On the simultaneous location of a service facility and a rapid transit line

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Abstract

In this paper we provide a finite set of candidates to be one of the endpoints of an optimal solution for the problem of locating simultaneously a service facility and a rapid transit line.

1 Introduction

The problem of locating simultaneously a service facility and a rapid transit line was introduced by Espejo and Rodríguez-Chía [1]. In that paper, the authors present a set of candidates to be one of the endpoints of an optimal transit line as well as a solution procedure for this problem. Later, Díaz-Bañez et al. [2] provided an alternative description of the endpoints of an optimal solution for this problem because the previous description does not work in general. In this paper, we give a finer description of this set, actually we provide a set of candidates of finite cardinality to be one of the endnodes of the optimal segment.

A complete motivation and justification of above mentioned model is given in [1]. Therefore, we proceed directly with the formulation of the problem. Formally, consider $(\mathbb{R}^2, || \cdot ||_1)$ and let $A = \{a_1, \ldots, a_M\}$ be the set of demand points, $\omega = \{\omega_1, \ldots, \omega_M\} \subseteq \mathbb{R}^+$ be the set representing the intensity of the demand in the elements of the set A and x be the service facility to be located. The rapid transit line, represented by a segment of given length, ℓ , is defined by two extreme points, e and e, which represent the entrance (access) and the exit, respectively. The distance from e to e through the rapid transit line is defined as e0, where e1 is the length of the rapid transit line and e2 is a constant greater than or equal to 1. (Note that it means that the speed within the rapid transit line is greater than or equal to in the plane). Obviously, the closest extreme point of the segment to the service facility should be the one representing the exit, that is, $||x - s||_1 \le ||x - e||_1$. In [1] is proved that an optimal location for service facility is placed on the exit extreme point of the optimal segment for the problem of locating simultaneously a service facility and a rapid transit line. Therefore, the model can be reformulated as the problem of locating a rapid transit line (segment) such that one its endpoints is the service facility. Thus, the distance between a demand point and the service facility is given by the function

$$d_{e,x}(a_i, x) = \min\{||x - a_i||_1, ||e - a_i||_1 + \ell/k\}.$$

Therefore, the problem can be formulated as follows:

min
$$f(e,x) = \sum_{i=1}^{M} w_i d_{e,x}(a_i, x)$$
 (1)
s.t. $||x - e||_2 = \ell$.

Observe that for $\ell=0$, the classical (weighted) rectilinear median problem in the plane is obtained. Let W be the solution set of this problem for the case $\ell=0$ (usually called median, me). Let I be the set of intersection points (intersection points of the fundamental lines, in this case, horizontal and vertical lines at demand points, see [3]) and $I_W := I \cap W$. An interesting result obtained in [1] states that the endpoints of an optimal segment are in opposite quadrants with respect to some $t \in I_W$, see [1] for further details.

In general, the four quadrants defined by a point $q \in \mathbb{R}^2$ will be denoted by C_q^i , $i = 1, \ldots, 4$ (to simplify the notation, the quadrants defined by the origin will be denoted by C^i , $i = 1, \ldots, 4$). In what follows, we study the case $|I_W| = 1$ (notice that in this case |W| = 1, that is, the median is unique), the remaining cases will be analyzed analogously. Without loss of generality and for a better understanding, it is assumed that the median point is the origin, i.e., $I_W = me = (0,0)$ and in addition, we solve the problem where e is located at the first quadrant defined by the origin, C^1 (the analysis of the remaining cases can be analogously addressed). Hence, x will be in C^3 . In general, the circumference centered at point x of radius ℓ will be denoted by S_x (to simplify the notation, the circumference centered at the origin will be denoted by S). Let \overline{S} denote the circle centered at the origin and radius ℓ , $\overline{S}^i := \overline{S} \cap C^i$, $S_x^i := S_x \cap C^i$, $C^i(A) := C^i \cap A$, for i = 1, 2, 3, 4.

In order to analyze Problem (1), we recall the definition of captation region:

Definition 1.1 For a given segment with endpoints e and x, the set $CR_{e,x} := \{a = (a_1, a_2) \in A : ||x - a||_1 \ge ||e - a||_1 + \ell/k\}$ is called captation region associated with e. The boundary of $CR_{e,x}$ is defined as $\partial CR_{e,x} = \{a \in CR_{e,x} : ||x - a||_1 = ||e - a||_1 + \ell/k\}$ and the relative interior of CR_e as $ri(CR_{e,x}) = CR_{e,x} \setminus \partial CR_{e,x}$.

Note that $a_i \in CR_e$ if and only if $d_e(x, a_i) = \min\{||x - a_i||_1, ||e - a_i||_1 + \ell/k\} = ||e - a_i||_1 + \ell/k$. Therefore, all demand points that belong to the captation region associated with $e \in S^1$ will use the rapid transit line to achieve the service facility. The captation region associated with e and x is given by (see [1] for further details):

- if
$$x_1 \le e_1 \le \bar{e}_1$$
,

$$CR_{e,x} = \{a = (a_1, a_2) \in A : a_1 \ge x_1, \ a_2 \ge h_{e,x}^+, \ a_1 + a_2 \ge c_{e,x}\} \cup \{a = (a_1, a_2) \in A : a_1 < x_1, \ a_2 \ge h_{e,x}^-\},$$
- if $\bar{e}_1 < e_1 < \bar{\bar{e}}_1$,

$$CR_{e,x} = \{ a = (a_1, a_2) \in A : a_1 \ge v_{e,x}^+, \ a_2 \ge h_{e,x}^+, \ a_1 + a_2 \ge c_{e,x} \},$$

- if $\bar{e}_1 \le e_1 \le x_1 + \ell$,

$$CR_{e,x} = \{a = (a_1, a_2) \in A : a_2 \geq x_2, \ a_1 \geq v_{e,x}^+, \ a_1 + a_2 \geq c_{e,x}\} \cup \{a = (a_1, a_2) \in A : a_2 < x_2, \ a_1 \geq v_{e,x}^-\},$$
 where $h_{e,x}^- := \frac{e_1 + e_2 + \ell/k}{2} + \frac{x_2 - x_1}{2}, \ h_{e,x}^+ := \frac{-e_1 + e_2 + \ell/k}{2} + \frac{x_1 + x_2}{2}, \ v_{e,x}^- := \frac{e_1 + e_2 + \ell/k}{2} + \frac{x_1 - x_2}{2}, \ v_{e,x}^+ := \frac{e_1 - e_2 + \ell/k}{2} + \frac{x_1 + x_2}{2}, \ c_{e,x} := \frac{e_1 + e_2 + \ell/k}{2} + \frac{x_1 + x_2}{2} \text{ and } \bar{e} = (\bar{e}_1, \bar{e}_2), \ \bar{e} = (\bar{e}_1, \bar{e}_2) \in S_x \text{ such that } (\bar{e}_2 - x_2) - (\bar{e}_1 - x_1) = \frac{\ell}{k} (\bar{e}_2 - x_2 > \bar{e}_1 - x_1) \text{ and } (\bar{e}_1 - x_1) - (\bar{e}_2 - x_2) = \frac{\ell}{k} (\bar{e}_1 - x_1 > \bar{e}_2 - x_2).$ The boundary of $CR_{e,x}$ is defined as $\partial CR_{e,x} = \{a \in CR_{e,x} : ||a - x||_1 = ||a - e||_1 + \ell/k\}$. A graphical representation of these captation regions for the case $(x_1, x_2) = (0, 0)$ is given by Figure 1.

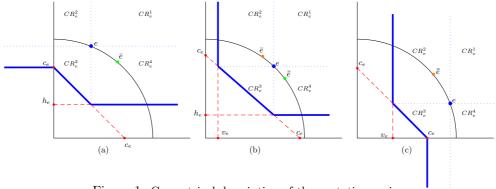


Figure 1: Geometrical description of the captation region

2 Dominant solutions

In this section, we characterize a finite set of candidates to be one of the endpoints of an optimal solution of Problem (1).

Theorem 2.1 Let e^* and x^* be the endpoints of an optimal segment of Problem (1) with e^* restricted to the first quadrant. Then, x^* and e^* , satisfy one of the following conditions:

- i) Either $x^* \in \bar{S}^3 \cap I$ or $e^* \in \bar{S}^1 \cap I$ (Figure 2 shows an example of the possible locations of the endpoints of an optimal segment).
- ii) $x_k^* = a_{jk}$ and $e_{k'}^* = a_{j'k'}$ for some $j, j' \in \{1, \dots, M\}$ and $k, k' \neq k \in \{1, 2\}$, such that, the angle of the segment with the positive direction of the x-edge, θ , satisfies that $\tan(\theta) = -\frac{w_{e^*, x^*}^b}{w_{e^*, x^*}^a}$ with $w_{e^*, x^*}^b > 0$ (and consequently $w_{e^*, x^*}^a < 0$), where w_{e^*, x^*}^a and w_{e^*, x^*}^b are defined in the proof of Lemma 2.1.

Remark 2.1 The description given by [2] states that either one of the endpoints of an optimal segment is on an intersection point or both endpoints are placed on fundamental lines at two demand points, one vertical and the other horizontal. Observe that we have with this description a infinite many number of candidates to be one of the endpoints of an optimal segment. However, our description is finer than the one in [2] because we have reduced this set of candidates to a finite set.

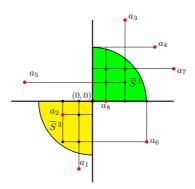


Figure 2: Possible locations of e and x.

In order to prove this result, we will show that for any segment of length ℓ defined by $x \notin \bar{S}^3 \cap I$ and $e \notin \bar{S}^1 \cap I$, such that, x and e do not satisfy condition ii) of Theorem 2.1, we can find another segment

of extreme points x' and e' such that $f(x', e') \leq f(x, e)$. Observe that if $x \notin \bar{S}^3 \cap I$ ($e \notin \bar{S}^1 \cap I$), then $x_1 \neq a_{i1}$ or $x_2 \neq a_{i2}$, $\forall a_i \in A$ ($e_1 \neq a_{i1}$ or $e_2 \neq a_{i2}$, $\forall a_i \in A$). Firstly, the case $x_1 \neq a_{i1}$ and $e_1 \neq a_{i1}$, $\forall a_i \in A$ will be considered (Subsection 2.1); secondly, $x_1 \neq a_{i1}$ and $e_2 \neq a_{i2}$, $\forall a_i \in A$ (Subsection 2.2). The remaining cases can be analogously studied.

2.1 Case $x_1 \neq a_{i1}$ and $e_1 \neq a_{i1}$, $\forall a_i \in A$.

Let x', e', x'' and e'' be such that (see Figure 3)

$$x' = x + (-\lambda, 0); \quad e' = e + (-\lambda, 0); \quad x'' = x + (\lambda, 0); \quad e'' = e + (\lambda, 0),$$
 (2)

where λ is a small enough positive value satisfying that:

- i) $CR_{e,x} \setminus \partial CR_{e,x} = CR_{e',x'} \setminus \partial CR_{e,x} = CR_{e'',x''} \setminus \partial CR_{e,x}$ and $C_x^i(A) = C_{x'}^i(A) = C_{x''}^i(A)$, $i = 1, \ldots, 4$.
- ii) $a_{i1} \neq e'_1$ and $a_{i1} \neq e''_1$, for any $a_i \in A$.

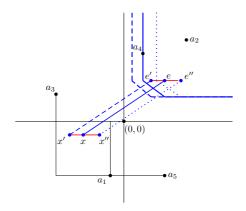


Figure 3: $e_1 \neq a_{i1}, \forall a_i \in A$.

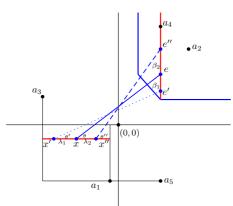


Figure 4: $e_2 \neq a_{i2}, \forall a_i \in A$.

Theorem 2.2 Let e, x be the extreme points of a segment of length ℓ such that $x_1 \neq a_{i1}$ and $e_1 \neq a_{i1}$, $\forall a_i \in A$. Then, $f(e', x') \leq f(e, x)$ or $f(e'', x'') \leq f(e, x)$, where e', x', e'', x'' were given by (2).

Proof:

Since $||x - a_i||_1 \le ||a_i - e||_1 + \ell/k$, for all $a_i \in C_x^3(A)$, the objective function can be expressed as:

$$f(e,x) = \sum_{a_i \notin CR_{e,x}} \omega_i ||x - a_i||_1 + \sum_{a_i \in CR_{e,x}} \omega_i (||e - a_i||_1 + \ell/k) = \sum_{a_i \in (C_x^2(A) \setminus CR_{e,x}) \cup C_x^3(A)} \omega_i ||x - a_i||_1 + \sum_{a_i \in (C_x^1(A) \cup C_x^4(A)) \setminus CR_{e,x}} \omega_i (||e - a_i||_1 + \ell/k) + \sum_{a_i \in CR_{e,x} \setminus \partial CR_{e,x}} \omega_i (||e - a_i||_1 + \ell/k).$$

From (2), the definition of λ and $e_1 \neq a_{i1}, \forall a_i \in A$, we obtain the following:

- (1) $\forall a_i \in C_x^2(A) \setminus \text{ri}(CR_{e,x}) \cup C_x^3(A)$: $||x' a_i||_1 = ||x a_i||_1 \lambda$ and $||x'' a_i||_1 = ||x a_i||_1 + \lambda$.
- (2) $\forall a_i \in (C_x^1(A) \cup C_x^4(A)) \setminus CR_{e,x}$: $||x' a_i||_1 = ||x a_i||_1 + \lambda$ and $||x'' a_i||_1 = ||x a_i||_1 \lambda$.
- (3) $\forall a_i \in CR_{e,x} \backslash \partial CR_{e,x}$:

(a)
$$a_{i1} < e_1$$
: $||e' - a_i||_1 = ||e - a_i||_1 - \lambda$ and $||e'' - a_i||_1 = ||e - a_i||_1 + \lambda$.

(b)
$$a_{i1} > e_1$$
: $||e' - a_i||_1 = ||e - a_i||_1 + \lambda$ and $||e'' - a_i||_1 = ||e - a_i||_1 - \lambda$.

(4) $\forall a_i \in \partial CR_{e,x} \setminus C_x^2(A)$:

(a)
$$a_{i1} > e_1$$
: $||e' - a_i||_1 + \ell/k = ||e - a_i||_1 + \lambda + \ell/k = ||x - a_i||_1 + \lambda = ||x' - a_i||_1$ and $||e'' - a_i||_1 + \ell/k = ||e - a_i||_1 - \lambda + \ell/k = ||x - a_i||_1 - \lambda = ||x'' - a_i||_1$.

(b)
$$a_{i1} < e_1$$
: $||e' - a_i||_1 + \ell/k = ||e - a_i||_1 - \lambda + \ell/k = ||x - a_i||_1 - \lambda < ||x - a_i||_1 + \lambda = ||x' - a_i||_1$
and $||e'' - a_i||_1 + \ell/k = ||e - a_i||_1 + \lambda + \ell/k = ||x - a_i||_1 + \lambda > ||x - a_i||_1 - \lambda = ||x'' - a_i||_1$.

Hence, we get that

$$f(x',e') - f(x,e) = (\omega_{e,x}^1 - \omega_{e,x}^2 - \omega_{e,x}^3 + \omega_{e,x}^4 + \omega_{e,x}^+ - \omega_{e,x}^- + \omega_{e,x}^{\delta+} - \omega_{e,x}^{\delta-}) \lambda = \Delta',$$

$$f(x'',e'') - f(x,e) = (-\omega_{e,x}^1 + \omega_{e,x}^2 + \omega_{e,x}^3 - \omega_{e,x}^4 - \omega_{e,x}^+ + \omega_{e,x}^- - \omega_{e,x}^{\delta-} - \omega_{e,x}^{\delta-}) \lambda = \Delta'',$$

where

$$\begin{array}{llll} \omega_{e,x}^{1} &= \sum_{a_{j} \in C_{x}^{1}(A) \backslash CR_{e,x}} \omega_{j}; & \omega_{e,x}^{2} &= \sum_{a_{j} \in C_{x}^{2}(A) \backslash \dot{\Gamma}\dot{I}(CR_{e,x})} \omega_{j}; & \omega_{e,x}^{3} &= \sum_{a_{j} \in C_{x}^{3}(A)} \omega_{j}; & \omega_{e,x}^{4} &= \sum_{a_{j} \in C_{x}^{4}(A) \backslash CR_{e,x}} \omega_{j}; \\ \omega_{e,x}^{+} &= \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in \partial CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta-} &= \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} < e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta-} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta-} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta-} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta-} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta-} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; & \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A): a_{j} > e_{1}} \omega_{j}; \\ \omega_{e,x}^{\delta+} &= \sum_{a_{j} \in CR_{e,x} \backslash C_{x}^{2}(A):$$

Since $\Delta' = -\Delta'' - 2\lambda \omega_{e,x}^{\delta-}$, we conclude that:

- If
$$\Delta'' \leq 0$$
, then $f(x'', e'') \leq f(x, e)$.

- Otherwise,
$$\Delta' \leq 0$$
 and $f(x', e') \leq f(x, e)$, and the result follows.

2.2 Case $x_1 \neq a_{i1}$ and $e_2 \neq a_{i2}$, $\forall a_i \in A$.

Let x', e', x'' and e'' be such that (see Figure 4)

$$x' = x + (-\lambda_1, 0); \quad e' = e + (0, -\beta_1); \qquad x'' = x + (\lambda_2, 0); \quad e'' = e + (0, \beta_2),$$
 (3)

where $\lambda_1, \lambda_2, \beta_1, \beta_2 \in \mathbb{R}^+$ are small enough values satisfying that

i)
$$CR_{e,x}\backslash\partial CR_{e,x}=CR_{e',x'}\backslash\partial CR_{e,x}=CR_{e'',x''}\backslash\partial CR_{e,x}$$
 and $C_x^i(A)=C_{x'}^i(A)=C_{x''}^i(A),\ i=1,\ldots,4.$

ii) $e'_2 \neq a_{i2}$ and $e''_2 \neq a_{i2}$, for any $a_i \in A$.

In order to prove that either $f(e', x') \leq f(e, x)$ or $f(e'', x'') \leq f(e, x)$, we will use similar arguments as in the previous case, before that we give some technical results.

Proposition 2.1 Let θ , θ' and θ'' be the angles defined by the segment of extreme points e and x, e' and x', e'' and x'', and the horizontal line, respectively (see Figure 4), where e', x', e'' and x'' were defined in (3). Then

$$\frac{\beta_1}{\lambda_1} = \frac{2\ell \sin\left(\frac{\theta - \theta'}{2}\right) \cos\left(\frac{\theta + \theta'}{2}\right)}{2\ell \sin\left(\frac{\theta' + \theta}{2}\right) \sin\left(\frac{\theta - \theta'}{2}\right)} = \frac{1}{\tan(\frac{\theta + \theta'}{2})}, \qquad \frac{\beta_2}{\lambda_2} = \frac{2\ell \sin\left(\frac{\theta'' - \theta}{2}\right) \cos\left(\frac{\theta'' + \theta}{2}\right)}{2\ell \sin\left(\frac{\theta + \theta''}{2}\right) \sin\left(\frac{\theta'' - \theta}{2}\right)} = \frac{1}{\tan(\frac{\theta'' + \theta}{2})}.$$

Moreover,

- i) If $\lambda_1 = \lambda_2$ ($\beta_1 = \beta_2$) then $\beta_2 < \beta_1$ ($\lambda_2 > \lambda_1$).
- ii) If $\theta' < \theta < \theta'' \le \pi/4$ $(\pi/4 \le \theta' < \theta < \theta'')$, then $\lambda_1 < \beta_1$ and $\lambda_2 < \beta_2$ $(\lambda_1 > \beta_1$ and $\lambda_2 > \beta_2)$.

Proof:

It is consequence of $\lambda_1 = \ell(\cos\theta' - \cos\theta)$, $\beta_1 = \ell(\sin\theta - \sin\theta')$, $\lambda_2 = \ell(\cos\theta - \cos\theta'')$, $\beta_2 = \ell(\sin\theta'' - \sin\theta)$, and that $0 < \theta' < \theta < \theta'' \le \frac{\pi}{2}$ and the tangent function is monotonous in $(-\frac{\pi}{2}, \frac{\pi}{2})$. \square

Lemma 2.1 Let e', x', e'', x'' be given by (3). Then,

$$f(x', e') - f(x, e) = \Delta'$$
 and $f(x'', e'') - f(x, e) = \Delta''$,

where $\Delta' = \omega_{e,x}^a \lambda_1 + \omega_{e,x}^b \beta_1 + d'$ and $\Delta'' = -\omega_{e,x}^a \lambda_2 - \omega_{e,x}^b \beta_2 + d''$, with $\omega_{e,x}^a, \omega_{e,x}^b, d', d'' \in \mathbb{R}$, $d', d'' \leq 0$.

Proof:

Let e', x', e'', x'' be given by (3).

1. If $a_i \in \partial CR_{e,x} \setminus C_x^2(A)$, with $a_{i2} \leq e_2$, then:

$$||e'-a_i||_1 + \ell/k = ||e-a_i||_1 - \beta_1 + \ell/k = ||x-a_i||_1 - \beta_1 < ||x-a_i||_1 + \lambda_1 = ||x'-a_i||_1, \quad \text{i.e. } a_i \in CR_{e',x'},$$

$$||e''-a_i||_1 + \ell/k = ||e-a_i||_1 + \beta_2 + \ell/k = ||x-a_i||_1 + \beta_2 > ||x-a_i||_1 - \lambda_2 = ||x'-a_i||_1, \quad \text{i.e. } a_i \notin CR_{e'',x''}.$$

- 2. If $a_i \in \partial CR_{e,x} \setminus C_x^2(A)$, with $a_{i2} > e_2$, using similar arguments and by Proposition 2.1, it can be proved that:
 - (a) $\theta' < \theta < \theta'' \le \pi/4$, then $a_i \in CR_{e'',x''}$ and $a_i \notin CR_{e',x'}$.
 - (b) $\pi/4 < \theta' < \theta < \theta''$, then $a_i \notin CR_{e'',x''}$ and $a_i \in CR_{e',x'}$.
 - (c) $\theta' < \theta = \pi/4 < \theta''$, then $a_i \notin CR_{e',x'}$ and $a_i \notin CR_{e'',x''}$.

Consider the following notation:

$$\begin{array}{llll} \omega_{e,x}^{1} & = \sum_{a_{j} \in C_{x}^{1}(A) \backslash CR_{e,x}} \omega_{j}; & \omega_{e,x}^{2} & = \sum_{a_{j} \in C_{x}^{2}(A) \backslash \dot{\Gamma}\dot{I}(CR_{e,x})} \omega_{j}; & \omega_{e,x}^{3} & = \sum_{a_{j} \in C_{x}^{3}(A)} \omega_{j}; & \omega_{e,x}^{4} & = \sum_{a_{j} \in C_{x}^{4}(A) \backslash CR_{e,x}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta+} & = \sum_{a_{j} \in \partial CR_{e,x} \backslash C_{x}^{2}(A): a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in \partial CR_{e,x} \backslash C_{x}^{2}(A): a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in \partial CR_{e,x} \backslash C_{x}^{2}(A): a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; & \omega_{e,x}^{\delta-} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j}; \\ \omega_{e,x}^{+} & = \sum_{a_{j} \in CR_{e,x} \backslash \partial CR_{e,x}: a_{j} \geq e_{2}} \omega_{j};$$

Using similar arguments as Theorem 2.2, we can obtain an expression of $f(x', e') - f(x, e) = \Delta'$ and $f(x'', e'') - f(x, e) = \Delta''$ as follows:

(a) if $\theta' < \theta < \theta'' \le \frac{\pi}{4}$,

$$\Delta' = (\omega_{e,x}^{1} - \omega_{e,x}^{2} - \omega_{e,x}^{3} + \omega_{e,x}^{4})\lambda_{1} + (\omega_{e,x}^{+} - \omega_{e,x}^{-})\beta_{1} - \omega_{e,x}^{\delta-}\beta_{1} + \omega_{e,x}^{\delta+}\lambda_{1}$$

$$= (\omega_{e,x}^{1} - \omega_{e,x}^{2} - \omega_{e,x}^{3} + \omega_{e,x}^{4})\lambda_{1} + (\omega_{e,x}^{+} - \omega_{e,x}^{-} + \omega_{e,x}^{\delta+})\beta_{1} - \omega_{e,x}^{\delta-}\beta_{1} + \omega_{e,x}^{\delta+}(\lambda_{1} - \beta_{1});$$

$$\Delta'' = (-\omega_{e,x}^{1} + \omega_{e,x}^{2} + \omega_{e,x}^{3} - \omega_{e,x}^{4})\lambda_{2} + (-\omega_{e,x}^{+} + \omega_{e,x}^{-})\beta_{2} - \omega_{e,x}^{\delta-}\lambda_{2} - \omega^{\delta+}\beta_{2}$$

$$= (-\omega_{e,x}^{1} + \omega_{e,x}^{2} + \omega_{e,x}^{3} - \omega_{e,x}^{4})\lambda_{2} + (-\omega_{e,x}^{+} + \omega_{e,x}^{-} - \omega_{e,x}^{\delta+})\beta_{2} - \omega_{e,x}^{\delta-}\lambda_{2}.$$

Therefore, Δ' and Δ'' can be expressed as $\Delta' = \omega_{e,x}^a \lambda_1 + \omega_{e,x}^b \beta_1 + d_1$ and $\Delta'' = -\omega_{e,x}^a \lambda_2 - \omega_{e,x}^b \beta_2 + d_2$, where $\omega_{e,x}^a = \omega_{e,x}^1 - \omega_{e,x}^2 - \omega_{e,x}^3 + \omega_{e,x}^4$, $\omega_{e,x}^b = \omega_{e,x}^+ - \omega_{e,x}^- + \omega_{e,x}^{\delta+}$, $d_1 = -\omega_{e,x}^{\delta-} \beta_1 + \omega_{e,x}^{\delta+} (\lambda_1 - \beta_1)$ and $d_2 = -\omega_{e,x}^{\delta-} \lambda_2$. Since $\lambda_1 < \beta_1$ (see Proposition 2.1), then $d_1, d_2 \leq 0$.

(b) if $\frac{\pi}{4} \leq \theta' < \theta < \theta''$,

$$\Delta' = (\omega_{e,x}^{1} - \omega_{e,x}^{2} - \omega_{e,x}^{3} + \omega_{e,x}^{4})\lambda_{1} + (\omega_{e,x}^{+} - \omega_{e,x}^{-} + \omega_{e,x}^{\delta+})\beta_{1} - \omega_{e,x}^{\delta-}\beta_{1};$$

$$\Delta'' = (-\omega_{e,x}^{1} + \omega_{e,x}^{2} + \omega_{e,x}^{3} - \omega_{e,x}^{4})\lambda_{2} + (-\omega_{e,x}^{+} + \omega_{e,x}^{-} - \omega_{e,x}^{\delta+})\beta_{2} - \omega_{e,x}^{\delta-}\lambda_{2} + \omega_{e,x}^{\delta+}(\beta_{2} - \lambda_{2}).$$

Hence, $\Delta' = \omega_{e,x}^a \lambda_1 + \omega_{e,x}^b \beta_1 + d_1'$ and $\Delta'' = -\omega_{e,x}^a \lambda_2 - \omega_{e,x}^b \beta_2 + d_2'$, where $d_1' = -\omega_{e,x}^{\delta-} \beta_1$ and $d_2' = -\omega_{e,x}^{\delta-} \lambda_2 + \omega_{e,x}^{\delta+} (\beta_2 - \lambda_2)$. By Proposition 2.1, $\lambda_2 > \beta_2$ and therefore, $d_1', d_2' \leq 0$.

(c) If $\theta' < \theta = \frac{\pi}{4} < \theta''$, using similar arguments to the two previous cases, we obtain that $\Delta' = \omega_{e,x}^a \lambda_1 + \omega_{e,x}^b \beta_1 + d_1$ and $\Delta'' = -\omega_{e,x}^a \lambda_2 - \omega_{e,x}^b \beta_2 + d_2'$. By Proposition 2.1, $\lambda_2 > \beta_2$, $\beta_1 > \lambda_1$ and therefore, $d_1, d_2' \leq 0$.

Theorem 2.3 Let e, x be the extreme points of a segment of length ℓ such that $x_1 \neq a_{i1}$, $e_2 \neq a_{i2}$, $\forall a_i \in A$ as well as x and e do not satisfy condition ii) of Theorem 2.1. Then, $f(e', x') \leq f(e, x)$ or $f(e'', x'') \leq f(e, x)$, where e', x', e'', x'' were given by (3).

Proof:

By Lemma 2.1, $f(x', e') - f(x, e) = \Delta'$ and $f(x'', e'') - f(x, e) = \Delta''$, where $\Delta' = \omega_{e,x}^a \lambda_1 + \omega_{e,x}^b \beta_1 + d'$ and $\Delta'' = -\omega_{e,x}^a \lambda_2 - \omega_{e,x}^b \beta_2 + d''$, with $d', d'' \leq 0$.

- 1. If $\omega_{e,x}^b \leq 0$, we choose $\lambda_1 = \lambda_2 = \lambda$. In this case, $\beta_2 < \beta_1$ (Proposition 2.1), then $\Delta' + \Delta'' = \omega_{e,x}^b(\beta_1 \beta_2) + d' + d'' \leq 0$ and the result follows.
- 2. If $\omega_{e,x}^b > 0$, we distinguish:
 - (a) If $\omega_{e,x}^a \geq 0$, we choose $\beta_1 = \beta_2 = \beta$. In this case, $\lambda_2 > \lambda_1$ (Proposition 2.1), then $\Delta' + \Delta'' = \omega_{e,x}^a(\lambda_1 \lambda_2) + d'' + d'' \leq 0$.
 - (b) If $\omega_{e,x}^a < 0$, let θ , θ' and θ'' be the angles of the segments defined by e and x, e' and x', and e'' and x'', and the horizontal line, respectively.
 - i) If $\frac{\omega_{e,x}^b}{-\omega_{e,x}^a} < \tan{(\theta)}$, since $\theta \theta'$ is small enough then $\frac{\omega_{e,x}^b}{-\omega_{e,x}^a} \le \tan{\left(\frac{\theta + \theta'}{2}\right)}$, equivalently, $\frac{-\omega_{e,x}^a}{\omega_{e,x}^b} \ge \frac{1}{\tan\left(\frac{\theta + \theta'}{2}\right)}$. By Proposition 2.1, we get that $\frac{\beta_1}{\lambda_1} = \frac{1}{\tan\left(\frac{\theta + \theta'}{2}\right)}$. Then $\frac{\beta_1}{\lambda_1} \le \frac{-\omega_{e,x}^a}{\omega_{e,x}^b}$, i.e., $\omega_{e,x}^a \lambda_1 + \omega_{e,x}^b \beta_1 \le 0$. Therefore, $\Delta' \le 0$.
 - ii) If $\frac{\omega_{e,x}^b}{-\omega_{e,x}^a} > \tan(\theta)$, since $\theta'' \theta$ is small enough then $\frac{\omega_{e,x}^b}{-\omega_{e,x}^a} \ge \tan\left(\frac{\theta + \theta''}{2}\right)$. Using similar arguments as in the previous case, we obtain that $\Delta'' \le 0$.

Thus, we have found a movement where the objective function does not increase except for the case $\frac{\omega_{e,x}^b}{-\omega_{e,x}^a} = \tan{(\theta)}$ with $\omega_{e,x}^b > 0$ and $\omega_{e,x}^a < 0$, i.e., the case described by condition ii) of Theorem 2.1 or whenever one the endpoints of the optimal segment is an intersection point. Therefore the result follows.

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