CHICOM: A code of tests for comparing unweighted and weighted histograms and two weighted histograms

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Abstract

A Fortran-77 program for calculating test statistics to compare weighted histogram with an unweighted histogram and two histograms with weighted entries is presented. The code calculates test statistics for cases of histograms with normalized weights of events and unnormalized weights of events.

Keywords: homogeneity test, fit Monte Carlo distribution to data, comparison experimental and simulated data, data interpretation

PROGRAM SUMMARY

Program Title: CHICOM
Journal Reference:
Catalogue identifier:
Licensing provisions: none
Programming language: Fortran-77
Computer: Any Unix/Linux workstation or PC with a Fortran-77 compiler.
Classification: 4.13, 11.9, 16.4, 19.4
External routines/libraries used: FPLSOR (M103) [1] and BRENT [2]
Nature of problem: The program calculates test statistics for comparing two weighted histograms and an unweighted histogram with a weighted one.
Solution method: Calculation of test statistics is done according formulas presented in Ref. [3].
Running time: 0.001 sec for 5 bins histogram.

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1. Introduction

A histogram with m bins for a given probability density function p(x) is used to estimate the probabilities p_i that a random event belongs in bin i:

$$p_i = \int_{S_i} p(x) dx, \ i = 1, \dots, m.$$
 (1)

Integration in (1) is carried out over the bin S_i and $\sum_{1}^{m} p_i = 1$. A histogram can be obtained as a result of a random experiment with the probability density function p(x).

A frequently used technique in data analysis is the comparison of two distributions through the comparison of histograms. The hypothesis of homogeneity [1] is that the two histograms represent random values with identical distributions. It is equivalent to there existing m constants $p_1, ..., p_m$, such that $\sum_{i=1}^{m} p_i = 1$, and the probability of belonging to the *i*th bin for some measured value in both experiments is equal to p_i .

Let us denote the number of random events belonging to the *i*th bin of the first and second histograms as n_{1i} and n_{2i} , respectively. The total number of events in the histograms are equal to $n_j = \sum_{i=1}^m n_{ji}$, where j = 1, 2.

As shown in [1] the statistic

$$\frac{1}{n_1 n_2} \sum_{i=1}^{m} \frac{(n_2 n_{1i} - n_1 n_{2i})^2}{n_{1i} + n_{2i}} \tag{2}$$

has approximately a χ^2_{m-1} distribution if hypothesis of homogeneity is valid.

Weighted histograms are often obtained as a result of Monte-Carlo simulations. References [2, 3, 4] are examples of research on high-energy physics, statistical mechanics, and astrophysics using such histograms. To define a weighted histogram let us write the probability p_i (1) for a given probability density function p(x) in the form

$$p_i = \int_{S_i} p(x)dx = \int_{S_i} w(x)g(x)dx,$$
(3)

where

$$w(x) = p(x)/g(x) \tag{4}$$

is the weight function and g(x) is some other probability density function. The function g(x) must be > 0 for points x, where $p(x) \neq 0$. The weight w(x) = 0 if p(x) = 0, see Ref. [5]. Because of the condition $\sum_i p_i = 1$ further we will call the above defined weights normalized weights as opposed to the unnormalized weights $\check{w}(x)$ which are $\check{w}(x) = const \cdot w(x)$.

The histogram with normalized weights was obtained from a random experiment with a probability density function g(x), and the weights of the events were calculated according to (4). Let us denote the total sum of the weights of the events in the *i*th bin of the histogram with normalized weights as

$$W_i = \sum_{l=1}^{n_i} w_i(l),$$
 (5)

where n_i is the number of events at bin *i* and $w_i(l)$ is the weight of the *l*th event in the *i*th bin. The total number of events in the histogram is equal to $n = \sum_{i=1}^{m} n_i$, where *m* is the number of bins. The quantity $\hat{p}_i = W_i/n$ is the estimator of p_i with the expectation value $E[\hat{p}_i] = p_i$. Note that in the case where g(x) = p(x), the weights of the events are equal to 1 and the histogram with normalized weights is the usual histogram with unweighted entries.

Let us introduce notations need for the description of tests for comparing histograms:

- $W_{ji} = \sum_{l=1}^{n_{ji}} w_{ji}(l)$ the total sum of the weights of the events in the *i*th bin of the *j*th the histogram with normalized weights;
- $r_{ji} = \sum_{l=1}^{n_{ji}} w_{ji}(l) / \sum_{l=1}^{n_{ji}} w_{ji}^2(l)$ estimator of the ratio of moments in the *i*th bin of the *j*th histogram with normalized weights.

And the same quantities we introduce for the histograms with unnormalized weighted entries:

- $\check{W}_{ji} = \sum_{l=1}^{n_{2i}} \check{w}_{ji}(l)$
- $\check{r}_{ji} = \sum_{l=1}^{n_{ji}} \check{w}_{ji}(l) / \sum_{l=1}^{n_{ji}} \check{w}_{ji}^2(l)$

Notice that $W_{ji} = n_{ji}$ and $r_{ji} = 1$ for histograms with unweighted entries.

Three types of statistics used for comparing histograms are presented at Ref [6].

Histograms with normalized weighted entries.

Let us introduce the statistic

$${}_{1}X_{k}^{2} = \sum_{j=1}^{2} \frac{1}{n_{j}} \sum_{i \neq k} \frac{r_{ji}W_{ji}^{2}}{p_{i}} + \sum_{j=1}^{2} \frac{1}{n_{j}} \frac{(n_{j} - \sum_{i \neq k} r_{ji}W_{ji})^{2}}{1 - \sum_{i \neq k} r_{ji}p_{i}} - \sum_{j=1}^{2} n_{j}.$$
 (6)

with the sums in (6) extending over all bins *i* except one bin *k*. In the equation (6), the probabilities p_i are unknown, and estimators \hat{p}_i of the probabilities are found by minimization of (6). We denote by ${}_1\hat{X}_k^2$ the value of ${}_1X_k^2$ after substitution of the estimators \hat{p}_i into (6). As shown in [6], the statistic

$$_{1}X^{2} = \operatorname{Med}\left\{_{1}\hat{X}_{1}^{2}, \,_{1}\hat{X}_{2}^{2}, \dots, \,_{1}\hat{X}_{m}^{2}\right\}$$
(7)

approximately has a χ^2_{m-1} distribution if the hypothesis of homogeneity is valid.

Histograms with unnormalized weighted entries.

Let us introduce the statistic

$${}_{2}X_{k}^{2} = \sum_{j=1}^{2} \frac{s_{kj}^{2}}{n_{j}} + 2\sum_{j=1}^{2} s_{kj}, \qquad (8)$$

where

$$s_{kj} = \sqrt{\sum_{i \neq k} \check{r}_{ji} p_i \sum_{i \neq k} \check{r}_{ji} \check{W}_{ji}^2 / p_i} - \sum_{i \neq k} \check{r}_{ji} \check{W}_{ji}.$$
(9)

Again estimators \hat{p}_i of unknown probabilities p_i are found by minimization of (8). We denote by ${}_2\hat{X}_k^2$ the value of ${}_2X_k^2$ after substitution of the estimators \hat{p}_i into (8). As shown in [6], the statistic

$${}_{2}X^{2} = \operatorname{Med}\left\{{}_{2}\hat{X}^{2}_{1}, {}_{2}\hat{X}^{2}_{2}, \dots, {}_{2}\hat{X}^{2}_{m}\right\}$$
(10)

approximately has a χ^2_{m-2} distribution if the hypothesis of homogeneity is valid.

Histograms with normalized and unnormalized weighted entries. Let us introduce the statistic

$${}_{3}X_{k}^{2} = \frac{1}{n_{1}}\sum_{i \neq k} \frac{r_{1i}W_{1i}^{2}}{p_{i}} + \frac{1}{n_{1}} \frac{(n_{1} - \sum_{i \neq k} r_{1i}W_{1i})^{2}}{1 - \sum_{i \neq k} r_{1i}p_{i}} - n_{1} + \frac{s_{k2}^{2}}{n_{2}} + 2s_{k2}.$$
 (11)

We denote by ${}_{3}\hat{X}_{k}^{2}$ the value of ${}_{3}X_{k}^{2}$ after substitution of the estimators \hat{p}_{i} into (11). As shown in [6], the statistic

$$_{3}X^{2} = \operatorname{Med}\left\{{}_{3}\hat{X}^{2}_{1}, {}_{3}\hat{X}^{2}_{2}, \dots, {}_{3}\hat{X}^{2}_{m}\right\}$$
(12)

approximately has a χ^2_{m-2} distribution if the hypothesis of homogeneity is valid.

The chi-square approximation is asymptotic. This means that the critical values may not be valid if the expected frequencies are too small. The use of the chi-square test is inappropriate if any expected frequency is < 1, or if the expected frequency is < 5 in > 20% of the bins for either histogram. This restriction observed in the usual chi-square test [7] is quite reasonable for the proposed test.

Information for readers. Recently, another paper dedicated to weighted histograms has been published in "Computer Physics Communication", see Ref. [9]. The same author has presented a program for goodness of fit test for histograms with weighted and unweighted entries. The test is used in a data analysis for comparison theoretical frequencies with frequencies represented by histogram.

2. Computer program

CHICOM is a subroutine which can be called from the Fortran programs for calculating test statistics ${}_{1}X^{2}$, ${}_{2}X^{2}$ and ${}_{3}X^{2}$.

Usage

CALL CHICOM(AEX, ERAEX, NEV, AMC, ERAMC, NMC, NCHA, MODE, STAT, NDF, IFAIL)

Input Data

AEX – one dimensional real array of first weighted histogram content

ERAEX – one dimensional real array of histogram content for entries of first histogram with squares of weights.

NEV – number of events in the first histogram n_1

AMC – one dimensional real array of second weighted histogram content

ERAMC – one dimensional real array of histogram content for entries of second histogram with squares of weights.

NMC – number of events in the second histogram n_2

NCHA – number of bins m

MODE – equal 1 for both histograms with normalized weights, equal 2 for both histograms with unnormalized weights equal 3 for first histogram with normalized weights and the second with unnormalized weights

Output data

STAT – test statistic

NDF – number of degree of freedom l of the χ_l^2 distribution if hypothesis H_0 is true (will be l = m - 1 or l = m - 2)

IFAIL – will be > 0 if calculation is not successful.

3. Test run

We take a distribution:

$$p(x) \propto \frac{2}{(x-10)^2+1} + \frac{1}{(x-14)^2+1}$$
 (13)

defined on the interval [4, 16] and representing two so-called Breit-Wigner peaks [8]. Three cases of the probability density function g(x) are considered

$$g_1(x) = p(x) \tag{14}$$

$$g_2(x) = 1/12 \tag{15}$$

$$g_3(x) \propto \frac{2}{(x-9)^2+1} + \frac{2}{(x-15)^2+1}$$
 (16)

Distribution $g_1(x)$ (14) results in a histogram with unweighted entries, while distribution $g_2(x)$ (15) is a uniform distribution on the interval [4, 16]. Distribution $g_3(x)$ (16) has the same form of parametrization as p(x) (13), but with different values for the parameters.

Three cases were considered:

	First histogram		Second histogram		
N⁰	type of weight	weight	type of weight	weight	
1	normalized	$p(x)/g_1(x) = 1$	normalized	$p(x)/g_1(x) = 1$	
2	unnormalized	$0.5p(x)/g_2(x)$	unnormalized	$2p(x)/g_3(x)$	
3	normalized	$p(x)/g_1(x) = 1$	unnormalized	$0.5p(x)/g_3(x)$	

For each case histograms with 5 bins were created by simulation 500 entries for first histogram and 1000 entries for the second one. The results of the calculations are presented below.

Test 1

INPUT

AEX	11.0000	58.0000	234.0000	102.0000	95.0000
ERAEX	11.0000	58.0000	234.0000	102.0000	95.0000
NEV	500				
AMC	30.0000	119.0000	439.0000	182.0000	230.0000
ERAMC	30.0000	119.0000	439.0000	182.0000	230.0000
NMC	1000				
NCHA	5				
MODE	1				

OUTPUT

STAT NDF IFAIL	4.7391 4 0		(p-valu	ne = 0.3151)	
Test	t 2				
	LΝ	PUT			
AEX	9.3018	22.8871	122.0670	51.6786	46.2622
ERAEX	0.8026	7.7173	142.7876	27.7087	28.5724
NEV	500				
AMC	68.9455	213.5029	898.8528	397.7258	419.0171
ERAMC	108.3022	229.3163	3697.7102	1455.0262	699.6888
	1000				
MODE	5				
IIODL	2				
	01	UTPUT			
OTAT	1 0111		(p. volu	$n_0 = 0.5011$	
NDF	1.9111 3		(p-van	le = 0.3911)	
TFATI.	0				
	-				
Test	t 3				
	I	NPUT			
AEX	17.0000	53.0000	225.0000	101.0000	104.0000
ERAEX	17.0000	53.0000	225.0000	101.0000	104.0000
NEV	500				
AMC	14.2303	53.9921	204.9794	111.6337	101.1128
ERAMC	5.4897	14.5935	198.6223	103.7259	40.9275
NMC	1000				
NCHA	5				
MUDE	3				

OUTPUT

STAT	1.4431	(p-value = 0.6955)
NDF	3	
IFAIL	0	

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