From Data to Probability Densities without Histograms Bernd A. Berg^{a,b} and Robert C. Harris^b

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Abstract

When one deals with data drawn from continuous variables, a histogram is often inadequate to display their probability density. It deals inefficiently with statistical noise, and binsizes are free parameters. In contrast to that, the empirical cumulative distribution function (obtained after sorting the data) is parameter free. But it is a step function, so that its differentiation does not give a smooth probability density. Based on Fourier series expansion and Kolmogorov tests, we introduce a simple method, which overcomes this problem. Error bars on the estimated probability density are calculated using a jackknife method. We give several examples and provide computer code reproducing them. You may want to look at the corresponding figures 4 to 9 first.

Key words: Display of data, Probability densities, Histograms, Continuous variables, Cumulative distribution functions.

1. Introduction

One is often confronted with displaying an empirical probability density (PD) f(x) of a continuous variable x. Most commonly this is done using histograms. While that is entirely appropriate and parameter free when x is confined to discrete values, in case of a continuous variable x, one is faced with choosing a binsize, or even several binsizes. This is a frustrated problem: On one side one would like the binsize infinitesimally small, so that the resolution of the curve becomes perfect. On the other side the binsize has to be sufficiently big so that statistical fluctuations do not destroy the smoothness of the underlying curve. Here we do not attempt to fine-tune the binsize parameter(s), but propose to by-pass the entire problem by using a method based on the cumulative distribution function (CDF)

$$F(x) = \int_{-\infty}^{x} dx' f(x') . \qquad (1)$$

F(x) is a monotonically increasing function as f(x) cannot be negative. In a range with f(x) > 0, F(x) is strictly monotone.

Given a time series of n real numbers (data), a parameter free estimate of the CDF, called empirical CDF (ECDF), is well-known [1,2]: The step function $\overline{F}(x)$ which, after sorting the data, increases by 1/n at each data point. Unfortunately, that does not help directly in getting a good estimate of the probability density. The derivative of a step function is a sum of Dirac delta functions, whereas the probability density is often known to be a smooth function, as will be assumed in the

Preprint submitted to Computer Physics Communications

forthcoming. So it appears that one needs some kind of interpolation of the CDF before taking the derivative. This is no fun, as one has to decide whether the interpolation of 2, 3, 4, or k points will work best. In contrast, plotting a histogram of the data is simple and robust. However, it is tedious to guess a smooth function from a histogram. Typically, either the statistical errors on the bins are small, but the resolution of the function is bad, or the resolution is good, but the statistical errors are large.

Let $F_0(x)$ be the straight line, which is zero for $x \leq x_{\pi_1}$ and one for $x \geq x_{\pi_n}$, where x_{π_1} is the smallest and x_{π_n} the largest data point. After subtracting $\overline{F}_0(x)$ from $\overline{F}(x)$, a function is left over, which is zero for $x \leq x_{\pi_0}$ and $x \geq x_{\pi_n}$, and inbetween well-suited for a Fourier series expansion. This leads to the desired smooth approximation as long as the expansion is sufficiently short, but will imitate every wiggle of the data, when carried too far. Therefore, one needs a cut-off criterion. We provide this by using the Kolmogorov [3,1,2]test, which tells us whether the difference between the ECDF and an analytical approximation of the CDF is explained by chance. The result is a welldefined, smooth empirical estimate of the PD. Our approach is so simple and straightforward, that one can hardly imagine that it is original, but we have not seen it in use before. Whether there exists previous literature on it or not, it is certainly desirable to bring it to the attention of a general science community. With this paper we also distribute startup software, which should help to bridge initial barriers against applying the approach.

In the next section we review the CDF, its empirical estimate, and other preliminaries. Section 3 explains our construction of PDs and gives numerical examples. We consider independent events sampled from (1) a Gaussian distributions, (2) a Cauchy distribution, and data from (3) a Markov chain Monte Carlo simulation, which creates an autocorrelated time series. Summary and conclusions follow in section 4. The appendix gives a listing of the main routine and explains Web access to Fortran 77 code, which reproduces the examples of this paper.



Fig. 1. ECDF from 100 Gaussian random numbers versus exact Gaussian CDF.

2. Cumulative Distribution Functions

Assume we generate n random numbers x_1, \dots, x_n according to a distribution function F(x). We may re-arrange the x_i in increasing order. Denoting the smallest value by x_{π_1} , the next smallest by x_{π_2} , etc., we arrive at

$$x_{\pi_1} \le x_{\pi_2} \le \ldots \le x_{\pi_n} \tag{2}$$

where π_1, \ldots, π_n is a permutation of $1, \ldots, n$. As long as the data are not yet created, the x_{π_i} are random variables, afterwards they are data. An estimator for the distribution function F(x) is the ECDF

$$\overline{F}(x) = \frac{i}{n} \quad \text{for} \quad x_{\pi_i} \le x < x_{\pi_{i+1}}, \tag{3}$$

with $i = 0, 1, \ldots, n-1, n$ and the definitions $x_{\pi_0} = -\infty$, $x_{\pi_{n+1}} = +\infty$. Confidence limits can be obtained from the bimodal distribution: With x_q defined by $F(x_q) = q, 0 \le q \le 1$, the probability to find k data points with values $x_{\pi_1} \le \ldots \le x_{\pi_k} < x_q$ and n - k data points with $x_q < x_{\pi_{k+1}} \le \ldots \le x_{\pi_n}$ is given by

$$b_n(k,q) = \left(\frac{n}{k}\right) q^k (1-q)^{n-k} .$$
 (4)

Figure 1 shows an ECDF from 100 Gaussian distributed random numbers together with the exact CDF. Using Marsaglia [4] random numbers, the data were created for the probability density



Fig. 2. Peaked ECDF from 100 Gaussian random numbers versus exact Gaussian peaked CDF.

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{5}$$

and sorted with the heapsort algorithm. The CDF is in this case determined by the error function:

$$G(x) = \int_{-\infty}^{x} dx' g(x') = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) .$$
 (6)

The computer code used was that of Ref. [2].

In contrast to histograms as estimators of a PD from data, the ECDF has the advantage that no free parameters are involved in its definition, but the probability density of events is encoded in its slope. This makes it often impossible to read off from graphs like Fig. 1 high probability regions, in particular, the point of maximum likelihood. An ECDF is well-suited for determining confidence intervals, however, and that can be further improved by switching from the CDF to the peaked CDF [2] defined by

$$F_p(x) = \begin{cases} F(x) \text{ for } F(x) \le \frac{1}{2}; \\ 1 - F(x) \text{ for } F(x) > \frac{1}{2}. \end{cases}$$
(7)

By construction the maximum of the peaked CDF $F_p(x)$ is at the median $x_{\frac{1}{2}}$ and $F_p(x_{1/2}) = 1/2$. Therefore, $F_p(x)$ has two advantages over the CDF F(x): The median is clearly exhibited and the accuracy of the ordinate is doubled. It looks a bit like a PD, but it is in essence still the integrated PD.

The peaked ECDF $\overline{F}_p(x)$ is defined in the same way, just replacing F(x) in Eq. (7) by $\overline{F}(x)$. Fig-



Fig. 3. Peaked ECDF from 10 000 Gaussian random numbers versus exact Gaussian peaked CDF. The arrows indicate 70% and 95% confidence intervals.

ure 2 displays the peaked ECDF and peaked CDF for the same data as used in Fig. 1. While in these two figures deviations of the estimates from the exact function due to statistical deviations are clearly visible, the picture changes dramatically when one increases the number of data by a factor of 100. Figure 3 shows the peaked ECDF from 10 000 Gaussian random numbers together with the exact peaked CDF. A difference is no longer visible to the naked eye. Note that for large n use of a fast sorting algorithm is mandatory. For a heapsort the CPU time scales with $n \log_2(n)$. For details see, e.g., Ref. [2].

Estimation of confidence intervals is also illustrated in Fig. 3: Just pick the desired likelihood on the ordinate and follow the arrows to their termination points. Call such a point x_y , then the probability for $x < x_y$ is given by the value on the ordinate for arrows emerging from the left, and the probability $x > x_y$ for arrows emerging from the right. Using one arrow from the left, and one arrow from the right, the probability content for the interval between these x values is obtained by subtracting the appropriate numbers from one (70% for the x values from the two inner and 95% for the x values from the two outer arrows of Fig. 3).

Do the empirical and exact CDFs of our figures agree? The Kolmogorov test [3] answers this question. It returns the probability Q, that the difference between the analytical CDF and an ECDF from statistically independent data is due to chance. If the analytical CDF is exactly known and the data are indeed sampled from this distribution, Q is a uniformly distributed random variable in the range 0 < Q < 1. Turned around, if one is not sure about the exact CDF, or the data, or both, and Q is small (say, $Q < 10^{-6}$) one concludes that the difference between the proposed CDF and the data is not due to chance, but that one of the assumptions made is false.

Kolmogorov's ingenious test relies on no more than the maximum difference \triangle between the ECDF and the CDF. For the one-sided tests the relationship to Q is analytically known [5,2] for any value of n. In practice one prefers the twosided test for which \triangle is defined by

$$\Delta = \max_{x} \left| F(x) - \overline{F}(x) \right| . \tag{8}$$

An exact analytical conversion into Q is then not known, but an asymptotic expansion due to Stephens [6] gives satisfactory results for $n \ge 4$, which is for practically all applications sufficient. The test yields Q = 0.19 for the data of Fig. 1 and 2, and Q = 0.78 for the data of Fig. 3. Both values are consistent with the assumption that the difference between the data and the exact CDF is due to chance.

3. Probability Densities

We would like to construct an empirical probability density (EPD) from an ECDF given by Eq. (3). The first idea, that comes to mind, is to differentiate smooth interpolations of $\overline{F}(x)$. This turns out to be tedious as long as one is unable to find a simple, generic rule, which determines the optimal interpolation for the purpose at hand, and we do not pursue this line.

The method we propose consists of two well-defined steps.

- (i) Define as an initial approximation to $\overline{F}(x)$ a differentiable, monotonically increasing function $F_0(x)$, with details as specified below.
- (ii) Fourier expand the remainder until the Kolmogorov test yields $Q \ge Q_{\text{cut}} = 1/2$ (there may be some flexibility in lowering Q_{cut}).

For $F_0(x)$ we require

$$F_0(x) = 0 \quad \text{for} \quad x \le a \,, \tag{9}$$

$$F_0(x) = 1 \quad \text{for } x \ge b \,, \tag{10}$$

where the interval [a, b] has to lie within the range of the data:

$$x_{\pi_1} \le a < b \le x_{\pi_n} \tag{11}$$

For PDs with support on a compact interval, or with fast fall-off like for a Gaussian distribution, the natural choice is $a = x_{\pi_1}$ and $b = x_{\pi_n}$. In case of slow fall-off, like for a Cauchy distributions, or other distributions with outliers, one has to restrict the analysis to [a, b] regions, which are sufficiently populated by data. This can be interpreted as considering instead of f(x) the PD

$$f_{ab} = \begin{cases} c f(x) & \text{for } a \le x \le b; \\ 0 & \text{otherwise.} \end{cases}$$
(12)

Here the constant c is defined by the normalization $\int dx f_{ab}(x) = 1$ and empirically obtained by the left-out probability content of $\overline{F}(x)$, i.e., $c = n/n_{ab}$, where n_{ab} is the number of data in [a, b] and n the total number of data. We denote the CDF of $f_{ab}(x)$ by $F_{ab}(x)$ and its ECDF by $\overline{F}_{ab}(x)$. After calculating from $\overline{F}_{ab}(x)$ an estimate $\overline{f}_{ab}(x)$ of $f_{ab}(x)$, the estimate of f(x) is for $x \in [a, b]$ given by $\overline{f}(x) = c^{-1}\overline{f}_{ab}(x)$.

For $x \in [a, b]$ we restrict our choice of $F_0(x)$ in this paper to the straight line,

$$F_0(x) = \frac{x-a}{b-a} \quad \text{for} \quad a \le x \le b \,. \tag{13}$$

More elaborate definitions will likely give improvements in a number of situations (we comment on that in the conclusions), but would at the present state just discourage applications. Our point here is to show that good results are already obtained with the definition (13).

Once $F_0(x)$ is defined, the remainder of the ECDF is given by

$$R(x) = \overline{F}_{ab}(x) - F_0(x), \qquad (14)$$

which we expand into the Fourier series

$$R(x) = \sum_{i=1}^{m} d(i) \sin\left(\frac{i\pi(x-a)}{b-a}\right) .$$
 (15)

The cosine terms are not present due to the boundary conditions R(a) = R(b) = 0. The Fourier coefficients follow from

$$d(i) = \sqrt{\frac{2}{b-a}} \int_{a}^{b} dx R(x) \sin\left(\frac{i\pi(x-a)}{b-a}\right) \quad (16)$$

Because in our case R(x) is the difference of a step function and a linear function, the integrals over the flat regions of the step function are easily calculated, and the integrals (16) are solved by adding them up.

The Fourier expansion (14) is useless for a too large value of m, because it will then reproduce all statistical fluctuations of the data. To get around this problem, we perform the two-sided Kolmogorov test first between $\overline{F}_{ab}(x)$ and $F_0(x)$ (m = 0), and then each time m is incremented by $m \to m + 1$. Once $Q \ge Q_{cut} = 1/2$ is reached for the Kolmogorov Q, we know that the difference between the data and our analytical approximation is explained by statistical fluctuations. The other way round, the only information left in the data is statistical noise. Therefore, the expansion is terminated at that point. The thus obtained smooth estimate of the CDF,

$$F_{\text{estimate}}(x) = F_0(x) + R(x), \qquad (17)$$

yields $\overline{f}_{ab}(x)$ by differentiation, and our final estimate of the desired PD is: $\overline{f}(x) = n_{ab}\overline{f}_{ab}(x)/n$.

One likes to attach error bars to the estimate of the PD. We do this by dividing the (unsorted) original data into jackknife [7,8] bins and repeat the analysis for each bin. Comparing the function values thus obtained at selected points, error bars follow in the usual jackknife way (see [2] for technical details).

3.1. Gaussian distribution

Figure 4 shows a histogram of 51 bins for 2000 random numbers generated according to the Gaussian distribution (the error bars follow follow from the variance p(1-p) of the bimodal distribution (4) with p = h(i)/n).

Figure 5 shows the estimate $\overline{g}(x)$ obtained from the same data with the method described in this



Fig. 4. Histogram from 2000 Gaussian random numbers.



Fig. 5. Estimate $\overline{g}(x)$ of the Gaussian PD from the same data as used in Fig. 4.

section. We used $a = x_{\pi_1}$ and $b = x_{\pi_n}$. For the estimate from all 2 000 data, Q = 0.97 was reached with m = 4 (Q = 0.056 with m = 3). Twenty jackknife bins were used to calculate the error bars. As all results of this section, the analysis is fully reproducible with the programs provided on the Web as described in the appendix.

3.2. Cauchy distribution

We consider the Cauchy distribution defined by the PD

$$f_c(x) = \frac{1}{\pi (1+x^2)}$$
(18)

which leads to the CDF



Fig. 6. Histogram from 20\,000 Cauchy distributed random numbers.



Fig. 7. Estimate $\overline{f}_c(x)$ of the Cauchy PD from the same data as used in Fig. 6.

$$F_c(x) = \int_{-\infty}^x f_c(x') \, dx' = \frac{1}{2} + \frac{1}{\pi} \, \tan^{-1}(x) \, . \quad (19)$$

Due to the slow $\sim x^{-2}$ fall-off of the Cauchy PD, neither its mean nor its variance are defined.

Therefore, it comes to no surprise that we encounter considerably more difficulties in applying our method for Cauchy data than for Gaussian data or for the data of our next example. To overcome instabilities, the analysis has to be restricted to the central region and far more data than before are needed for stable estimates. We ended up with using 20 000 Cauchy distributed random number and restricting the analysis to

$$a = x_{\pi_{3001}}$$
 and $b = x_{\pi_{17000}}$,

which came out to be a = -1.984 and b = 1.916. Figure 6 depicts the 51-bins histogram of the data in this region. The estimate of the PD $\overline{f}_c(x)$ with our method is then shown in Fig. 7. As before, twenty jackknife bins were used for the error estimates and for the estimate from all data Q = 0.77was obtained for m = 2 ($Q < 10^{-14}$ for m = 1).

3.3. Double peak from Markov Chain Monte Carlo

Markov chain Monte Carlo (MCMC) simulations are widely employed in physics and other disciplines. Data in a MCMC time series are autocorrelated, which makes straightforward application of the Kolmogorov test questionable. Here we illustrate for an example from lattice gauge theory (LGT) a way to deal with this problem.

In 4D U(1) LGT, a double peak in the action has been observed on symmetric lattices [9]. This is characteristic for a first order phase transition, although the situation in 4D U(1) LGT is somewhat questionable due to the weakness of the transition and other circumstances.

For this paper we have used the biased Metropolis-Heatbath algorithm of Ref. [10] to generate U(1) data at $\beta = 1.007$ on an 8⁴ lattice. These are parameters appropriate for producing a double peak. In LGT and statistical physics a standard unit for the time series is one "sweep", which corresponds for sequential updating (as used in our simulations) to updating each dynamical variable once. We have generated a time series of 2 560 000 sweeps, which takes about twenty hours on a 2GHz PC.

We calculated the integrated autocorrelation time τ_{int} of our time series and found $\tau_{\text{int}} \approx 1280$ sweeps. As the effective number of statistically independent data in an autocorrelated time series is given by [2] n/τ_{int} , our measurement of τ_{int} implies that we have generated approximately $2560\ 000/1\ 200 = 2\ 000$ independent measurements. In the spirit of data reduction, we then selected 2\ 000\ action values, separated by steps of 1\ 200\ sweeps, from the time series.

For these 2000 measurements, the histogram with 51 entries is plotted in Fig. 8, where s is the action density. Figure 9 shows the estimate $\overline{f}(s)$



Fig. 8. Histogram from $2\,000$ effectively independent action measurements in U(1) lattice gauge theory.



Fig. 9. Estimate $\overline{f}(s)$ of the PD from the same data as used in Fig. 8.

obtained from the same data with our method, using $a = x_{\pi_1}$ and $b = x_{\pi_n}$. For the estimate from all 2 000 data Q = 0.93 was reached with m = 5(Q = 0.026 with m = 4), and twenty jackknife bins were used to calculate the error bars. The improvement from Fig. 8 to Fig. 9 is as good as the corresponding improvement for the Gaussian distribution.

This result sheds new light on an effect, which is well-known to workers in MCMC simulations, but has to our knowledge not been satisfactorily explained, the amazing smoothing, which one obtains when one includes all autocorrelated data in the histogram. In our case this is 640 000 data points, as we have taken measurements every 4 sweeps. The histogram error bars then fall almost on top of the jackknife error bars of Fig. 9. How can that be when there is little or no additional information in the autocorrelated data? Part of the answer appears to be that most of this information is already in the statistically independent data, but is usually not exploited.

4. Summary and Conclusions

To someone used to plotting histogram, our method may appear difficult, but indeed it is not. True, first one has to sort the data to calculate their ECDF, then from it the empirical PD, using Fourier series expansion and Kolmogorov tests, and finally jackknife error bars need to be calculated, but all these steps are standard. Besides the subroutine shown in the appendix, another short subroutine, and the main program, we had no programming to do. All other routines were already available in the Web based Fortran 77 code of Ref. [2]. Once the archive described in the appendix is downloaded from the Web, and the program is up and running, the extra effort compared to plotting histograms is almost negligible. And the differences between Fig. 4 and Fig. 5, Fig. 6 and Fig. 7, and finally Fig. 8 and Fig. 9 speak for themselves.

It is astonishing that these results could be obtained with a simple straight line (13) as initial approximation for the CDF. There is certainly space for improvement at the price of giving up the presently achieved simplicity. For our examples one and two, the exact CDFs are known and pointed out by Eqs. (6) and (19). Obviously, one does not test our method by constructing $F_0(x)$ from them in these examples, but they could be good choices, when one has reasons to expect them to be a close, albeit not exact, approximation of the investigated distribution. For instance, for the double peak of our third example, a good initial guess could be a sum of two Gaussians, and one could start off by fitting their CDF to the data. Afterwards care has to be taken that Eqs. (9)and (10) remain valid. That this can always be achieved follows from the interpretation (12) of that requirement. We abstained from investigating a double Gaussian $F_0(x)$, because it starts to get tedious and there is nothing really to improve on our result. But, one can easily imagine that there are more complicated situations, accompanied by limited statistics, where an improvement of $F_0(x)$ becomes essential. If the problem for which this happens is important too, it will become worthwhile to explore more $F_0(x)$ functions.

With our $Q_{\text{cut}} = 1/2$ rule, we are slightly overexpanding the Fourier transformation (15). In the average Q should be 1/2, but all our final values are $\geq 1/2$. That gives some flexibility to lower Q_{cut} , which should be used with discretion in situations were the m of the Fourier expansion (15) appears to be too large. A warning right away: The only situation in which we did not see a rapid approach towards $Q > Q_{\text{cut}}$ turned out to be one in which we had not noticed that the underlying distribution was discrete, while the Kolmogorov test did notice that.

We did not develop a statistically rigorous approach. We address physicists and others, who do not hesitate to use whatever works, not those, who want to forbid numerical recipes. We rely on the assumption that the Fourier series (15) would be rapidly convergent, when the ECDF in Eq. (14) would be replaced by the corresponding (unknown) exact CDF. That is in spirit similar to picking a primer in Bayesian statistics, when a rigorous one is not known, a procedure, which can modify the probability content of confidence intervals.

Acknowledgments: BB likes to thank Alexei Bazavov for useful discussions. This work was in part supported by the U.S. Department of Energy under contract DE-FG02-97ER41022.

Appendix A. Computer Code

An archive with Fortran 77 example runs can presently be downloaded from the website of BB. Google Bernd Berg, or go directly to

http://people/scs.fsu.edu/~berg/ .

Take from there the research link and download the gzipped archive

STMC_CDFtoPD.tgz.

Unfold the archive. (If instructions for that are needed, they can be found on the website of Ref. [2], which is also linked on the main website of BB.) Go then to the folder

STMC_CDFtoPD/Work/CDFtoPDexamples.

All three examples of this paper can be run as special cases of the program cdf_to_pd.f in this folder, and more instructions are given in its readme.txt file. To show that at its heart our method is indeed quite simple, we list in the following our main subroutine.

SUBROUTINE CDF_PD(IUO,NDAT,SDAT,Fxct) C Bernd Berg, Robert Harris Dec 16 2007 C Transforms an empirical cumulative distribution function (CDF) into C a corresponding probability density using and initial function plus C Fourier series expansion for the CDF C On INPUT: C IUO C NDAT Write unit, unchanged on exit. Number of input data, unchanged on exit. C SDAT Sorted input data, unchanged on exit C INTERNAL. Cut-off. Fourier expansion is terminated for Q>Qcut. C Qcut cumulative distribution function. C NMAX Maximum number of terms in the Fourier series. C On OUTPUT: C Fxct Exact CDF (means here analytical approximation of the CDF). include '../../Libs/Fortran/implicit.sta include '../../Libs/Fortran/constants.par PARAMETER(Qcut=HALF,NMAX=100) ! Change also in functions DIMENSION SDAT(NDAT).Fxct(NDAT) COMMON /CDFProb/ XMIN, XRANGE, DN(NMAX), M ! Expansion parameters. С XMIN=SDAT(1) ! Initializations. XRANGE=SDAT(NDAT)-SDAT(1) DO J=1.NDAT Fxct(J)=(SDAT(J)-SDAT(1))/XRANGE END DO DO K=1.NMAX DN(K)=ZERO ENDDO DO M=1.NMAX ! Integration for the Fourier series coefficients: DO I=1,NDAT-1 X1 = (SDAT(T) - SDAT(1)) / XRANGEX2=(SDAT(I+1)-SDAT(1))/XRANGE DN(M)=DN(M)+(ONE*I/NDAT-X1)*COS(M*PI*X1)/(M*PI) DN(M)=DN(M)-(ONE*I/NDAT-X2)*COS(M*PI*X2)/(M*PI) DN(M)=DN(M)+SIN(M*PI*X1)/(M*M*PI*PI) DN(M)=DN(M)-SIN(M*PI*X2)/(M*M*PI*PI) ENDDO DN(M)=DN(M)*TWO DO K=1,NDAT ! CDF in Fourier series approximations: XRANGE1=SDAT(K)-SDAT(1) Fxct(K)=Fxct(K)+DN(M)*SIN(M*PI/XRANGE*XRANGE1) ENDDO CALL KOLM2_AS(NDAT,Fxct,DEL,Q) ! Kolmogorov test. IF(Q.GT.Qcut) GOTO 1 ENDDO WRITE(IUO,'(/," CDF_PD failed M,Q =",I6,G12.3)') M,Q STOP "CDF_PD: Expansion failed."
WRITE(IU0,'(/," CDF_PD: Final M,Q =",I6,G12.3,/)') M,Q 1 C RETURN

```
END
```

With exception of the jackknife routine

dat_to_datj.f,

to be found in Libs/Fortran of the package, all other subroutines used are from the code of Ref. [2] and for convenience included here. For the Kolmogorov test in the form of Stephens the routine kolm2_as.f of [2] has been modified to abort in case of no convergence. (Note also that the related routine kolm2_as2.f of [2], which is not used in this paper, does not work for large values of the input arguments N_1 , N_2 due to bad coding of a multiplication of N_1 and N_2 , which can be easily corrected.) All routines are copyrighted by their authors, who are listed in one of the first lines of each routine. Limited permission of their use is given under the conditions stated on the Fortran download page of Ref. [2].

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