q with $x(g'_3) \leq x(q) \leq x(c^*)$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_3)| \leq r$; see Figure 8(c).
- (iv) If $c' \in Q_3$ and $\Pi(c, c_2)$ intersects the y-axis at a point q with $y(g'_4) \leq y(q) \leq y(c^*)$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_4)| \leq r$; see Figure 8(d).
- (v) If $c' \in Q_4$ and $\Pi(c, c_3)$ intersects the x-axis at a point q with $x(c^*) \leq x(q) \leq x(g'_1)$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_1)| \leq r$; see Figure 8(e).
- (vi) If $c' \in Q_4$ and $\Pi(c, c_3)$ intersects the y-axis at a point q with $y(g'_4) \leq y(q) \leq y(c^*)$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_4)| \leq r$; see Figure 8(f).



Figure 8: Illustration of Lemma 5: (a) Item (i), (b) Item (ii), (c) Item (iii), (d) Item (iv), (e) Item (v), and (f) Item (vi).

Proof. We prove Item (i), the proofs of the other five items are symmetric.

Notice that $|\Pi(c,q)| + 1 \leq r$. Moreover, by Corollary 1, the polygon does not intersect $\overline{c^*g_1}$; see Figure 9.

- If $x(c^*) \le x(q) \le 1$, then, $|c^*q| \le 1$; see Figure 9(a). Thus, $|\Pi(c,c^*)| \le |\Pi(c,q)| + |qc^*| \le |\Pi(c,q)| + 1 \le r$.
- If $1 < x(q) \le x(g'_1)$, then $|\underline{qg'_1}| \le 1$; see Figure 9(b). Moreover, by the definition of g'_1 , the polygon does not intersect $\overline{qg'_1}$. Thus, $|\Pi(c,g'_1)| \le |\Pi(c,q)| + |qg'_1| \le |\Pi(c,q)| + 1 \le r$.



Figure 9: Illustration of the proof of Lemma 5, Item (i): (a) $x(c^*) \le x(q) \le 1$, and (b) $1 \le x(q) \le x(q'_1)$.

Lemma 6. Let $D \in \mathcal{D}$ be a disk centered at c with radius r.

- (i) If $c' \in Q_1$, $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(q) > x(g_1^+)$, $\Pi(c, c^*)$ intersects the segment $\overline{z_1g_2}$ and the polygon intersects the segment $\overline{z_2g_2}$, then $|\Pi(c, g_1^+)| \leq r$; see Figure 10(a).
- (ii) If $c' \in Q_1$, $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(q) > x(g_1^+)$, $\Pi(c, c^*)$ intersects the segment $\overline{z_1g_2}$ and the polygon does not intersect the segment $\overline{z_2g_2}$, then $|\Pi(c, g_1^+)| \leq r$ or $|\Pi(c, g_2)| \leq r$; see Figure 10(b).
- (iii) If $c' \in Q_3$, $\Pi(c, c_2)$ intersects the y-axis at a point q with $y(q) < y(g_4^-)$, and $\Pi(c, c^*)$ intersects the segment $\overline{z_3g_3}$, then, $|\Pi(c, g_4^-)| \leq r$ or $|\Pi(c, g_3)| \leq r$; see Figure 10(c).
- (iv) If $c' \in Q_4$, $\Pi(c, c_3)$ intersects the x-axis at a point q with $x(q) > x(g_1^-)$, and $\Pi(c, c^*)$ intersects the segment $\overline{z_4g_4}$, then, $|\Pi(c, g_1^-)| \leq r$ or $|\Pi(c, g_4)| \leq r$; see Figure 10(d).

Proof. We prove Items (i) and (ii), the proofs of the other two items are symmetric to the proof of Item (ii).

Let ℓ_v be the vertical line passing through g_1^+ . Since $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(q) > x(g_1^+)$, $\Pi(c, c^*)$ intersects ℓ_v at a point p and the polygon cannot intersect the segment



Figure 10: Illustration of Lemma 6: (a) Item (i), (b) Item (ii), (c) Item (iii), and (d) Item (iv).

 pg_1^+ . Let $b = (b_x, 2)$ be the point on $\Pi(c, c^*)$; see Figure 12. We distinguish between two cases. **Case 1:** $b_x \leq \frac{3}{2}$.

Proof of item (i): Since the polygon intersects the segment $\overline{z_2g_2}$, we have $y(q^*) \leq y(g_1^+)$. Thus, the angle $\angle (p, g_1^+, q^*)$ is the largest in the triangle $\triangle (p, g_1^+, q^*)$; see Figure 11. Thus, $|pg_1^+| \leq |\Pi(p, q^*)|$. Therefore, $|\Pi(c, g_1^+)| \leq |\Pi(c, p)| + |pg_1^+| \leq |\Pi(c, p)| + |\Pi(q^*)| = |\Pi(c, q^*)| \leq r$. **Proof of item (ii):**

- If the polygon does not intersect the segment $\overline{g_2b}$, then, by Observation 6, $|bg_2| \le |bq^*|$, and thus $|\Pi(c,g_2)| = |\Pi(c,b)| + |bg_2| \le |\Pi(c,b)| + |bq^*| \le |\Pi(c,q^*)| \le r$; see Figure 12(a).
- Otherwise, the polygon intersects the segment g_2b .
 - If the polygon intersects the disk D', then g_1^+ is defined as the intersection of the sweeping horizontal line ℓ_h with D', and thus $y(q^*) \leq y(g_1^+)$; see Figure 12(b). Thus, $|pg_1^+| \leq |pq^*|$. Therefore, since the polygon does not intersect the segment $\overline{pg_1^+}$, we have $|\Pi(c,g_1^+)| \leq |\Pi(c,p)| + |pg_1^+| \leq |\Pi(c,p)| + |\Pi(p,q^*)| = |\Pi(c,q^*)| \leq r$.



Figure 11: $b_x \leq \frac{3}{2}$ and the polygon intersects the segment $\overline{z_2g_2}$.



Figure 12: $b_x \leq \frac{3}{2}$: (a) The polygon does not intersect the segment $\overline{g_2b}$. (b) The polygon intersects the disk D'.

- Otherwise, let ℓ'_3 be the horizontal line that is tangent to D^* at the point (0,1) and let D_b be the disk centered at b and is tangent to ℓ'_3 . Let ℓ_b be a tangent line of D_b and D^* as depicted in Figure 13(b). Let g be the intersection point of ℓ_b with D' in Q_1 . Since g is below g_1^+ on the boundary of D', we have $|\Pi(c, g_1^+)| \leq |\Pi(c, g)|$. Therefore, to prove that $|\Pi(c, g_1^+)| \leq r$, it is sufficient to prove that $|\Pi(c, g)| \leq r$.

Let ℓ'_v be the vertical line passing through g. Since $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(q) > x(z_1)$, $\Pi(c, c^*)$ intersects ℓ'_v at a point p' and the polygon cannot intersect the segment $\overline{p'g}$. Let b_x denote the x-coordinate of b, and notice that the coordinates of $g = (g_x, g_y)$ depend on b_x . To compute the coordinates of g, we compute the intersection point between the tangent line ℓ_b with D' in Q_1 . The equation of ℓ_b is



Figure 13: $b_x \leq \frac{3}{2}$ and the polygon intersects the segment $\overline{g_2b}$ but not D'.

 $y = \frac{2}{b_x}x - \frac{\sqrt{4+b_x^2}}{b_x}$, and the equation of D' is $(x-1)^2 + y^2 = 1$. Hence, we have

$$g_x = \frac{b_x^2 + 2\sqrt{4 + b_x^2} + 2b_x\sqrt{\sqrt{4 + b_x^2} - 1}}{4 + b_x^2} \text{ and } g_y = \frac{2b_x - b_x\sqrt{4 + b_x^2} + 4\sqrt{\sqrt{4 + b_x^2} - 1}}{4 + b_x^2}$$

Since $\frac{b_x^2 + 2\sqrt{4 + b_x^2 + 2b_x}\sqrt{\sqrt{4 + b_x^2 - 1}}}{4 + b_x^2} - b_x > 0$, for every $0 < b_x \le \frac{3}{2}$, we have $b_x \le g_x$, for every $0 < b_x \le \frac{3}{2}$.

Let $g' = (g_x, 2)$ and notice that the polygon does not intersect the segment $\overline{g'g}$; see Figure 13(b). Moreover, since the polygon does not intersect the segment $\overline{p'g}$, we have $|\Pi(c,g')| \leq |\Pi(c,b)|$. We now claim that

$$|g'g| = 2 - g_y < \sqrt{4 + b_x^2} - 1 = |bq^*|$$
, for every $0 < b_x \le g_x$.

That is, $3 - \sqrt{4 + b_x^2} < g_y = \frac{2b_x - b_x\sqrt{4 + b_x^2} + 4\sqrt{\sqrt{4 + b_x^2} - 1}}{4 + b_x^2}$. To see the correctness of this inequality, we need to show that $(3 - \sqrt{4 + b_x^2})(4 + b_x^2) < 2b_x - b_x\sqrt{4 + b_x^2} + 4\sqrt{\sqrt{4 + b_x^2} - 1}$. This is true since the left side of this inequality has maximum value equals 4, when $b_x = 0$, and the right side of this inequality has minimum value equals 4, when $b_x = 0$, for each $0 < b_x \leq \frac{3}{2}$. Therefore, we have $|\Pi(c,g)| \leq |\Pi(c,g')| + |g'g| \leq |\Pi(c,b)| + |bq^*| \leq |\Pi(c,q^*)| \leq r$.

Case 2: $b_x > \frac{3}{2}$. We show that in this case we have $|\Pi(c, g_1^+)| \leq r$, which proves both Items (i) and (ii). Let $a = (a_x, a_y)$ be the intersection point of $\Pi(c, c^*)$ with the segment $\overline{z_1g_2}$. Since $\Pi(p, c^*)$ intersects ℓ_v , we have $a_x \geq b_x > \frac{3}{2}$. Since the polygon does not intersect the segment $\overline{ag_1^+}$, we have $|\Pi(c, g_1^+)| \leq |\Pi(c, a)| + |ag_1^+|$. Thus, it is sufficient to prove that $|ag_1^+| \leq |aq^*|$, for each $\frac{3}{2} \leq a_x \leq x(z_1)$.

Let ℓ'_3 be the horizontal line passing through point (0,1) and let D_a be the disk centered at a and is tangent to ℓ'_3 . Let ℓ_a be a tangent line to D_a and D^* as depicted in Figure 14(b). Let g be the intersection point of ℓ_a with D' in Q_1 . We distinguish between two cases.



Figure 14: $b_x > \frac{3}{2}$ and $y(g) \le y(g_1^+)$.

Case 2.1: $y(g) \le y(g_1^+)$ (i.e., g is below g_1^+ on the boundary of D'). Since $1 \le x(g_1^+) \le 2$, we have $|ag_1^+| \le |ag|$. Therefore, to prove the lemma, it is sufficient to prove that $|ag| \le |aq^*|$, for each $\frac{3}{2} \le a_x \le x(z_1)$.

Since a is on the segment $\overline{z_1g_2}$ and the equation of the line passing through z_1 and g_2 is $y = \frac{4\sqrt{2}-5}{3}x + 2$, we have $a_y = \frac{(4\sqrt{2}-5)a_x}{3} + 2$. Notice that x(g) and y(g) (the coordinates of g) depend on a_x . To compute these coordinates, we compute the intersection point between the tangent line ℓ_a with D' in Q₁. The equation of ℓ_a is $y = \left(x + \frac{3}{4\sqrt{2}-5}\right)m - \frac{6}{(4\sqrt{2}-5)a_x} - 1$, where

$$m = \frac{1}{8(5\sqrt{2}-6)a_x} \left((12\sqrt{2}-15)a_x + 18 - \sqrt{-3((120\sqrt{2}-171)a_x^2 - (1712\sqrt{2}-2420)a_x + 480\sqrt{2}-684)} \right).$$

Let t be the point on the segment $\overline{z_1g_2}$, where $x(t) = \frac{3}{2}$. We prove that $|ag| \leq |aq^*|$, for each $\frac{3}{2} \leq a_x \leq x(z_1)$ by dividing the segment $\overline{tz_1}$ into 7 intervals defined by the points t_0, t_1, \ldots, t_7 , where $x(t_0) = x(t) = \frac{3}{2}$, $x(t_1) = 1.52$, $x(t_2) = 1.56$, $x(t_3) = 1.63$, $x(t_4) = 1.74$, $x(t_5) = 1.9$, $x(t_6) = 2.15$, and $x(t_7) = x(z_1) = \frac{3(2+\sqrt{2})}{4}$. For each $1 \leq i \leq 7$, we compute the intersection point g'_i of l_a with the disk D', where $a = t_i$, and we show that $|ag'_i| = \sqrt{(a_x - x(g'_i))^2 + (a_y - y(g'_i))^2} \leq \sqrt{a_x^2 + a_y^2} - 1 = |aq^*|$, for each $x(t_{i-1}) \leq a_x \leq x(t_i)$.

• For i = 1, we have $g'_1 = (1.8033, 0.5955)$, and thus

$$\sqrt{(a_x - 1.8033)^2 + (a_y - 0.5955)^2} \le \sqrt{a_x^2 + a_y^2} - 1$$
, for each $1.5 \le a_x \le 1.52$

• For i = 2, we have $g'_2 = (1.8152, 0.5792)$, and thus

$$\sqrt{(a_x - 1.8152)^2 + (a_y - 0.5792)^2} \le \sqrt{a_x^2 + a_y^2} - 1$$
, for each $1.52 \le a_x \le 1.56$.

• For i = 3, we have $g'_3 = (1.8347, 0.5507)$, and thus

$$\sqrt{(a_x - 1.8347)^2 + (a_y - 0.5507)^2} \le \sqrt{a_x^2 + a_y^2} - 1, \text{ for each } 1.56 \le a_x \le 1.63$$

• For i = 4, we have $g'_4 = (1.8623, 0.5063)$, and thus

$$\sqrt{(a_x - 1.8623)^2 + (a_y - 0.5063)^2} \le \sqrt{a_x^2 + a_y^2} - 1$$
, for each $1.63 \le a_x \le 1.74$.

• For i = 5, we have $g'_5 = (1.8966, 0.4429)$, and thus

$$\sqrt{(a_x - 1.8966)^2 + (a_y - 0.4429)^2} \le \sqrt{a_x^2 + a_y^2} - 1$$
, for each $1.174 \le a_x \le 1.9$.

• For i = 6, we have $g'_6 = (1.9376, 0.3478)$, and thus

$$\sqrt{(a_x - 1.9376)^2 + (a_y - 0.3478)^2} \le \sqrt{a_x^2 + a_y^2} - 1$$
, for each $1.9 \le a_x \le 2.15$.

• For i = 7, we have $g'_7 = (1.9787, 0.2053)$, and thus

$$\sqrt{(a_x - 1.9787)^2 + (a_y - 0.2053)^2} \le \sqrt{a_x^2 + a_y^2} - 1, \text{ for each } 2.15 \le a_x \le \frac{3(2 + \sqrt{2})}{4}.$$

These inequalities hold since the function $\sqrt{a_x^2 + a_y^2} - 1 - \sqrt{(a_x - x(g'_i))^2 + (a_y - y(g'_i))^2}$ is monotonic in the interval $x(t_{i-1}) \leq a_x \leq x(t_i)$, and has minimum value when $a_x = x(t_{i-1})$, for each $1 \leq i \leq 7$. Thus, for each $a_x \leq x(t_i)$, where $1 \leq i \leq 7$, we have $|ag| \leq |ag'_i|$. This proves that $|ag| \leq |aq^*|$.



Figure 15: $b_x > \frac{3}{2}$ and $y(g') > y(g_1^+)$.

Case 2.2: $y(g) > y(g_1^+)$ (i.e., g is above g_1^+ on the boundary of D'). Observe that this case can happen only if the polygon intersects the line ℓ'_3 . Recall that ℓ_v is the vertical line passing

through \underline{g}_1^+ , p is the intersection point of $\Pi(c, c^*)$ with ℓ_v , and the polygon does not intersect the segment \overline{pg}_1^+ . Let ℓ_h be the horizontal line passing through g_1^+ . Since the polygon intersects ℓ'_3 , we have $y(q^*) \leq y(g_1^+)$, i.e., q^* is below $\underline{\ell}_h$; see Figure 15. Thus, the angle $\angle(p, g_1^+, q^*) \geq \frac{\pi}{2}$, and, since the polygon does not intersect \overline{pg}_1^+ , we have $|pg_1^+| \leq |\Pi(p, q^*)|$. Therefore, $|\Pi(c, g_1^+)| \leq |\Pi(c, p)| + |pg_1^+| \leq |\Pi(c, p)| + |\Pi(p, q^*)| = |\Pi(c, q^*)| \leq r$.

Lemma 7. Let $D \in \mathcal{D}$ be a disk centered at c with radius r.

- (i) If $c' \in Q_1$, $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(q) > x(g_1^+)$, and $\Pi(c, c^*)$ intersects the segment $\overline{z_1g_1}$, then $|\Pi(c, g_1^+)| \leq r$; see Figure 16(a).
- (ii) if $c' \in Q_3$, $\Pi(c, c_2)$ intersects the y-axis at a point q with $y(q) < x(g_4^-)$, and $\Pi(c, c^*)$ intersects the segment $\overline{z_3g_4}$, then $|\Pi(c, g_4^-)| \leq r$; see Figure 16(b).
- (iii) If $c' \in Q_4$, $\Pi(c, c_3)$ intersects the x-axis at a point q with $x(q) > x(g_1^-)$, and $\Pi(c, c^*)$ intersects the segment $\overline{z_4g_1}$, then $|\Pi(c, g_1^-)| \le r$; see Figure 16(c).





Figure 16: Illustration of Lemma 7: Item (i), (b) Item (ii), and (c) Item (iii).

Proof. We prove Item (i), the proofs of the other items are symmetric.

Let ℓ_v be the vertical line passing through g_1^+ , and let p be the intersection point of $\Pi(c, c^*)$ with ℓ_v ; see Figure 17. We distinguish between two cases.

Case 1: $y(p) \ge y(g_1^+)$; see Figure 17(a). Let *a* be the intersection point of $\Pi(c, c^*)$ with $\overline{z_1g_1}$. By the definition of z_1 , we have $|ag_1| \le |aq^*|$, and thus $|\Pi(c, g_1)| \le r$. Moreover, since *p* is above g_1^+ , g_1^+ is inside the pseudo-triangle $\triangle(c, q^*, g_1)$; see Figure 17(a). Thus, by Observation 3, *D* contains g_1^+ , and therefore $|\Pi(c, g_1^+)| \le r$.



Figure 17: Illustration of the proof of Lemma 7, Item (i): (a) $y(p) \ge y(g_1^+)$, (b) $y(p) < y(g_1^+)$ and $\Pi(c, c_3)$ intersects ℓ_t above g_1^+ , and (c) $y(p) < y(g_1^+)$ and $\Pi(c, c_3)$ intersects ℓ_t below g_1^+ .

Case 2: $y(p) < y(g_1^+)$; see Figure 17(b). Let ℓ_t be the line that is tangent to D^* and passes through g_1^+ , and observe that $\Pi(c, c_3)$ intersects this line.

• If $\Pi(c, c_3)$ intersects ℓ_t above g_1^+ , then g_1^+ is inside the pseudo-triangle $\triangle(c, q^*, q_3)$; see Figure 17(b). Thus, by Observation 3, D contains jg_1^+ , and therefore $|\Pi(c, g_1^+)| \leq r$.

• If $\Pi(c, c_3)$ intersects ℓ_t below g_1^+ , then let ℓ_h be the horizontal line passing through g_1^+ ; see Figure 17(c). Let a be the intersection point of $\Pi(c, c_3)$ with the boundary of D', and let b be the intersection point of $\Pi(c, c_3)$ with ℓ_h . Observe that $x(b) \leq x(g_1^+) \leq x(a)$ and $y(b) = y(g_1^+) \geq y(a)$. Hence, the angle $\angle (a, g_1^+, b)$ is the largest in the triangle $\triangle (a, g_1^+, b)$. Thus, $|ag_1^+| \leq |ab| \leq |\Pi(a, q_3)|$. Therefore, $|\Pi(c, g_1^+)| \leq |\Pi(c, a)| + |ag_1^+| \leq |\Pi(c, a)| + |\Pi(a, q_3)| = |\Pi(c, q_3)| \leq r$.

Lemma 8. Let $D \in \mathcal{D}$ be a disk centered at c with radius r.

- (i) If $c' \in Q_1$ and $\Pi(c, c^*)$ does not intersect the segments $\overline{z_1g_1}$ nor $\overline{z_1g_2}$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^+)| \leq r$; see Figure 18(a).
- (ii) If $c' \in Q_3$, $\Pi(c, c^*)$ does not intersect the segments $\overline{z_3g_4}$ nor $\overline{z_3g_3}$, and $\Pi(c, c_2)$ intersects the negative y-axis, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_4^-)| \leq r$; see Figure 18(b).
- (iii) If $c' \in Q_4$, $\Pi(c, c^*)$ does not intersect the segments $\overline{z_4g_4}$ nor $\overline{z_4g_1}$, and $\Pi(c, c_3)$ intersects the positive x-axis, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^-)| \leq r$; see Figure 18(c).





Figure 18: Illustration of Lemma 8: (a) Item (i), (b) Item (ii), and (c) Item (iii).

Proof. We prove Item (i), the proofs of the other two items are symmetric.

Since $c' \in Q_1$, the path $\Pi(c, c_1)$ intersects the positive x-axis at a point q. If $x(c^*) \leq x(q) \leq x(g_1^+)$, then by Lemma 5, $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^-)| \leq r$. Otherwise, $x(g_1^+) < x(q) \leq x(z_1)$. Let ℓ_h be the horizontal line passing through g_1^+ .

- If $\Pi(c, c_1)$ intersects ℓ_h , then let p be this intersection point; see Figure 19(a). Thus, $x(g_1^+) \leq x(p) \leq x(z_1)$ and the polygon does not intersect the segment $\overline{g_1^+p}$. Let q_1 be the intersection point of $\Pi(c, c_1)$ with ℓ_1 . Since, $|pg_1^+| = |pq| + (x(p) 2)$, x(p) < 3, and $|pq_1| = |pq| + 1$, we have $|pg_1^+| < |pq_1| \leq |\Pi(p, q_1)|$. Therefore, since $|\Pi(c, q_1)| \leq r$, we have $|\Pi(c, g_1^+)| \leq |\Pi(c, p)| + |pg_1^+| \leq |\Pi(c, p)| + |\Pi(p, q_1)| = |\Pi(c, q_1)| \leq r$.
- If $\Pi(c, c_1)$ does not intersect ℓ_h , then let ℓ_t be the tangent to D^* with positive slope that passes through g_1^+ ; see Figure 19(b).
 - If $\Pi(c, c_3)$ intersects ℓ_t above g_1^+ , then g_1^+ is inside the pseudo-triangle $\triangle(c, q^*, q_3)$. Thus, by Observation 3, D contains g_1^+ , and therefore $|\Pi(c, g_1^+)| \leq r$.
 - If $\Pi(c, c_3)$ intersects ℓ_t below g_1^+ , then let ℓ_h be the horizontal line passing through g_1^+ . Let a be the intersection point of $\Pi(c, c_3)$ with the boundary of D', and let b be the intersection point of $\Pi(c, c_3)$ with ℓ_h . Observe that $x(b) \leq x(g_1^+) \leq x(a)$ and $y(b) = y(g_1^+) \geq y(a)$. Hence, the angle $\angle(a, g_1^+, b)$ is the largest in the triangle $\triangle(a, g_1^+, b)$. Thus, $|ag_1^+| \leq |ab| \leq |\Pi(a, q_3)|$. Therefore, $\Pi(c, g_1^+) \leq |\Pi(c, a)| + |ag_1^+| \leq |\Pi(c, a)| + |\Pi(a, q_3)| = |\Pi(c, q_3)| \leq r$.



Figure 19: Illustration of the proof of Lemma 8, Item (i): (a) $\Pi(c, c_1)$ intersects ℓ_h , and (b) $\Pi(c, c_1)$ does not intersect ℓ_h .

Lemma 9. Let $D \in \mathcal{D}$ be a disk centered at c with radius r.

- (i) If $c' \in Q_1$ and $g_1^- \neq g_1$, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^-)| \leq r$; see Figure 20(a).
- (ii) If $c' \in Q_4$, $g_1^+ \neq g_1$ and $\Pi(c, c_3)$ intersects the positive x-axis, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^+)| \leq r$; see Figure 20(b)



(iii) If $c' \in Q_4$, $g_4^- \neq g_4$ and $\Pi(c, c_3)$ intersects the negative y-axis, then $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_4^-)| \leq r$; see Figure 20(c).

Figure 20: Illustration of Lemma 9: (a) Item (i), (b) Item (ii), and (c) Item (iii).

Proof. We prove Item (i), the proofs of the other two items are symmetric.

Since $c' \in Q_1$, $\Pi(c, c_1)$ intersects the positive x-axis at a point q. If $x(c^*) \leq x(q) \leq x(g_1^-)$, then, by Lemma 5, $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^-)| \leq r$. Otherwise, $x(q) > x(g_1^-)$. Let ℓ_t be the line of negative slope that is tangent to D^* and passes through g_1^- , and observe that if $g_1^- \neq g_1$, then $\Pi(c, c_1)$ intersects this line; see Figure 21.

- If $\Pi(c, c_1)$ intersects ℓ_t below g_1^- , then g_1^- is inside the pseudo-triangle $\triangle(c, q^*, q_1)$; see Figure 21(a). Thus, by Observation 3, D contains g_1^- , and therefore $|\Pi(c, g_1^-)| \leq r$.
- If $\Pi(c, c_1)$ intersects ℓ_t above g_1^- , then let ℓ_h be the horizontal line passing through g_1^- ; see Figure 21(b). Let *a* be the intersection point of $\Pi(c, c_1)$ with the boundary of *D'*, and let *b* be the intersection point of $\Pi(c, c_1)$ with ℓ_h . Observe that $x(b) \leq x(g_1^-) \leq x(a)$ and y(b) =

 $y(g_1^-) \leq y(a)$. Hence, the angle $\angle (a, g_1^-, b)$ is the largest in the triangle $\triangle (a, g_1^-, b)$. Thus, $|ag_1^-| \leq |ab| \leq |\Pi(a, q_1)|$. Therefore, $|\Pi(c, g_1^-)| \leq |\Pi(c, a)| + |ag_1^-| \leq |\Pi(c, a)| + |\Pi(a, q_1)| = |\Pi(c, q_1)| \leq r$.



Figure 21: Illustration of the proof of Lemma 9, Item (i): (a) $\Pi(c, c_1)$ intersects ℓ_t below g_1^- , and (b) $\Pi(c, c_1)$ intersects ℓ_t above g_1^- .

Lemma 10. Let $D \in \mathcal{D}$ be a disk centered at c with radius r and $c' \in Q_2$, such that $\Pi(c, c_1)$ intersects the x-axis at a point q with $x(q) < x(z_2)$, and $\Pi(c, c^*)$ intersects the segment $\overline{z_2g_2}$. If $\alpha_2 > \frac{\pi}{3}$, then $|\Pi(c, g_2^+)| \leq r$.

Proof. Let ℓ_v be the vertical line passing through g_2^+ and let p be the intersection point of $\Pi(c, c^*)$ with ℓ_v ; see Figure 22.

- If $y(p) \ge y(g_2^+) = 1$, then, since $y(q^*) \le 1$, the angle $\angle (p, g_2^+, q^*)$ is the largest in the triangle $\triangle (p, g_2^+, q^*)$; see Figure 22(a). Since the polygon does not intersect $\overline{pg_2^+}$, we have $|pg_2^+| \le |\Pi(p, q^*)|$. Therefore, $|\Pi(c, g_2^+)| \le |\Pi(c, p)| + |pg_2^+| \le |\Pi(c, p)| + |\Pi(p, q^*)| = |\Pi(c, q^*)| \le r$.
- If $y(p) < y(g_2^+)$, then consider the path $\Pi(c, c_2)$ and notice that, since $\alpha_2 > \frac{\pi}{3}$, this path intersects ℓ_v at a point a; see Figure 22(b). If $y(a) \ge y(g_2^+)$, then g_2^+ is inside the pseudo triangle $\triangle(c, q^*, q_2)$, and by Observation 3, D contains g_2^+ . Otherwise, $0 \le y(a) < y(g_2^+)$. In this case, $|ag_2^+| \le 1$, and, since $\alpha_2 > \frac{\pi}{3}$, we have $|\Pi(a, q_2)| > 1$. Moreover, since the polygon does not intersect $\overline{ag_2^+}$, we have $|ag_2^+| < |\Pi(a, q_2)|$. Therefore, $|\Pi(c, g_2^+)| \le |\Pi(c, a)| + |ag_2^+| < |\Pi(c, a)| + |\Pi(a, q_2)| \le r$.

Lemma 11. Let $D \in \mathcal{D}$ be a disk centered at c with radius r and $c' \in Q_4$, such that $\Pi(c, c_3)$ intersects the x-axis at a point q where $x(q) > x(g_1)$. If $\alpha_3 > \frac{\pi}{6}$, then

• if $\Pi(c, c^*)$ intersects $\overline{z_4g_1}$, then $|\Pi(c, g_1)| \leq r$; and



Figure 22: Illustration of the proof of Lemma 10: (a) $\Pi(c, c^*)$ intersects ℓ_v above g_2^+ , and (b) $\Pi(c, c^*)$ intersects ℓ_v below g_2^+ .

• if $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$, then $|\Pi(c, g_1)| \leq r$ and $|\Pi(c, g_1^-)| \leq r$.

Proof. Let ℓ_t be the line tangent to D^* with a positive slope that passes through g_1 , and notice that the acute angle between ℓ_t and the x-axis is $\frac{\pi}{3}$. Since $\alpha_3 > \frac{\pi}{6}$, $\Pi(c, c_3)$ intersects ℓ_t . Moreover, since $x(q) > x(g_1)$, g_1 is inside the pseudo-triangle $\triangle(c, q^*, q_3)$; see Figure 23(a). Thus, by Observation 3, D contains g_1 , and therefore $|\Pi(c, g_1)| \leq r$. Let g = (-1, -1), and notice that g_1^- is on the small arc g_1g of D' between g_1 and g; see Figure 23(b). If $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$, then g_1g is contained in the pseudo-triangle $\triangle(c, q^*, q_3)$. Thus, by Observation 3, D contains both g_1 and g_1^- . Therefore, $|\Pi(c, g_1)| \leq r$ and $|\Pi(c, g_1^-)| \leq r$.



Figure 23: Illustration of the proof of Lemma 11: (a) $\Pi(c, c^*)$ intersects $\overline{z_4g_1}$, and (b) $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$.

The following lemma and its proof is symmetric to Lemma 11.

Lemma 12. Let $D \in \mathcal{D}$ be a disk centered at c with radius r and $c' \in Q_4$, such that $\Pi(c, c_3)$ intersects the y-axis at a point q where $y(q) < y(g_4)$. If $\alpha_3 \leq \frac{\pi}{3}$, then

- if $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$, then $|\Pi(c, g_4)| \leq r$; see Figure 24(a); and
- if $\Pi(c, c^*)$ intersects $\overline{z_4g_1}$, then $|\Pi(c, g_4)| \leq r$ and $|\Pi(c, g_4^+)| \leq r$; see Figure 24(b).



Figure 24: Illustration of Lemma 12: (a) $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$, and (b) $\Pi(c, c^*)$ intersects $\overline{z_4g_1}$.

3 The Algorithm

In this section, we show how to compute a set S of five points that pierce all the disks of \mathcal{D} . The algorithm, in a high-level description, works as follows. It first initializes S by $\{c^*\}$. Then, it goes over the segments $\overline{z_i g_i}$, $\overline{z_i g_{i+1}}$, for each i = 1, 2, 3, 4 (in a fixed order), and, for each segment, it checks whether the polygon intersects the segment, and adds to S a point $g'_i \in \{g_i, g_i^+, g_i^-\}$.

Recall that α_2 (resp., α_3) is the acute angle between ℓ_2 (resp., ℓ_3) and the *x*-axis, and notice that at most one of them is greater than $\frac{\pi}{3}$. We distinguish between three cases:

- (i) $\alpha_2 > \frac{\pi}{3};$
- (ii) $\alpha_3 > \frac{\pi}{3};$
- (iii) $\alpha_2 \leq \frac{\pi}{3}$ and $\alpha_3 \leq \frac{\pi}{3}$.

Notice that Case (i) and Case (ii) are symmetric. In Algorithm 1, we describe how to compute S in Case (i), and, in Algorithm 2, we describe how to compute S in Case (iii).

Algorithm 1 Compute S when $\alpha_2 > \frac{\pi}{3}$

1: $g'_1 \leftarrow g_1, g'_2 \leftarrow g_2, g'_3 \leftarrow g_3, g'_4 \leftarrow g_4$ 2: if P does not intersect $\overline{z_1g_1}$ then if P intersects $\overline{z_1g_2}$ or $\overline{z_2g_2}$ then 3: $g_1' \leftarrow g_1^+$ 4: if P intersects $\overline{z_2g_2}$ then 5: $g_2' \leftarrow g_2^+$ 6: else 7: if P intersects $\overline{z_4g_4}$ then 8: $g_1' \leftarrow g_1^-$ 9: 10: if P does not intersect $\overline{z_2g_3}$ then if P intersects $\overline{z_2g_2}$ then 11: $g_2' \leftarrow g_2^+$ 12:if P does not intersect $\overline{z_3g_4}$ then 13:if P intersects $\overline{z_3q_3}$ then 14:15: $g'_4 \leftarrow g_4^-$ 16: return $S = \{c^*, g'_1, g'_2, g'_3, g'_4\}$

Algorithm 2 Compute S when $\alpha_2 \leq \frac{\pi}{3}$ and $\alpha_3 \leq \frac{\pi}{3}$

1: $g'_1 \leftarrow g_1, g'_2 \leftarrow g_2, g'_3 \leftarrow g_3, g'_4 \leftarrow g_4$ 2: if P does not intersect $\overline{z_1g_1}$ then if P intersects $\overline{z_1g_2}$ then 3: $g_1' \leftarrow g_1^+$ 4: 5: if P does not intersect $\overline{z_2g_3}$ then if P intersects $\overline{z_2g_2}$ then 6: $g'_3 \leftarrow g_3^-$ 7: 8: if P does not intersect $\overline{z_1g_1}$, $\overline{z_1g_2}$, $\overline{z_2g_2}$ nor $\overline{z_2g_3}$ then if P intersects $\overline{z_3g_4}$ then 9: $g'_3 \leftarrow g^+_3$ 10: if P intersects $\overline{z_4g_4}$ then 11: $g_1' \leftarrow g_1^-$ 12:13: **return** $S = \{c^*, g_1', g_2', g_3', g_4'\}$

4 Correctness

Let $D \in \mathcal{D}$ be a disk with center c and radius r. For each $i \in \{1, 2, 3\}$, let q_i be the intersection point of the path $\Pi(c, c_i)$ with the line ℓ_i . Let q^* be the intersection point of the path $\Pi(c, c^*)$ with the boundary of D^* , and let c' be the point on $\Pi(c, c^*)$, such that the edge (c', c^*) is the last edge in $\Pi(c, c^*)$. That is, c' is the first point on $\Pi(c, c^*)$ that is visible from c^* . We prove that the set $S = \{c^*, g'_1, g'_2, g'_3, g'_4\}$ (that is computed by the algorithm) pierces all the disks of \mathcal{D} . In the proof, we distinguish between three cases: (i) $\alpha_2 > \frac{\pi}{3}$; (ii) $\alpha_3 > \frac{\pi}{3}$; and (iii) $\alpha_2 \leq \frac{\pi}{3}$ and $\alpha_3 \leq \frac{\pi}{3}$. Following the algorithm, we show in Section 4.1 the proof for Case (i) and in Section 4.2 the proof for Case (ii) (since Case (i) and Case (ii) are symmetric).

4.1 Case (i): $\alpha_2 > \frac{\pi}{3}$

Let $D \in \mathcal{D}$ be a disk with center c and radius r. We show that D is pierced by at least one of the points of S. We distinguish between four cases according to which quadrant c' belongs to.

4.1.1 $c' \in Q_1$

We prove that D is pierced by at least one of the points g'_1, g'_2 , or c^* . We distinguish between four cases.

Case 1: The polygon does not intersect $\overline{z_1g_1}$, $\overline{z_1g_2}$, $\overline{z_2g_2}$, nor $\overline{z_4g_4}$. In this case, $g'_1 = g_1$ and $g'_2 = g_2$, and by Lemma 4, D is pierced by at least one of the points g'_1 , g'_2 , and c^* .

Case 2: The polygon intersects $\overline{z_1g_1}$; see Figure 25. In this case, $g'_1 = g_1$. Consider the path $\Pi(c, c_1)$ and notice that this path intersects the positive x-axis. Let q be this intersection point. Thus, $|\Pi(c,q)| + 1 \le r$.

- (i) If $x(c^*) \le x(q) \le x(g_1)$, then by Lemma 5, Item (i), D contains at least one of the points c^* or g_1 ; see Figure 25(a).
- (ii) If $x(g_1) < x(q) \le x(z_1)$, then, since $x(g_1) = 2$ and $x(z_1) < 3$, we have $|qg_1| < 1$. Since q is the intersection point of $\Pi(c, c_1)$ with the x-axis, the polygon does not intersect $\overline{qg_1}$. Thus, $|\Pi(c, g_1)| \le |\Pi(c, q)| + |qg_1| \le |\Pi(c, q)| + 1 \le r$. Therefore, D contains g_1 .
- (iii) If $x(q) > x(z_1)$, then consider the path $\Pi(c, c^*)$ and let p be the intersection point of $\Pi(c, c^*)$ with $\overline{z_1g_1}$; see Figure 25(b). Thus, the polygon does not intersect $\overline{pg_1}$, and, by Observation 5, we have $|pg_1| \leq |pq^*| \leq |\Pi(p, q^*)|$. Thus, $|\Pi(c, g_1)| \leq |\Pi(c, p)| + |pg_1| \leq |\Pi(c, p)| + |pq^*| \leq |\Pi(c, p)| + |\Pi(p, q^*)| = |\Pi(c, q^*)| \leq r$. Therefore, D contains g_1 .



Figure 25: Illustration of the proof of Case 2. (a) $x(c^*) \leq x(q) \leq x(z_1)$, and (b) $x(q) > x(z_1)$.

Case 3: The polygon does not intersect $\overline{z_1g_1}$ but intersects $\overline{z_1g_2}$ or $\overline{z_2g_2}$. In this case, $g'_1 = g_1^+$. Consider the path $\Pi(c, c_1)$ and notice that this path intersects the positive x-axis at a point q.

- (i) If $x(c^*) \leq x(q) \leq x(g_1^+)$, then, by Lemma 5, Item (i), $|\Pi(c,c^*)| \leq r$ or $|\Pi(c,g_1^+)| \leq r$, and therefore D contains c^* or g_1^+ .
- (ii) If $x(q) > x(g_1^+)$ and $\Pi(c, c^*)$ intersects the segment $\overline{z_1g_2}$, then,

- if the polygon intersects the segment $\overline{z_2g_2}$, then, by Lemma 6, Item (i), $|\Pi(c, g_1^+)| \leq r$, and therefore D contains g_1^+ ; and
- if the polygon does not intersect the segment $\overline{z_2g_2}$, then, in this case, $g'_2 = g_2$, and, by Lemma 6, Item (ii), $|\Pi(c, g_1^+)| \leq r$ or $|\Pi(c, g_2)| \leq r$, and therefore D contains g_1^+ or g_2 .
- (iii) If $x(q) > x(g_1^+)$ and $\Pi(c, c^*)$ intersects the segment $\overline{z_1g_1}$, then, by Lemma 7, Item (i), $|\Pi(c, g_1^+)| \le r$, and therefore D contains g_1^+ .
- (iv) If $x(q) > x(g_1^+)$ and $\Pi(c, c^*)$ does not intersect the segments $\overline{z_1g_1}$ nor $\overline{z_1g_2}$, then, by Lemma 8, Item (i), $|\Pi(c, c^*)| \le r$ or $|\Pi(c, g_1^+)| \le r$, and therefore D contains c^* or g_1^+ .

Case 4: The polygon does not intersect $\overline{z_1g_1}$, $\overline{z_1g_2}$ nor $\overline{z_2g_2}$ but intersects $\overline{z_4g_4}$. In this case, $g'_1 = g_1^-$, and thus, by Lemma 9, Item (i), $|\Pi(c, g_1^-)| \leq r$ or $|\Pi(c, c^*)| \leq r$, and therefore *D* contains g_1^- or c^* .

4.1.2 $c' \in Q_2$

We prove that D is pierced by at least one of the points g'_2 , g'_3 , or c^* . We distinguish between three cases.

Case 1: The polygon does not intersect $\overline{z_2g_2}$ nor $\overline{z_2g_3}$. In this case, $g'_2 = g_2$ and $g'_3 = g_3$, and by Lemma 4, D is pierced by at least one of the points g'_2 , g'_3 , or c^* .

Case 2: The polygon intersects $\overline{z_2g_3}$. In this case, $g'_3 = g_3$ and D contains at least one of the points c^* or g_3 (the proof is symmetric to Case 2 in Section 4.1.1); see Figure 26.



Figure 26: Case 2: (a) $x(g_3) \le x(q) \le x(c^*)$, and (b) $x(q) < x(z_2)$.

Case 3: The polygon does not intersect $\overline{z_2g_3}$ but intersects $\overline{z_2g_2}$. In this case, $g'_2 = g_2^+ = (-1, 1)$ and $g'_3 = g_3$. Consider the path $\Pi(c, c_1)$ and notice that it intersects the negative x-axis at a point q. Thus, $|\Pi(c,q)| + 1 \le r$.

(i) If $x(g_3) \le x(c^*)$, then, by Lemma 5, Item (ii), we have $|\Pi(c, c^*)| \le r$ or $|\Pi(c, g_3)| \le r$, and therefore D contains c^* or g_3 .

- (ii) If $x(z_2) \le x(q) \le x(g_3)$, then, since $x(z_2) > -3$ and $x(g_3) = -2$, we have $|qg_3| < 1$. Since the polygon does intersect $\overline{qg_3}$, we have $|\Pi(c,g_3)| \le |\Pi(c,q)| + |qg_3| < |\Pi(c,q)| + 1 \le r$. Therefore, D contains g_3 .
- (iii) If $x(q) < x(z_2)$, then consider the path $\Pi(c, c^*)$, and notice that this path intersects either $\overline{z_2g_2}$ or $\overline{z_2g_3}$.
 - If $\Pi(c, c^*)$ intersects $\overline{z_2g_2}$, then, since $\alpha_2 > \frac{\pi}{3}$, by Lemma 10, we have $|\Pi(c, g_2^+)| \le r$, and therefore D contains g_2^+ .
 - If $\Pi(c, c^*)$ intersects $\overline{z_2g_3}$, then let p be this intersection point. Since the polygon does not intersect $\overline{pg_3}$, by Observation 5, we have $|pg_3| \leq |pq^*| \leq |\Pi(p, q^*)|$. Thus, $|\Pi(c, g_3)| \leq |\Pi(c, p)| + |pg_3| \leq |\Pi(c, p)| + |\Pi(p, q^*)| = |\Pi(c, q^*)| \leq r$. Therefore, D contains g_3 .

4.1.3 $c' \in Q_3$

We prove that D is pierced by at least one of the points g'_3 , g'_4 , or c^* .

Case 1: The polygon does not intersect $\overline{z_3g_3}$ nor $\overline{z_3g_4}$. In this case, $g'_3 = g_3$ and $g'_4 = g_4$, and by Lemma 4, D is pierced by at least one of the points g'_3 , g'_4 , and c^* .

Case 2: The polygon intersects $\overline{z_3g_4}$. In this case $g'_3 = g_3$ and $g'_4 = g_4$. Consider the path $\Pi(c, c_2)$, and notice that it intersects either the negative y-axis or the negative x-axis. Let q be this intersection point. Thus, $|\Pi(c,q)| + 1 \leq r$.

(i) If $\Pi(c, c_2)$ intersects the negative y-axis, then D contains at least one of the points c^* or g_4 (the proof is symmetric to Case 2 in Section 4.1.1); see Figure 27.



Figure 27: $\Pi(c, c_2)$ intersects the y-axis (a) $y(g_4) \le y(q) \le y(c^*)$, and (b) $y(q) < y(g_4)$.

- (ii) If $\Pi(c, c_2)$ intersects the negative x-axis and $x(g_3) \leq x(q) \leq x(c^*)$, then, by Lemma 5, Item (iii), D contains at least one of the points c^* or g_3 ; see Figure 28(a).
- (iii) If $\Pi(c, c_2)$ intersects the negative x-axis and $x(q) < x(g_3)$, then consider the path $\Pi(c, c^*)$ and notice that, since $\alpha_2 > \frac{\pi}{3}$ and $x(q) < x(g_3)$, this path intersects $\overline{z_3g_3}$ at a point p and the

polygon does not intersect $\overline{pg_3}$; see Figure 28(b). Hence, by Observation 5, we have $|pg_3| \leq |pq^*| \leq |\Pi(p,q^*)|$. Thus, $|\Pi(c,g_3)| \leq |\Pi(c,p)| + |pg_3| \leq |\Pi(c,p)| + |\Pi(p,q^*)| = |\Pi(c,q^*)| \leq r$. Therefore, D contains g_3 .



Figure 28: Case 2: (a) $\Pi(c, c_2)$ intersects the x-axis and $x(g_3) \leq x(q) \leq x(c^*)$, and (b) $\Pi(c, c_2)$ intersects the x-axis and $x(q) < x(g_3)$.

Case 3: The polygon does not intersect $\overline{z_3g_4}$ but intersects $\overline{z_3g_3}$. In this case $g'_3 = g_3$ and $g'_4 = g_4^-$. Consider the path $\Pi(c, c_2)$ and notice that it intersects either the negative y-axis or the negative x-axis. Let q be this intersection point. Thus, $|\Pi(c, q)| + 1 \leq r$.

(i) $\Pi(c, c_2)$ intersects the negative x-axis, then D contains at least one of the points c^* and g_3 (the proof is symmetric to the proof of Items (ii) and (iii) in the previous case); see Figure 29.



Figure 29: $\Pi(c, c_2)$ intersects the x-axis. (a) $x(g_3) \le x(q) \le x(c^*)$, and (b) $x(q) < x(g_3)$.

- (ii) $\Pi(c, c_2)$ intersects the negative y-axis and $y(g_4^-) \leq y(q) \leq y(c^*)$, then, by Lemma 5, Item (iv), we have $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_4^-)| \leq r$, and therefore D contains c^* or g_4^- .
- (iii) If $y(q) < y(g_4^-)$ and $\Pi(c, c^*)$ intersects the segment $\overline{z_3g_3}$, then, by Lemma 6, Item (iii), $|\Pi(c, g_4^-)| \leq r$ or $|\Pi(c, g_3)| \leq r$, and therefore D contains g_4^- or g_3 .
- (iv) If $y(q) < y(\overline{g_4})$ and $\Pi(c, c^*)$ intersects the segment $\overline{z_3g_4}$, then, by Lemma 7, Item (ii), $|\Pi(c, \overline{g_4})| \leq r$, and therefore D contains $\overline{g_4}$.
- (v) If $y(q) < y(\overline{g_4})$ and $\Pi(c, c^*)$ does not intersect the segments $\overline{z_3g_4}$ nor $\overline{z_3g_3}$, then, by Lemma 8, Item (ii), $|\Pi(c, \overline{g_4})| \leq r$, and therefore D contains $\overline{g_4}$.

4.1.4 $c' \in Q_4$

We prove that D is pierced by at least one of the points g'_1 , g'_4 , or c^* . Consider the path $\Pi(c, c_3)$, and notice that it intersects either the positive x-axis or the negative y-axis at a point q. The point q is on the positive x-axis.

Case 1: $x(c^*) \leq x(q) \leq x(g'_1)$. By Lemma 5, Item (v), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_1)| \leq r$, and therefore D contains c^* or g'_1 .

Case 2: $x(q) > x(g'_1)$. We distinguish between three cases. **Case 2.1:** $g'_1 = g_1$.

- (i) If the polygon intersects $\overline{z_1g_1}$, then $\Pi(c, c_3)$ intersects $\overline{z_1g_1}$ at a point p. Thus, g_1 is inside the pseudo-triangle $\triangle(c, q^*, p)$, and, by Observation 3, D contains g_1 .
- (ii) If the polygon intersects $\overline{z_{4}g_1}$, then *D* contains at least one of the points c^* or g_1 (the proof is symmetric to the proof of Case 2 in Section 4.1.1).
- (iii) If the polygon does not intersect $\overline{z_1g_1}$ nor $\overline{z_4g_1}$, then, since $g'_1 = g_1$, the polygon does not intersect $\overline{z_4g_4}$. If $g'_4 = g_4$, then, by Lemma 4, D is pierced by at least one of the points g'_1, g'_4 , and c^* . Otherwise, $g'_4 = g_4^-$. In this case, the polygon intersects $\overline{z_3g_3}$, and, by Observation 7, we have $\alpha_3 > \frac{\pi}{5}$. Thus, by Lemma 11, $|\Pi(c,g_1)| \leq r$, and therefore D contains g_1 .

Case 2.2: $g'_1 = g^+_1 \neq g_1$. By Lemma 9, Item (ii), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g^+_1)| \leq r$, and therefore *D* contains c^* or g^+_1 . **Case 2.3:** $g'_1 = g^-_1$.

- (i) If $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$, then, if $g'_4 = g_4$, then, by Lemma 6, Item (iv), $|\Pi(c, g_1^-)| \leq r$ or $|\Pi(c, g_4)| \leq r$, and therefore D contains g_1^- or g_4 . Otherwise, $g'_4 = g_4^-$. In this case, the polygon intersects $\overline{z_3g_3}$, and, by Observation 7, we have $\alpha_3 > \frac{\pi}{5}$. Thus, by Lemma 11, $|\Pi(c, g_1)| \leq r$, and therefore D contains g_1 .
- (ii) If $\Pi(c, c^*)$ intersects $\overline{z_4g_1}$, then by Lemma 7, Item (iii), $|\Pi(c, g_1^-)| \leq r$, and therefore D contains g_1^- .
- (iii) If $\Pi(c, c^*)$ does not intersect $\overline{z_4g_1}$ nor $\overline{z_4g_4}$, then, by Lemma 8, Item (iii), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^-)| \leq r$, and therefore D contains c^* or g_1^- .

The point q is on the negative y-axis.

Case 1: $y(g'_4) \leq y(q) \leq y(c^*)$. By Lemma 5, Item (vi), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_4)| \leq r$, and therefore D contains c^* or g'_4 .

Case 2: $y(q) < y(g'_4)$. We distinguish between two cases.

Case 2.1: $g'_4 = g_4^- \neq g_4$. By Lemma 9, Item (iii), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_4^-)| \leq r$, and therefore D contains c^* or g_4^- .

Case 2.2: $g'_4 = g_4$. Then, since $\alpha_3 \leq \frac{\pi}{3}$, by Lemma 12, $|\Pi(c, g_4)| \leq r$, and therefore D contains g_4 .

4.2 Case (iii): $\alpha_2 \leq \frac{\pi}{3}$ and $\alpha_3 \leq \frac{\pi}{3}$

Let $D \in \mathcal{D}$ be a disk with center c and radius r. We show that D is pierced by at least one of the points of S. Notice that in Algorithm 2, Q_1 is symmetric to Q_2 and Q_3 is symmetric to Q_4 . Therefore, we show the correctness for the cases where $c' \in Q_1$ and $c' \in Q_4$.

4.2.1 $c' \in Q_1$

We prove that D is pierced by at least one of the points g'_1 , g'_2 , or c^* . We distinguish between four cases.

Case 1: The polygon does not intersect $\overline{z_1g_1}$, $\overline{z_1g_2}$, nor $\overline{z_4g_4}$. In this case, $g'_1 = g_1$ and $g'_2 = g_2$, and by Lemma 4, D contains at least one of the points g'_1, g'_2 , and c^* .

Case 2: The polygon intersects $\overline{z_1g_1}$; see Figure 25. In this case, $g'_1 = g_1$, and D contains at least one of the points c^* or g_1 . The proof is the same as in Case 2 of Section 4.1.1.

Case 3: The polygon does not intersect $\overline{z_1g_1}$, but intersects $\overline{z_1g_2}$. In this case, $g'_1 = g_1^+$ and $g'_2 = g_2$, and D contains at least one of the points g_2 , g_1^+ and c^* . The proof is the same as in Case 3 of Section 4.1.1.

Case 4: The polygon does not intersect $\overline{z_1g_1}$ nor $\overline{z_1g_2}$ but intersects $\overline{z_4g_4}$. In this case $g'_1 = g_1^-$ and $g'_2 = g_2$, and D contains at least one of the points c^* or g_1^- . The proof is the same as in Case 4 of Section 4.1.1.

4.2.2 $c' \in Q_4$

We prove that D is pierced by at least one of the points g'_1, g'_4 , and c^* . Notice that in this case where both α_2 and α_3 are less or equal to $\frac{\pi}{3}$, Algorithm 2 does not change g_4 , thus $g'_4 = g_4$. Consider the path $\Pi(c, c_3)$, and notice that it intersects either the positive x-axis or the negative y-axis at a point q.

The point q is on the positive x-axis.

Case 1: $x(c^*) \leq x(q) \leq x(g'_1)$. By Lemma 5, Item (v), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g'_1)| \leq r$, and therefore D contains c^* or g'_1 .

Case 2: $x(q) > x(g'_1)$. We distinguish between three cases. **Case 2.1:** $g'_1 = g_1$.

- (i) If the polygon intersects $\overline{z_1g_1}$, then $\Pi(c, c_3)$ intersects $\overline{z_1g_1}$ at a point p. Thus, g_1 is inside the pseudo-triangle $\triangle(c, q^*, p)$, and, by Observation 3, D contains g_1 .
- (ii) If the polygon intersects $\overline{z_4g_1}$, then *D* contains at least one of the points c^* or g_1 (the proof is symmetric to the proof of Case 2 in Section 4.1.1).

(iii) If the polygon does not intersect $\overline{z_1g_1}$ nor $\overline{z_4g_1}$, then, since $g'_1 = g_1$, the polygon does not intersect $\overline{z_4g_4}$. Since $g'_4 = g_4$, by Lemma 4, D is pierced by at least one of the points g'_1, g'_4 , and c^* .

Case 2.2: $g'_1 = g^+_1 \neq g_1$. By Lemma 9, Item (ii), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g^+_1)| \leq r$, and therefore *D* contains c^* or g^+_1 . **Case 2.3:** $g'_1 = g^-_1$.

- (i) If $\Pi(c, c^*)$ intersects $\overline{z_4g_4}$, then, since $g'_4 = g_4$, by Lemma 6, Item (iv), we have $|\Pi(c, g_1^-)| \leq r$ or $|\Pi(c, g_4)| \leq r$, and therefore D contains g_1^- or g_4 .
- (ii) If $\Pi(c, c^*)$ intersects $\overline{z_4g_1}$, then by Lemma 7, Item (iii), $|\Pi(c, g_1^-)| \leq r$, and therefore D contains g_1^- .
- (iii) If $\Pi(c, c^*)$ does not intersect $\overline{z_4g_1}$ nor $\overline{z_4g_4}$, then, by Lemma 8, Item (iii), $|\Pi(c, c^*)| \leq r$ or $|\Pi(c, g_1^-)| \leq r$, and therefore D contains c^* or g_1^- .

The point q is on the negative y-axis.

Case 1: $y(g'_4) \leq y(q) \leq y(c^*)$. By Lemma 5, Item (vi), $|\Pi(c,c^*)| \leq r$ or $|\Pi(c,g'_4)| \leq r$, and therefore D contains c^* or g'_4 .

Case 2: $y(q) < y(g'_4)$. Since, $g'_4 = g_4$ and $\alpha_3 \leq \frac{\pi}{3}$, by Lemma 12, we have $|\Pi(c, g_4)| \leq r$, and therefore D contains g_4 .

5 Conclusion

We have shown that five points are sufficient to pierce a set of pairwise intersecting geodesic disks inside a polygon P. This improves the upper bound of 14, which was provided by Bose et al. [1]. This upper bound is very close to the lower bound for stabbing pairwise intersecting disks in the plane, which was proven to be four.

References

- P. Bose, P. Carmi, and T. C. Shermer. Piercing pairwise intersecting geodesic disks. *Computat. Geom.*, 98:101774, 2021.
- [2] P. Carmi, M. J. Katz, and P. Morin. Stabbing pairwise intersecting disks by four points. CoRR, abs/1812.06907, 2018.
- [3] L. Danzer. Zur lösung des Gallaischen problems über kreisscheiben in der Euklidischen ebene. Studia Sci. Math. Hungar, 21(1-2):111–134, 1986.
- [4] S. Har-Peled, H. Kaplan, W. Mulzer, L. Roditty, P. Seiferth, M. Sharir, and M. Willert. Stabbing pairwise intersecting disks by five points. *Discrete Math.*, 344(7):112403, 2021.
- [5] E. Helly. Über mengen konvexer körper mit gemeinschaftlichen punkten. Jahresber. Dtsch. Math.-Ver., 32:175–176, 1923.
- [6] E. Helly. Über systeme von abgeschlossenen mengen mit gemeinschaftlichen punkten. Monatshefte Math., 37(1):281–302, 1930.

- [7] R. Pollack, M. Sharir, and G. Rote. Computing the geodesic center of a simple polygon. *Discrete Comput. Geom.*, 4:611–626, 1989.
- [8] L. Stacho. Über ein problem für kreisscheiben familien. Acta Sci. Math. (Szeged), 26:273–282, 1965.
- [9] L. Stacho. A solution of Gallai's problem on pinning down circles. Mat. Lapok, 32(1-3):19-47, 1981/84.