# On path ranking in time-dependent graphs 

Tommaso Adamo, Gianpaolo Ghiani, Emanuela Guerriero, Dipartimento di Ingegneria per l'Innovazione, Università del Salento, Lecce, Italia


#### Abstract

In this paper we study a property of time-dependent graphs, dubbed "path ranking invariance". Broadly speaking, a time-dependent graph is "path ranking invariant" if the ordering of its paths (w.r.t. travel time) is independent of the start time. In this paper we show that, if a graph is path ranking invariant, the solution of a large class of time-dependent vehicle routing problems can be obtained by solving suitably defined (and simpler) time-independent routing problems. We also show how this property can be checked by solving a linear program. If the check fails, the solution of the linear program can be used to determine a tight lower bound. In order to assess the value of these insights, the lower bounds have been embedded into an enumerative scheme. Computational results on the time-dependent versions of the Travelling Salesman Problem and the Rural Postman Problem show that the new findings allow to outperform state-of-the-art algorithms.


Keywords: time-dependent routing, path ranking invariance.

## 1 Introduction

Vehicle routing is concerned with the design of "least cost" routes for a fleet of vehicles, possibly subject to side constraints, such as vehicle capacity or delivery time windows. According to Toth and Vigo (2014), the vast majority of the literature is based on the assumption that the data used to formulate the problems do not depend on time. Only in recent years there has been a flourishing of scholarly work in a time-dependent setting (Gendreau et al. (2015)).

The main goal of this paper is to study a fundamental property of time-dependent graphs that we call path ranking invariance. As detailed in the following, a time-dependent graph is path ranking invariant if the ordering of its paths w.r.t. travel duration is not dependent on the start travel time. We demonstrate that this property can be exploited to solve a large class of time-dependent routing problems including the Time-Dependent Travelling Salesman Problem (TDTSP) and the Time-Dependent Rural Postman Problem (TDRPP). We prove that if a graph is path ranking invariant, then the ordering of the solutions of these problems, w.r.t. travel duration, is the same as a suitably-defined time-independent counterpart. In order to determine sufficient conditions for path ranking invariance, we introduce a decision problem, named the Constant Traversal Cost Problem (CTCP). A decision problem is a problem with a yes-or-no answer (Arora and Barak (2009). We prove that if a time-dependent graph is a yes-instance of the CTCP, then it is path ranking invariant. Then, the decidability of the CTCP is demonstrated by devising a certificatechecking algorithm based on the solution of a linear program. Finally, we show that, if the CTTP feasibility check fails, our results can be used to determine an auxiliary path ranking invariant
graph, where the travel time functions are lower approximations of the original ones. Such less congested graph can be used to determine lower bounds. We evaluate the benefits of this new approach on both the TDTSP and the TDRPP.

The paper is organized as follows. Section 2 summaries the literature. In Section 3, we formally define the path ranking invariance property and discuss its relationship with optimality conditions of the Time-Dependent General Routing Problem (TDGRP) which includes the TDTSP and TDRPP as special cases. In Section 4 , we introduce a parameterized family of travel cost functions and define a set of sufficient conditions for path ranking invariance. In Section 5 , we define the Constant Traversal Time Problem. In particular, we prove the decidability of the CTCP by devising an algorithm that correctly decides if a time-dependent graph is a yes-instance. In Section 6, we define a lower bounding procedure. In Section 7, we discuss the benefits obtained when this new approach is embedded into state-of-the-art algorithms for the the TDTSP and TDRPP. Finally, some conclusions follow in Section 8 .

## 2 State of the art

The literature on time-dependent routing problems is quite scattered and disorganized. In this section, we present a brief review of the Time-Dependent Travelling Salesman Problem (TDTSP) and the Time-Dependent Rural Postman Problem (TDRPP) which are used in this paper to test the computational potential of the path ranking invariance property. For a complete survey on time-dependent routing, see Gendreau et al. (2015).

### 2.1 Time-Dependent Travelling Salesman Problem

Malandraki and Daskin (1992) were the first to address the TDTSP and proposed a Mixed Integer Programming (MIP) formulation. Then Malandraki and Dial (1996) devised an approximate dynamic programming algorithm while Li et al. (2005) developed two heuristics. Schneider (2002) proposed a simulated annealing heuristic and Harwood et al. (2013) presented some metaheuristics. Cordeau et al. (2014) derived some properties of the TDTSP as well as lower and upper bounding procedures. They also represented the TDTSP as MIP model for which they developed some families of valid inequalities. These inequalities were then used into a branch-and-cut algorithm that solved instances with up to 40 vertices. Arigliano et al. (2018) exploited some properties of the problem and developed a branch-and-bound algorithm which outperformed the Cordeau et al. (2014) branch-and-cut procedure. Melgarejo et al. (2015) presented a new global constraint that was used in a Constraint Programming approach. This algorithm was able to solve instances with up to 30 customers. Recently, Adamo et al. (2020) proposed a parameterized family of lower bounds, whose parameters are chosen by fitting the traffic data. When embedded into a branch-and-bound procedure, their lower bounding mechanism allows to solve to optimality a larger number of instances than Arigliano et al. (2018).

Variants of the TDTSP have been examined by Albiach et al. (2008), Arigliano et al. (2018), Montero et al. (2017) and Vu et al. (2020) (TDTSP with Time Windows), by Helvig et al. (2003) (Moving-Target TSP) and by Montemanni et al. (2007) (Robust TSP with Interval Data). Finally, it is worth noting that a scheduling problem, other than the above defined TDTSP, is also known as Time-Dependent Travelling Salesman Problem. It amounts to sequence a set of jobs on a single machine in which the processing times depend on the position of the jobs within the schedule (Picard and Queyranne (1978), Fox et al. (1980), Gouveia and Voß (1995), Vander Wiel
and Sahinidis (1996), Miranda-Bront et al. (2010), Stecco et al. (2008), Godinho et al. (2014)).

### 2.2 Time-Dependent Rural Postman Problem

Tan and Sun (2011) were the first to propose an exact algorithm for the TDRPP. They devised an integer linear programming model, based on an arc-path formulation enforced by the introduction of valid inequalities. The computational results showed that no instance was solved to optimality, with a relative gap between the best feasible solution and the lower bound equal to $3.16 \%$ on average. Calogiuri et al. (2019) provided both a lower bound and an upper bound with a worst-case guarantee. The proposed bounds were embedded into a branch-and-bound algorithm that was able to solve TDRPP instances with up to 120 arcs with a percentage of required arcs equal to $70 \%$.

When the set of required vertices is empty, but all arcs of the time-dependent graph have to be visited, the TDGRP reduces to the Time-Dependent Chinese Postman problem (TDCPP). An integer programming formulation for solving the TDCPP was proposed by Sun et al. (2011). Sun et al. (2011) also proved that the TDCPP is NP-hard and they proposed a dynamic programming algorithm for solving it. A linear integer programming formulation, namely the cycle-path formulation, was presented by Sun et al. (2015).

## 3 The path ranking invariance property

Let $G:=(V, A, \tau)$ be a directed and connected graph, where $V$ is the set of vertices, $A:=\{(i, j)$ : $i \in V, j \in V\}$ is the set of arcs. Moreover, let $\tau: A \times \mathbb{R}^{+} \rightarrow \mathbb{R}$ denote a function that associates to each arc $(i, j) \in A$ and starting time $t \in[0,+\infty)$ the traversal time when a vehicle leaves the vertex $i$ at time $t$. In particular, we suppose that it is given a planning horizon $[0, T]$ and the travel time functions are constant in the long run, that is $\tau(i, j, t):=\tau(i, j, T)$ with $t \geq T$. For the sake of notational simplicity, we use $\tau_{i j}(t)$ to designate $\tau(i, j, t)$. We suppose that traversal time $\tau_{i j}(t)$ satisfy the first-in-first-out (FIFO) property, i.e., leaving the vertex $i$ later implies arriving later at vertex $j$. In the following we denote with $\mathcal{T}_{\tau}(i, j)$, the ordered set of time instants corresponding to the breakpoints of $\tau_{i j}(t)$.

For any given path $p_{k}:=\left(i_{0}, i_{1}, \ldots, i_{k}\right)$, the corresponding duration $z_{\tau}\left(p_{k}, t\right)$ can be computed recursively as:

$$
\begin{equation*}
z\left(p_{k}, t\right):=z\left(p_{k-1}, t\right)+\tau_{i_{k-1} i_{k}}\left(z\left(p_{k-1}, t\right)\right) \tag{1}
\end{equation*}
$$

with the initialization $z\left(p_{0}, t\right):=0$.
Definition 3.1 (Path dominance rule). Given two paths $p^{\prime}$ and $p^{\prime \prime}$ of $G$ and their traversal time functions, $z\left(p^{\prime}, t\right)$ and $z\left(p^{\prime \prime}, t\right)$ respectively, we say that $p^{\prime}$ dominates $p^{\prime \prime}$, iff:

$$
\begin{equation*}
z\left(p^{\prime}, t\right) \geq z\left(p^{\prime \prime}, t\right) \quad \forall t \geq 0 \tag{2}
\end{equation*}
$$

Definition 3.2 (Path ranking invariance). A time-dependent graph $G$ is path ranking invariant, if the path dominance rule holds true for any pair of paths $p^{\prime}$ and $p^{\prime \prime}$ of $G$.

The importance of path ranking invariance property is related to its relationship with the optimality conditions of some classical time-dependent routing problems. Given a time-dependent graph $G:=(V, A, \tau)$, a set of required vertices $V_{R} \subseteq V$ and a set of required arcs $A_{R} \subseteq A$, let us denote with $\mathcal{P}$ a set of paths starting from and ending to a given vertex $i_{0}$ of $V$ and passing
through each required vertex $i \in V_{R}$ and each required arc $(i, j) \in A_{R}$ at least once. Given a starting time $t_{0}$, we focus on the Time-Dependent General Routing Problem aiming to determine the least duration path on $\mathcal{P}$, that is

$$
\begin{equation*}
\min _{p \in \mathcal{P}} z\left(p, t_{0}\right) \tag{3}
\end{equation*}
$$

For notational convenience, we model the Time-Dependent Travelling Salesman Problem as a special case of the compact formulation (3), where it is required that $G$ is complete, $A_{R}:=\emptyset$ and $V_{R}:=V$. Algorithms developed for the time-invariant counterpart of such routing problems are not able to consider time-varying travel times without essential structural modifications. Nevertheless, we observe that the absence of time constraints implies that time-varying travel times have an impact on the ranking of solutions of the routing problem (3), but they do not pose any difficulty for feasibility check of solutions. In particular, one can assert that there always exists a time-invariant (dummy) cost function $d: A \rightarrow \mathbb{R}^{+}$such that a least duration path of 3 is also a least cost path of the time-invariant instance of

$$
\begin{equation*}
\min _{p \in \mathcal{P}} \sum_{(i, j) \in p} d(i, j) \tag{4}
\end{equation*}
$$

where the notation $(i, j) \in p$ means that the $\operatorname{arc}(i, j) \in A$ is traversed by the path $p$.
Definition 3.3. A time-invariant cost function $d: A \rightarrow \mathbb{R}^{+}$is valid for $G$, if the least duration path of the time-dependent instance $(\mathcal{P}, \tau)$ of (3) is also a least cost path of the time-invariant instance $(\mathcal{P}, d)$ of (4), for any set $\mathcal{P}$ of paths defined on $G$.

If we are given a cost function valid for a time-dependent graph $G$, then we can solve various time-dependent routing problems defined on $G$, by exploiting algorithms developed for their timeinvariant counterpart. For example we can determine the least duration Hamiltonian circuit of $G$, by solving a simpler (yet NP-Hard) TSP where cost of $\operatorname{arc}(i, j) \in A$ is equal to $d(i, j)$. Similarly, the least duration solution of an instance of TDGRP defined on $G$ can be determined by solving a time-invariant GRP. Nevertheless, the main issue of such approach is: how to certificate that a cost function $d: A \rightarrow \mathbb{R}^{+}$is valid for a given time-dependent graph. Since the travel time functions are constant in the long run, the answer is quite straightforward for path ranking invariant graphs.

Remark 3.4. If the time-dependent graph $G$ is path ranking invariant, then it is valid for $G$ any cost function $d: A \rightarrow \mathbb{R}^{+}$, where each value $d(i, j)$ is proportional to the traversal time of arc $(i, j) \in A$ when the vehicle leaves vertex $i$ at time instant $T$, that is:

$$
\arg \min _{p \in \mathcal{P}} \sum_{(i, j) \in \mathcal{P}} \tau_{i j}(T)=\arg \min _{p \in \mathcal{P}} z(p)
$$

This explain our previous assertion about the relationship between the path ranking invariance property and the optimality conditions of the class of time-dependent routing problems (3). In the following sections we demonstrate that a class of path ranking invariant graphs is computable. In computability theory, a set of symbols is computable if there exists an algorithm that correctly decides whether a symbol belongs to such set. In particular, we aim to devise an algorithm that takes as input a time-dependent graph $G$ and either decides correctly that $G$ belongs to a class of path ranking invariant graphs or returns a constant cost function suitable to determine a lower bound on the optimal solution of (3).

## 4 A family of travel cost functions

In this section, we propose a parameterized family of travel cost functions and investigate its relationship with path ranking invariance property. The parameter of such a cost model is a step cost function. In particular, we are interested in time-dependent graphs, for which there exists $a$ step function generating a constant travel cost for each arc. In subsection 4.1, we prove that the existence of such step function is a sufficient condition for stating that the graph is path ranking invariant. In subsection 4.2, we discuss some properties of the proposed travel cost model. Such properties are exploited in Section 5 in order to devise an algorithm that, given as input a timedependent graph, correctly decides if there exists or not a step function generating a constant travel cost for each arc.

### 4.1 The model

Let $b: \mathcal{T}_{b} \rightarrow \mathbb{R}^{+}$denote a step function such that $\mathcal{T}_{b}$ is an ordered set of time instants and

$$
b(t):=b_{h} \quad t \in\left[t_{h}, t_{h+1}\right],
$$

with $b_{h}>0, t_{h} \in \mathcal{T}_{b}$ and $h=0, \ldots,\left|\mathcal{T}_{b}\right|-1$. For notational convenience, we also use $\mathbf{b}=$ $\left[\left(t_{0}, b_{0}\right), \ldots,\left(t_{\left|\mathcal{T}_{b}\right|-1}, b_{\left|\mathcal{T}_{b}\right|-1}\right)\right]$ to designate the step function $b(t)$. The step-function $\mathbf{b}$ is the input parameter of a family of travel cost functions $c_{i j}(t, \mathbf{b})$ defined as follows. The set of breakpoints $\mathcal{T}_{b}$ represents a partition of the planning horizon in $\left|\mathcal{T}_{b}\right|$ time intervals, whilst each value $b_{h}$ models the cost associated to one time unit spent traveling during the $h$-th time interval, with $h=0, \ldots,\left|\mathcal{T}_{b}\right|-1$. Since each unit cost is strictly positive, it never pays to wait. Given a time instant $t \geq 0$, the value $c_{i j}(t, \mathbf{b})$ represents the overall travel cost associated to $\operatorname{arc}(i, j) \in A$, when the vehicle leaves the vertex $i$ at time $t \in\left[t_{p}, t_{p+1}\right]$ and arrives at vertex $j$ at time $\left(t+\tau_{i j}(t)\right) \in\left[t_{q}, t_{q+1}\right]$, that is:

$$
\begin{equation*}
c_{i j}(t, \mathbf{b}):=\left(t_{p+1}-t\right) b_{p}+\sum_{h:=p+1}^{q-1}\left(t_{h+1}-t_{h}\right) b_{h}+\left(t+\tau_{i j}(t)-t_{q}\right) b_{q} \tag{5}
\end{equation*}
$$

with $\left(t_{p}, b_{p}\right),\left(t_{q}, b_{q}\right) \in \mathbf{b}$ and $p, q=0, \ldots,\left|\mathcal{T}_{b}\right|-1$. In Figure 1 it is reported a numerical example consisting of a given travel time function $\tau_{i j}(t)$ and an arbitrarily chosen step function $\mathbf{b}$, where the sets of breakpoints are $\mathcal{T}_{\tau}(i, j):=\{0.0,4.0,5.0\}$ and $\mathcal{T}_{b}:=\{0.0,1.0,2.0,3.0,4.0,5.0\}$. As shown in Figure 1, the selected step function $\mathbf{b}$ generates a travel cost function $c_{i j}(t, \mathbf{b})$ which has a constant value equal to 3 .

Let us denote with $\mathcal{L}_{C}$ the set of time-dependent graphs such that there exists a step function $\mathbf{b}^{*}$ generating a constant travel cost for each $\operatorname{arc}(i, j) \in A$.

Proposition 4.1. If a time-dependent graph $G$ belongs to $\mathcal{L}_{C}$, then it is path ranking invariant.

Proof. Since $\mathbf{b}^{*}$ is a step function, (5) can be rewritten as follows:

$$
\underline{c}_{i j}=\int_{t}^{t+\tau_{i j}(t)} b^{*}(\mu) d \mu .
$$

We observe that for each path $p_{k}$ defined on $G$ it is associated a travel cost computed as follows:

$$
\sum_{(i, j) \in p_{k}} \underline{c}_{i j}=\int_{t}^{z_{\tau}\left(p_{k}, t\right)} b^{*}(\mu)
$$

This implies that for any pair of paths $p$ and $p^{\prime}$ defined on $G$, it results that:

$$
\sum_{(i, j) \in p} \underline{c}_{i j} \leq \sum_{(i, j) \in p^{\prime}} \underline{c}_{i j} \Leftrightarrow z_{\tau}(p, t) \leq z_{\tau}\left(p^{\prime}, t\right)
$$

for $t \geq 0$. Since paths $p$ and $p^{\prime}$ have been arbitrarily chosen, the thesis is proven.


Figure 1: A continuous piecewise linear arc travel time function $\tau_{i j}$, the associated constant step function $b(t)$.

### 4.2 Properties of travel cost functions

We observe that, given a time-dependent graph $G=(V, A, \tau)$ and a step function $\mathbf{b}$, each output travel cost function $c_{i j}(t, \mathbf{b})$ is continuous piecewise linear function, with a number of breakpoints not greater than $\mathcal{T}_{\tau}(i, j)+2 \times\left|\mathcal{T}_{b}\right|$, with $(i, j) \in A$. In particular, we denote with $\mathcal{T}_{c}\left(i, j, \mathcal{T}_{b}\right)$ the corresponding set of breakpoints. From Proposition 4.1, it descends that we are interested in determining if there exists a step function $\mathbf{b}^{*}$ satisfying (6):

$$
\begin{equation*}
\mathcal{T}_{c}\left(i, j, \mathcal{T}_{b^{*}}\right)=\emptyset \tag{6}
\end{equation*}
$$

For this reason, we now provide both sufficient conditions and necessary conditions for asserting that a time instant $t$ is a breakpoint of a travel cost function $c_{i j}(t, \mathbf{b})$, with $(i, j) \in A$ and $t \geq 0$.

Let $\Gamma_{i j}(t)$ denote the arrival time at node $j$ when the vehicle starts to traverse the $\operatorname{arc}(i, j) \in A$ at time instant $t$, i.e. $\Gamma_{i j}(t)=t+\tau_{i j}(t)$. First from (5) we have that if a time instant $t$ is a breakpoint of $c_{i j}(t, \mathbf{b})$ then at least one of the following necessary conditions hold true: $t$ is a breakpoint of $b(t) ; \Gamma_{i j}(t)$ is a breakpoint of $b(t) ; t$ is a breakpoint of $\tau_{i j}(t)$. More formally:

$$
\begin{equation*}
t \in \mathcal{T}_{c}\left(i, j, \mathcal{T}_{b}\right) \quad \Rightarrow \quad t \in \mathcal{T}_{b} \vee t \in \mathcal{T}_{\tau}(i, j) \vee \Gamma_{i j}(t) \in \mathcal{T}_{b} \tag{7}
\end{equation*}
$$

As far as sufficient conditions is concerned, a time instant $t$ is a breakpoint of $c_{i j}(t, \mathbf{b})$ if the following three conditions hold: $t$ is a breakpoint of the step function $\mathbf{b} ; t$ is not a breakpoint of the travel time function $\tau_{i j}(t)$; the arrival time $\Gamma_{i j}(t)$ is not a breakpoint of $\mathbf{b}$. More formally:

$$
\begin{equation*}
t \in \mathcal{T}_{b} \wedge t \notin \mathcal{T}_{\tau}(i, j) \wedge \Gamma_{i j}(t) \notin \mathcal{T}_{b} \Rightarrow t \in \mathcal{T}_{c}\left(i, j, \mathcal{T}_{b}\right) \tag{8}
\end{equation*}
$$

The implication (8) has been demonstrated in the Theorem 4.1 .
Theorem 4.1. Let us suppose that we are given an arc $(i, j) \in A$, a step function $b(t)$ and one of its breakpoint $t_{p}$, that is $t_{p} \in \mathcal{T}_{b}$ with $p=1, \ldots,\left|\mathcal{T}_{b}\right|$. If both the conditions (a) and (b) hold true, then a breakpoint of $c_{i j}(t, \boldsymbol{b})$ also occurs at time instant $t_{p}$.
(a) The time instant $t_{p}$ is not a breakpoint of $\tau_{i j}(t)$, i.e. $t_{p} \notin \mathcal{T}_{\tau}(i, j)$.
(b) The arrival time $\Gamma_{i j}\left(t_{p}\right)$ is not a breakpoint of $b(t)$, that is $\Gamma_{i j}\left(t_{p}\right) \notin \mathcal{T}_{b}$

Proof. If we write Equation (5) for the time instant $t_{p}$, we obtain:

$$
\begin{equation*}
c_{i j}\left(t_{p}, \mathbf{b}\right)=\left(t_{p+1}-t_{p}\right) b_{p}+\sum_{\ell=p+1}^{q-1}\left(t_{\ell+1}-t_{\ell}\right) b_{\ell}+\left(t_{p}+\tau_{i j}\left(t_{p}\right)-t_{q}\right) b_{q} \tag{9}
\end{equation*}
$$

The thesis is proved if we demonstrate that there exists $\Delta>0$, such that

$$
\frac{c_{i j}\left(t_{p}, \mathbf{b}\right)-c_{i j}\left(t_{p}-\Delta, \mathbf{b}\right)}{\Delta} \neq \frac{c_{i j}\left(t_{p}+\Delta, \mathbf{b}\right)-c_{i j}\left(t_{p}, \mathbf{b}\right)}{\Delta}
$$

From the hypothesis it results that there exists a $\Delta>0$ such that the following conditions hold true. (1) The travel time function $\tau_{i j}(t)$ does not change its slope in the time interval $\left[t_{p}-\Delta, t_{p}+\Delta\right]$. (2) The step function $b(t)$ does not change its value during the time intervals $\left[\left(t_{p}-\Delta\right), t_{p}\right],\left[t_{p},\left(t_{p}+\Delta\right)\right]$ and $\left[\Gamma_{i j}\left(t_{p}-\Delta\right), \Gamma_{i j}\left(t_{p}+\Delta\right)\right]$. This implies that:

$$
\left[\left(t_{p}-\Delta\right), t_{p}\right] \subseteq\left[t_{p-1}, t_{p}\right] \wedge\left[t_{p},\left(t_{p}+\Delta\right)\right] \subseteq\left[t_{p}, t_{p+1}\right]
$$

and

$$
\left[\Gamma_{i j}\left(t_{p}-\Delta\right), \Gamma_{i j}\left(t_{p}+\Delta\right)\right] \subseteq\left[t_{q}, t_{q+1}\right]
$$

with $q>p$. Let us write Equation (5) for the time instant $\left(t_{p}-\Delta\right)$ :

$$
\begin{equation*}
c_{i j}\left(t_{p}-\Delta, \mathbf{b}\right)=\left(t_{p}-t_{p}+\Delta\right) b_{p-1}+\sum_{\ell=p}^{q-1}\left(t_{\ell+1}-t_{\ell}\right) b_{\ell}+\left(t_{p}-\Delta+\tau_{i j}\left(t_{p}-\Delta\right)-t_{q}\right) b_{q} \tag{10}
\end{equation*}
$$

By subtracting (10) from (9), we obtain:

$$
\begin{equation*}
\frac{c_{i j}\left(t_{p}, \mathbf{b}\right)-c_{i j}\left(t_{p}-\Delta, \mathbf{b}\right)}{\Delta}=-b_{p-1}+b_{q}+\frac{\tau_{i j}\left(t_{p}\right)-\tau_{i j}\left(t_{p}-\Delta\right)}{\Delta} b_{q} \tag{11}
\end{equation*}
$$

Similarly let us subtract (9) from (5) rewritten for time instant $\left(t_{p}+\Delta\right)$

$$
\frac{c_{i j}\left(t_{p}+\Delta, \mathbf{b}\right)-c_{i j}\left(t_{p}, \mathbf{b}\right)}{\Delta}=-b_{p}+b_{q}+\frac{\tau_{i j}\left(t_{p}+\Delta\right)-\tau_{i j}\left(t_{p}\right)}{\Delta} b_{q}
$$

Since $t_{p}$ is not a breakpoint of $\tau_{i j}(t)$, then we have that:

$$
\frac{c_{i j}\left(t_{p}, \mathbf{b}\right)-c_{i j}\left(t_{p}-\Delta, \mathbf{b}\right)}{\Delta}-\frac{c_{i j}\left(t_{p}+\Delta, \mathbf{b}\right)-c_{i j}\left(t_{p}, \mathbf{b}\right)}{\Delta}=-b_{p-1}+b_{p}
$$

As $t_{p}$ is a breakpoint of the given step function we have that $b_{p-1} \neq b_{p}$. The thesis is proved.

## 5 The Constant Traversal Cost Problem

As stated by Proposition 4.1, the membership of a graph $G$ in the set $\mathcal{L}_{C}$ implies its path ranking invariance. To ease the discussion, from now on, we refer to the decision problem corresponding to the set $\mathcal{L}_{C}$ as the Constant Traversal Cost Problem.

## Constant Traversal Cost Problem

Input: A time-dependent graph $G$.
Question: Is there a step function $\mathbf{b}^{*}$ such that the corresponding travel cost $c_{i j}\left(t, \mathbf{b}^{*}\right)$ associated to each arc $(i, j) \in A$ is constant?

In this section, we demonstrate that the CTCP is decidable. This goal is gained by demonstrating that there exists a computable function $\mathbf{1}_{C}(G)$ which is characteristic for the set $\mathcal{L}_{C}$ of time-dependent graphs, that is $\mathbf{1}_{C}(G):=1$ if $G \in \mathcal{L}_{C}$ and $\mathbf{1}_{C}(G):=0$ if $G \notin \mathcal{L}_{C}$.

### 5.1 Definition of the characteristic function

We denote with $\gamma\left(\mathcal{T}_{b}, \mathbf{b}\right)$ the maximum travel cost range of a graph $G$ with respect to a step cost function $\mathbf{b}$, defined as follows:

$$
\gamma(\mathbf{b}):=\max _{(i, j) \in A}\left(\max _{t \in[0, T]} c_{i j}(t, \mathbf{b})-\min _{t \in[0, T]} c_{i j}(t, \mathbf{b})\right)
$$

Let denote with $\gamma^{*}$ the min-max travel cost range of a graph $G$, that is:

$$
\begin{equation*}
\gamma^{*}:=\min _{\mathbf{b}}(\gamma(\mathbf{b}) \mid b(t)>\rho \quad \forall t \geq 0) \tag{12}
\end{equation*}
$$

By observing that a constant travel cost function has a range value zero, from $\sqrt{12}$ it descends the following Proposition.

Proposition 5.1. The boolean function (13) is a total characteristic function of $C T C P$.

$$
\mathbf{1}_{C}(G):=\left\{\begin{array}{ll}
1 & \text { if } \gamma^{*}=0  \tag{13}\\
0 & \text { otherwise }
\end{array} .\right.
$$

In order to assess the decidability of the CTCP, we have to devise an algorithm that decides correctly whether the optimal value $\gamma^{*}$ has value zero. We observe that the optimization problem (12) consists of two interdependent tasks: (a) determining the breakpoint set $\mathcal{T}_{b}$; (b) determining the values $\left(b_{0}, \ldots, b_{\left|\mathcal{T}_{b}\right|-1}\right)$ so that to minimize the corresponding maximum travel cost range. As far as the first task is concerned we limit the corresponding search space to a finite and discrete set $\boldsymbol{\Omega}$, such that its elements are potential breakpoints of $\mathbf{b}^{*}$, that is:

$$
\begin{equation*}
t \in \mathcal{T}_{b^{*}} \Rightarrow t \in \boldsymbol{\Omega} \tag{14}
\end{equation*}
$$

With the aim of determining a set $\boldsymbol{\Omega}$ satisfying these conditions, we associate to each arc $(i, j) \in A$ the set $\boldsymbol{\Omega}_{i j}$. In the definition of each $\boldsymbol{\Omega}_{i j}$ a key role is played by ordered sets of time instants, termed time sequences generated by a given time instant on arc $(i, j) \in A$.
Definition 5.1. An ordered set of time instants $\boldsymbol{\omega}:=\left\{\omega_{1}, \ldots, \omega_{L}\right\}$, is a time sequence generated by time instant $\omega_{1}$ on the arc $(i, j) \in A$ if the following conditions hold true:

$$
\begin{gathered}
\Gamma_{i j}^{-1}\left(\omega_{1}\right)<0 \\
\omega_{\ell}:=\Gamma_{i j}\left(\omega_{\ell-1}\right) \quad \ell=2, \ldots, L \\
\Gamma_{i j}\left(\omega_{L}\right)>T
\end{gathered}
$$

with $0 \leq \omega_{1} \leq \omega_{2} \leq \cdots \leq \omega_{L} \leq T$
We observe that there exists an infinite number of time sequences that can be generated on an $\operatorname{arc}(i, j) \in A$ : one for each time instant $t$ satisfying the first condition of Definition 5.1 For example for the travel time function in Figure 1 there exists one time sequence for each time instant belonging to the interval $[0,2[$. On the other hand, due to the FIFO property, the arrival time function $\Gamma_{i j}(t)$ is strictly increasing. This implies that each time instant $t \in[0, T]$ belongs to exactly one time sequence generated on the arc $(i, j) \in A$. We reference such time sequence with $\boldsymbol{\omega}_{i j}(t):=\left\{\omega_{i j 1}(t), \ldots, \omega_{i j L}(t)\right\}$. In the numerical example of Figure 1, both of the time instants 2.0 and 4.0 belong to the (unique) time sequence generated by 0.0 on $\operatorname{arc}(i, j)$, i.e. $\boldsymbol{\omega}_{i j}(2.0):=$ $\boldsymbol{\omega}_{i j}(4.0):=\{0.0,2.0,4.0\}$. Given a time-dependent graph $G=(V, A, \tau)$, the set $\boldsymbol{\Omega}_{i j}$ is defined as the union set of the time sequences $\boldsymbol{\omega}_{i j}(t)$, such that $t$ is a breakpoint of $\tau$, that is:

$$
\begin{equation*}
\boldsymbol{\Omega}_{i j}:=\bigcup_{t \in \mathcal{T}} \boldsymbol{\omega}_{i j}(t) \tag{15}
\end{equation*}
$$

with $\mathcal{T}:=\bigcup_{(i, j) \in A} \mathcal{T}_{\tau}(i, j)$ and $(i, j) \in A$. It is worth noting that since $\mathcal{T}$ is a subset of $\boldsymbol{\Omega}$, it is guaranteed that $\boldsymbol{\Omega}$ is not empty.

Theorem 5.2. The set $\boldsymbol{\Omega}$ defined in (16) is a finite and discrete set of potential breakpoints of $b^{*}(t)$.

$$
\begin{equation*}
\boldsymbol{\Omega}:=\bigcap_{(i, j) \in A} \boldsymbol{\Omega}_{i j} \tag{16}
\end{equation*}
$$

Proof. We first observe that due to the FIFO property, given a time instant $t \in T$, the corresponding $\boldsymbol{\omega}_{i j}(t)$ is a finite and discrete set of time instants. Therefore from 15), 16) and Definition 5.1, it results that $\Omega$ consists of up to $|A| \times|\mathcal{T}| \times \frac{T}{\tau_{\text {min }}}$ where $\tau_{\text {min }}$ is the minimum traversal time of $\tau$, that is $\tau_{\text {min }}:=\min _{(i, j) \in A}\left(\tau_{i j}(t) \mid t \geq 0\right)$. This demonstrates that $\boldsymbol{\Omega}$ is a finite and discrete set. We now prove that (16) defines a set of potential breakpoints for $\mathbf{b}^{*}$, that is:

$$
\begin{equation*}
t \in \mathcal{T}_{b^{*}} \Rightarrow t \in \boldsymbol{\Omega}_{i j} \tag{17}
\end{equation*}
$$

with $(i, j) \in A$.
We prove 17 by contradiction. Let us suppose that there exists one arc $(i, j) \in A$ and a breakpoint $t^{\prime}$ of $b^{*}(t)$ such that $t^{\prime} \notin \boldsymbol{\Omega}_{i j}$. We prove the thesis by demonstrating that:

$$
\begin{equation*}
t \in \boldsymbol{\omega}_{i j}\left(t^{\prime}\right) \wedge t \in \mathcal{T}_{b^{*}} \Rightarrow t \in \mathcal{T}_{c}\left(i, j, \mathbf{b}^{*}\right) \tag{18}
\end{equation*}
$$

which contradicts the hypothesis that $t^{\prime}$ is a breakpoint of a step function $\mathbf{b}^{*}$ generating a constant travel cost on arc $(i, j)$, that is $\mathcal{T}_{c}\left(i, j, \mathbf{b}^{*}\right)=\emptyset$. We first observe that, since a time instant belongs to exactly one time sequence generated on $\operatorname{arc}(i, j)$, then we have that $\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)$ shares no time instant with $\boldsymbol{\Omega}_{i j}$. This implies that no time instant belonging to $\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)$ is a breakpoint of the travel time function $\tau_{i j}(t)$, that is:

$$
\begin{equation*}
t_{\ell} \in \boldsymbol{\omega}_{i j}\left(t^{\prime}\right) \Rightarrow t_{\ell} \notin \boldsymbol{\Omega}_{i j} \Rightarrow t_{\ell} \notin \mathcal{T}_{\tau} \subseteq \boldsymbol{\Omega}_{i j} \tag{19}
\end{equation*}
$$

with $\ell=1, \ldots,\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|$.
From (8) and $\sqrt[19]{ }$ it results that $\sqrt{18}$ can be demonstrate if we prove by induction on $\ell$ that:

$$
\begin{equation*}
t_{\ell} \in \boldsymbol{\omega}_{i j}\left(t^{\prime}\right) \wedge t_{\ell} \in \mathcal{T}_{b^{*}} \Rightarrow \Gamma_{i j}\left(t_{\ell}\right) \notin \mathcal{T}_{b^{*}} \wedge t_{\ell} \notin \mathcal{T}_{\tau}(i, j) \tag{20}
\end{equation*}
$$

with $\ell=1, \ldots,\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|$.
Case $\ell=\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|$. In this case $t_{\ell}$ denotes the last element of the ordered set $\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)$. From Definition 5.1 it results that $\Gamma_{i j}\left(t_{\ell}\right)>T$. Since any step function is constant in the long run, it results that the arrival time $\Gamma_{i j}\left(t_{\ell}\right)$ is not a breakpoint of $\boldsymbol{b}^{*}$ for $\ell=\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|$. From 19 it results that (20) holds true for $\ell=\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|$.

Case $\ell \leq\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|-1$. We suppose by induction that $t_{\ell+1}$ satisfies 20 . Since $t_{\ell}$ is the predecessor
 tion hypothesis, it descends that $\Gamma_{i j}\left(t_{\ell}\right)$ cannot be a potential breakpoint of $b^{*}(t)$, i.e. $\Gamma_{i j}\left(t_{\ell}\right) \notin \mathcal{T}_{b^{*}}$. From 19 it results that 20 holds true for $\ell<\left|\boldsymbol{\omega}_{i j}\left(t^{\prime}\right)\right|$.

Given a constant value $\rho>0$, let us define the following restriction of 12 :

$$
\begin{equation*}
\bar{\gamma}:=\min _{\mathbf{b}}\left(\gamma(\mathbf{b}) \mid \mathcal{T}_{b} \subseteq \boldsymbol{\Omega} \wedge b(t) \geq \rho \quad \forall t \geq 0\right) \tag{21}
\end{equation*}
$$

Proposition 5.2. Given a time-dependent graph $G$, the travel cost range $\bar{\gamma}$ is equal to zero iff the $\gamma^{*}$ has value zero.

Proof. Let us denote with $\bar{\gamma}_{1}$ an upper bound of $\gamma^{*}$ defined as follows:

$$
\bar{\gamma}_{1}:=\min _{\mathbf{b}}\left(\gamma(\mathbf{b}) \mid \mathcal{T}_{b} \subseteq \boldsymbol{\Omega} \wedge b(t)>0 \quad \forall t \geq 0\right)
$$

We have that:

$$
\gamma^{*} \leq \bar{\gamma}_{1} \leq \bar{\gamma}
$$

We divide the demonstration in two parts.
Part I. First, we prove that $\left(\gamma^{*}=0 \Leftrightarrow \bar{\gamma}_{1}=0\right)$. Since $\bar{\gamma}_{1}$ is an upper bound of $\gamma^{*}$, then the necessity part is proved, that is $\bar{\gamma}_{1}=0 \Rightarrow \gamma^{*}=0$. As far as the sufficiency part is concerned, we observe that the main implication of Theorem 5.2 is that $\mathcal{T}_{b^{*}} \subseteq \boldsymbol{\Omega}$. This implies that $\gamma^{*}=0 \Rightarrow \bar{\gamma}_{1}=0$.
Part II. We now prove that $\left(\bar{\gamma}_{1}=0 \Leftrightarrow \bar{\gamma}=0\right)$. Since $\bar{\gamma}$ is an upper bound of $\bar{\gamma}_{1}$, then the necessity part is proved, that is $\bar{\gamma}=0 \Rightarrow \bar{\gamma}_{1}=0$. Therefore, we only need to prove that, when $\bar{\gamma}_{1}$ is equal to zero, it is always possible to determine a step cost function $\bar{b}(t) \geq \rho$ generating on each $\operatorname{arc}(i, j) \in A$ a constant travel cost function. Let us suppose that there exists a step cost function $0<b^{*}(t)<\rho$, generating travel constant cost functions, that is

$$
c_{i j}\left(t, \mathbf{b}^{*}\right)=\underline{c}_{i j}\left(\mathbf{b}^{*}\right)
$$

with $(i, j) \in A$ and $t \geq 0$. Since $b^{*}(t)>0$, then it is always possible to determine a positive number $\alpha>0$, such that:

$$
\alpha \times b^{*}(t) \geq \rho
$$

with $t \geq 0$. If we set $\bar{b}(t)$ equal to $\alpha \times b^{*}(t)$, then we have that

$$
c_{i j}(t, \overline{\mathbf{b}})=\alpha \times \underline{c}_{i j}\left(\mathbf{b}^{*}\right)
$$

with $t \geq 0$. This implies that

$$
\gamma\left(\overline{\mathbf{b}}, \mathcal{T}_{\bar{b}}\right)=\alpha \times \gamma\left(\mathbf{b}^{*}, \mathcal{T}_{b^{*}}\right)=0
$$

which proves the thesis.

From Theorem 5.2 and Proposition 5.2 , it follows that $\mathbf{1}_{C}(G)$ can be reformulated as follows:

$$
\mathbf{1}_{C}(G):= \begin{cases}1 & \text { if } \bar{\gamma}=0  \tag{22}\\ 0 & \text { otherwise }\end{cases}
$$

As illustrated in the following section, the main advantage of formulation 220 is that the travel cost range $\bar{\gamma}$ is the optimal objective function value of a Linear Programming (LP) problem.

### 5.2 Decidability of the CTCP

In order to assess the decidability of the CTCP, we outline an algorithm consisting of two steps. The former determines the set $\boldsymbol{\Omega}$ of potential breakpoints of $b^{*}(t)$. The latter determines the optimal objective function value of the optimization problem 21.

We recall that $\boldsymbol{\Omega}$ is the intersection of $|A|$ sets of time instants. Moreover each $\boldsymbol{\Omega}_{i j}$ is the union
of a finite number of time sequences $\boldsymbol{\omega}_{i j}(t)$, one for each travel time breakpoint $t \in \mathcal{T}$ and arc $(i, j) \in A$. Therefore, we only need to devise an iterative procedure that, given a time instant $t$ and a travel time function $\tau_{i j}(t)$, determines the ordered set $\boldsymbol{\omega}_{i j}(t)$ in a finite number of iteration steps. For this purpose we propose the Algorithm 1 consisting of two main steps.
(a) Firstly $\boldsymbol{\omega}_{i j}(t)$ is iteratively enriched by the arrival time $\Gamma_{i j}(t)$ associated to a start time equal to $t$, by the arrival time $\Gamma_{i j}\left(\Gamma_{i j}(t)\right)$ associated to a start time equal to $\Gamma_{i j}(t)$, etc, until no time instant less than or equal to $T$ can be generated.
(b) Finally, $\boldsymbol{\omega}_{i j}(t)$ is iteratively enriched by the start time $\Gamma_{i j}^{-1}(t)$ associated to an arrival time equal to $t$, by the start time $\Gamma_{i j}^{-1}\left(\Gamma_{i j}^{-1}(t)\right)$ associated to an arrival time equal to $\Gamma_{i j}^{-1}(t)$, etc, until no time instant greater than or equal to 0 can be generated.

For example by running Algorithm 1 iteratively for each distinct breakpoint of $\tau_{i j}(t)$ of Figure 1, we obtain the time sequences $\boldsymbol{\omega}_{i j}(4.0)=\{0.0,2.0,4.0\}$ and $\boldsymbol{\omega}_{i j}(5.0)=\{1.0,3.0,5.0\}$. Therefore the set $\boldsymbol{\Omega}_{i j}$ consists of the time instants $\{0.0,1.0,2.0,3.0,4.0,5.0\}$.

Proposition 5.3. Algorithm 1 converges in a finite number of iterations.
Proof. The thesis is proved by observing that each while-condition evaluates to false after a finite number of iterations. Indeed, due to the FIFO hypothesis, the arrival time function $\Gamma_{i j}(t)$ as well as the corresponding inverse function $\Gamma_{i j}^{-1}(t)$ are, respectively, strictly increasing and strictly decreasing functions, with $(i, j) \in A$.

```
Algorithm 1 Determining the timed sequence \(\boldsymbol{\omega}_{i j}(t)\)
    INPUT: A continuous piecewise linear FIFO travel time function \(\tau_{i j}(t)\) and a time instant \(t\).
    OUTPUT: The ordered set \(\boldsymbol{\omega}\) of time instants.
    \(\omega \leftarrow \emptyset\)
    \(\boldsymbol{\omega} \leftarrow t\)
    \(t^{\prime} \leftarrow t\)
    while \(\left(\Gamma_{i j}\left(t^{\prime}\right) \leq T\right) \wedge\left(\Gamma_{i j}\left(t^{\prime}\right) \notin \boldsymbol{\omega}\right)\) do
        \(\boldsymbol{\omega} \leftarrow \Gamma_{i j}\left(t^{\prime}\right)\)
        \(t^{\prime} \leftarrow \Gamma_{i j}\left(t^{\prime}\right)\)
    \(t^{\prime} \leftarrow t\)
    while \(\left(\Gamma_{i j}^{-1}\left(t^{\prime}\right) \geq 0\right) \wedge\left(\Gamma_{i j}^{-1}\left(t^{\prime}\right) \notin \boldsymbol{\omega}\right)\) do
        \(\boldsymbol{\omega} \leftarrow \Gamma_{i j}^{-1}\left(t^{\prime}\right)\)
        \(t^{\prime} \leftarrow \Gamma_{i j}^{-1}\left(t^{\prime}\right)\)
```

Once it has been determined the set $\boldsymbol{\Omega}$, it is possible to formulate an instance of the linear program (23)-30). A solution of such linear programming model represents the parameters of a constant piecewise function $y(t)$ and a continuous piecewise linear function $x_{i j}(t)$, with $(i, j) \in A$. We partition the time horizon into a finite number $|\boldsymbol{\Omega}|$ of time slots $\left[t_{h}, t_{h+1}\right](h=0, \ldots,|\boldsymbol{\Omega}|-1)$. The continuous variable $y_{h}$ represents the value of $y(t)$ during the $h-t h$ time interval, that is:

$$
y(t)=y_{h}
$$

with $t \in\left[t_{h}, t_{h+1}\right]$ and $h=0, \ldots,|\boldsymbol{\Omega}|-1$. The function $x_{i j}(t)$ is the continuous piecewise linear function corresponding to the linear interpolation of the points $\left(t_{i j k}, x_{i j k}\right)$, that is:

$$
x_{i j}\left(t_{i j k}\right)=x_{i j k},
$$

where $x_{i j k}$ is a continuous variable, with $t_{i j k} \in \boldsymbol{\Omega}_{i j}, k=0, \ldots,\left|\boldsymbol{\Omega}_{i j}-1\right|$ and $(i, j) \in A$. The continuous variables $\underline{x}_{i j}$ and $\bar{x}_{i j}$ represent, respectively, the maximum and minimum value of $x_{i j}(t)$, with $(i, j) \in A$. The continuous variable $\zeta_{i j}$ represents the range of the continuous piecewise linear function $x_{i j}(t)$, with $(i, j) \in A$. Finally the continuous variable $\zeta$ represents the max-min range value associated to the overall set of $\mathbf{x s}$ functions generated by the problem.

$$
\begin{equation*}
\zeta^{*}:=\min \zeta \tag{23}
\end{equation*}
$$

s.t.
$x_{i j k}=\sum_{h=0}^{|\boldsymbol{\Omega}|-1} a_{i j k h} \times y_{h} \quad k=0, \ldots,\left|\boldsymbol{\Omega}_{i j}\right|-1, \quad(i, j) \in A$
$\zeta \geq \bar{x}_{i j}-\underline{x}_{i j}$

$$
\begin{equation*}
(i, j) \in A \tag{24}
\end{equation*}
$$

$\underline{x}_{i j} \leq x_{i j k}$
$k=0, \ldots,\left|\boldsymbol{\Omega}_{i j}\right|-1, \quad(i, j) \in A$
$\bar{x}_{i j} \geq x_{i j k}$
$k=0, \ldots,\left|\boldsymbol{\Omega}_{i j}\right|-1, \quad(i, j) \in A$
$y_{h} \geq \rho$
$h=0, \ldots,|\boldsymbol{\Omega}|-1$
$x_{i j k} \geq 0$
$k=0, \ldots,\left|\boldsymbol{\Omega}_{i j}\right|-1, \quad(i, j) \in A$
$\zeta \geq 0$

The objective function (23) states that the optimization model aims to determine a constant stepwise function $y^{*}(t)$, such that it is minimized the maximum range value of the corresponding xs functions. Constraints (24) state the relationship between $y(t)$ and $x_{i j}(t)$ at time instant $t_{i j k} \in \boldsymbol{\Omega}_{i j}$. In particular each coefficient $a_{i j k h}$ represents the time spent on the arc $(i, j) \in A$ during period [ $\left.t_{h}, t_{h+1}\right]$ if the start time is $t_{i j k} \in \boldsymbol{\Omega}_{i j}$, i.e.

$$
a_{i j k h}= \begin{cases}\min \left(t_{h+1}-t_{h}, \max \left(0, \Gamma_{i j}\left(t_{i j k}\right)-t_{h}\right)\right) & \text { if } k \leq h  \tag{31}\\ 0 & \text { if } k>h\end{cases}
$$

with $h=0, \ldots,|\boldsymbol{\Omega}|-1, k=0, \ldots,\left|\boldsymbol{\Omega}_{i j}\right|-1$. Constraints 25 state the relationship between the objective function $\zeta$ and the range value of $x_{i j}(t)$, modeled as the difference between $\bar{x}_{i j}$ and $\underline{x}_{i j}$. Constraints (26) and 27 state the relationship between $\underline{x}_{i j}, \bar{x}_{i j}$ and the continuous variables $x_{i j k}$. Constraints (28) state that the constant stepwise linear function $y(t)$ has to be greater or equal than the input parameter $\rho>0$. Constraints (29) and (30) provide the non-negative conditions of the remaining decision variables.

Theorem 5.3. Given a time-dependent graph $G$, the optimal solution of the linear program (23)(30) is also optimal for the optimization problem (21.)

Proof. Let $\left(\mathcal{T}_{\mathbf{y}^{*}}, \mathbf{y}^{*}\right)$ denote the parameters of the constant stepwise function $y^{*}(t)$ optimal for the linear program (23)-(30). We observe that $y^{*}(t)$ can be used to generate two continuous piecewise
linear functions for each $\operatorname{arc}(i, j) \in A$. The former is the travel cost function $c_{i j}\left(t, \mathbf{y}^{*}\right)$. The latter is the continuous piecewise linear $x_{i j}^{*}(t)$ defined as the linear interpolation associated to the optimal values $x_{i j k}^{*}$ with $k=0, \ldots\left|\boldsymbol{\Omega}_{i j}\right|-1$ and $(i, j) \in A$. We want to prove that $c_{i j}\left(t, \mathbf{y}^{*}\right)=x_{i j}^{*}(t)$ for $t \geq 0$. We start by observing that the right hand side of constraints (24) corresponds to the right-hand side of (5) when $t=t_{i j k} \in \boldsymbol{\Omega}_{i j}$, that is $c_{i j}\left(t_{i j k}, \mathbf{y}^{*}\right)=x_{i j}^{*}\left(t_{i j k}\right)$, with $(i, j) \in A$ and $k=0, \ldots\left|\boldsymbol{\Omega}_{i j}-1\right|$. Therefore, the thesis is proved if we demonstrate that all breakpoints of each travel cost function $c_{i j}\left(t, \mathbf{y}^{*}\right)$ belong to $\boldsymbol{\Omega}_{i j}$, with $(i, j) \in A$. From (7) we have that:

$$
\begin{equation*}
t \in \mathcal{T}_{c}\left(i, j, \mathbf{y}^{*}\right) \quad \Rightarrow \quad t \in \mathcal{T}_{y^{*}} \vee t \in \mathcal{T}_{\tau}(i, j) \vee \Gamma_{i j}(t) \in \mathcal{T}_{y^{*}} . \tag{32}
\end{equation*}
$$

Since there is no constraint stating that $y_{h}^{*} \neq y_{h+1}^{*}$, then the breakpoints set $\mathcal{T}_{y^{*}}$ is a subset of each $\boldsymbol{\Omega}_{i j}$, with $(i, j) \in A$. Moreover, from the definition of $\boldsymbol{\Omega}_{i j}$, we have that all breakpoints of $\tau_{i j}(t)$ belongs to it. Finally we observe that from the definition of $\boldsymbol{\Omega}_{i j}$, it descends that:

$$
\Gamma_{i j}(t) \in \mathcal{T}_{y^{*}} \subseteq \boldsymbol{\Omega}_{i j} \Rightarrow \boldsymbol{\omega}_{i j}\left(\Gamma_{i j}(t)\right) \subseteq \boldsymbol{\Omega}_{i j} \Rightarrow t \in \boldsymbol{\Omega}_{i j}
$$

Therefore, the implication (32) can be rewritten as:

$$
t \in \mathcal{T}_{c}\left(i, j, \mathcal{T}_{y^{*}}\right) \Rightarrow t \in \boldsymbol{\Omega}_{i j}
$$

which proves the thesis.
From Proposition 5.3 and Theorem 5.3 it descends that, given a time-dependent graph $G=$ ( $V, A, \tau$ ), the value of the characteristic function $\mathbf{1}_{C}(G)$ can be computed as follows.
Step 1]. Determine $\boldsymbol{\Omega}$ by iteratively running Algorithm 1, for each arc $(i, j) \in A$ and each breakpoint of the travel time function $\tau_{i j}(t)$.
Step 2. Solve the linear program $(23)-\sqrt{30}$. The value the function $\mathbf{1}_{C}(G)$ takes on as output is

$$
\mathbf{1}_{C}(G):= \begin{cases}1 & \text { if } \quad \zeta^{*}=0 \\ 0 & \text { otherwise }\end{cases}
$$

## 6 A lower bounding procedure

If the CTCP feasibility check fails, then the optimal solution of the linear program (23)-(30) can be used to determine "good" lower bounds for the optimal solutions of the class of time-dependent routing problems (3).

We preliminarily define a parameterized family of travel time functions $\underline{\tau}(\mathbf{b})$, where the parameter is the step function $\mathbf{b}$. In particular, given a time-dependent graph $G=(V, A, \tau)$, the travel time function $\underline{\tau}_{i j}(t, \mathbf{b})$ we generate is a lower approximation of the original travel time function $\tau_{i j}(t)$, that is:

$$
\underline{\tau}_{i j}(t, \mathbf{b}) \leq \tau_{i j}(t),
$$

for each $t \geq 0$ and $(i, j) \in A$. We start by observing that, given a step function $\mathbf{b}$, it is univocally associated to each arc $(i, j) \in A$ the travel cost function $c_{i j}(t, \mathbf{b})$. In order to generate $\tau_{i j}(t, \mathbf{b})$, we first approximate such travel cost function to its minimum value $\underline{c}_{i j}(\mathbf{b})=\min _{t} c_{i j}(t, \mathbf{b})$. Then $\underline{\tau}_{i j}(t, \mathbf{b})$ is defined as the output function of the travel time model proposed by Ichoua et al. (2003a) (IGP model).

Definition 6.1. The travel time function $\underline{\tau}_{i j}(t, \boldsymbol{b})$ is the continuous piecewise linear function generated by IGP model, where the input parameters are the step function $\boldsymbol{b}$ and the constant value $\underline{c}_{i j}(b)$.

According to the IGP model, given a start time $t$ the travel time value $\underline{\tau}_{i j}(t, \mathbf{b})$ is computed by an iterative procedure (Algorithm 2). The relationship between the input parameters and the output value of Algorithm 2 can be expressed in a compact fashion as follows:

$$
\begin{equation*}
\underline{c}_{i j}(\mathbf{b})=\int_{t}^{t+\underline{\tau}_{i j}(t, \mathbf{b})} b(\mu) d \mu . \tag{33}
\end{equation*}
$$

```
    time \(t\)
    \(k \leftarrow p: t_{p} \leq t \leq t_{p+1}\)
    \(d \leftarrow \underline{c}_{i j}(\mathbf{b})\)
    \(t^{\prime} \leftarrow t+d / b_{p}\)
    while \(t^{\prime}>t_{k+1}\) do
        \(d \leftarrow d-b_{k}\left(t_{k+1}-t\right)\)
        \(t \leftarrow t_{k+1}\)
        \(t^{\prime} \leftarrow t+d / b_{k+1}\)
        \(k \leftarrow k+1\)
    return \(t^{\prime}-t\)
```

Algorithm 2 Computing the travel time $\underline{\tau}_{i j}(t, \mathbf{b})$
INPUT: A step function $\mathbf{b}=\left[\left(t_{0}, b_{0}\right), \ldots,\left(t_{\left|\mathcal{T}_{b}-1\right|}, b_{\left|\mathcal{T}_{b}-1\right|}\right)\right]$, a constant cost $\underline{c}_{i j}(\mathbf{b})$ and the start

Proposition 6.1. The travel time function $\underline{\tau}_{i j}(t, \boldsymbol{b})$ is a lower approximation of the original travel time function $\tau$.

Proof. We observe that a compact formulation of (5) is the following:

$$
c_{i j}(t, \mathbf{b})=\int_{t}^{t+\tau_{i j}(t)} b(\mu) d \mu
$$

From (33) and the definition of $\underline{c}_{i j}(\mathbf{b})$, it results that:

$$
\underline{c}_{i j}(\mathbf{b}) \leq c_{i j}(t, \mathbf{b}) \Leftrightarrow \underline{\tau}_{i j}(t, \mathbf{b}) \leq \tau_{i j}(t),
$$

with $t \geq 0$, which proves the thesis.
Proposition 6.2. Given a time-dependent graph $G$ and a step function $\boldsymbol{b}$, the time depedent graph $\underline{G}_{b}=(V, A, \underline{\tau}(\boldsymbol{b}))$ is path ranking invariant.

Proof. Since bis a step function, from (33) it results that if the right-hand side of (5) is evaluated w.r.t $\underline{\tau}_{i j}(t, \mathbf{b})$, then we obtain the constant value $\underline{c}_{i j}(\mathbf{b})$, for any time instant $t \geq 0$ and $\operatorname{arc}(i, j) \in A$. This implies that the time-dependent graph $\underline{G}_{\mathbf{b}}=(V, A, \underline{\tau}(\mathbf{b}))$ is a yes instance of the Constant Traversal Cost Problem, which proves the thesis.

The family of lower approximations $\underline{\tau}(\mathbf{b})$ gives rise to a parameterized family of lower bounds $\underline{z}(\mathbf{b})$, defined as follows. We denote with $\underline{z}\left(p_{k}, t, \mathbf{b}\right)$ the traversal time of a path $p_{k}$ at time instant $t$ on the time-dependent graph $\underline{G}_{\mathbf{b}}=(V, A, \underline{\tau}(\mathbf{b}))$, that is

$$
\begin{equation*}
\underline{z}\left(p_{k}, t, \mathbf{b}\right)=\underline{z}\left(p_{k-1}, t, \mathbf{b}\right)+\underline{\tau}_{i_{k-1} i_{k}}\left(\underline{z}\left(p_{k-1}, t\right), \mathbf{b}\right), \tag{34}
\end{equation*}
$$

with the initialization $\underline{z}\left(p_{0}, t, \mathbf{b}\right)=0$. Since $\underline{G}_{\mathbf{b}}$ is less congested than $G$, the duration of a path is shorter on $\underline{G}_{\mathbf{b}}$, that is $\underline{z}\left(p_{k}, t, \mathbf{b}\right) \leq z\left(p_{k}, t\right)$. Given a set $\mathcal{P}$ of paths defined on $(V, A)$, we compute the lower bound $\underline{z}(\mathbf{b})$ as the duration of the quickest path of $\mathcal{P}$ on $\underline{G}_{\mathbf{b}}$, that is:

$$
\underline{z}(\mathbf{b})=\min _{p \in \mathcal{P}} \underline{z}(p, t, \mathbf{b}) \leq \min _{p \in \mathcal{P}} z(p, t) .
$$

The main implication of Proposition 6.2 is that the lower bound $\underline{z}(\mathbf{b})$ can be computed by solving a time-invariant routing problem. In particular, we have that:

$$
\underline{z}(\mathbf{b})=\underline{z}\left(\underline{p}_{\mathbf{b}}^{*}, t, \mathbf{b}\right),
$$

where

$$
\underline{p}_{\mathbf{b}}^{*}=\arg \min _{p \in \mathcal{P}} \sum_{(i, j) \in p} \underline{c}_{i j}(\mathbf{b}) .
$$

In determining $\underline{p}_{\mathbf{b}}^{*}$ we exploit that a cost function valid for $\underline{G}_{\mathbf{b}}$ is $d(i, j)=\underline{c}_{i j}$, with $(i, j) \in A$. Indeed, since $\underline{\tau}_{i j}(t, \mathbf{b})$ is constant in the long run, we have that $\underline{\tau}_{i j}(T, \mathbf{b})=\underline{c}_{i j} / b_{\left|\mathcal{T}_{\mathbf{b}}\right|-\mathbf{1}}$.

In order to find the best (larger) lower bound, the following problem has to be solved:

$$
\begin{equation*}
\max _{\mathbf{b}} \underline{z}(\mathbf{b}) \tag{35}
\end{equation*}
$$

Unfortunately, this problem is nonlinear, nonconvex and non differentiable. So it is quite unlikely to determine its optimal solution with a moderate computational effort. Instead, we aim to find a good lower bound as follows. We observe that the tightness of the lower bound $\underline{z}(\mathbf{b})$ clearly depends on the maximum fitting deviation between the original travel time function $\tau$ and its lower approximation $\underline{\tau}(\mathbf{b})$. It is worth noting that we generate $\underline{\tau}_{i j}(\mathbf{b})$ by approximating travel cost function $c_{i j}(t, \mathbf{b})$ to its minimum value $c_{i j}(\mathbf{b})$, with $(i, j) \in A$. The maximum fitting deviation of such travel cost approximation is the maximum range value $\gamma(\mathbf{b})$ defined in Section 5.1. If $\gamma(\mathbf{b})$ is equal to zero, then both travel time function $\underline{\tau}(\mathbf{b})$ and each travel cost $\underline{c}_{i j}(\mathbf{b})$ are perfect fit, with $(i, j) \in A$. In this case, the original graph $G$ is path ranking invariant and $\underline{z}(\mathbf{b})$ is the best (larger) lower bound. Otherwise the value of $\gamma(\mathbf{b})$ represents a measurement of the distance from this special case.

For these reasons we heuristically solve (35), by determining the optimal step function $\mathbf{y}^{*}$ of the linear program (23)-30), which represents the step function with the min-max value of travel cost range. In particular, we observe that the coefficient $c_{i j}\left(\mathbf{y}^{*}\right)$ are also determined by the optimal solution of such linear program. Indeed the optimal piecewise linear function $x_{i j}^{*}(t)$ corresponds to the travel cost function $c_{i j}\left(t, \mathbf{y}^{*}\right)$, and, therefore, its minimum value corresponds the optimal value of the decision variable $\underline{x}_{i j}$, namely $\underline{x}_{i j}^{*}$ with $(i, j) \in A$. Summing up the proposed lower bounding procedure consists of three main steps.

- STEP 1. Solve the linear program (23)-(30). Set the step function bequal to $\mathbf{y}^{*}$. Similarly we set the coefficient $\underline{c}_{i j}$ to $\underline{x}_{i j}^{*}$ for each $(i, j) \in A$.
- STEP 2. Determine $\underline{p}_{\mathbf{b}}^{*}$ as the least cost solution of the following time-independent routing problem:

$$
\min _{p \in \mathcal{P}} \sum_{(i, j) \in p} \underline{c}_{i j}(\mathbf{b})
$$

- STEP 3. Compute the lower bound $\underline{z}(\mathbf{b})$ by evaluating $\underline{p}_{\mathbf{b}}^{*}$ w.r.t. $\underline{\tau}(\mathbf{b})$, that is:

$$
\underline{z}(\mathbf{b})=\underline{z}\left(\underline{p}_{\mathbf{b}}^{*}, t_{0}, \mathbf{b}\right)
$$

It is worth noting that due to a huge number of travel time breakpoints, the linear program (23)- 30 might not be solved in a reasonable amount of time (e.g., in realistic time-dependent graph). In this cases we update the first step of the lower bounding procedure as follows. We solve a smaller instance of the linear program, where each $\boldsymbol{\Omega}_{i j}$ is set equal to a given (unique) set of time instant $\mathcal{B}$.

It results that each optimal continuous piecewise linear function $x_{i j}^{*}(t)$ is a surrogate function of the travel cost $c_{i j}\left(t, \mathbf{y}^{*}\right)$. This implies that during the first step we compute the coefficient $\underline{c}_{i j}\left(\mathbf{y}^{*}\right)$ by enumerating all breakpoints of $c_{i j}\left(t, \mathbf{y}^{*}\right),(i, j) \in A$.

We finally observe that since the path $\underline{p}_{\mathrm{b}}^{*}$ belongs to the set of feasible paths $\mathcal{P}$, we also generate a parameterized family of upper bound $\overline{\bar{z}}(\mathbf{b})$ obtained by evaluating $\underline{p}_{\mathbf{b}}^{*}$ w.r.t. the original travel time function $\tau$ :

$$
\bar{z}(\mathbf{b}):=z\left(\underline{p}_{\mathbf{b}}^{*}, t_{0}\right) .
$$

## 7 Computational Results

We have tested our lower bounding procedure on two well-known vehicle routing problems: the Time Dependent Travelling Salesman Problem and the Time Dependent Rural Postman Problem. The state-of-the-art algorithms for such time-dependent routing problems are, to the best of our knowledge, Adamo et al. (2020)(TDTSP) and Calogiuri et al. (2019) (TDRPP). Both contributions gave rise to lower bounding procedures which were used in an enumerative scheme. It is worth noting that the idea of a lower bound based on a less congested graph $\underline{G}=(V, A, \underline{\tau})$ is also the idea underlying the lower bounds proposed by Adamo et al. (2020) and Calogiuri et al. (2019). Figure 2 reports an example concerning a time-dependent travel time function $\tau_{i j}(t)$ associated with an arc of a time-dependent graph $G$, consisting of both time-invariant arcs and time-dependent arcs. The Figure also shows the $\underline{\tau}_{i j}(t)$ obtained by our approach along with the lower approximations of $\tau_{i j}(t)$ obtained by applying the approach proposed, respectively, by Calogiuri et al. (2019) and Adamo et al. (2020). In particular, Calogiuri et al. (2019) assumed that the original traversal time $\tau_{i j}(t)$ was generated from the model by Ichoua et al. $\left.\mid 2003 \mathrm{~b}\right)$ as the time needed to traverse an arc $(i, j) \in A$ of length $L_{i j}$ at step-wise speed $v_{i j}(t)$, when the vehicle leaves the vertex $i$ at time $t$. Then Calogiuri et al. (2019) exploited the speed decomposition proposed by Cordeau et al. (2014), where given a partition of the planning horizon in $H$ time periods, the speed value $v_{i j h}$ of $\operatorname{arc}(i, j) \in A$ during period $h$ is express as:

$$
v_{i j h}=\delta_{i j h} f_{h} u_{i j}
$$

where

- $u_{i j}$ is the maximum travel speed across $\operatorname{arc}(i, j) \in A$, i.e., $u_{i j}=\max _{h=0, \ldots, H-1} v_{i j h}$;
- $f_{h}$ belongs to $\left.] 0,1\right]$ and is the best (i.e., lightest) congestion factor during $h-t h$ interval, i.e., $f_{h}=\max _{(i, j) \in A} v_{i j h} / u_{i j}$;
- $\delta_{i j h}$ belongs to $\left.] 0,1\right]$ and represents the degradation of the congestion factor of arc $(i, j)$ in the $h-t h$ interval with respect to the less-congested arc in the same period.

In Calogiuri et al. (2019) the lower approximation $\underline{\tau}$ was defined as the traversal time of the vehicle evaluated w.r.t. the most favourable congestion factor during each interval $h$-th, i.e., $\underline{v}_{i j h} . \leftarrow f_{h} u_{i j}$. The main drawback of this approach is that, if a subset of arcs are time-invariant, then traffic congestion factors $f_{h}$ are all equal to 1 . In these cases, the lower approximation $\underline{\tau}$ determined by Calogiuri et al. (2019) are constant and equal to $L_{i j} / u_{i j}$ (see Figure 2). Adamo et al. (2020) enhanced the speed decomposition of Cordeau et al. (2014), by determining its parameters through a fitting procedure of the traffic data. As shown in the example reported in Figure 2, by applying our approach, we get a lower approximation $\tau$ that fits the original $\tau$ better than the travel time approximation determined by Adamo et al. (2020). Indeed, the fitting procedure by Adamo et al. (2020) aims to minimize the deviation between the (original) speed values $v_{i j h}$ and the most favourable speed value $\underline{v}_{i j h}$, during some (not necessarily consecutive) periods. As shown in Figure 2 this approach do not assure that there exists at least one time instant closing the fitting deviation between the lower approximation and the original travel time $\tau$, which, on the contrary, it is guaranteed by our procedure.
As discussed in the following subsections, computational results show that tighter approximation travel times implies tighter lower bounds for both TDTSP instances and TDRPP instances. In particular, when embedded into a branch-and-bound procedure, the proposed lower bounding mechanism allows to solve to optimality a larger number of instances than the state-of-the-art algorithms.


Figure 2: Comparing the $\underline{\tau}$ functions determined by, respectively, our lower bounding procedure, Cordeau et al. (2014), Adamo et al. (2020)

### 7.1 Results on the TDTSP

The procedure illustrated in Section 6 can be used to determine a "good" lower bound for TDTSP as follows. We run the first step of the proposed lower bounding procedure and solve the linear program (23)-(30). Then we run the second and third steps of the lower bounding procedure. In particular the solution $\underline{p}_{\mathrm{b}}^{*}$ corresponds to a first hamiltonian circuit determined by solving an Asymmetric TSP with arc costs equal to $d_{i j}=\min _{t \geq 0} x_{i j}^{*}(t)$. The lower bound is embedded into the branch-and-bound scheme presented by Arigliano et al. (2018) that we sketch here. First we initialize the subproblem queue $Q$ with the original problem $P$ and set the upper bound $U B=z\left(\underline{p}_{\mathbf{y}^{*}}^{*}, t_{0}\right)$. The instances of the Asymmetric TSP are solved by means of Carpaneto and Toth (1980). At a generic step, let $\tilde{P}$ be the subproblem extracted from $Q$ and characterized by a set of branching constraints corresponding to a fixed path $p_{k}=\left(0=v_{0}, v_{1}, v_{2}, \ldots v_{k}\right)$, and possibly some forbidden arcs. Then we compute its lower bound, named $L B_{1}$. In particular we run the proposed procedure by omitting the first initialization step: the step function $\mathbf{y}^{*}$ and the functions $x_{i j}^{*}(t)$ are inherited by the parent
problem, whilst the coefficient $\underline{c}_{i j}$ is updated as follows:

$$
\underline{c}_{i j}=\min _{t \in\left[z\left(p_{k}, t_{0}\right), U B\right]} x_{i j}^{*}(t),
$$

with $(i, j) \in A$. Hence, we examine the (integer) optimal solution of the $\underline{\tilde{P}}$ relaxation:

$$
\begin{equation*}
\left(0=v_{0}, v_{1}, \ldots, v_{k}, v_{k+1}, \ldots, v_{k^{\prime}}, v_{k^{\prime}+1}, \ldots, v_{n+1}=0\right) \tag{36}
\end{equation*}
$$

which also gives an upper bound $U B_{1}$ on the optimal solution of $\tilde{P}$, obtained by evaluating 36 w.r.t. the original travel time function $\tau$. If $U B_{1}<U B$, then we determine the upper bound as $U B=U B_{1}$. If $U B_{1}=L B_{1}$, then this solution is optimal for problem $\tilde{P}$. Otherwise, if $L B_{1}<U B$, the procedure branches as follows. Let $k^{\prime} \in\{k+1, \ldots, n\}$ be the index such that its traversal time evaluated w.r.t. $\underline{\tau}$ is equal to the original traversal time, i.e. $\underline{z}\left(p_{k^{\prime}}, t_{0}\right)=z\left(p_{k^{\prime}}, t_{0}\right)$. We create $k^{\prime}-k+1$ subproblems as follows:

$$
\begin{gather*}
\psi_{v_{k}, v_{k+1}}=1, \psi_{v_{k+1}, v_{k+2}}=1, \psi_{v_{k+2}, v_{k+3}}=1, \ldots, \psi_{v_{k^{\prime}}, v_{k^{\prime}+1}}=1  \tag{37}\\
\ldots  \tag{38}\\
\psi_{v_{k}, v_{k+1}}=1, \psi_{v_{k+1}, v_{k+2}}=1, \psi_{v_{k+2}, v_{k+3}}=0  \tag{39}\\
\psi_{v_{k}, v_{k+1}}=1, \psi_{v_{k+1}, v_{k+2}}=0  \tag{40}\\
\psi_{v_{k}, v_{k+1}}=0
\end{gather*}
$$

where $\psi_{i j}$ is a binary variable equal to 1 if and only if $\operatorname{arc}(i, j) \in A$ is in the solution, while the arcs having $\psi_{i j}$ equal to 0 are forbidden.

We compared the obtained results with the results obtained in Adamo et al. (2020). Both procedures are implemented in $\mathrm{C}++$ and run on MacBook Pro with a 2.33-GHz Intel Core 2 Duo processor and 4 GB of memory. Linear programs are solved with Cplex 12.9. We consider the same instances generated in Adamo et al. (2020) and we impose a time limit of 3600 s . Two scenarios are given: a first traffic pattern A in which a limited traffic zone is located in the center; a second traffic pattern B in which a heaviest traffic congestion is situated in the center. In Adamo et al. (2020) a fundamental role was played by $\Delta$ which represents the worst degradation $\delta_{i j h}$ of the congestion factor of any arc $(i, j) \in A$ over the entire planning horizon, i.e. $\Delta=\min _{h=0, \ldots, H-1}\left(\min _{(i, j) \in A} \delta_{i j h}\right)$. For Adamo et al. (2020) hard instances are characterized by low value of $\Delta$ in traffic pattern B. The results for the two scenarios are shown in Tables 1 and 2 in which 30 instances are generated for each combination of $|V|=15,20,25,30,35,40,45,50$ and $\Delta=0.90,0.80,0.70$. The headings are as follows:

- $O P T$ : number of instances solved to optimality out of 30 ;
- $U B_{I} / L B_{F}$ : average ratio of the initial upper bound value $U B_{I}$ the best lower bound $L B_{F}$ available at the end of the search;
- $G A P_{I}$ : average initial optimality gap $\frac{U B_{I}-L B_{I}}{L B_{I}} \%$;
- $G A P_{F}$ : average final optimality gap $\frac{U B_{F}-L B_{F}}{L B_{F}} \%$;
- $N O D E S$ : average number of nodes;
- TIME: average computing time in seconds.

Except for columns $O P T$, we report results on two distinct rows: the first row is the average across instances solved to optimality, and the second row is the average for the remaining instances. For the sake of conciseness, the first or the second row has been omitted whenever none or all instances are solved to optimality. For columns NODES and TIME we report only averages for instances that are solved to optimality. Computational results show that our algorithm is capable of solving 923 instances out of 1440 instances while the procedure by Adamo et al. (2020) solved only 566 problems. The improvement is remarkable for hard instances characterized by the traffic pattern B. Here, the Adamo et al. (2020) algorithm solves 229 out of 720 instances while our algorithm procedure succeeds on 505 instances. This can be explained by the quality of the new lower bound as well as by the low computational effort spent to solve the linear program 23 - 30 . In particular, we formulate a reduced-size instance of such linear model by including in the set $\mathcal{B}$ only 75 time instants. Moreover we set the value of $\rho$ equal to $1 / \min _{h=0, \ldots,|\mathcal{B}|-1}\left(t_{h+1}-t_{h}\right)$. We finally observe that the average computational time is 438 seconds for our procedure and 710 seconds for Adamo et al. (2020). Indeed, in spite of a higher initial gap, the new lower bound reduces the overall number of visited branch-and-bound nodes: on average our procedure processes 413 nodes (i.e. ATSP instances) less than Adamo et al. (2020).
Table 1: Pattern A

Table 2：Pattern B

|  |  | 㒸 | $\begin{aligned} & A \\ & H \\ & 0 \\ & 0 \\ & 0 \end{aligned}$ |  |  | $\begin{aligned} & \underset{\sim}{\underset{\sim}{2}} \\ & \underset{\sim}{\infty} \\ & \underset{\sim}{\infty} \\ & \hline \end{aligned}$ |  | $\begin{aligned} & 4 \\ & 0 \\ & 0 \\ & 0 \\ & 0 \\ & 0 \end{aligned}$ |  |  |  | $\begin{aligned} & \vec{N} \\ & A \\ & A \\ & A \\ & \end{aligned}$ | $\begin{aligned} & \infty \\ & \underset{y}{\infty} \\ & \underset{\sim}{\infty} \\ & \underset{\sim}{2} \end{aligned}$ |  |  | $\left.\begin{aligned} & \infty \\ & \infty \\ & \infty \\ & \dot{\infty} \\ & \dot{0} \\ & \underset{\sim}{0} \end{aligned} \right\rvert\,$ |  |  |  | $\left\lvert\, \begin{gathered} \infty \\ 0 \\ \\ \\ \hline \end{gathered}\right.$ | $\left\lvert\, \begin{gathered} \text { a } \\ \text { in } \\ \dot{y} \end{gathered}\right.$ |  |  | $\begin{aligned} & 0 \\ & 0 \\ & \text { in } \\ & \underset{\sim}{i} \end{aligned}$ | （ |
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| $\left\|\begin{array}{c} 0 \\ 0 \end{array}\right\| \begin{aligned} & 2 \\ & 4 \\ & 4 \\ & 4 \\ & 4 \end{aligned}$ | $5$ | $\underset{\exists}{\otimes}$ | on |  |  | 礐 |  |  |  | $\bigcirc$ |  | $\left\lvert\, \begin{gathered} \infty \\ \infty \\ 0 \\ \\ \\ 0 \end{gathered}\right.$ |  | $\begin{cases}\substack{0 \\ \infty \\ \infty \\ \rightarrow \\ \hline}\end{cases}$ | $\begin{gathered} f \\ 4 \\ 0 \end{gathered}$ |  | － |  |  | ® |  | $\begin{array}{ll} 0 & 1 \\ 0 \\ 0 \\ 0 & 0 \\ 0 \end{array}$ | Na | $\mathfrak{l l}$ | － |
|  | $\left\{\begin{array}{l} 0 \\ 0 \\ 0 \\ 0 \end{array}\right.$ |  | $\begin{array}{cc} 8 & 2 \\ 0 & \infty \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}$ | $\left.\left\lvert\, \begin{array}{ccc} 0 & 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right.\right)$ |  | $\left\|\begin{array}{ll} 8 & 1 \\ 0 & 1 \\ 0 & -1 \end{array}\right\|$ | $\begin{array}{cc} 8 & 7 \\ 8 & \vec{x} \\ 0 & 0 \\ 0 & 10 \end{array}$ | $\left(\begin{array}{cc} 8 & \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \\ 0 & -1 \end{array}\right.$ |  |  |  | $\left[\begin{array}{cc} 8 & 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$ | $\left\{\begin{array}{ll\|} \hline 0 & 0 \\ 0 & 0 \\ 0 \\ 0 \end{array}\right.$ | $\left\{\begin{array}{l} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right.$ |  | $\mathfrak{B}$ | 名 | ? | $\left.\right\|_{0} ^{0}$ | $\begin{array}{ll} \hat{Q}_{0} \\ 0 \\ 0 & 1 \end{array}$ |  | $\left\lvert\, \begin{array}{cc} 8 & \infty \\ 0 \\ 0 \\ 0 \end{array}\right.$ | $0 .$ | $\begin{array}{ll} 8 & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right) \end{array}$ | 景 |
|  | $\left\{\begin{array}{l} \infty \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right.$ |  | $$ | $\left\|\begin{array}{cc} n_{0} & 0 \\ 0 \\ 0 \\ 1 \end{array}\right\|$ |  | $\left\{\begin{array}{l} x_{0}^{2} \\ 0 \\ 0 \\ \\ \end{array}\right.$ |  |  | $\left\lvert\, \begin{array}{ll} \substack { n \\ \begin{subarray}{c}{2{ n \\ \begin{subarray} { c } { 2 } } \\ {\dot{0}} \\ {\hline} \\ \hline \end{array}\right.$ |  | $\underset{\substack{\infty \\ ⿱ 丷 ⿱ 一 ⿱ ㇒ ⿴ 囗 ⿱ 一 一 寸, ~}}{ }$ | $\left.\begin{array}{ll} 7 & 0 \\ 7 & 0 \\ 0 & 0 \\ m & \infty \end{array} \right\rvert\,$ |  | $\begin{cases}4 \\ x & 5 \\ 1 & 0 \\ \text { co }\end{cases}$ |  | $\begin{cases}\infty & 0 \\ 0 & 0 \\ \infty \\ 0 \\ i \\ i\end{cases}$ |  | $8: \substack{i \\ 0 \\ i \\ i \\ i}$ | $\left\lvert\, \begin{aligned} & \infty \\ & \infty \\ & \infty \\ & -i \end{aligned}\right.$ | $\begin{gathered} n \\ \infty \\ \infty \\ 0 \\ -1 \end{gathered}$ |  | $\begin{array}{cc} 0 \\ 0 \\ \\ \\ \hline \end{array}$ | $\begin{aligned} & 9 \\ & 8 \\ & 8 \end{aligned}$ | $\mathfrak{y}$ | $\xrightarrow{\circ}$ |
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| $0$ | $\left.5 \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 \end{array}\right]$ |  |  | $\left\|\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 \\ 0 & 0 \end{array}\right\|$ | $\left.\left\lvert\, \begin{array}{cc} 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right.\right)$ |  | $\left[\begin{array}{ll} \substack{n \\ \\ 0 \\ 0 \\ 0 \\ 0} & \infty \end{array}\right.$ | $\begin{array}{cc} \substack{8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0} \\ 0 & 1 \\ 0 \end{array}$ |  | $\left.\left\lvert\, \begin{array}{ccc} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \infty \end{array}\right.\right)$ |  |  |  |  |  |  | $\left(\begin{array}{c} \infty \\ \infty \\ \infty \\ \infty \end{array}\right.$ | $\mathfrak{B}$ |  | $\begin{array}{ll} 10 & \mathrm{~N} \\ 0 \\ 0 \\ -0 \\ \hline \end{array}$ | $=0$ | $\left\|\begin{array}{cc} \infty & 0 \\ 0 \\ 0 \\ -\underbrace{}_{1} \end{array}\right\|$ |  | $\mathfrak{c}$ | － |
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### 7.2 Results on the TDRPP

Calogiuri et al. (2019) devised lower and upper bounds for the TDRPP. The proposed bounds were embedded in a branch-and-bound algorithm. In particular, they enhanced the branching scheme of the algorithm by Arigliano et al. (2018) in order to deal with quickest paths linking distinct required arcs.
We embedded our bounds in such optimal solution algorithm. Both procedures are implemented in $\mathrm{C}++$ and run on MacBook Pro-with a $2.33-\mathrm{GHz}$ Intel Core 2 Duo processor and 4 GB of memory. Linear programs are solved with Cplex 12.9. In particular the algorithm by Ávila et al. (2015) was used to solve the time-invariant routing problem at the second step of the proposed lower bounding procedure.
We consider the instances that Calogiuri et al. (2019) generated in two phases. First, a complete graph $G=(V, A, \tau)$ was generated where the $|V|$ locations are randomly generated as reported in Cordeau et al. (2014). For each possible value of $|V|$ in the set $\{20,40,60\}$. The set of arcs A was determined by selecting $|V|+\alpha \times|V|$ arcs from $A$ as follows. The set A was first initialized with the $|V|$ arcs of an Hamiltonian tour on $G$. Then a number of arcs $\alpha \times|V|$ are added, where $\alpha \in\{0.2 ; 0.6 ; 1\}$. The required arcs set $R$ was generated by randomly selecting a percentage $\beta$ of arcs in $A$, with $\beta \in\{0.3 ; 0.5 ; 0.7\}$. By combining 30 instances for each value of $|V|$, three values of $|V|$, three values of $\alpha$ and three values of $\beta$, Calogiuri et al. (2019) obtained 810 different combinations of $(G, R)$. For each combination $(G, R)$, they generated 3 different traffic patterns, characterized by three different values of traffic congestion, i.e. $\Delta=\{0.7,0.8,0.9\}$. Results are reported in Tables 34 and 5 with the same column headings of TDTSP results. Computational results show that our algorithm is capable of solving 1772 instances out of 2430 instances while the procedure by Calogiuri et al. (2019) solved only 1618 problems. The improvement is remarkable for computational time, which was on average equal to 98 seconds for our procedure and 189 seconds for Calogiuri et al. (2019). This can be explained by the quality of the new lower bound as well as by the low computational effort spent to solve the linear program $(23)-(30)$, which was formulated by including in the set $\mathcal{B}$ only 75 time instants. Indeed, the new lower bound reduces the overall number of visited branch-and-bound nodes: on average our procedure processes about 4394 nodes (i.e. RPP instances) less than Calogiuri et al. (2019).

## 8 Conclusion

This paper has introduced a key property of time-dependent graphs that we have called "path ranking invariance". We have shown that this property can be exploited in order to solve a large class of time-dependent routing problems, including the Time-Dependent Travelling Salesman Problem and the Time-Dependent Rural Postman Problem. We have also proved that the path ranking invariance can be checked by solving a decision problem, named Constant Traversal Cost Problem. When such check fails, we have defined a new family of parameterized lower and upper bounds, whose parameters can be chosen by solving a linear program. Computational results show that, used in a branch-and-bound algorithm, this mechanism outperforms state-of-the-arts optimal algorithms. Future work will focus on adapting the new ideas to other time-dependent routing problems.

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Table 3: Pattern B. $\beta=0.3$

Table 4: Pattern B. $\beta=0.5$

Table 5: Pattern B. $\beta=0.7$


