

# CONSTRUCTING A SPACE FROM THE SYSTEM OF GEODESIC EQUATIONS

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## Abstract

Given a space it is easy to obtain the system of geodesic equations on it. In this paper the inverse problem of reconstructing the space from the geodesic equations is addressed. A procedure is developed for obtaining the metric tensor from the Christoffel symbols. The procedure is extended for determining if a second order quadratically semi-linear system can be expressed as a system of geodesic equations, provided it has terms only quadratic in the first derivative apart from the second derivative term. A computer code has been developed for dealing with larger systems of geodesic equations.

# 1. Introduction

The development of geometry originates in its applications for map making [1], but even more from its use in kinematics and dynamics [2]. As such, it is of interest to look at the interplay of geometry and dynamics. The path of a test particle in a flat space is a straight line. The curved space generalization of the straight line is a geodesic. Thus test particles in curved spaces move along geodesics. Given a space one easily obtains the system of geodesic equations on it [3]. In principle it should be easy to obtain the space from the geodesic equations. This would be of physical relevance as actual observations would not provide the space but would provide the paths followed by “test particles”. For example, in general relativity, one assumes some matter-energy distribution and then solves the Einstein equations to obtain the metric tensor, using which one obtains the geodesic equations, which give the paths for test particles [4]. However, one does not really know the matter-energy distribution in any actual situation, but only the observed paths of particles. Consequently, it would be of interest to be able to determine the metric directly from the geodesic equations.

Though simple in principle, the problem is complicated by the fact that the Christoffel symbols are non-linear combinations of the metric tensor and its first derivative. As such, a system of highly non-linear first order partial differential equations would have to be solved to obtain the metric tensor. The problem can be reduced enormously in complexity by contracting the Christoffel symbols with the metric tensor, to obtain a system of first order linear partial differential equations. In general, even this is very complicated to solve. Further, we would need to check compatibility of the solutions obtained.

The procedure adopted here uses the skew symmetry of the covariant form of the Riemann tensor in the first two indices, and the symmetry under interchange of the first and second pair of indices to provide a system of linear equations that can be solved simultaneously. If the system does not decouple we finally have to solve  $n$  partial first order differential equations for one function of  $n$  variables. Consequently an arbitrary constant appears in the solution. If it decouples we need to solve correspondingly more differential equations and hence more arbitrary constants appear. These constants generally get determined by inserting the solutions back into the equations for the metric. In principle it is possible that they may not be fully evaluated and lead to a class of metrics. In this paper a specific prescription is provided to reconstruct the space from the geodesic equations. The metric constructed is unique (up to some multiplicative constants appearing in the solution). For two variables the procedure will be explained in detail in the next section. However, for larger systems the procedure is still too complicated to be implemented by hand. In section 3 we have provided a general discussion of the general case and the logic of the computer code to obtain the metric from the Christoffel symbols. Further, if we only have a system of second order quadratically semi-linear ordinary differential equations (ODEs) given, that have only the quadratic term, we would not know whether they could, consistently, be regarded as a system of geodesic equations. In section 4 we give a brief discussion of how the code can check whether the system can, or cannot, be regarded as a system of geodesic equations. In section 5 there are some specific examples

given to illustrate the use of the general procedure. In the last section we have given a brief summary and discussion of the results.

## 2. A System of Two Equations for Two Variables

The essential principle for obtaining the metric from the Christoffel symbols may be seen directly by considering a system of two geodesic equations for two functions of one variable. However, the general procedure involves additional complications that will be discussed later. For the system of two equations

$$x'' = a(x, y)x'^2 + 2b(x, y)x'y' + c(x, y)y'^2, \quad (1)$$

$$y'' = d(x, y)x'^2 + 2e(x, y)x'y' + f(x, y)y'^2, \quad (2)$$

we can read off the Christoffel symbols as the the negative of the coefficients of the quadratic terms. Thus

$$\Gamma_{11}^1 = -a, \Gamma_{12}^1 = -b, \Gamma_{22}^1 = -c, \Gamma_{11}^2 = -d, \Gamma_{12}^2 = -e, \Gamma_{22}^2 = -f. \quad (3)$$

Note that for a general second-order quadratically semi-linear system of ODEs, the coefficients cannot be assumed to be expressible as Christoffel symbols. However, if we are given the system of equations as geodesic equations, we can assume that the coefficients are so expressible. For a known metric tensor the Christoffel symbols are then given by

$$\Gamma_{jk}^i = \frac{1}{2}g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}). \quad (4)$$

Now construct the Riemann tensor from these Christoffel symbols

$$R_{jkl}^i = \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{ml}^i \Gamma_{jk}^m - \Gamma_{mk}^i \Gamma_{jl}^m, \quad (5)$$

where the Einstein summation convention, that repeated indices are summed over, has been used. Note that the tensor is skew in the last two indices,  $k$  and  $l$ . As such, when they are equal the tensor is trivially zero. Thus, without loss of generality, we can set  $k = 1, l = 2$ . Putting the tensor into fully covariant form it is skew in its first two indices as well. Using the metric tensor to lower the index of the curvature tensor, we obtain the two linear relations for the metric coefficients:

$$g_{11}R_{112}^1 + g_{12}R_{112}^2 = 0, \quad (6)$$

$$g_{12}R_{212}^1 + g_{22}R_{212}^2 = 0. \quad (7)$$

There are various possibilities for the Riemann tensor components being zero or non-zero. Not all possibilities are consistently allowed. Apart from the case of a flat space,  $R_{jkl}^i = 0$ ,

only two possibilities survive: (a) when  $i = j$ ,  $R^i_{jkl} = 0$ ; (b) when  $i = j$ ,  $R^i_{jkl} \neq 0$ . In case (b) these equations can be used to write  $g_{11}$  and  $g_{22}$  in terms of  $g_{12}$

$$g_{11} = -\frac{R^2_{112}}{R^1_{112}}g_{12} := Ag_{12}, \quad (8)$$

$$g_{22} = -\frac{R^2_{212}}{R^1_{212}}g_{12} := Bg_{12}. \quad (9)$$

To make the procedure easier to see, write  $g_{11} = p(x, y)$ ,  $g_{12} = q(x, y)$  and  $g_{22} = r(x, y)$ . In case (b) using eqs.(7) and (8) we get  $p$  and  $r$  in terms of  $q$ . Then, writing the Christoffel symbols explicitly we obtain the differential equation for  $q$

$$q_x = (Ab + a + Bd + e)q, q_y = (Ac + b + Be + f)q. \quad (10)$$

The solution for  $q$  is provided by integrating eq.(9) relative to  $x$  and  $y$  and comparing the arbitrary functions of integration

$$q(x, y) = \alpha(y)\exp\left(\int(Ab + a + Bd + e)dx\right) = \beta(x)\exp\left(\int(Ac + b + Be + f)dy\right). \quad (11)$$

There would appear to be an arbitrary constant still left. This disappears on using the resulting  $p, q, r$  in the expression for the Christoffel symbols. (Remember that the inverse metric contains the functions as well as their first derivatives.)

In case (a)  $q = 0$ . We now get two sets of two partial differential equations for  $p$  and  $r$ , which can be solved to give

$$p(x, y) = \gamma(y)\exp\left(\int 2a(x, y)dx\right) = \delta(x)\exp\left(\int 2c(x, y)dy\right); \quad (12)$$

$$r(x, y) = \mu(y)\exp\left(\int 2d(x, y)dx\right) = \nu(x)\exp\left(\int 2f(x, y)dy\right). \quad (13)$$

There now appear to be two arbitrary constants appearing which are determined by inserting the expressions back into the Christoffel symbols. If the constant(s) remain, we obtain a class of metrics for the same geodesics.

Note the remarkable fact that the algebraic symmetry properties of the ‘geometric’ intrinsic curvature tensor, from the ‘ODE’ point of view, are just the compatibility criteria for being able to obtain the metric from the system of geodesic equations.

Finally, there remains the case that the space is flat. The metric tensor can certainly be set as the flat space metric tensor in Cartesian coordinates and hence the metric is “reconstructed”. However, this would not be the metric tensor in the coordinates used. We could now solve the full system of six linear first order partial differential equations for the three functions  $p, q, r$  of two variables  $x, y$ . The compatibility is now guaranteed. There are other, neater, methods available as well [5].

### 3. The General Procedure

Whereas, in principle there is nothing new when we have more than two variables, the problem arises because there are many more possibilities now. To see how the equations proliferate, consider the three variable case. We now have 18 Christoffel symbols for a system of three geodesic equations. These lead to six independent components of  $R_{ijkl}$ . There are now three sets of constraining equations each of which has three possible index choices. As such, we have 9 linearly dependent equations for the 6 metric coefficients. If all the components are non-zero, we have enough equations to be able to obtain the metric coefficients from three differential equations. In fact if we have five distinct components non-zero, we could solve the system. However, we have many possibilities between this case and the flat metric. For 4 variables there are 10 independent metric coefficients, 20 linearly independent components of  $R_{ijkl}$  and 40 Christoffel symbols for a system of four geodesic equations. This appears to be a very heavily over-determined system and compatibility checks would become really long.

The proliferation of equations would have rapidly rendered the problem intractable were it not for the availability of computer codes to solve such systems, for many more variables, going through *all* possibilities. We have constructed such a computer code that enables us to solve the problem in full generality. It is given at: [www.cam.wits.ac.za/inverse](http://www.cam.wits.ac.za/inverse).

The logic of the code is as follows. We first differentiate the Christoffel symbols  $\Gamma_{jk}^i$  relative to all the dependent variables and combine them to form the curvature tensor of *valence* [1, 3], i.e. with one upper and three lower indices,  $R_{jkl}^i$ . Next we use the symmetry properties of the covariant form of the Riemann tensor, namely

$$g_{im}R_{jkl}^m = -g_{jm}R_{ikl}^m, \quad (14)$$

and

$$g_{im}R_{jkl}^m = g_{km}R_{lij}^m. \quad (15)$$

Since (14) is skew in  $i, j$ , there are  $n^3(n-1)/2$  linearly independent equations. Further, (15) are  $n^4$  equations. There are only  $n(n+1)/2$  independent components of  $g_{ij}$ . As such, the system must be grossly over-determined. However, if the  $\Gamma_{jk}^i$  are, indeed, Christoffel symbols, they must be consistent. As such, one can use the first  $n(n+1)/2 - 1$  of them to obtain all the  $g_{ij}$  in terms of one of them (say  $g_{11}$ ). It is to be noted that since the system is homogeneous, there can be no non-trivial determination for *all* the metric coefficients from here. One now writes the equation for the metric tensor in terms of the given Christoffel symbols as

$$g_{ik,j} + g_{jk,i} - g_{ij,k} = 2g_{il}\Gamma_{jk}^l. \quad (16)$$

With the full set of equations for all  $n^2(n+1)/2$  independent Christoffel symbols we can reduce the equations to a system of  $n$  first order linear partial differential equations for one function (say  $g_{11}$ ) of  $n$  variables. We can now solve these and obtain the full metric tensor.

It may happen that the system of equations has a rank less than  $n(n+1)/2 - 1$ . If it is one less, we will need to solve the system for  $n(n+1)/2 - 2$  components of the metric tensor and then solve partial differential equations for the last *two* metric coefficients. Similarly, if the rank is  $p$  less, we would solve the system for  $n(n+1)/2 - p - 1$  and then solve the remaining partial differential equations for those  $p$  components. Note that there would be no need to re-check compatibility of the solutions, other than to determine the arbitrary constants arising from the solution of the differential equations.

## 4. Consistency Criteria for Systems of Geodesic Equations

So far we have taken it for granted that the system given is for geodesics. It may happen that one obtains a system of equations that look formally like the system of geodesic equations, in that they can be written as

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0, \quad (17)$$

but that they cannot be regarded as a system of geodesic equations. The point is that there is no check that the system of partial differential equations (16) is internally consistent. To check this we would require that (14) and (15) form a consistent set of equations. *This is still not enough!* We also need to check that the first Bianchi identities are satisfied, namely

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0. \quad (18)$$

If these are satisfied we can, indeed, regard the given system of equations as a system of geodesic equations and possibly use the results of theorems on global linearizability of the system to obtain the solution [5].

The computer code we have prepared can be used to obtain the metric tensor if the system is known to be of geodesic equations and can be used to check the consistency of the system as geodesic equations.

## 5. Computation

The algorithm is implemented as follows: Specify the order of the geodesic equation via **n**. Assume  $g_{ij} = g_{ji}$  and  $R^i_{jnn} = 0$  by enforcing the following rules

$$\mathbf{SetAttributes}[\mathbf{g}, \mathbf{Orderless}]; \quad (19)$$

$$\mathbf{R}[\mathbf{i}_-, \mathbf{j}_-, \mathbf{k}_-, \mathbf{l}_-] := \mathbf{0}; \mathbf{i} == \mathbf{j} \&\& \mathbf{k} == \mathbf{l}; \quad (20)$$

in Mathematica. We next introduce the lists **SkewSymmetry** and **Symmetry** which caters for all the possible combinations of **i,j,k** and **l** when summing over the repeated index in (14) and (15). The independent variables are represented by **X** in **eq1[i-,j-,k-,l-]** and **eq2[i-,j-,k-,l-]**. The matrices which are then formed when mapping **eq1[i-,j-,k-,l-]** and **eq2[i-,j-,k-,l-]** to their corresponding lists **Symmetry** and **SkewSymmetry**, are stored in **SYM** and **SkewSYM** respectively. We construct the  $g_{ij}$  metric tensors with **gcomponents**, which is then used in conjunction with **CoefficientArrays** to construct the matrices **ASym** and **ASkewSym**. By choosing  $n(n+1)/2 - 1$  rows from each using the input from **ChooseEqns** these two matrices are used to form the matrix **A** and vector **b**. The vector **sol** then solves the metric tensors in terms of  $g_{11}$  by using **LinearSolve** in conjunction with the matrix **A** and vector **b**. The overdetermined system of linear partial differential equations, **EqnSet16**, are then used to solve for  $g_{11}$  by using **DSolve**. The order of the problem dictates the number of arbitrary functions which will then have to be solved subsequently.

The implementation of the  $n = 3$  case is given in the Appendix.

The code has been checked for the following examples.

- 1. Systems of two equations:** (a) geodesics on a sphere; (b) a linearizable system given in [5]; (c) a non-geodesic system (in [5]).
- 2. Systems of three equations:** (a) flat space; (b) a 3-sphere; (c) linearizable (in [5]); (d) non-geodesic (in [5]).
- 3. Physical four dimensional Lorentzian systems:** (a) the Reissner-Nordström system (geodesics for a point charged mass, in which the charge could be taken to be zero to get the Schwarzschild system); (b) the Kerr system (for a rotating point mass, in which the rotation could be taken to be zero to reduce to the Schwarzschild system).

## 6. Summary and Discussion

We have shown explicitly how to construct the metric from the geodesic equations. In other words, if we knew the geodesics *globally* we could reconstruct the full *manifold* with the metric on it and if we know them locally we can reconstruct the metric and hence the space, locally. It is remarkable that the purely geometric entity measuring curvature should provide, when looked at from the viewpoint of differential equations, the compatibility conditions for the system to be regarded as describing geodesics. The significance of this representation is that it provides a procedure to reduce the system of equations as follows. We use the conjecture of [6] that for  $m$ -dimensional sections of constant curvature it will have an  $so(m+1)$  symmetry algebra. We can use the geometric information to choose the sections of constant curvature to decouple the system of geodesic equations (using the procedure of [6]). These  $m$  geodesic equations would then be completely solved and we would have only a system of  $n - m$  coupled equations to be solved. Notice the heavy utilization of purely geometric considerations.

We solved the problem explicitly for the case of a system of two geodesic equations analytically. The only problem arose in the case of the flat space where we anyhow *know* the metric in Cartesian coordinates. If we want to express the metric in the given coordinates, it may not be so easy. However, the metric coefficients are directly determinable by solving decoupled first order partial differential equations for each of the metric coefficients. There are also more elegant methods available [5].

For larger systems one needs a computer code. Such a code was developed and its logic is given here. The code is available at: [www.cam.wits.ac.za/inverse](http://www.cam.wits.ac.za/inverse). Some examples illustrate the use of the code. The code further provides a check for the given system of equations to be consistently regarded as a system of geodesic equations.

This approach is of importance as it provides a geometric method for solving systems of ODEs [5]. Further developments may be possible by embedding the space in higher dimensional spaces in which they become larger quadratically semi-linear systems. Attempts to convert cubically semi-linear systems to quadratically semi-linear systems in a higher dimension are in progress [7]. This procedure is the inverse of that adopted by Aminova and Aminov [8], in which they use the geodesic re-parametrization symmetry ( $\partial/\partial s$ ) to reduce the system by one dimension. Further, it would be possible to reduce the order of a higher order system by increasing the number of variables. Thus we could possibly use the same techniques for higher order ODEs by embedding in correspondingly higher dimensions. This technique could also be tried to reduce from higher degree equations to two. It would be of great interest if the approach could be extended to PDEs as well.

## Appendix

We implement our code for  $\mathbf{n} = \mathbf{3}$ . The skew symmetry of  $R_{jkl}^i$  implies that without loss of generality, in both (21) and (22) below we have  $(k, l) = \{(1, 2), (1, 3), (2, 3)\}$ , while in (22)  $(i, j) = \{(1, 2), (1, 3), (2, 3)\}$ . The equations (21) and (22) are our skew symmetry and symmetry equations respectively and are generalized as

$$g_{j1}R_{ikl}^1 + g_{i1}R_{jkl}^1 + g_{j2}R_{ikl}^2 + g_{i2}R_{jkl}^2 + g_{j3}R_{ikl}^3 + g_{i3}R_{jkl}^3 = 0, \quad (21)$$

$$g_{i1}R_{jkl}^1 - g_{k1}R_{lij}^1 + g_{i2}R_{jkl}^2 - g_{k2}R_{lij}^2 + g_{i3}R_{jkl}^3 - g_{k3}R_{lij}^3 = 0. \quad (22)$$

From (21) we obtain the nine equations which is generated by `eq2[i_, j_, k_, l_]`

$$g_{11}R_{112}^1 + g_{12}R_{112}^2 + g_{13}R_{112}^3 = 0, \quad (23)$$

$$g_{11}R_{113}^1 + g_{12}R_{113}^2 + g_{13}R_{113}^3 = 0, \quad (24)$$

$$g_{11}R_{123}^1 + g_{12}R_{123}^2 + g_{13}R_{123}^3 = 0, \quad (25)$$

$$g_{12}R_{212}^1 + g_{22}R_{212}^2 + g_{23}R_{212}^3 = 0, \quad (26)$$

$$g_{12}R_{213}^1 + g_{22}R_{213}^2 + g_{23}R_{213}^3 = 0, \quad (27)$$

$$g_{12}R_{223}^1 + g_{22}R_{223}^2 + g_{23}R_{223}^3 = 0, \quad (28)$$

$$g_{13}R_{312}^1 + g_{23}R_{312}^2 + g_{33}R_{312}^3 = 0, \quad (29)$$

$$g_{13}R_{313}^1 + g_{23}R_{313}^2 + g_{33}R_{313}^3 = 0, \quad (30)$$

$$g_{13}R_{323}^1 + g_{23}R_{323}^2 + g_{33}R_{323}^3 = 0. \quad (31)$$

From (22) we obtain the six equations which is generated by **eq1**[**i**-, **j**-, **k**-, **l**-]

$$g_{11} \left( R_{213}^1 - R_{312}^1 \right) + g_{12} \left( R_{213}^2 - R_{312}^2 \right) + g_{13} \left( R_{213}^3 - R_{312}^3 \right) = 0, \quad (32)$$

$$g_{11}R_{223}^1 + g_{12} \left( R_{223}^2 - R_{312}^1 \right) - g_{22}R_{312}^2 + g_{13}R_{223}^3 - g_{23}R_{312}^2 = 0, \quad (33)$$

$$g_{11} \left( R_{312}^1 - R_{213}^1 \right) + g_{12} \left( R_{312}^2 - R_{213}^2 \right) + g_{13} \left( R_{312}^3 - R_{213}^3 \right) = 0, \quad (34)$$

$$g_{11}R_{323}^1 + g_{12} \left( R_{323}^2 - R_{313}^1 \right) - g_{22}R_{313}^2 - g_{23}R_{313}^3 + g_{13}R_{323}^3 = 0, \quad (35)$$

$$g_{11}R_{223}^1 - g_{12} \left( R_{312}^1 - R_{223}^2 \right) - g_{22}R_{312}^2 + g_{13}R_{223}^3 - g_{23}R_{312}^3 = 0, \quad (36)$$

$$-g_{11}R_{323}^1 + g_{12} \left( R_{313}^1 - R_{323}^2 \right) + g_{22}R_{313}^2 + g_{23}R_{313}^3 - g_{13}R_{323}^3 = 0. \quad (37)$$

We choose (26), (27), (31), (32) and (33) to solve for for  $g_{12}$ ,  $g_{13}$ ,  $g_{22}$ ,  $g_{23}$  and  $g_{33}$ . This choice is done by **ChooseEqns** = {**4, 5, 9, 10, 11**} in our code, which is then used to relate these metrics to the  $g_{11}$  metric. Here are the relations

$$\begin{aligned} \Delta &= R_{212}^3 R_{223}^3 \left( R_{213}^2 \right)^2 - \\ &\left( -R_{212}^1 \left( R_{312}^3 \right)^2 + R_{212}^1 R_{213}^3 R_{312}^3 + \left( R_{312}^2 R_{212}^3 + R_{212}^2 R_{213}^3 \right) R_{223}^3 \right. \\ &\quad \left. + R_{223}^2 R_{212}^3 \left( R_{213}^3 - R_{312}^3 \right) \right) R_{213}^2 - R_{213}^1 R_{212}^2 \left( R_{312}^3 \right)^2 + \\ &R_{213}^3 \left( R_{212}^2 R_{223}^2 R_{213}^3 + R_{312}^2 \left( -R_{213}^1 R_{212}^3 + R_{212}^1 R_{213}^3 + R_{212}^2 R_{223}^3 \right) \right) + \\ &\quad R_{312}^1 \left( R_{213}^2 R_{212}^3 - R_{212}^2 R_{213}^3 \right) \left( R_{213}^3 - R_{312}^3 \right) + \\ &\left( R_{213}^1 R_{312}^2 R_{212}^3 + \left( R_{212}^2 \left( R_{213}^1 - R_{223}^2 \right) - R_{212}^1 R_{312}^2 \right) R_{213}^3 \right) R_{312}^3, \end{aligned} \quad (38)$$

$$g_{12} = g_{11} \left( \left( R_{213}^2 R_{212}^3 - R_{212}^2 R_{213}^3 \right) \left( \left( R_{312}^1 - R_{213}^1 \right) R_{223}^3 + R_{223}^1 \left( R_{213}^3 - R_{312}^3 \right) \right) \right) / \Delta, \quad (39)$$

$$\begin{aligned} g_{13} &= g_{11} \left( \left( R_{312}^2 R_{212}^3 - R_{212}^2 R_{312}^3 \right) R_{213}^2 + \right. \\ &\left( - \left( R_{212}^2 R_{223}^2 + R_{212}^1 R_{312}^2 \right) R_{213}^3 + R_{213}^2 \left( R_{223}^2 R_{212}^3 + R_{212}^1 R_{312}^3 \right) + \right. \\ &\quad \left. R_{312}^1 \left( R_{212}^2 \left( R_{213}^3 + R_{312}^3 \right) - \left( R_{213}^2 + R_{312}^2 \right) R_{212}^3 \right) \right) R_{213}^1 + \\ &\quad \left. R_{312}^2 \left( R_{213}^2 R_{212}^3 - R_{212}^2 R_{213}^3 \right) + \right. \end{aligned}$$

$$R_{223} \left( R_{213}^2 - R_{312}^2 \right) \left( R_{212}^2 R_{213}^3 - R_{213}^2 R_{212}^3 \right) + R_{312} \left( \left( R_{212}^2 R_{223}^2 + R_{212} R_{312}^2 \right) R_{213}^3 - R_{213}^2 \left( R_{223}^2 R_{212}^3 + R_{212} R_{312}^3 \right) \right) / \Delta, \quad (40)$$

$$g_{22} = g_{11} \left( R_{213}^1 R_{212}^3 - R_{212}^1 R_{213}^3 \right) \left( \left( R_{213}^1 - R_{312}^1 \right) R_{223}^3 + R_{223}^1 \left( R_{312}^3 - R_{213}^3 \right) \right) / \Delta, \quad (41)$$

$$g_{23} = -g_{11} \left( R_{213}^1 R_{212}^2 - R_{212}^1 R_{213}^2 \right) \left( \left( R_{213}^1 - R_{312}^1 \right) R_{223}^3 + R_{223}^1 \left( R_{312}^3 - R_{213}^3 \right) \right) / \Delta, \quad (42)$$

$$\begin{aligned} g_{33} = & g_{11} \left( \left( R_{212}^2 R_{323}^2 R_{223}^3 + R_{323}^1 \left( R_{212}^2 R_{312}^3 - R_{312}^2 R_{212}^3 \right) \right) \left( R_{213}^1 \right)^2 + \right. \\ & \left. \left( -R_{323}^2 \left( R_{212}^1 R_{213}^2 R_{223}^3 + R_{223}^1 R_{212}^2 \left( R_{213}^3 - R_{312}^3 \right) \right) + \right. \\ & R_{323}^1 \left( \left( R_{212}^2 R_{223}^2 + R_{212}^1 R_{312}^2 \right) R_{213}^3 - R_{213}^2 \left( R_{223}^2 R_{212}^3 + R_{212}^1 R_{312}^3 \right) \right) + \\ & R_{312} \left( R_{323}^1 \left( \left( R_{213}^2 + R_{312}^2 \right) R_{212}^3 - R_{212}^2 \left( R_{213}^3 + R_{312}^3 \right) \right) - R_{212}^2 R_{323}^2 R_{223}^3 \right) R_{213}^1 + \\ & \left. \left( R_{312}^1 \right)^2 R_{323}^1 \left( R_{212}^2 R_{213}^3 - R_{213}^2 R_{212}^3 \right) + R_{223}^1 \left( R_{323}^1 \left( R_{213}^2 - R_{312}^2 \right) \left( R_{213}^2 R_{212}^3 \right. \right. \right. \\ & \left. \left. \left. - R_{212}^2 R_{213}^3 \right) + R_{212}^1 R_{213}^2 R_{323}^2 \left( R_{213}^3 - R_{312}^3 \right) \right) + \right. \\ & \left. R_{312}^1 \left( R_{212}^1 R_{213}^2 R_{323}^2 R_{223}^3 + R_{323}^1 \left( R_{213}^2 \left( R_{223}^2 R_{212}^3 + R_{212}^1 R_{312}^3 \right) - \right. \right. \right. \\ & \left. \left. \left. \left( R_{212}^2 R_{223}^2 + R_{212}^1 R_{312}^2 \right) R_{213}^3 \right) \right) \right) / \Delta. \quad (43) \end{aligned}$$

Substituting into (16) we obtain a system of differential equations for the  $R_{jkl}^i$  in terms of the Christoffel symbols which is represented in our code by **EqnSet16**. Imposing Christoffel symbols we can determine the  $R_{jkl}^i$  from the system of ordinary differential equations and hence the metric. On solving for  $g_{11}$  we get

$$g_{11}(x_1, x_2, x_3) = 0 \quad (44)$$

$$\begin{aligned} g_{11}(x_1, x_2, x_3) = & e^{\int_1^{x_1} \Gamma_{11}^1(K_1, x_2, x_3) dK_1} \int_1^{x_1} e^{-\int_1^{K_2} \Gamma_{11}^1(K_1, x_2, x_3) dK_1} \\ & \left( s_2(K_2, x_2, x_3) \Gamma_{11}^2(K_2, x_2, x_3) + s_3(K_2, x_2, x_3) \Gamma_{11}^3(K_2, x_2, x_3) \right) dK_2 \\ & + e^{\int_1^{x_1} \Gamma_{11}^1(K_1, x_2, x_3) dK_1} c_1[x_2, x_3] \quad (45) \end{aligned}$$

We can then subsequently solve for the arbitrary function  $c_1[x_2, x_3]$ .

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