# Bottleneck Matching in the Plane* 

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#### Abstract

We present an algorithm for computing a bottleneck matching in a set of $n=2 \ell$ points in the plane, which runs in $O\left(n^{\omega / 2} \log n\right)$ deterministic time, where $\omega \approx 2.37$ is the exponent of matrix multiplication.


Let $P$ be a set of $n=2 \ell$ points in the plane, and let $G=(P, E)$ denote the complete Euclidean graph over $P$, which is an undirected weighted graph with $P$ as its set of vertices, and the weight of an edge $e=(p, q) \in E$ is $\operatorname{dist}(p, q)$, where dist denotes the Euclidean distance. For a perfect matching $M$ in $G$, let $\lambda(M)$ denote the length of a longest edge in $M$. A perfect matching $M$ is a bottleneck matching of $P$ if $\lambda(M) \leq \lambda\left(M^{\prime}\right)$, for any other perfect matching $M^{\prime}$ in $G$.

In this note, we present an algorithm for computing a bottleneck matching $M^{*}$ of $P$, with running time $O\left(n^{\omega / 2} \log n\right)=O\left(n^{1.187}\right)$, where $\omega<2.3728596$ is the exponent of matrix multiplication [4, 10]. All previous algorithms for computing $M^{*}$ are based on the algorithm of Gabow and Tarjan [9, which computes a bottlenck maximum matching in a (general) graph with $n$ vertices and $m$ edges in time $O\left((n \log n)^{1 / 2} m\right)$. These algorithms apply Gabow and Tarjan's algorithm to a subgraph $G^{\prime}$ of $G$, with only $O(n)$ edges, that is guaranteed to contain $M^{*}$. The running time of these algorithms is therefore $O\left(T(n)+n^{3 / 2} \log ^{1 / 2} n\right)$, where $T(n)$ is the time needed to compute some suitable subgraph $G^{\prime}$. In the work of Chang et al. [7, $T(n)=O\left(n^{2}\right)$, and in the subsequent work of Su and Chang [11, $T(n)=$ $O\left(n^{5 / 3} \log n\right)$, which is the best previously-known bound for computing $M^{*}$; see below for further details concerning possible choices of $G^{\prime}$. In contrast to the previous algorithms, we replace the algorithm of Gabow and Tarjan (which immediately sets a lower bound of $\Omega\left(n^{3 / 2} \log ^{1 / 2} n\right)$ on the running time) by an algorithm of Bonnet et al. 5 for computing a maximum matching (not necessarily bottleneck) in an intersection graph of geometric objects in the plane, and use it as a decision procedure in our search for $M^{*}$.

We first observe that $\lambda^{*}:=\lambda\left(M^{*}\right)$ is the distance between some pair of points in $P$. For a real number $r>0$, let $G_{r}$ denote the graph that is obtained from $G$ by retaining only the edges of length at most $r$. That is, $G_{r}=\left(P, E_{r}\right)$, where

$$
E_{r}=\{(p, q) \in P \times P \mid p \neq q \text { and } \operatorname{dist}(p, q) \leq r\} .
$$

[^0]$G_{r}$ is also the intersection graph of the set of disks of radius $r / 2$ centered at the points of $P$, and by suitably scaling the configuration, it is the unit-disk graph over $P$ (see [8]). Bonnet et al. [5] presented an $O\left(n^{\omega / 2}\right)$-time algorithm for computing a maximum matching in $G_{r}$. Their algorithm can therefore also be used to determine whether $G_{r}$ contains a perfect matching, which is the case if and only if the number of edges in the returned maximum matching is $\ell$. Let $r^{*}>0$ be the smallest value for which $G_{r^{*}}$ contains a perfect matching. Then, by applying the algorithm of Bonnet et al. to $G_{r^{*}}$, we obtain a bottleneck matching $M^{*}$ of $P\left(\right.$ with $\left.\lambda^{*}=r^{*}\right)$.

Since $r^{*}$ is the distance between some pair of points in $P$, we can perform a binary search in the (implicit) set of the $\binom{n}{2}$ distances determined by pairs of points in $P$, using the algorithm of Bonnet et al. to resolve comparisons, thereby obtaining $r^{*}$ and $M^{*}$. To perform the binary search, we use a distance selection algorithm (such as in [2), whose running time is $O^{*}\left(n^{4 / 3}\right)$ (where the $O^{*}(\cdot)$ notation hides subpolynomial factors), to find the next distance to test. Since $\omega / 2<4 / 3$, we obtain a bottleneck matching algorithm for $n=2 \ell$ points in the plane with running time $O^{*}\left(n^{4 / 3}\right)$, which is already significantly faster than the best known such algorithm.

However, we can do better. If we are given a set $D$ of $m$ candidate distances that includes $r^{*}$, we can obtain, in total time $O\left(m+n^{\omega / 2} \log m\right)$, a bottleneck matching of $P$, by performing a binary search in $D$. Our goal is, therefore, to find such a candidate set $D$ of size $m=O\left(n^{\omega / 2}\right)$ (in fact, our set will be of size $\left.O(n)\right)$ that can also be generated efficiently. A natural choice of $D$ is the set of lengths of a suitable subgraph $G^{\prime}$ of $G$ that can be shown to contain a bottleneck matching of $P$.

Chang et al. [7] proved that the 17-relative neighborhood graph of $P$, denoted as 17RNG $(P)$, contains a bottleneck matching of $P$. This is a graph over $P$, where $(p, q)$ is an edge if and only if the number of points of $P$ whose distance from both $p$ and $q$ is smaller than $\operatorname{dist}(p, q)$ is at most 16. As shown in [7], the number of edges in $17 \mathrm{RNG}(P)$ is $O(n)$, so we could use the set of distances corresponding to these edges as our candidate set $D$. However, we do not know how to compute $17 \mathrm{RNG}(P)$ efficiently. The best known algorithm for computing the $k$ RNG, for any fixed integer $k$, of a set of $n$ points, is due to Su and Chang [11], and it runs in $O\left(n^{5 / 3} \log n\right)$ time (with the constant of proportionality depending on $k$ ). This bound can be improved, using more recent range searching techniques ${ }^{1}$ but it seems unlikely that such an improvement will be near the bound we are after.

Instead, we consider another graph, known as the 17-geographic neighborhood graph of $P$, denoted as 17GNG $(P)$, which has also been studied in [7 and is defined as follows. Partition the plane into six closed wedges $W_{1}, \ldots, W_{6}$, each with opening angle $60^{\circ}$, by the three lines through the origin with slopes $0, \pm \sqrt{3}$. The graph $17 \mathrm{GNG}(P)$ consists of all the edges $(p, q) \in P \times P$, with $p \neq q$, for which there exists $1 \leq i \leq 6$ such that $q \in p+W_{i}$, where $p+W_{i}$ is $W_{i}$ translated by $p$, so that there are at most 16 points $q^{\prime} \in p+W_{i}$ with $\operatorname{dist}\left(p, q^{\prime}\right)<\operatorname{dist}(p, q)$. (When the slope of the line through $p$ and $q$ is in $\{0, \pm \sqrt{3}\}$ there are two such wedges $W_{i}$.) See Figure 1 for an illustration. As shown in [7, 17GNG( $P$ ) is a supergraph of $17 \mathrm{RNG}(P)$, and the number of its edges is still $O(n)$. We show below that $17 \mathrm{GNG}(P)$ can be computed in $O\left(n \log ^{3} n\right)$ time. Once we have computed 17GNG $(P)$, we perform a binary search on the set of distances corresponding to its edge set, as described above, to obtain $M^{*}$, in total time $O\left(n^{\omega / 2} \log n\right)$.

[^1]Before continuing, we remark that there is actually no need to compute $17 \mathrm{GNG}(P)$. All we need is the set of lengths of its edges, over which we will run binary search to find $M^{*}$ using the algorithm of [5]. Nevertheless, for the sake of completeness, and since this is a result of independent interest, we present an algorithm that actually constructs $17 \mathrm{GNG}(P)$ efficiently. The algorithm is relatively simple when the points of $P$ are in general position, but is somewhat more involved when this is not the case.


Figure 1: The wedges $p+W_{1}, \ldots, p+W_{6}$ and the edges that are added to $2 \mathrm{GNG}(P)$ 'due' to $p$.

Computing $17 \mathrm{GNG}(P)$ in general position. Begin by assuming that $P$ is in general position. Actually, the only condition that we require is that no three points form an isosceles triangle. Computing 17GNG $(P)$ in this case is relatively simple, since informally, the ranges that arise in computing $17 \mathrm{GNG}(P)$ are wedges with fixed orientations of their bounding rays. In contrast, the ranges that arise in computing $17 \mathrm{RNG}(P)$ are lenses formed by the intersection of two congruent disks, which are more expensive to process.

For each $p \in P$, we compute the set $N_{i}^{17}(p)$, for $i=1, \ldots, 6$, which consists of the points that have at most 16 other points in $P \cap\left(p+W_{i}\right)$ that are closer to $p$. When $P$ is in general position, these are the 17 closest points to $p$ in $P \cap\left(p+W_{i}\right)$, see Figure 1. However, when this is not the case, $\left|N_{i}^{17}(p)\right|$ could be much larger, see Figure 2, (If $\left|P \cap\left(p+W_{i}\right)\right| \leq 17$, then $N_{i}^{17}(p)=P \cap\left(p+W_{i}\right)$.) We will later discuss the extension of the algorithm and analysis to degenerate situations.

We construct six similar data structures, where the $i$ 'th structure is used to compute the collection of sets $\left\{N_{i}^{17}(p) \mid p \in P\right\}$. The $i$ 'th structure is an augmented two-level 'orthogonal' range searching structure, where each level stores the points of (suitable canonical subsets of) $P$ in their order in the direction perpendicular to a bounding ray of $W_{i}$. For each node $v$ in the second level of the structure, we compute the 17th-order Voronoi diagram $\operatorname{Vor}_{17}\left(Q_{v}\right)$ of $Q_{v}$, the canonical set of $v$, and store it at $v$ as a point-location structure. (Recall that $\operatorname{Vor}_{17}\left(Q_{v}\right)$ partitions the plane into maximal regions, each with a fixed set of 17 closest sites.) The total size of the data structure is thus $O\left(n \log ^{2} n\right)$, and it can be computed in total time $O\left(n \log ^{3} n\right)$, where the Voronoi diagrams are computed by the
algorithm of Chan and Tsakalidis [6], which takes $O\left(\left|Q_{v}\right| \log \left|Q_{v}\right|\right)$ deterministic time per diagram. (See also the earlier algorithm of Agarwal et al. 3], which is randomized and slightly less efficient.) In fact, since 17 is a constant, we can compute the diagram in a simpler incremental manner, starting with the standard diagram and adding one index at a time. This also takes $O\left(\left|Q_{v}\right| \log \left|Q_{v}\right|\right)$ time per diagram.


Figure 2: When $P$ is not in general position $\left|N_{i}^{2}(p)\right|$ could be larger than 2. Here, $\left|N_{1}^{2}(p)\right|=$ $2,\left|N_{2}^{2}(p)\right|=5$, and $\left|N_{3}^{2}(p)\right|=3$.

Now, for each $p \in P$, we perform a query in the $i$ 'th structure to obtain the set $N_{i}^{17}(p)$. We initialize this set to be empty. We then identify the $O\left(\log ^{2} n\right)$ second-level nodes, such that $P \cap\left(p+W_{i}\right)$ is the disjoint union of their canonical sets. Next, we visit these nodes, one at a time, and at each node $v$ we perform a query with $p$ in $\operatorname{Vor}_{17}\left(Q_{v}\right)$, and update $N_{i}^{17}(p)$ accordingly, in constant time. A query thus takes $O\left(\log ^{3} n\right)$ time for each $p \in P$, for a total of $O\left(n \log ^{3} n\right)$ time. (The latter bound follows since, in general position, each set $N_{i}^{17}(p)$ is of constant size (at most 17)). The overall output of this procedure is the desired graph 17GNG $(P)$.

Handling degeneracies. When $P$ is not in general position, some modifications are required. As already observed, in this case $\left|N_{i}^{17}(p)\right|$ might be much larger than 17 (it could be even as large as $n-1$ ), although the total number of edges in $17 \mathrm{GNG}(P)$ remains $O(n)$ [7]. The problematic situation is when, for some $p \in P$ and some $1 \leq i \leq 6$, and for some canonical set $Q$ of $P \cap\left(p+W_{i}\right)$, there exists a large subset $Q^{\prime}$ of $Q$, all of whose points lie on a circle centered at $p$, with at most 16 points of $Q$ inside the circle. In that case all points of $Q^{\prime}$ form with $p$ potential edges of $17 \mathrm{GNG}(P)$, but to retain the aforementioned running time, we cannot afford to spend more than $O(\log n)$ time while processing $Q$ (when querying with $p$ ), unless they are indeed edges of $17 \mathrm{GNG}(P)$, in which case we may spend additional $O\left(\left|Q^{\prime}\right|\right)$ time. However, at this point we do not know whether they are edges of 17GNG $(P)$, because other canonical sets might generate with $p$ shorter edges that exclude the potentially numerous longest edges involving $Q$ from the graph ${ }^{2}$

To address this issue, we modify the query with $p$ in the $i$ 'th structure as follows. As before, we first identify the $O\left(\log ^{2} n\right)$ second-level nodes of the structure, such that $P \cap\left(p+W_{i}\right)$ is the disjoint union of their canonical sets. The next stage consists of two rounds, where in each round we process each of these $O\left(\log ^{2} n\right)$ nodes. The sole purpose of the first round is to find the distances $d_{1}<d_{2}<\cdots<d_{k}$, where $k \leq 17$, between $p$ and the

[^2]17 closest points to $p$ among the points in $P \cap\left(p+W_{i}\right)$. (We write $k \leq 17$, since the same distance may be attained by several points.) More precisely, if we were to sort the points in $P \cap\left(p+W_{i}\right)$ by their distance from $p$, resolving ties arbitrarily, then $d_{1}, \ldots, d_{k}$ would be the distinct distances corresponding to the first 17 points in the sequence. We now describe how to implement the first round in total $O\left(\log ^{3} n\right)$ time, for each fixed $p$ and $i$, that is, in overall $O\left(n \log ^{3} n\right)$ time.

We use the following notation. For a (two-dimensional open) cell $C$ of $\operatorname{Vor}_{17}\left(Q_{v}\right)$, for some second-level node $v$, denote by $P(C)$ the set of the (uniquely defined) 17 closest points of $Q_{v}$ to any point in $C$. Moreover, for any point $p$ in the closure of $C$, put $d_{p}(C)=$ $\max \{\operatorname{dist}(p, q) \mid q \in P(C)\}$.

For fixed $p$ and $i$, we maintain a sorted sequence $A=A_{p, i}$ of the 17 closest points to $p$ among the points of $P \cap\left(p+W_{i}\right)$ that were encountered so far; initially $A$ is empty. (Ties are broken arbitrarily, and we limit the length of $A$ to 17 even when there are additional points, not in $A$, whose distance from $p$ is equal to that between $p$ and the last point in A.) At each second-level node $v$ of the structure, we locate $p$ in $\operatorname{Vor}_{17}\left(Q_{v}\right)$, and proceed as follows. Let $C$ be a cell of the diagram such that $p$ lies in the closure of $C$ (the cell is not unique when $p$ lies on an edge or is a vertex of the diagram). Obtaining $C$ from the point location query is straightforward in all cases, and takes constant time. We update $A$ (or initialize it, if $v$ is the first visited node) by examining the points in $P(C)$. Specifically, we sort the set $P(C)$ by distance to $p$, and either copy the resulting sequence into $A$, when $v$ is the first processed node, or otherwise merge it with $A$, retaining only the first 17 elements.

Put $\Delta_{p, i}=\{\operatorname{dist}(p, q) \mid q \in A\}$, and note that $\left|\Delta_{p, i}\right| \leq 17$, where a strict inequality is possible. We have:

Lemma. After processing all the $O\left(\log ^{2} n\right)$ canonical sets that form $P \cap\left(p+W_{i}\right), \Delta_{p, i}$ is the set of distances between $p$ and the 17 closest points to $p$ in $P \cap\left(p+W_{i}\right)$.

Proof. We argue, by induction, that after processing some of the canonical sets, $\Delta_{p, i}$ is the set of distances between $p$ and the 17 closest points to $p$ in the union of these sets. The claim holds trivially initially, so assume that it holds just before processing some canonical set $Q$.

Let $C$ be the cell of $\operatorname{Vor}_{17}(Q)$ that is obtained from the point location query, so $p$ lies in the closure of $C$. The claim is easy to argue when $p$ lies in the interior of $C$, so the interesting case is when $p \in \partial C$. We observe that in this case, if $C^{\prime}$ is any other cell that contains $p$ (necessarily on its boundary) then $d_{p}\left(C^{\prime}\right)=d_{p}(C)$ and

$$
P\left(C^{\prime}\right) \backslash\left\{q \in P\left(C^{\prime}\right) \mid \operatorname{dist}(p, q)<d_{p}\left(C^{\prime}\right)\right\}=P(C) \backslash\left\{q \in P(C) \mid \operatorname{dist}(p, q)<d_{p}(C)\right\}
$$

This follows directly from the definition of the diagram, since otherwise, either $P(C)$ or $P\left(C^{\prime}\right)$ (or both) would not be the correct answer to the query with $p$. In other words, in this case it does not matter which of the cells that are adjacent to $p$ we use to update $A$. This property implies that the invariant is preserved after processing $Q$, and it therefore completes the induction step in this case too.

In the second round ${ }^{3}$ we iterate again over all points $p$ and indices $i$. For a fixed pair $p$ and $i$, we retrieve the $O\left(\log ^{2} n\right)$ canonical sets that form $P \cap\left(p+W_{i}\right)$. For each of these

[^3]sets $Q$, we locate $p$ in $\operatorname{Vor}_{17}(Q)$ and compute $d_{p}(C)$ in constant time, for some (single) cell $C$ of the diagram whose closure contains $p$ (recall that this value is independent of $C$ ). We then compute the set $\Delta_{p, i}(C)$ of distances between $p$ and the points of $P(C)$ that belong to $\Delta_{p, i}$, i.e., $\Delta_{p, i}(C)=\left\{d \in \Delta_{p, i} \mid d=\operatorname{dist}(p . q)\right.$ for some $\left.q \in P(C)\right\}$. Two cases can arise:
(i) $d_{p}(C) \notin \Delta_{p, i}(C)$. In this case the subset $P_{p}(C)$ of $P(C)$ of the points whose distances to $p$ are in $\Delta_{p, i}(C)$ is independent of the cell $C$ (as long as $C$ contains $p$ in its closure). We then simply find $P_{p}(C)$ and add to $17 \mathrm{GNG}(P)$ all the (at most 16) edges $(p, q)$ for $q \in P_{p}(C)$.
(ii) $d_{p}(C) \in \Delta_{p, i}(C)$. In this case we iterate over all the cells $C$ incident to $p$, and add all the edges $(p, q)$, for $q \in \bigcup_{C} P(C)$. This does not affect the asymptotic bound on the running time: if there are $k$ such cells, we process $O(k)$ points but add $\Theta(k)$ edges to $17 \mathrm{GNG}(P)$, as is easily checked. (If $p$ is adjacent to $k$ cells then there are $k$ points in $Q$ at distance $d_{p}(C)$ from $p$, and all of them form with $p$ edges of $17 \mathrm{GNG}(P)$.)

By applying this procedure to each $p$ and $i$ and to each corresponding canonical set $Q$, we obtain the desired graph 17GNG( $P$ ).

Wrapping it up. We have shown that one can compute $17 \mathrm{GNG}(P)$ in $O\left(n \log ^{3} n\right)$ time. Having computed $17 \mathrm{GNG}(P)$, we continue as before, running a binary search over its $O(n)$ edge lengths, using the algorithm of Bonnet et al. [5] to guide the search. In summary, we obtain:

Theorem. A bottleneck matching of a set of $n=2 \ell$ points in the plane can be computed in $O\left(n^{\omega / 2} \log n\right)$ deterministic time.

Abu-Affash et al. [1] presented an algorithm that, given a perfect matching $M$ of $P$, returns in $O(n \log n)$ time a non-crossing perfect matching $N$ of $P$, such that $\lambda(N) \leq$ $2 \sqrt{10} \lambda(M)$. We thus obtain:

Corollary. A non-crossing perfect matching $N$ of a set of $n=2 \ell$ points in the plane with $\lambda(N) \leq 2 \sqrt{10} \lambda^{*}$ can be computed in $O\left(n^{\omega / 2} \log n\right)$ deterministic time, where $\lambda^{*}$ is the length of a longest edge in a bottleneck matching of $P$.

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[^1]:    ${ }^{1}$ The best improvement we have found runs in $O^{*}\left(n^{7 / 5}\right)$ time.

[^2]:    ${ }^{2}$ This issue disappears if we only maintain the lengths of the edges of $17 \mathrm{GNG}(P)$, as all these longest edges have the same length.

[^3]:    ${ }^{3}$ The second round is not needed if we are interested only in the lengths of the edges of $17 \mathrm{GNG}(P)$.

