

The *Invar* tensor package: Differential invariants of Riemann

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Abstract

The long standing problem of the relations among the scalar invariants of the Riemann tensor is computationally solved for all $6 \cdot 10^{23}$ objects with up to 12 derivatives of the metric. This covers cases ranging from products of up to 6 undifferentiated Riemann tensors to cases with up to 10 covariant derivatives of a single Riemann. We extend our computer algebra system *Invar* to produce within seconds a canonical form for any of those objects in terms of a basis. The process is as follows: (1) an invariant is converted in real time into a canonical form with respect to the permutation symmetries of the Riemann tensor; (2) *Invar* reads a database of more than $6 \cdot 10^5$ relations and applies those coming from the cyclic symmetry of the Riemann tensor; (3) then applies the relations coming from the Bianchi identity, (4) the relations coming from commutations of covariant derivatives, (5) the dimensionally-dependent identities for dimension 4, and finally (6) simplifies invariants that can be expressed as product of dual invariants. *Invar* runs on top of the tensor computer algebra systems *xTensor* (for *Mathematica*) and *Canon* (for *Maple*).

Key words: Riemann tensor, tensor calculus, Mathematica, Maple, computer algebra

PACS: 02.70.Wz, 04.20.-q, 02.40.Ky

Program summary

Title of program: Invar Tensor Package v. 2.0

Catalogue identifier:

Program obtainable from: (submitted to Computer Physics Communications)

<http://www.lncc.br/~portugal/Invar.html> (Maple version) and

<http://metric.iem.csic.es/Martin-Garcia/xAct/Invar/> (Mathematica)

Reference in CPC to previous version: Computer Physics Communications 177 (2007) 640–648

Catalogue identifier of previous version: ADSP

Does the new version supersede the original program?: Yes. The previous version (1.0) only handled algebraic invariants. The current version (2.0) has been extended to cover differential invariants as well.

Computers: Any computer running Mathematica versions 5.0 to 6.0 or Maple versions 9 to 11

Operating systems under which the new version has been tested: Linux, Unix, Windows XP, MacOS

Programming language: Mathematica and Maple

Memory required to execute with typical data: 100 Mb

No. of bits in a word: 64 or 32

No. of processors used: 1

No. of bytes in distributed program, including test data, etc.: Code < 1Mb; database: 40 Mb; 13 expanded cases order 12 at commutation step: 250 Mb.

Distribution format: Unencoded compressed tar file

Nature of physical problem: Manipulation and simplification of scalar polynomial expressions formed from the Riemann tensor and its covariant derivatives.

Method of solution: Algorithms of computational group theory to simplify expressions with tensors that obey permutation symmetries. Tables of syzygies of the scalar invariants of the Riemann tensor.

Restrictions on the complexity of the problem: The present version only handles scalars, but not expressions with free indices.

Typical running time: One second to fully reduce any monomial of the Riemann tensor up to degree 7 or order 10 in terms of independent invariants.

1 Introduction

Extracting information from the Riemann curvature tensor of a metric field on a manifold is not an easy task due to its complicated algebraic structure. The problem can be handled when restricting to the algebraic invariants of the Riemann tensor, for which the problem of giving all of them in terms of a basis has been recently solved, after decades of continued work [1,2,3]. However, this is frequently not enough and the differential invariants of Rie-

mann are then required. For example, Bonnor [4] has shown that the acceleration in the ‘photon rocket’ metric does not affect the algebraic Riemann invariants, but changes the differential invariants. In this way it is possible to introduce singularities in the curvature which are unnoticed by the algebraic curvature invariants [5]. Differential invariants of Riemann are also required to compute the different loop-orders of renormalization of the Einstein-Hilbert Lagrangian [6], or in dealing with its generalizations, like the Lagrangian $L(R, \nabla^2 R, \dots \nabla^{2n} R)$ proposed in [7], or the general diffeomorphism invariant Lagrangian $L(g_{ab}, R_{bcde}, \nabla_{a_1} R_{bcde}, \dots, \nabla_{(a_1} \dots \nabla_{a_m)} R_{bcde}, \dots)$ analyzed in [8].

In spite of the large amount of effort dedicated to the algebraic invariants, very little has been said about how to manipulate large families of differential invariants, except for the important work by Fulling *et al.* [9], because the problem is much more complicated. Here we shall apply and generalize the techniques we proposed in [1] to cover this case, extending our tensor computer algebra system *Invar* to handle both the algebraic and the differential invariants of the Riemann tensor.

2 The problem

As in [1] we shall work on a manifold of dimension d with a metric field g_{ab} and its associated structures: the (torsionless) Levi-Civita connection ∇_a , its Riemann tensor R_{abcd} and (when restricting to $d = 4$) the totally antisymmetric tensor ϵ_{abcd} , such that $\nabla_e g_{ab} = 0$ and $\nabla_e \epsilon_{abcd} = 0$.

Our main objective is constructing a basis of independent *invariants* (that is, scalars formed from contraction of several Riemann tensors and their ∇ -derivatives) and *dual invariants* (those also having an ϵ_{abcd} tensor), along with all polynomial relations giving any other *dependent* invariant in terms of those in the basis. The restricted algebraic case without derivatives of Riemann has been essentially solved, both from the computational point of view [1] and, after a long series of contributions, from the theoretical point of view [2,3]. The general case is much more complicated and remains, however, nearly unexplored. The only article known to us in this direction is the work by Fulling *et al.* [9] in which, using Young tableaux techniques, they were able to compute numbers of independent *nondual* invariants in the basis for a variety of cases, in different dimensions. They also gave the members of the basis for the simpler of those cases, but no expansions of the dependent invariants were provided, rendering the result unpractical. This article fills in that important gap by recomputing the basis of invariants (hence confirming for the first time the results in [9]) together with a database of expressions of any other independent invariant in terms of the basis.

3 Notations

Following Fulling *et al.* [9] we separate the set of all monomial invariants of degree n (the number of Riemann tensors) in subsets $\mathcal{R}_{\{\lambda_1, \dots, \lambda_n\}}$, where λ_i is the differentiation order of the i -th Riemann tensor, assuming the tensors have been sorted such that $\lambda_i \leq \lambda_{i+1}$. An n -tuple $\{\lambda_1, \dots, \lambda_n\}$ will be referred to as a *case*, with the corresponding invariants having $N = 4n + \sum_{i=1}^n \lambda_i$ indices and order $\Lambda = 2n + \sum_{i=1}^n \lambda_i$, the number of derivatives of the metric, not to be confused with the total number of derivatives of the Riemann tensors, which is $\sum_{i=1}^n \lambda_i$. Both N and Λ are always even numbers because we only consider scalar expressions, with all indices paired among them. For example an invariant of the case $\{0, 1, 3\}$, hence with $N = 16$ indices and order $\Lambda = 10$, is

$$R_{abcd} \nabla_e R^{ecfg} \nabla^a \nabla_f \nabla_h R^{bdh}{}_g. \quad (1)$$

Sets of dual invariants and dual cases will be denoted with an asterisk, as in $\mathcal{R}_{\{\lambda_1, \dots, \lambda_n\}}^*$ or $\{\lambda_1, \dots, \lambda_n\}^*$. The algebraic cases considered in [1] are denoted here as $\{0, .^n., 0\}$ for degree n .

We shall see that commutation of derivatives produces relations among invariants of different cases, because it converts second derivatives into additional Riemann tensors. This operation changes both the degree n and the differentiation orders λ_i of the Riemann tensors, but not the total order Λ of metric derivatives. This allows a simple classification of the relations, which are all homogeneous in Λ . Following again ref. [9] we give results up to $\Lambda = 12$, which consistently fills the gap among the algebraic cases of degrees 1 to 7 (that is $\Lambda = 2$ to $\Lambda = 14$) in [1]. Concerning duals, we give results up to $\Lambda = 8$, also consistent with the algebraic degrees 1 to 5 of [1]. See tables 1, 2, 3 for the actual list of cases considered in this investigation, already sorted by their corresponding values of Λ : there are 48 nondual cases and 15 dual cases to treat. In the following, when talking about total numbers of invariants and relations, we always include the 7 nondual and 5 dual algebraic cases already considered in [1].

4 Algorithms

Relations among Riemann invariants are a consequence of the symmetries obeyed by the Riemann tensor and its ∇ -derivatives. There are six different symmetries we can exploit and so we proceed in six respective consecutive *steps*, each producing new relations and therefore decreasing the number of

independent invariants, as shown in tables 1, 2 and 3 (note that steps 3 and 4 are new with respect to [1]):

- (1) Permutation symmetries. The Riemann tensor obeys the following symmetries under permutations of indices:

$$R_{bacd} = -R_{abcd}, \quad R_{cdab} = R_{abcd}. \quad (2)$$

As explained in [1] this type of tensor symmetry, and the induced symmetries in tensor products, can be efficiently handled using fast algorithms for manipulation of permutation groups [10]. In our system, each monomial is converted in real time to its canonical form. For example, the canonical form of invariant (1) is

$$-R^{abcd} R_a{}^{efg}{}_{;e} R_{bcf}{}^h{}_{;hgd} \equiv -I_{\{0,1,3\},2595}. \quad (3)$$

This process is very fast: see Figure 1 for an histogram of timings of a case with 28 indices. In this way, we enormously reduce the number of invariants we need to control. There are $16! \sim 2 \cdot 10^{13}$ invariants for case $\{0,1,3\}$. As shown in Table 1, there are 3237 different canonical forms (column ‘Canon’), and if we remove products of invariants of lower degree, this number is further reduced to 3099 (column ‘Invars’), which are indexed from $I_{\{0,1,3\},1}$ to $I_{\{0,1,3\},3099}$. The whole column ‘Invars’ has been constructed by canonicalization of a supercomplete set of more than 20 million permutations in 15 hours, giving 640 119 different nondual canonical forms and 7698 canonical duals. Those canonical forms were then sorted within each case according to the following priorities: i) invariants with more (differentiated) Ricci scalars are sorted first; ii) invariants with more Ricci tensors are sorted first; iii) more Laplacians first; iv) ordering based on indices. The indexing of algebraic invariants in the first version of *Invar* has been preserved for backwards compatibility.

We do not consider at this step symmetries coming from permutation of covariant derivatives acting on scalar expressions. They will be treated in step 4.

- (2) Cyclic symmetry. The Riemann tensor obeys the multiterm symmetry $R_{a[bcd]} = 0$. For each canonical invariant after step 1 we generate several cyclic relations by replacing R_{abcd} by its 3-index antisymmetrized part in all possible inequivalent ways. For example, from the canonical form (3) we get three cyclic relations (already canonicalized):

$$\begin{aligned} R^{abcd} R_a{}^{efg}{}_{;e} (2 R_{bcf}{}^h{}_{;hgd} + R_{cdf}{}^h{}_{;hgb}) &= 0, \\ R^{abcd} R_a{}^{efg}{}_{;e} (R_{bcf}{}^h{}_{;hgd} - R_{bfc}{}^h{}_{;hgd} + R_b{}^h{}_{cf;hgd}) &= 0, \\ R^{abcd} (R_a{}^{efg}{}_{;e} R_{bcf}{}^h{}_{;hgd} + R_a{}^{efg}{}_{;f} R_{bcg}{}^h{}_{;hed} - R_a{}^{efg}{}_{;f} R_{bce}{}^h{}_{;hgd}) &= 0. \end{aligned}$$

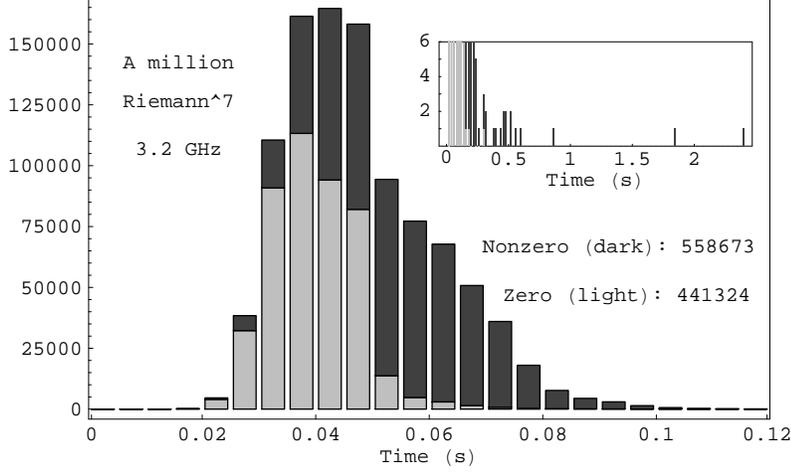


Fig. 1. Histogram of timings of canonicalization with *xTensor* [11] of a million algebraic invariants of degree 7. The average is below 0.05s and only two cases (with exceptionally high symmetry groups involved) take more than a second.

In terms of indexed invariants those are respectively

$$\begin{aligned} 2I_{\{0,1,3\},2595} + I_{\{0,1,3\},2610} &= 0, \\ I_{\{0,1,3\},2595} - I_{\{0,1,3\},2601} + I_{\{0,1,3\},2757} &= 0, \\ I_{\{0,1,3\},2595} - I_{\{0,1,3\},2634} + I_{\{0,1,3\},2640} &= 0, \end{aligned}$$

allowing to express $I_{\{0,1,3\},2610}$, $I_{\{0,1,3\},2757}$ and $I_{\{0,1,3\},2640}$ in terms of lower-index invariants.

- (3) Bianchi identity. The Riemann tensor obeys the multiterm symmetry $R_{ab[cd;e]} = 0$. (Note that this, together with the previous permutation and cyclic symmetries, make the tensor $R_{abcd;e}$ vanish under antisymmetrization of any set of 3, 4 or 5 indices.) Again, from every one of the independent invariants after step 1, two equations are constructed by replacing $\nabla_a R_{bcde}$ by $\nabla_{[a} R_{bc]de}$ and $\nabla_{[a} R_{de]bc}$. The results are then simplified using the information of steps 1 and 2. For example from the canonical invariant (3) we get a zero equation and

$$R^{abcd}(R_a{}^{efg}{}_{;e} R_{bcf}{}^h{}_{;hgd} + R_a{}^e{}^{f;g} R_{bcf}{}^h{}_{;hgd} - R_a{}^e{}^{f;g} R_{bcg}{}^h{}_{;hfd}) = 0.$$

In terms of indexed invariants and using step 2 this is

$$I_{\{0,1,3\},1445} - I_{\{0,1,3\},1451} + I_{\{0,1,3\},2595} = 0,$$

so that $I_{\{0,1,3\},2595}$ is no longer an independent invariant.

- (4) Commutation of derivatives. Given any tensor $T^{a_1 \dots a_n}{}_{b_1 \dots b_m}$ we have

$$\begin{aligned} \nabla_d \nabla_c T^{a_1 \dots a_n}{}_{b_1 \dots b_m} - \nabla_c \nabla_d T^{a_1 \dots a_n}{}_{b_1 \dots b_m} &= \\ &= \sum_{k=1}^n R_{cde}{}^{ak} T^{a_1 \dots e \dots a_n}{}_{b_1 \dots b_m} - \sum_{k=1}^m R_{cdbk}{}^e T^{a_1 \dots a_n}{}_{b_1 \dots e \dots b_m}. \end{aligned}$$

We generate new equations by exchanging the order of all consecutive covariant derivatives in the canonical invariants after step 1. As we have already mentioned, in this step different cases are mixed because second covariant derivatives are converted into Riemann tensors, and so we find equations involving cases with the same order Λ but larger degree, that is $\{0, 1, 3\}$ and $\{0, 0, 1, 1\}$ in our example (3): under commutation of ∇_g and ∇_d and after the use of steps 1, 2 and 3 we get

$$\begin{aligned} & -I_{\{0,1,3\},535} + I_{\{0,1,3\},536} + I_{\{0,1,3\},538} - I_{\{0,1,3\},539} \\ & + I_{\{0,1,3\},541} - I_{\{0,1,3\},542} - I_{\{0,1,3\},547} + I_{\{0,1,3\},548} \\ & + I_{\{0,0,1,1\},385} - I_{\{0,0,1,1\},386} - I_{\{0,0,1,1\},394} + I_{\{0,0,1,1\},397} = 0. \end{aligned}$$

The equations at this step, once expanded in terms of independent invariants, become very long, many of them having several thousand terms for $\Lambda = 12$. This is because in step 4 we still have 6368 independent invariants, without even counting the 1639 objects from the algebraic degree-7 case. The hardest case to build is $\{10\}$, taking nearly 300 hours of CPU time and containing equations with up to 5617 terms (independent invariants).

Our results in columns ‘Commute’ of tables 1 and 2 perfectly coincide with those of column ‘Total’ of appendix A of Fulling *et al.* [9] after dealing with the fact that they count product invariants. Note that they do not study dual invariants and so everything in table 3 is new.

- (5) Dimensionally dependent identities (also called Lovelock type identities): antisymmetrization in $d + 1$ indices in dimension d gives zero [13]. We generate a large number of those equations for dimension 4 by antisymmetrization of random groups of 5 indices in each independent invariant after step 1.

Again, our results in column ‘4D’ of tables 1 and 2 agree with those of Fulling *et al.* [9]. They give no results for dual invariants.

- (6) Duals. A product of two ϵ tensors can be given as a linear combination of products of δ ’s and hence a product of two dual invariants can be expressed as a linear combination of nondual invariants. We have combined the dual invariants in pairs in all possible ways, obtaining 51 relations which allow decomposing 51 independent invariants after step 5 into products of dual invariants.

We have chosen this order of use of symmetries because only steps 1–5 are signature independent, only steps 1–4 are dimension independent, steps 1–3 do not mix cases, steps 1–2 do not involve derivatives, and finally step 1 employs only permutation symmetries, which must be used first. Note the importance of not decomposing Riemann in its Weyl and Ricci parts, which would make the process dimension-dependent right from the outset.

Case	Canon	Invars	Cyclic	Bianchi	Commute	4D	Duals
{0}	1	1	1	1	1	1	1
{0,0}	4	3	2	2	2	2	2
{2}	2	2	2	1	1	1	1
{0,0,0}	13	9	5	5	5	3	3
{0,2}	14	12	9	5	3	3	3
{1,1}	12	12	9	4	4	4	4
{4}	12	12	11	6	1	1	1
{0,0,0,0}	57	38	15	15	15	4	3
{0,0,2}	119	99	48	27	15	10	10
{0,1,1}	137	125	63	23	23	17	17
{0,4}	138	126	84	47	3	3	3
{1,3}	138	138	95	32	5	5	5
{2,2}	89	86	59	23	7	7	7
{6}	105	105	90	50	1	1	1
{0,0,0,0,0}	288	204	54	54	54	5	3
{0,0,0,2}	1193	1020	313	175	79	26	25
{0,0,1,1}	1922	1749	564	194	194	76	74
{0,0,4}	1647	1473	648	361	17	12	12
{0,1,3}	3237	3099	1387	442	53	42	42
{0,2,2}	1735	1622	727	244	46	34	34
{1,1,2}	1641	1617	741	143	67	52	52
{0,6}	1770	1665	1025	570	3	3	3
{1,5}	1770	1770	1115	362	5	5	5
{2,4}	1770	1746	1093	356	9	9	9
{3,3}	962	962	612	211	9	9	9
{8}	1155	1155	945	525	1	1	1

Table 1

Number of independent non-dual invariants after the different steps of simplification: 0) ‘Canon’: canonical invariants including products of lower degree; 1) ‘Invars’: canonical invariants without products; 2) ‘Cyclic’: invariants after imposing the Cyclic symmetry; 3) ‘Bianchi’: invariants after imposing the Bianchi identity; 4) ‘Commute’: invariants after commuting covariant derivatives; 5) ‘4D’: invariants after imposing all possible dimensionally dependent identities for dimension 4; and 6) ‘Duals’: independent invariants after decomposition in products of duals.

5 Implementation

The implementation of the new database of relations among the differential invariants closely follows that of our previous version of *Invar* [1]. In particular the commands given in appendices C and D of [1] are still valid, with minimal changes, the most important of them being the fact that now the different cases are identified by the case list of orders $\{0,1,3\}$ in the *Mathematica* version and $[0,1,3]$ in the *Maple* version rather than an integer degree. Therefore an invariant is now denoted as `RInv[{0,1,3}, 2595]` or `DualRInv[{1,3}, 920]` in *Mathematica* and `RInv[[0,1,3], 2595]` or `DualRInv[[1,3], 920]`

Case	Canon	Invars	Cyclic	Bianchi	Commute	4D	Duals
{0,0,0,0,0,0}	2070	1613	270	270	270	8	4
{0,0,0,0,2}	14 408	12 722	2495	1371	549	66	58
{0,0,0,1,1}	29 427	27 022	5439	1725	1725	245	228
{0,0,0,4}	21 750	19 617	5622	3094	99	37	36
{0,0,1,3}	64 635	60 984	17 662	5440	577	242	240
{0,0,2,2}	33 252	30 974	9030	2861	445	169	165
{0,1,1,2}	64 500	62 465	18 272	3226	1235	505	503
{1,1,1,1}	5684	5606	1733	210	210	86	83
{0,0,6}	27 675	25 590	10 600	5840	17	12	12
{0,1,5}	54 930	53 160	22 330	6938	55	44	44
{0,2,4}	54 930	52 764	22 063	6861	93	72	72
{1,1,4}	27 540	27 396	11 695	2121	83	66	66
{0,3,3}	27 986	27 024	11 402	3597	68	53	53
{1,2,3}	54 930	54 654	23 255	4204	212	171	171
{2,2,2}	9280	9104	3879	715	66	49	49
{0,8}	26 670	25 515	14 910	8225	3	3	3
{1,7}	26 670	26 670	15 855	5000	5	5	5
{2,6}	26 670	26 460	15 675	4950	9	9	9
{3,5}	26 670	26 670	15 855	5000	11	11	11
{4,4}	13 685	13 607	8111	2615	12	12	12
{10}	15 120	15 120	11 970	6615	1	1	1
{0,0,0,0,0,0,0}	19 610	16 532	1639	1639	1639	7	3

Table 2

Number of independent invariants for the cases with order $\Lambda = 12$, plus the algebraic case of degree 7. See caption of table 1 for the meaning of the column headers.

Dual case	Canon	Invars	Cyclic	Bianchi	Commute	4D
{0}*	1	1	0	0	0	0
{0,0}*	5	4	1	1	1	1
{2}*	3	3	0	0	0	0
{0,0,0}*	35	27	6	6	6	2
{0,2}*	63	58	13	5	1	1
{1,1}*	36	36	9	2	2	2
{4}*	32	32	11	4	0	0
{0,0,0,0}*	288	232	40	40	40	1
{0,0,2}*	1059	967	212	98	29	6
{0,1,1}*	1095	1047	236	54	54	13
{0,4}*	920	876	285	128	1	1
{1,3}*	920	920	296	60	2	2
{2,2}*	484	478	163	37	3	3
{6}*	435	435	220	95	0	0
{0,0,0,0,0}*	3031	2582	330	330	330	2

Table 3

Number of independent dual invariants after the different steps of simplification. See the caption of table 1 for the meaning of the column headers. Note that there is no final column 'Duals' in this table.

in *Maple*. The old notation `RInv[3, 9]` for an algebraic invariant is automatically translated to its new notation `RInv[{0,0,0}, 9]` in *Mathematica* and `RInv[[0,0,0], 9]` in *Maple*.

The system now handles derivatives of tensorial expressions. The covariant derivative ∇_a is treated as an operator and denoted `CD[-a]` in our tensor computer algebra systems. Hence, we can represent the invariant

$$I_{\{1,1\},2} \equiv R^{ab}{}_{;c} R_c{}^d{}_{;e} = -R^{ab}{}_{;a} R_{;b}$$

as `CD[c][R[a,b,-a,-b]]*CD[-e][R[-c,d,-c,e]]` in *Mathematica*, and as `CD[c](R[a,b,-a,-b])*CD[-e](R[-c,d,-c,e])` in *Maple*.

Covariant derivatives in *xTensor* and *Canon* are linear operators and automatically obey the standard rules [14], in particular the Leibnitz rule for products. The canonicalization algorithms consistently manipulate derivatives and only one comment is required, concerning how the different Riemann tensors are sorted in a given invariant: we follow the convention of having tensors with more covariant derivatives on the right to help the canonicalizer, which starts placing indices on the left. That is why we chose the convention $\lambda_i \leq \lambda_{i+1}$ for the cases $\{\lambda_1, \dots, \lambda_n\}$, opposite to the choice by Fulling *et al.* [9].

As a simple example of use of the main command `RiemannSimplify` we can check the interesting relation

$$R^{abcd;e} R_{be}{}^{fg;hi} R_{cfdg;ih} = \frac{1}{8} R^{abcd;e} R_{ab}{}^{fg;hi} R_{cdfg;ih},$$

in which the rearrangement of the lower indices produces a factor 1/8. This is rule number 2868 in the file of Bianchi relations among the invariants of case $\{2,2,2\}$. We can check it in *Mathematica* using the sentence

```
expr = CD[-a]@CD[e]@R[a,b,c,d] * CD[i]@CD[h]@R[-b,-e,f,g]
      * CD[-h]@CD[-d]@R[-c,-f,-g,-i]
      - 1/8 * CD[-e]@CD[e]@R[a,b,c,d] * CD[i]@CD[h]@R[-a,-b,f,g]
      * CD[-h]@CD[-i]@R[-c,-d,-f,-g];
RiemannSimplify[ expr ]
```

0

where `f@x` is a *Mathematica* shorthand for `f[x]`. The corresponding *Maple* expression can be obtained replacing each `f@x` by `f(x)`. The final call would be

```
RiemannSimplify( expr )
```

0

The database of relations is arranged as follows: there is a file for each step and case (so $6 \times 48 + 5 \times 15 = 363$ files in total) containing the relations at that step among the invariants of that case and all previous cases. The full database takes more than 1.5 Gbytes of memory and even more once it is read by the computer algebra systems. Most of it comes from the 13 last cases of order 12 at the commutation step (step 4), and so we provide alternative smaller files in which the rules have not been fully expanded in terms of independent invariants of previous cases. The user can configure the system to read either the fully expanded files or the non-expanded files in those cases. If the latter are chosen then repeated (up to four times) use of the simplification functions might be needed to fully expand an invariant, resulting in a slower process. With non-expanded files we have 365Mbytes of data.

6 Conclusions

We have successfully extended our tensor computer algebra package *Invar* to handle differential invariants of the Riemann tensor up to 12 derivatives of the metric. Unlike the problem of algebraic invariants, to which a great deal of effort has been dedicated in the last decades, the problem of differential invariants remained almost unexplored due to its much more difficult character: two new sources of equations come into play, namely the Bianchi identity and the non-commutation of covariant derivatives. This article has completely solved the problem for those cases most likely to be needed in current computations. The problem has been solved in the most useful way: by providing a database with all equations coming from multiterm symmetries, such that any invariant can be immediately looked up without the need of inefficient intermediate computations.

The new database is much larger than its previous version, now containing 645 625 relations for 647 817 canonical invariants (counting both dual and nondual invariants). This represents an increase by a factor larger than 30 with respect to the 21 221 relations for 21 246 invariants in the old database. But this investigation is not only a quantitative extension of [1]. We have also extended our algorithms to handle derivatives and their associated symmetries: Bianchi and commutation. The latter has been especially hard, producing equations with thousands of terms, such that the database has increased in size by a factor larger than 250 with respect to the previous version.

Invar runs on top of the multipurpose tensor computer algebra systems *xTensor* [11] for *Mathematica* and *Canon* [12] for *Maple*.

A final step is required to construct a computer algebra system for generic treatment of the Riemann tensor: manipulation of expression with free indices. We are currently analyzing this extension.

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