# GroupMath: A Mathematica package for group theory calculations 

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Abstract
GroupMath is a Mathematica package which performs several calculations related to semi-simple Lie algebras and the permutation groups, both of which are important in particle physics as well as in other areas of research.

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## 1. Introduction

Calculations involving Lie algebras and their representations are ubiquitous in several fields of science, including high energy physics. It is therefore worth having tools that perform this kind of computations automatically, with fast and reliable results. That is the goal of GroupMath, the Mathematica package presented in this document.

This software has its origins in another piece of code - Susyno [1] - whose main purpose has no obvious connection with group theory. Indeed the aim of this latter program is to calculate the renormalization group equations for supersymmetric models. Doing so requires generating the superpotential and the soft-supersymmetric breaking potential in a fully expanded form, and for the user's convenience the program calculates these quantities automatically from the model's defining properties: its gauge symmetry group and the superfields transforming as representations of that group. Several functions associated to Lie algebras were created for this purpose, constituting a significant part of the whole program.

Given that it can be useful by itself, from the very beginning the group theory code of Susyno was made directly accessible to the user, and eventually it was used in other programs (SARAH [2], PyR@TE [3, 4] and Sym2Int [5, 6]). However, over time and with successive increments to this part of the program, it became clear that it would be best to have a separate program for this Lie algebra code. Hence GroupMath was created.

The finite permutation groups $S_{n}$ are also useful to our understanding of the representations of Lie algebras (see for example [6]), hence GroupMath contains several function related to these groups and their representations. It is also conceivable that future versions of the program might contain code related to other finite groups.

This is by no means the only available tool for performing group theory calculations. A likely incomplete list of other programs includes LiE [7], GAP [8], Affine.m [9] and LieART [10, 11], the latter two being Mathematica packages just like GroupMath. Furthermore, the reader also has at its disposal several references containing a large amount of pre-computed Lie algebra data $[12,13]$.

The rest of this work describes the program and how to use it. The available functions are also documented in the program's built-in documentation, which can be accessed directly from within Mathematica. Additionally, usage of the code requires some knowledge of Lie algebras (but not much), therefore a brief review of this topic is included in appendix. It
contains a minimalist explanation of what are simple algebras, Cartan matrices, Dynkin coefficients, weights and others important concepts (further clarifications are provided directly in the main text when each function is presented). More details can be found in Lie algebra textbooks [12, 14-17]. It is perhaps appropriate to warn the reader that, for convenience, Lie algebras are also referred to as Lie groups in the remainder of the text, even though strictly speaking they are not the same thing. Furthermore, a representation is a map from a group's elements to some vector space, but it is also very convenient to call the vector space itself a representation (as physicists usually do).

## 2. Installation

An up-to-date version of the program is available at

```
renatofonseca.net/groupmath
```

The downloaded folder GroupMath, once unpacked, must be placed in the Applications subfolder of one of the two special directories defined by the Mathematica variables \$BaseDirectory and \$UserBaseDirectory. Normally these correspond to the paths
/usr/share/Mathematica/Applications
/home/(computer name)/.Mathematica/Applications
on Linux systems, and
$C: \mid$ ProgramData $\backslash$ Mathematica $\backslash$ Applications
C: $\backslash$ Users $\backslash$ (computer name) $\backslash$ AppData $\backslash$ Roaming $\backslash$ Mathematica $\backslash$ Applications
on Windows machines. After placing the folder GroupMath in one of these locations, the program is loaded by typing

$$
\ll \text { GroupMath }
$$

in Mathematica's front end.

## 3. Lie groups

The format used by the program for Lie groups and its representations will now be explained. The various computations which can be performed with them are described afterwards.

## Working with Lie groups

In order to specify a simple Lie group, its Cartan matrix much be provided. The user can write these matrices, but this is neither recommended nor necessary, since they have already
been associated to the variables SU2, SU3, ..., SU32, S03, S05, S06, ..., S032, SP4, SP6, ..., SP32, E6, E7, E8, F4 and G2. For example:
$\ln [\mathrm{f}$ : $=\mathbf{G} \mathbf{G}$ // MatrixForm
Out[ 0 ]//MatrixForm=

$$
\left(\begin{array}{cc}
2 & -3 \\
-1 & 2
\end{array}\right)
$$

For even larger groups of the $S U(n), S O(n)$ and $S p(2 n)$ families, there is a function which will return the corresponding Cartan Matrix for any $n$ :

## CartanMatrix[<"group family name">,<family index>]

Returns the Cartan matrix of a simple group, given its name
The $U(1)$ group is associated with an empty list \{\}, and for convenience this value has already been assigned to the variable U1.

```
ln[v]:= U1
Out[0]= { }
```

Several of the program's function also work with semi-simple and reductive Lie groups, which are products of simple Lie groups and, perhaps, $U(1)$ 's. In this case, each of the factor groups should be indicated in a list: for example $\{S U 3, S U 2, \mathrm{U} 1\}$ represents the group $S U(3) \times S U(2) \times U(1)$.

The format just mentioned applies not only to input but also to the output of some functions (for instance when calculating maximal subgroups). This makes it easy to chain several functions together. However, the Cartan matrix notation is not ideal for human reading: it is not immediately obvious that $\{\{\{2,-1\},\{-1,2\}\},\{\{2\}\},\{ \}\}$ stands for the group $S U(3) \times S U(2) \times U(1)$. Therefore, in some circumstances it is useful to have a function which does the reverse of CartanMatrix:

## CMtoName[<group>]

Converts a group, given in the Cartan Matrix notation, to a name

```
In[f]:= CMtoName[{{{2, -1}, {-1, 2}}, {{2}}, {}}]
Out[0]= {SU3, SU2, U1}
```

The output is a string (or list of strings in the case of non-simple groups).

## Working with representations of Lie groups

An irreducible representation of a simple Lie group is specified by its Dynkin coefficients, which are lists of $m$ non-negative integers, where $m$ equals the group's rank. Therefore, $\{1,2,0,0\}$ is an example of a valid $S U(5)$ representation, since this group has rank 4. The representation of a $U(1)$ is specified by a charge, which is just a real number.

In the case of reductive Lie groups, a list of the representations of each of the factor groups is required, so $\{\{1,2,0,0\}, 3,\{1\}\}$ is a valid $S U(5) \times U(1) \times S U(2)$ representation.

Just like CMtoName for groups, it is possible to convert a representation to a name:
RepName[<group>, <representation>]
Computes a string-like name of a Lie group representation.

```
In[-]:= RepName[SU3, {0, 1}]
    RepName[{SU5, U1, SU2}, {{1, 2, 0, 0}, 3, {1}}]
Out[0]= \overline{3}
Out[0]= 175' }\otimes3\otimes
```

This is such a useful conversion that several GroupMath functions containing Lie group representations in the output have an option UseName which by default is False but, when set to True, it automatically converts Dynkin coefficients into names (several examples will be provided later). The convention used to name representations is the one in [10, 12], with two caveats: GroupMath associates the Dynkin indices $\{0,2\}$ to the $S U(3)$ sextet 6 (not $\{2,0\}$ ), and $\{0,0,0,2,0\}=\mathbf{1 2 6}$ for the $S O(10)$ group (not $\{0,0,0,0,2\}$ ).

Having said this, in many cases there is an easier way to specify a representation $R$ of dimension $d$. If the name of $R$ does not contain primes nor bars, then the user may provide only its dimension $d$; if the name of $R$ does contain a bar (but no primes), then it is associated to the number $-d$, where $d$ is again the dimension of the representation:

```
In[\sigma]:= RepName[SU3, -3]
Out[0]= \overline{3}
```

The convertion from this simplified input notation to Dynkin coefficients is done internally by the following function, which the user can also call directly:

## SimpleRepInputConversion[<group>,<rep in the simplified notation>]

Returns the Dynkin coefficients of the representation.

```
In[-]:= SimpleRepInputConversion[S010, 16]
    RepName[S010, %]
    SimpleRepInputConversion[S010, -16]
    RepName[S010, %]
Out[0]= {0, 0, 0, 0, 1}
Out[0]= 16
Out[\rho]= {0, 0, 0, 1, 0}
Out[0]= =
```

It is worth stressing that this simplified input notation does not work for representations with primes in their names; the only way of providing them as input is with Dynkin coefficients. As for $S O(8)$ representations which can have sub-indices, it is best to avoid the simplified input notation since it is not straightforward to describe which representation is being selected by the function SimpleRepInputConversion.

Finally, representations of the $S U(n)$ groups are often labeled by partitions or Young tableaux. It is not possible to use directly the latter objects to specify $S U(n)$ representation in GroupMath, but there are two functions which convert back and forth between partitions and Dynkin coefficients of $S U(n)$ groups (ConvertPartitionToDynkinCoef and ConvertToPartitionNotation). They are described later.

## Properties of a group

The list of simple complex Lie algebras and many of their important properties can be found in the literature. It might therefore seem of little interest to include the computation of those properties in a computer code. However, in some cases it might useful to have an automated way of accessing this information.

An important feature of a simple Lie group is its adjoint representation. Its Dynkin coefficients are given by the Adjoint function:

## Adjoint[<group>]

```
ln[`]:= Adjoint [S010]
Out[-0]={0, 1, 0, 0, 0}
```

The positive roots are another important feature which is computed by the following function:

## PositiveRoots[<group>]

```
In[\rho]:= PositiveRoots [SU3]
Out[\rho]= {{2, -1}, {-1, 2}, {1, 1}}
```


## Properties of a representation

The list of all representations of a simple Lie group up to some size can be obtained with the function RepsUpToDimN, while the function RepsUpToDimNNoConjugates only returns those representations with no bars in their name (for example the $\overline{\mathbf{3}}$ and $\overline{\mathbf{6}}$ of $S U(3)$ would be omitted). An option UseName can be used with these two functions.

## RepsUpToDimN[<simple Lie group>,<max dimension>]

```
In[\mp@code{]:= RepsUpToDimN[E6, 2000]}
    RepsUpToDimN[E6, 2000, UseName }->\mathrm{ True]
Out[r]= {{0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0},
        {0, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, 0}, {0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 2, 0},
        {2, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 1, 0}, {1, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 1, 1}}
```



## RepsUpToDimNNoConjugates[<simple Lie group>,<max dimension>]

```
In[o]:= RepsUpToDimNNoConjugates [E6, 2000]
    RepsUpToDimNNoConjugates[E6, 2000, UseName }->\mathrm{ True]
Out[r]= {{0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1},
        {0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 2, 0}, {1, 0, 0, 0, 1, 0}, {1, 0, 0, 0, 0, 1}}
Out[\rho]= {1, 27, 78, 351, 351', 650, 1728}
```

To compute the Dynkin coefficients of the complex conjugate of a representation, one can use ConjugateIrrep:

```
ConjugateIrrep[<group>,<representation>]
```

```
In[\varnothing]:= ConjugateIrrep[SU3, {1, 0}]
    ConjugateIrrep[SU5, {1, 0, 0, 1}]
Out[0]= {0, 1}
Out[0]={1, 0, 0, 1}
```

In general, a representation can be real (R), pseudo-real (PR) or complex (C). This information can be obtained with the following function:

```
Type0fRepresentation[<group>,<representation>]
```

```
    TypeOfRepresentation[SU2, 2]
    TypeOfRepresentation[SU2, 3]
    TypeOfRepresentation [SU3, 3]
Out[-]= PR
Out[0]= R
Out[0]= C
```

The DimR function returns the size of a representation of a Lie group, using the Weyl character formula. If the group is a product of several factors, this function computes a list with the sizes the different representations of each factor group (rather than the overall size of the representation).

```
DimR[<group>,<representation>]
```

```
In[f]:= DimR[SU5, {0, 1, 0, 0}]
Out[-]= 10
In[\rho]:= DimR[{SU2, SU3, U1}, {{1}, {1, 0}, 10}]
Out[-]={2, 3, 1}
```

Two other important numbers which characterize a representation $R$ are the Casimir invariant $C(R)$ and the Dynkin index $S(R)$. They are related to the algebra generator matrices $T^{c}$ as follows: ${ }^{1}$

$$
\begin{align*}
C(R) \mathbb{1} & =T^{c} T^{c}  \tag{1}\\
S(R) \delta^{a b} & =\operatorname{Tr}\left(T^{a} T^{b}\right) . \tag{2}
\end{align*}
$$

As such, $S(R)$ times the dimension of the algebra is the same as $C(R)$ times the dimension of the representation $R$. The functions Casimir and DynkinIndex compute these two numbers (if the group is not simple, they operate analogously to DimR). Nevertheless, it is worth noting that the answer depends on the normalization taken for the algebra generators; the one used by the program corresponds to $S(R)=1 / 2$ for the fundamental representation of the $S U(N)$ groups. ${ }^{2}$

[^0]
## DynkinIndex[<group>,<representation>]

```
In[~]:= Casimir[SU3, 3]
    DynkinIndex[SU3, 3]
Out[0]= 午
Out[-]= 支
```

Another important quantity is $a(R)$ appearing in the relation

$$
\begin{equation*}
a(R) d^{a b c}=\operatorname{Tr}\left(\left\{T^{a}, T^{b}\right\} T^{c}\right) \tag{3}
\end{equation*}
$$

where $d^{a b c}$ is a symmetric tensor which is fixed for a given group. The number $a(R)$ is important in the study of anomalies in quantum field theory, as it appears in the expression of amplitudes of 1-loop diagrams with three external gauge boson lines, hence the name triangular anomaly. It can be computed with the function TriangularAnomalyValue, which applies the algorithm described in [18]. If the group is not simple, $a$ is calculated for each of the factor groups, and furthermore there might exist mixed anomalies where one or more of the group generators in expression (3) correspond to a $U(1)$, while the others belong to another factor group. Hence, if the group is not simple the function TriangularAnomalyValue returns a list of numbers, corresponding to each of these $a$ values. By using the option Verbose -> True, the meaning of each number is spelled out.

TriangularAnomalyValue[<group>,<representation>]
$\operatorname{In}[-]:=$ TriangularAnomalyValue [\{SU3, SU2, U1\}, \{3, 2, 1/6\}]
Out $[0]=\left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{36}, 1\right\}$

```
In[f]:= TriangularAnomalyValue[{SU3, SU2, U1}, {3, 2, 1/6}, Verbose }->\mathrm{ True]
    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX-
        \XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX-
        XXX
    >>> The input group has the following factors: {SU3, SU2, U1}.
    >>> There are 4 anomalies to consider: {SU3' U1, SU2 ' U1, U13, SU3 3}}
    >>> The values of the anomalies given
    by the TriangularAnomalyValue function follow this order.
    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX-
         
Out[v]={\frac{1}{6},\frac{1}{4},\frac{1}{36},1}
```

As explained in appendix, the weights $\omega$ of a simple Lie algebra with rank $m$ can be seen as lists of $m$ integers $\omega_{i}=2\left\langle w, \alpha_{i}\right\rangle /\left\langle\alpha_{i}, \alpha_{i}\right\rangle$. They can be computed with the Weights functions, which returns a list with elements \{<weight>,<multiplicity>\}. This format is justified by the fact that weights can have degeneracy, with some values appearing repeated.

## Weights[<group>, <representation>]

```
In[-]:= Weights[SU3, {1, 1}]
Out[0]={{{1, 1}, 1}, {{2, -1}, 1}, {{-1, 2}, 1},
    {{0, 0}, 2}, {{1, -2}, 1}, {{-2, 1}, 1}, {{-1, -1}, 1}}
```

In the case of the $\{1,2,3,4\}$ irrep of $S U(5)$, which has dimension 198540, we can check that the sum of the multiplicities of all weights adds up to this number:

```
ln[㿟: rep = {1, 2, 3, 4};
    DimR[SU5, rep]
    weights = Weights[SU5, rep];
    Total[weights[[All, 2]]]
Out[-]= 198450
Out[-]= 198450
```

Out of all weights of a representation, there are a few whose $m$ coefficients $\omega_{i}$ are all non-negative: they are called dominant weights. They can be obtained, as in the following example, by using the Weights function.

```
ln[\rho]:= Cases [Weights[SU4, {1, 2, 1}], x_/; x[[1]] == Abs[x[[1]]]]
Out[r]= {{{1, 2, 1}, 1}, {{2, 0, 2}, 1}, {{2, 1, 0}, 2},
    {{0, 1, 2}, 2}, {{0, 2, 0}, 4}, {{1, 0, 1}, 5}, {{0, 0, 0}, 7}}
```

An alternative which is better suited for very large irreps is to use DominantWeights:
DominantWeights[<group>, <representation>]

```
In[f]:= DominantWeights[SU4, {1, 2, 1}]
Out[0]={{{1, 2, 1}, 1}, {{2, 0, 2}, 1}, {{2, 1, 0}, 2},
    {{0, 1, 2}, 2}, {{0, 2, 0}, 4}, {{1, 0, 1}, 5}, {{0, 0, 0}, 7}}
```

Through the action of the Weyl group (see the appendix), every weight can be associated to a dominant one, which is computed by this function:

```
DominantConjugate[<group>,<weight>]
```

```
In[\odot]:= DominantConjugate[SU3, {-5, 2}]
Out[-]= {{2, 3}, 2}
```

The first component of the output is the dominant weight, and the second indicates the minimum number of Weyl reflections needed to make the original weight dominant. On this topic, there are two other functions worth mentioning: ReflectWeight and WeylOrbit. The first performs the $m$ elementary reflections on a weight; the second returns all weights in an orbit of the Weyl group which contains some dominant weight $\omega$ indicated by the user. ${ }^{3}$

```
ReflectWeight[<group>, <weight>, <index i of the elementary reflection>]
```

```
In[ ]:= ReflectWeight[F4, {-5, 2, 2, 1}, 1]
    ReflectWeight[F4, {-5, 2, 2, 1}, 2]
    ReflectWeight[F4, {-5, 2, 2, 1}, 3]
    ReflectWeight [F4, {-5, 2, 2, 1}, 4]
Out[-]={5,-3, 2, 1}
Out[-]= {-3, -2, 6, 1}
Out[\rho]= {-5, 4, -2, 3}
Out[r]= {-5, 2, 3, -1 }
```


## WeylOrbit[<group>,<dominant weight>]

[^1]```
In[`]:= WeylOrbit[G2, {5, 7}]
Out[0]= {{5, 7}, {-5, 22}, {12, -7}, {17, -22}, {-12, 29}, {-17, 29},
    {17, -29}, {12, -29}, {-17, 22}, {-12, 7}, {5, -22}, {-5, -7}}
```

The weights of each irreducible representation of a simple Lie group obey some modular equation (or two in the case of the $S O(2 n)$ groups). For example, in the case of $S U(3)$, the value of $\omega_{1}+2 \omega_{2}(\bmod 3)$ is the same for any weight $\left\{\omega_{1}, \omega_{2}\right\}$ of an irrep. Therefore, this number is characteristic of each irreducible representation and it defines classes. This class-number can be computed as follows:

## ConjugacyClass[<group>, <representation>]

```
In[~]:= ConjugacyClass[SU3, 3]
    ConjugacyClass[SU3, -3]
    ConjugacyClass[SU3, 8]
Out[0]= {1}
Out[0]= {2}
Out[0]={0,
```

For consistency ConjugacyClass returns a list because, for $S O(2 n)$ groups, there are two class-numbers rather than one. The modulus in the congruence relation(s) which define the various classes (which is 3 in the case of $S U(3)$ ) depends only on the simple Lie group, and it can be obtained with the following function:

```
ConjugacyClassGroupModIndices[<group>]
```

```
In[`]:= ConjugacyClassGroupModIndices [SU3]
Out[0]= {3}
```

Let us now go over the decomposition of a product of irreps into irreducible components of the group. For example, $\mathbf{2} \times \mathbf{2}=\mathbf{1}+\mathbf{3}$ in $S U(2)$ and $\mathbf{3 \times 3 \times 3}=\mathbf{1}+\mathbf{8}+\mathbf{8}+\mathbf{1 0}$ in $S U(3)$. Using the algorithm described in [20] GroupMath is capable of quickly doing these operations, even for a high number of factors and for very large irreps. The relevant function, ReduceRepProduct, works with simple and reductive groups, accepting the option UseName->True.

```
ReduceRepProduct[<group>,<list of representation>]
```

```
In[\sigma]:= ReduceRepProduct[SU3, {3, -3}]
    ReduceRepProduct[SU3, {3, -3}, UseName }->\mathrm{ True]
Out[\sigma]= {{{1, 1}, 1}, {{0, 0}, 1}}
Out[~]={{8, 1}, {1, 1}}
```

The output is a list with items of the form \{<irrep>, <multiplicity>\}. There is no limit to the number of representations to be multiplied. For example, the product $3 \times 3 \times 8 \times$ $\mathbf{8} \times \mathbf{8} \times \overline{\mathbf{3}} \times \mathbf{8}$ of $S U(3)$ decomposes as follows

```
In[\rho]:= ReduceRepProduct[SU3, {3, 3, 8, 8, 8, -3, 8}]
Out[f]= {{{6, 5}, 1}, {{7, 3}, 4}, {{5, 4}, 26}, {{4, 6}, 5}, {{3, 5}, 64}, {{6, 2}, 48},
    {{4, 3}, 166}, {{2, 7}, 9}, {{1, 6}, 66}, {{2, 4}, 260}, {{5, 1}, 176},
    {{3, 2}, 434}, {{0, 8}, 5}, {{0, 5}, 137}, {{1, 3}, 448}, {{8, 1}, 5},
    {{7, 0}, 27}, {{4, 0}, 235}, {{2, 1}, 510}, {{0, 2}, 297}, {{1, 0}, 217}}
```

and even larger examples, such as the product of ten 120s of $S O(10)$, can easily be handled (note that $\{0,0,1,0,0\}=\mathbf{1 2 0}$ ):

```
In[f]:= ReduceRepProduct[S010, ConstantArray[{0, 0, 1, 0, 0}, 10]]
    {{{0, 0, 10, 0, 0}, 1}, {{0, 1, 8, 1, 1}, 9},
        {{1, 0, 8, 0, 2}, 45}, {{1, 0, 8, 2, 0}, 45}, {{0, 2, 8, 0, 0}, 45},
Out[`]= ...1815..., {{1, 0, 0, 6, 8}, 2058}, {{1, 0, 0, 8, 6}, 2058},
        {{0,0,0, 7, 7}, 11130}, {{0, 0, 0, 5, 9}, 4032}, {{0, 0, 0, 9, 5}, 4032}}
```

| large output show less | show more | show all | set size limit... |
| :--- | :--- | :--- | :--- | :--- |

As noted already above, the function ReduceRepProduct also works for semi-simple groups and $U(1)$ 's. Here is one such case:

```
ln[ [ ]:= Q = {3, 2, 1/6};
    uc = {-3, 1, -2/3};
    H={1, 2, 1/2};
    ReduceRepProduct [{SU3, SU2, U1}, {Q, uc, H}, UseName }->\mathrm{ True]
Out[0]={{8\otimes3\otimes0,1},{8\otimes1\otimes0,1},{1\otimes3\otimes0,1},{1\otimes1\otimes0,1}}
```

What ReduceRepProduct does not do is show the permutation symmetry associated to each irreducible component of a product of equal representations. For example, returning to the product of two $S U(2)$ doublets, it is known that the singlet is anti-symmetric under exchange of the two doublets, while the triplet is symmetric: $\mathbf{2 \times 2}=\mathbf{1}_{A}+\mathbf{3}_{S}$. This extra information is computed by the functions PermutationSymmetry and PermutationSymme-
tryOfInvariants described in section 4.

## Basis-dependent computations

All properties discussed up to now do not depend on a choice of basis for the representation matrices $T^{a}$. We shall now consider computations performed by the program which are basis dependent.

It is important to keep in mind that if $T^{a}$ are unitary representation matrices obeying the Casimir and Dynkin index relations (1) and (2), then the same is true for the set of matrices

$$
\begin{equation*}
O_{a b} U^{\dagger} T^{b} U \tag{4}
\end{equation*}
$$

if $O$ is an orthogonal matrix and $U$ is unitary. Therefore, a basis choice implies picking an $O$ matrix for each simple group, and a $U$ matrix for each of its irreducible representations. GroupMath's basis choice ensures that the $T^{a}$ have the following properties:

1. They are Hermitian: $T^{a}=\left(T^{a}\right)^{\dagger}$.
2. They obey equations (1) and (2) with $C(R)$ and $S(R)$ as given by the Casimir and DynkinIndex functions.
3. The representation matrices of different irreps of a same group are consistent, meaning that the structure constants $f_{a b c}$ appearing in the relation $\left[T^{a}, T^{b}\right]=f_{a b c} T^{c}$ are the same for all irreducible representations.
4. There is a maximum number of diagonal matrices $T^{a}$ (their number equals the group's rank). The matrices which are not diagonal come in pairs of the form $L+L^{T}$ and $i\left(L-L^{T}\right)$, for some strictly lower triangular matrices $L$.
5. The relation $T^{a}\left(R^{*}\right)=-\left[T^{a}(R)\right]^{*}$ holds for irreducible representations $R$ and $R^{*}$ which are the complex conjugate of each other, ensuring that $\left\{\exp \left[i \varepsilon_{a} T^{a}(R)\right]\right\}^{*}=$ $\exp \left[i \varepsilon_{a} T^{a}\left(R^{*}\right)\right]$.

Note that even for real representations (for example the triplet of $S U(2)$ ), we are allowed to choose a basis where $\left[\exp \left(i \varepsilon_{a} T^{a}\right)\right]^{*} \neq \exp \left[\exp \left(i \varepsilon_{a} T^{a}\right)\right]$. In fact, a real basis would require the $T^{a}$ to be anti-symmetric, and therefore non-diagonal; since the GroupMath basis always ensures that there is a maximal number of diagonal $T^{a}$ 's, we may deduce that it is never a real basis.

With this said, arguably the most important basis-dependent quantity that the program calculates are the $T^{a}$ matrices themselves:

[^2]\[

$$
\begin{aligned}
& \ln [\triangleright]:=\text { MatrixForm /@ RepMatrices[SU2, 4] } \\
& \operatorname{Out}\left[[]=\left\{\left(\begin{array}{cccc}
0 & \frac{\sqrt{3}}{2} & 0 & 0 \\
\frac{\sqrt{3}}{2} & 0 & 1 & 0 \\
0 & 1 & 0 & \frac{\sqrt{3}}{2} \\
0 & 0 & \frac{\sqrt{3}}{2} & 0
\end{array}\right),\left(\begin{array}{cccc}
0 & -\frac{i \sqrt{3}}{2} & 0 & 0 \\
\frac{i \operatorname{l} \sqrt{3}}{2} & 0 & -\dot{\mathbb{i}} & 0 \\
0 & \dot{\mathbb{i}} & 0 & -\frac{i \sqrt{3}}{2} \\
0 & 0 & \frac{i \sqrt{3}}{2} & 0
\end{array}\right),\left(\begin{array}{cccc}
\frac{3}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{3}{2}
\end{array}\right)\right\}\right.
\end{aligned}
$$
\]

In[o]:= MatrixForm / @ RepMatrices [SU3, 3]

$$
\begin{aligned}
\text { Out }[0]= & \left\{\left(\begin{array}{lll}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 \\
0 & 0 & 0
\end{array}\right),\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & 0 & \frac{i}{2} \\
0 & 0 & 0 \\
-\frac{i}{2} & 0 & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & -\frac{i}{2} & 0 \\
\frac{i}{2} & 0 & 0 \\
0 & 0 & 0
\end{array}\right),\right. \\
& \left.\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -\frac{i}{2} \\
0 & \frac{i}{2} & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0
\end{array}\right),\left(\begin{array}{ccc}
\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & -\frac{1}{2 \sqrt{3}} & 0 \\
0 & 0 & -\frac{1}{2 \sqrt{3}}
\end{array}\right),\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2}
\end{array}\right)\right\}
\end{aligned}
$$

The output of the RepMatrices function is a list of matrices which are not immediately displayed on screen because they are in Mathematica's SparseArray format. In order to view their entries one must use the Normal or MatrixForm commands. Note also that in the second example shown above, the output is similar but not exactly equal to one-half the Gell-Mann matrices. That is a consequence of the fact that no hard-coding of special cases was done: the RepMatrices function uses the same generic algorithm to build the output independently of the group or the irreducible representation (the algorithm is described in appendix B. 1 of [21]).

The function RepMatrices also works with semi-simple and $U(1)$ groups:

$$
\begin{aligned}
& \operatorname{In}[\cdot]:=\operatorname{MatrixForm} / @ \operatorname{RepMatrices}[\{\text { SU2, SU2, U1 }\},\{\{1\},\{1\}, 10\}] \\
& \text { Out }[0]=\left\{\left(\begin{array}{llll}
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0
\end{array}\right),\left(\begin{array}{cccc}
0 & 0 & -\frac{i}{2} & 0 \\
0 & 0 & 0 & -\frac{i}{2} \\
\frac{i}{2} & 0 & 0 & 0 \\
0 & \frac{i}{2} & 0 & 0
\end{array}\right),\left(\begin{array}{ccc}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 \\
0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 \\
0 & -\frac{1}{2}
\end{array}\right),\right. \\
& \left.\left(\begin{array}{cccc}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0
\end{array}\right),\left(\begin{array}{cccc}
0 & -\frac{i}{2} & 0 & 0 \\
\frac{i}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{i}{2} \\
0 & 0 & \frac{i}{2} & 0
\end{array}\right),\left(\begin{array}{cccc}
\frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{1}{2}
\end{array}\right),\left(\begin{array}{cccc}
10 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 \\
0 & 0 & 10 & 0 \\
0 & 0 & 0 & 10
\end{array}\right)\right\}
\end{aligned}
$$

In this example, the irreducible representation is 4-dimensional and the first three matrices correspond to the first $S U(2)$ factor, the next three correspond to the second $S U(2)$ factor,
and finally the $U(1)$ representation matrix comes last.
Note that for real or pseudo-real representations, the simplified input $d$ and $-d$ gets converted to the same Dynkin coefficients, so the output of RepMatrices is unchanged by the minus sign:

```
In[-]:= MatrixForm /@ RepMatrices[SU2, 2]
    MatrixForm /@ RepMatrices[SU2, -2]
Out[-]={([\begin{array}{cc}{0}&{\frac{1}{2}}\\{\frac{1}{2}}&{0}\end{array}),(\begin{array}{cc}{0}&{-\frac{i}{2}}\\{\frac{i}{2}}&{0}\end{array}),(\begin{array}{cc}{\frac{1}{2}}&{0}\\{0}&{-\frac{1}{2}}\end{array})}
Out[\rho]={([\begin{array}{ll}{0}&{\frac{1}{2}}\\{\frac{1}{2}}&{0}\end{array}),(\begin{array}{cc}{0}&{-\frac{i}{2}}\\{\frac{i}{2}}&{0}\end{array}),(\begin{array}{cc}{\frac{1}{2}}&{0}\\{0}&{-\frac{1}{2}}\end{array})}
```

What this means is that conjugation of the matrices of real or pseudo-real representations cannot be achieved with a minus sign in the input. Instead, the user must manually do this operation $\left(T^{a} \rightarrow-\left(T^{a}\right)^{T}\right)$ :

```
In[-]:= MatrixForm[-Transpose[#]] & /@ RepMatrices[SU2, 2]
Out[\rho]={(\begin{array}{cc}{0}&{-\frac{1}{2}}\\{-\frac{1}{2}}&{0}\end{array}),(\begin{array}{cc}{0}&{-\frac{i}{2}}\\{\frac{i}{2}}&{0}\end{array}),(\begin{array}{cc}{-\frac{1}{2}}&{0}\\{0}&{\frac{1}{2}}\end{array})}
```

As mentioned in appendix, for a simple group with rank $r$ one can find $3 r$ elements $\left\{e_{i}, f_{i}, h_{i}\right\}$ with $i=1, \cdots, r$ which obey the relations (27)-(29). The function RepMinimalMatrices computes explicit representation matrices for them. Note that these matrices by themselves do not form a basis of the algebra, and they do not obey the properties listed earlier (which apply only to the output of RepMatrices).

[^3]```
In[v]:= matrices = RepMinimalMatrices[G2, {0, 1}];
    Map [MatrixForm , matrices, {2}]
Out[~]={{([\begin{array}{lllllll}{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{1}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{1}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\end{array}),(\begin{array}{lllllll}{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{1}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{1}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\end{array}),(\begin{array}{lllllll}{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{1}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{-1}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{1}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{-1}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\end{array})},,
{(\begin{array}{ccccccc}{0}&{1}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{\sqrt{}{2}}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{\sqrt{}{2}}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{1}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\end{array}),(\begin{array}{cccccccc}{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{1}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{\sqrt{}{2}}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{\sqrt{}{2}}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{1}&{0}\end{array}),(\begin{array}{ccccccccc}{1}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{-1}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{2}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{0}\\{0}&{0}&{0}&{0}&{-2}&{0}&{0}\\{0}&{0}&{0}&{0}&{0}&{1}&{0}\\{0}&{0}&{0}&{0}&{0}&{0}&{-1}\end{array})}},
```

The output is a list of representation matrices for the elements $\left\{\left\{e_{1}, f_{1}, h_{1}\right\},\left\{e_{2}, f_{2}, h_{2}\right\}, \cdots\right\}$ using again the SparseArray format. In other words, this function returns a list of size $r$ where each element itself is a list of 3 matrices. Just like RepMatrices, RepMinimalMatrices also works with reductive groups.

When it comes to the adjoint representation of a group, there is a subtlety worth knowing. One can compute its matrices just as any other representation:

$$
\begin{aligned}
& \operatorname{In}[\rho]:=\text { MatrixForm /@ RepMatrices[SU2, Adjoint [SU2] ] } \\
& \operatorname{Out}[\circ]=\left\{\left(\begin{array}{ccc}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & -\frac{i}{\sqrt{2}} & 0 \\
\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
0 & \frac{i}{\sqrt{2}} & 0
\end{array}\right),\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right)\right\}
\end{aligned}
$$

However, these matrices are not directly comparable with the structure constants appearing in the relation $\left[T^{a}, T^{b}\right]=f_{a b c} T^{c}$ because of a basis difference. In particular, note that the matrices $\left(\widetilde{T}^{a}\right)_{b c} \equiv-i f_{a b c}$ are never diagonal, as opposed to the output of RepMatrices. In order to get the $\widetilde{T}^{a}$ one can use instead the GaugeRep function.

## GaugeRep[<group>]

$$
\begin{aligned}
& \ln [\circ]:=\text { MatrixForm /@ GaugeRep [SU2] } \\
& \operatorname{Out}[\cdot]=\left\{\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -\dot{\mathbb{i}} \\
0 & \dot{\mathbb{1}} & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & 0 & \dot{\mathbb{1}} \\
0 & 0 & 0 \\
-\dot{\mathbb{i}} & 0 & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & -\dot{\mathbb{1}} & 0 \\
\dot{\mathbb{i}} & 0 & 0 \\
0 & 0 & 0
\end{array}\right)\right\}
\end{aligned}
$$

With these representation matrices, one can write explicitly the components of an
irreducible representation from the product of the components of other representations. The numerical values appearing in these relations are usually called Clebsch-Gordan coefficients - at least for the case of the $S U(2)$ group. The closely related functions Invariants and IrrepInProduct compute these numbers.

Consider for example two doublets of $S U(2)$ with components labeled

$$
\begin{equation*}
\binom{\mathrm{a}[1]}{\mathrm{a}[2]} \text { and }\binom{\mathrm{b}[1]}{\mathrm{b}[2]} . \tag{5}
\end{equation*}
$$

The combination $\mathrm{a}[1] \mathrm{b}[2]-\mathrm{a}[2] \mathrm{b}[1]$ and any multiple of it is invariant under $S U(2)$. These are the types of expressions calculated by the Invariants function, which works for reductive groups and for products of an arbitrary numbers of representations.

## Invariants[<group>,<list of representations>]

```
In[f]:= Invariants[SU2, {2, 2}]
Out[0]={a[2]\timesb[1]-a[1] 直[2]}
In[-]:= Invariants[SU2, {2, 2, 3}]
Out[0]={\frac{\sqrt{}{2}\textrm{a}[2]\times\textrm{b}[2]\timesc[1]}{\mp@subsup{3}{}{1/4}}-\frac{\textrm{a}[2]\times\textrm{b}[1]\timesc[2]}{\mp@subsup{3}{}{1/4}}-
    a[1]\timesb[2]\timesc[2]
In[ø]:= Invariants[{SU2, SU3}, {{2, 3}, {2, - 3}}]
```



```
    a[1, 1] < b[2, 1] +a[1, 2] < b[2, 2] +a[1, 3] }\textrm{b}[2,3]
```

Several features of this function are noteworthy:

- Its output is a list of expressions (hence the curly brackets) because in general one might have more than one independent invariant.
- The letters a, b, c, d, ... are used to label the components of the first, second, third, fourth, ... representations, as ordered by the user. The basis for each of them is the one of RepMatrices. Numbers are added to each letter x to denote sequential components of the representation: $x[1], x[2], \ldots$. There is one index for each simple factor group so the components for the representation $(\mathbf{3}, \mathbf{2}, 7)$ of $S U(3) \times S U(2) \times U(1)$ are $\{\mathrm{x}[1,1]$, $x[1,2], x[2,1], x[2,2], x[3,1], x[3,2]\}$ and they are assumed to transform with the matrices RepMatrices[\{SU3, SU2, U1 \}, \{3,2,7\}].
- If some expression is a group invariant, so is any multiple of it. More generally, if there are several invariants one can make arbitrary linear combinations of them. In order to remove some of this arbitrariness, GroupMath orthonormalizes a set of independent
invariants as follows：${ }^{4}$ writing each invariant as $\kappa_{\alpha \beta \gamma \gamma \ldots}^{(i)} \mathrm{a}[\alpha] \mathrm{b}[\beta] \mathrm{c}[\gamma] \cdots$（with $i$ indexing each of them），the $\kappa^{(i)}$ numerical coefficients obey the relations

$$
\begin{equation*}
\left[\kappa_{\alpha \beta \gamma \ldots}^{(i)}\right]^{*} \kappa_{\alpha \beta \gamma \ldots}^{(j)}=\delta_{i j} \sqrt{\operatorname{dim}(\mathrm{a}) \operatorname{dim}(\mathrm{b}) \operatorname{dim}(\mathrm{c}) \cdots} \tag{6}
\end{equation*}
$$

where $\operatorname{dim}(\mathrm{x})$ is the dimension of the irreducible representation x ．For example，in the case of $\mathbf{2} \times \mathbf{2} \times \mathbf{3}$ in $S U(2)$ ，the only invariant must have coefficients normalized to $\left[\kappa_{\alpha \beta \gamma}^{(1)}\right]^{*} \kappa_{\alpha \beta \gamma}^{(1)}=2 \sqrt{3}$ ，and that is why we see an overall factor of $3^{1 / 4}$ in the output of Invariants．

It is possible to obtain just the numerical coefficients $\kappa_{\alpha \beta \gamma \ldots}^{(i)}$ rather than the expressions $\kappa_{\alpha \beta \gamma \ldots}^{(i)} \mathrm{a}[\alpha] \mathrm{b}[\beta] \mathrm{c}[\gamma] \cdots$ by using the option TensorForm－＞True：

```
In[-]:= result = Invariants [SU2, {2, 2}, TensorForm }->\mathrm{ True]
        Normal[result[[1]]]
```



```
Out[0]={{{0,-1},{1,0}}}
```

Using this options，the output is composed of two parts：the first one are the $\kappa_{\alpha \beta \gamma \ldots}^{(i)}$ coeffi－ cients，and the second is just the list of representation components used when TensorForm －＞False．

For complex representations，conjugation is simple：one just needs to indicate as input the conjugated irrep．

```
In[-]:= Invariants[SU4, {4, -4}]
Out[-0= {a[1] 直[1]+a[2]\timesb[2]+a[3]\timesb[3]+a[4]\timesb[4]}
```

However，for real and pseudo－real irreps this does not work．It is worth pointing out that， for those cases，using the simplified input $-d$ instead of $d$ does not help either，as both correspond to the same Dynkin coefficients：

```
In[-]:= Invariants[SU2, {2, 2}]
    Invariants[SU2, {2, -2}]
Out[0]={\mathbf{a[2] 人 b[1]-a[1] < b[2]}}
Out[0]={a[2]\timesb[1]-a[1] 直[2]}
```

[^4]In order to conjugate real and pseudo-real irreps (or even complex ones), there is an option Conjugations -> \{<True or False>, <True or False>, ...\} which will instruct the program to conjugate (True) or not (False) each of the irreps provided as input:

```
ln[v]:= Invariants[SU2, {2, 2}, Conjugations }->\mathrm{ {True, False}]
Out[0]={a[1]\timesb[1] +a[2]\timesb[2]}
```

Here the vector $(\mathrm{a}[1], \mathrm{a}[2])^{T}$ is assumed to transform with the matrices $-\left(T^{a}\right)^{T}$, where the $T^{a}$ matrices are the ones returned by RepMatrices[SU2, 2] (equal to one-half of the Pauli matrices).

The computational time increases rapidly with the number of representations being multiplied, as well as their size (memory usage can also be a problem). Nevertheless there is no limit for either of these quantities:


Let us now look at the function IrrepInProduct, which is very similar to Invariants. Consider the $S U(2)$ product $\mathbf{2} \times \mathbf{2} \times \overline{3}$ : if $\sqrt{2} \mathrm{a}[1] \mathrm{b}[1] \mathrm{c}[1]+(\mathrm{a}[2] \mathrm{b}[1]+\mathrm{a}[1] \mathrm{b}[2]) \mathrm{c}[2]$ $+\sqrt{2} \mathrm{a}[2] \mathrm{b}[2] \mathrm{c}[3]$ is an invariant, then surely the object

$$
\left(\begin{array}{c}
\sqrt{2} \mathrm{a}[1] \mathrm{b}[1]  \tag{7}\\
\mathrm{a}[2] \mathrm{b}[1]+\mathrm{a}[1] \mathrm{b}[2] \\
\sqrt{2} \mathrm{a}[2] \mathrm{b}[2]
\end{array}\right)
$$

transforms as a triplet $\mathbf{3}$ of $S U(2)$. The function IrrepInProduct picks out such combinations which transform irreducibly under a group.

IrrepInProduct[<group>, <list of reprs in product>,<repr to be picked out>]

```
In[\rho]:= IrrepInProduct[SU2, {2, 2}, 3]
Out[0]={{\sqrt{}{2}}\mathbf{a}[1]\times\textrm{b}[1],\textrm{a}[2]\times\textrm{b}[1]+\mathbf{a}[1]\times\textrm{b}[2],\sqrt{}{2}\mathbf{a}[2]\times\textrm{b}[2]}
```

Given the similarity between the two functions, the comments made in relation to Invariants apply to IrrepInProduct as well. But there are two differences. The first one is that IrrepInProduct does not perform a normalization of the expressions. The other difference is that to conjugate the representations in the product (the two doublets in the
example above) one should use the option ConjugateRepsInProduct, while ConjugateTargetRep conjugates the representation to be picked out. For instance, the combination of two doublets which transforms as $\overline{\mathbf{3}}$ (as opposed to $\mathbf{3}$ ) is the following:

```
In[ ]:= IrrepInProduct[SU2, {2, 2}, 3, ConjugateTargetRep }->\mathrm{ True]
Out[-]={{a[2]\timesb[2],- a[2]\timesb[1]
```

An even simpler example is this:

```
In[\rho]:= IrrepInProduct[SU2, {2, 1}, 2][[1]] // MatrixForm
IrrepInProduct[SU2, {2, 1}, 2,
            ConjugateTargetRep }->\mathrm{ True] [ [1]] // MatrixForm
```

Out[ 0 ]/MatrixForm=

$$
\binom{a[1] \times b[1]}{a[2] \times b[1]}
$$

Out[0 ]/MatrixForm=

$$
\binom{a[2] \times b[1]}{-a[1] \times b[1]}
$$

## Symmetry breaking

It is often important to know the subgroups $H$ of a Lie group $G$ and understand how the representations of $G$ behave when restricted to the elements of $H$. In many situations, an irreducible representation $R$ of $G$ decomposes into several irreducible representations of $H$; this information is often referred to as the branching rules of $R$.

Concerning the groups themselves, $H$ is said to be a maximal subgroup of $G$ if there is no other subgroup of $G$ containing $H$ (apart from $H$ and $G$ itself). For any subgroup $H$, by tracing all chains of subgroups $G_{0} \equiv G \rightarrow G_{1} \rightarrow G_{2} \rightarrow \cdots \rightarrow G_{q} \equiv H$ such that $G_{i}$ is a maximal subgroup of $G_{i-1}$, one can find all ways of embedding $H$ in $G$. GroupMath provides a function MaximalSubgroups which lists the maximal subgroups of a given group, and also Embeddings which can be used to find all the non-equivalent embeddings of any group in a larger one (obviating the need to track the chains of maximal subgroups just described). The output of both these functions contains the so called embedding projection matrices $P$ which are important for the calculation of branching rules. To be specific, a weight $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{p}\right)$ of $G$ is associated with a weight $\omega^{\prime}=\left(\omega_{1}^{\prime}, \omega_{2}^{\prime}, \cdots, \omega_{p^{\prime}}^{\prime}\right)$ of $H$, with

$$
\begin{equation*}
\omega_{i}^{\prime}=P_{i j} \omega_{j} . \tag{8}
\end{equation*}
$$

Since the rank of $G(=p)$ is necessarily larger or equal to the one of $H\left(=p^{\prime}\right)$, the projection matrix $P$ cannot have more rows than columns.

Once a projection matrix matrix $P$ associated to a particular embedding of $H$ in $G$ is known (with the help of the above functions, or by any other means), it can be supplied to
the function DecomposeRep which finds the branching rules of any irreducible representation of $G$. ${ }^{5}$

## MaximalSubgroups[<group>]

```
In[ []:= maxSubgroups = MaximalSubgroups [F4];
    {CMtoName[#[[1]]], MatrixForm[#[[2]]]} & /@ maxSubgroups
```



```
{{SP6, SU2},(\begin{array}{cccc}{0}&{0}&{0}&{1}\\{0}&{0}&{1}&{0}\\{0}&{1}&{0}&{0}\\{-2}&{-3}&{-2}&{-1}\end{array})},{{SU2},(\begin{array}{lllll}{22}&{42}&{30}&{16}\end{array})},{{G2,SU2},(\begin{array}{lllll}{0}&{1}&{0}&{0}\\{1}&{0}&{1}&{0}\\{4}&{8}&{6}&{4}\end{array})}}
```

The output is a list with the format \{<subgroup> ,<projection matrix>\}. The <subgroup> is a list of Cartan matrices, which can be converted to a human-friendly string with the CMtoName function. There is also a function DisplayEmbeddings which automatically prints a list with elements $\{<$ subgroup>, <projection matrix>\} in a grid:

## Embeddings[<group>, <subgroup>]

[^5]```
In[॰]:= embeddings = Embeddings[S010, {SU3, SU2, U1}];
DisplayEmbeddings[embeddings]
```

```
Out[0]=}(\begin{array}{l}{{\mathrm{ SU3, SU2, U1, U1 }}}\\{\begin{array}{c}{1}\\{0}\end{array}2}\end{array}
```

The output format of Embeddings is as the same as the one of MaximalSubgroups. Concerning $U(1)$ 's, arbitrary linear combinations of their charges can be made to produce the reduction $U(1)^{m} \rightarrow U(1)^{m^{\prime} \leq m}$, so this process is controlled by continuous degrees of freedom (the coefficients of the linear combinations). As such, Embeddings leaves this to the user: the function will always return the maximum number of allowed $U(1)$ 's. In light of this, the only impact of indicating U1'S as factors of the desired subgroup is that embeddings which cannot accommodate a sufficient number of $U(1)$ 's are ignored. For example:

```
In[~]:= embeddings2 = Embeddings[S010, {SU3, SU2, U1, U1}];
    DisplayEmbeddings [embeddings2]
```

Out[ $\left.]=\left|\begin{array}{c}\{\text { SU3, }\end{array}=\right| \begin{array}{ccccc}1 & 2 & 2 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1\end{array}\right)$

Note that the group $G$ does not need to be simple (an input such as Embeddings[\{S010,E6\}, $\{\mathrm{SU} 3, \mathrm{SU} 2, \mathrm{U} 1, \mathrm{U} 1\}]$ is valid).

The embedding information (a subgroup and its associated projection matrix) can then be passed on to DecomposeRep.

[^6]\[

$$
\begin{aligned}
& \left.\ln [\cdot]:=\text { DecomposeRep[S010, -16, \{SU3, SU2, U1, U1 }\},\left(\begin{array}{ccccc}
1 & 2 & 2 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
2 & 0 & -2 & -1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{array}\right) \text {, UseName } \rightarrow \text { True }\right] \\
& \text { Out }[0]=\{3 \otimes 2 \otimes-1 \otimes 0, \overline{3} \otimes 1 \otimes 1 \otimes 1, \overline{3} \otimes 1 \otimes 1 \otimes-1,1 \otimes 2 \otimes 3 \otimes 0,1 \otimes 1 \otimes-3 \otimes 1,1 \otimes 1 \otimes-3 \otimes-1\} \\
& \ln [\cdot]:=\operatorname{DecomposeRep}\left[\text { S010, }-16,\{\text { SU3, SU2, U1 }\},\left(\begin{array}{ccccc}
1 & 2 & 2 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
2 & 0 & -2 & -1 & -1
\end{array}\right) \text {, UseName } \rightarrow \text { True }\right] \\
& \text { Out }[0]=\{3 \otimes 2 \otimes-1, \overline{3} \otimes 2 \otimes 1,1 \otimes 2 \otimes 3,1 \otimes 2 \otimes-3\}
\end{aligned}
$$
\]

It is possible to omit the projection matrix. In that case, the function DecomposeRep will not return any data; instead, it will print a table with the decomposition of a given representation under all possible embeddings of the subgroup:

```
In[~]:= DecomposeRep[S010, -16, {SU3, SU2, U1}]
    There are 2 non-equivalent ways of embedding {SU3, SU2, U1} in SO10.
Under each of them, the representation
    -16 decomposes as follows (x1 is a free real number):
\begin{tabular}{|c|c|c|c|c|c|}
\hline Embedding & \multicolumn{5}{|c|}{ Decomposition } \\
\hline\(\# 1\) & \(\mathbf{3} \otimes \mathbf{2} \otimes-\mathbf{1}\) & \(\overline{\mathbf{3}} \otimes \mathbf{1} \otimes(1+\mathrm{x} 1)\) & \(\overline{\mathbf{3}} \otimes \mathbf{1} \otimes(1-\mathrm{x} 1)\) & \(\mathbf{1} \otimes \mathbf{2} \otimes 3\) & \(\mathbf{1} \otimes \mathbf{1} \otimes(-3+\mathrm{x} 1)\) \\
\(\mathbf{1} \otimes \mathbf{1} \otimes(-3-\mathrm{x} 1)\) \\
\hline\(\# 2\) & \(\mathbf{3} \otimes \mathbf{2} \otimes-1\) & \(\overline{\mathbf{3} \otimes \mathbf{2} \otimes 1}\) & \(\mathbf{1} \otimes \mathbf{2} \otimes 3\) & \(\mathbf{1} \otimes \mathbf{2} \otimes-3\) & \\
\\
\hline
\end{tabular}
```

The $\mathrm{x} 1, \mathrm{x} 2, \ldots$ variables which may appear are free parameters which control the linear combinations of $U(1)$ 's to be preserved; they show up only when it is possible to embed more $U(1)$ 's than those requested by the user.

There are several important remarks to be made about the functions just mentioned (in particular MaximalSubgroups and Embeddings). Let us go through them one by one.

A maximal subalgebra $H$ of some $G$ can be classified as regular if its root space is generated by a subset of the root vectors of $G$ and if the Cartan subalgebra of $H$ is contained in the one of $G .{ }^{6}$ If this is not the case, $H$ is a special maximal subalgebra. By default the output of the function MaximalSubgroups does not distinguish these two types of subalgebras, but it will do so if the option RplusS -> True is used.

Thanks to the seminar works [22, 23], for any Lie group $G$ it is rather straightforward to compile a list of all subgroups $\left\{H_{1}, H_{2}, \cdots, H_{i}\right\}$ which can possibly be maximal subgroups. However, some of the $H_{i}$ in such list might not be maximal (they might be contained in an another subgroup $H_{j \neq i}$ in the list) and it is considerably harder to establish which subgroups are actually maximal.

To appreciate this difficulty, note that it is not enough to compare the size of various subgroups $H_{i}$. For example, even though it is obviously possible to embed an $S U(2)$ group

[^7]in $S U(2) \times U(1)$, both are listed as maximal subgroups of $S U(3)$. The reason behind this seemingly contradictory situation is that it is possible to embed $S U(2)$ in $S U(3)$ in two inequivalent ways. In fact, under the subgroup $S U(2)_{1} \subset S U(2) \times U(1) \subset S U(3)$ a triplet of $S U(3)$ decomposes as a doublet plus a singlet $(\mathbf{3} \rightarrow \mathbf{2}+\mathbf{1})$, while the triplet is still irreducible under $S U(2)_{2} \subset S U(3)(\mathbf{3} \rightarrow \mathbf{3})$.

The concept of inequivalent embeddings of a subgroup which was just invoked also requires further explanation. An embedding $\phi$ of a group $H$ in $G$ is a function which associates the elements of $H$ to distinct elements of $G$ while preserving the structure of $H$ ( $\phi$ is said to be an injective homomorphism). Two embeddings of a group $H$ in $G-\phi_{1}$ and $\phi_{2}$ - are said to be linearly equivalent (henceforth just equivalent) if for any representation $\rho$ of $G$ the maps $\rho \circ \phi_{1}$ and $\rho \circ \phi_{2}$ are isomorphic representations of $H$, which is the same as saying that they lead to the same branching rules. ${ }^{7}$

When compiling lists of embeddings with the functions MaximalSubgroups and Embeddings, the program assumes this definition of equivalence, with the following caveat. Two embeddings might also be considered to be equivalent for the following reasons:

- A symmetry ${ }^{8}$ of the group or the subgroup. Consider for the example the embedding of $S U(4) \times S U(3)$ in $S U(7)$ associated with the branching rule $\mathbf{7} \rightarrow(\mathbf{4}, 1)+(\mathbf{1}, \mathbf{3})$. Clearly, there is another embedding under which it is the $\mathbf{7}$ branching in this way, and therefore $\mathbf{7} \rightarrow(\overline{\mathbf{4}}, 1)+(\mathbf{1}, \overline{\mathbf{3}})$. Furthermore, by swapping the representations of each of the subgroup factors - $S U(4)$ and $S U(3)$ - by their conjugates, we should also have the embeddings $\mathbf{7} \rightarrow(\mathbf{4}, \mathbf{1})+(\mathbf{1}, \overline{\mathbf{3}})$ and $\mathbf{7} \rightarrow(\overline{4}, 1)+(\mathbf{1}, \mathbf{3})$. In this way, from one branching rule one can immediately infer the existence of three others. We have complex conjugated representations, and this operation is associated with a symmetry of the $S U(n)$ groups: graphically, the Dynkin diagram is flipped by this operation. There is also such a $Z_{2}$ symmetry for the groups $S O(2 n)$ and $E(6)$, although in the case of the $S O(4 n)$ groups it is not associated to complex conjugation. Note also that the Dynkin diagram of $S O(8)$ is symmetric under the larger $S_{3}$ group of permutations.
- Permutations of equal factor groups. Consider the embedding of $S U(2) \times S U(2)$ in $S p(6)$ associated with the branching rule $\mathbf{6} \rightarrow(\mathbf{4}, \mathbf{1})+(\mathbf{1}, \mathbf{2})$. By swapping the two $S U(2)$ 's of the subgroup, we may trivially derive the embedding $\mathbf{6} \rightarrow(\mathbf{1}, \mathbf{4})+(\mathbf{2}, \mathbf{1})$. An analogous consideration also applies to equal factors, if there is any, of the group itself.

These are considered to be trivial variations of a single embedding by GroupMath, hence they are factored out in the outputs. The same criteria seems to have been adopted in reference [12] which lists the maximal subgroups of the simple Lie groups up to rank 8, and by [13] which extends this list up to groups of rank 20. A comparison of the output of MaximalSubgroups with these two references reveals the following differences. With regards

[^8]to the first reference [12], the program does not indicate $S U(2) \times S U(2) \times S U(2)$ as a maximal subgroup of $S O(12)$ while $S p(6)$ does appear on the list of maximal subgroups of $S p(14)$. In both instances, the problem seems to lie with [12] as have been reported in the literature [24]. On the other hand, comparing with table 226 of the second version of [13], GroupMath indicates the followings additional special maximal subgroups: $S O(5)$ for the groups $S p(20)$ and $S O(30)$, and $S O(8)$ for $S O(28)$.

The calculations performed by MaximalSubgroups are time consuming, but fortunately this function only requires one input: a simple group $G$. Hence MaximalSubgroups was pre-run on all $G$ 's up to rank 25 , and its results were saved to a file which is loaded on startup (for larger groups which were not pre-computed the running time can be long).

Up to version 1.0, GroupMath would find all embeddings of some subgroup $H$ in a group $G$ by automating the algorithm which is also used in computations done by hand: all chains of maximal subgroups $G_{0} \equiv G \rightarrow G_{1} \rightarrow G_{2} \rightarrow \cdots \rightarrow G_{q} \equiv H$ would be calculated, and duplicated embeddings obtained in this way would be eliminated. Unfortunately, this method can be very time consuming when the ranks of $G$ and $H$ are very different. Therefore, the function Embeddings ${ }^{9}$ works differently in the latest version of the program. It implements the idea described in [25] for the particular case $G=S U(n)$ and $H=S U(3) \times S U(2) \times U(1)$ but which can be extended for any other classical group $G$, and any $H$. The algorithm will not be explained in detail here; nevertheless we shall consider the example $G=S U(14)$ and subgroup $H=S U(5)$. Rather than dealing with an enormous number of chains of maximal subgroups from $G$ to $H$, we might consider instead the possible branching rules of the fundamental representation 14 :

$$
\begin{align*}
\mathbf{1 4} & \rightarrow \mathbf{1 0}+4 \times \mathbf{1} \\
& \rightarrow 2 \times \mathbf{5}+4 \times \mathbf{1} \\
& \rightarrow \mathbf{5}+\overline{\mathbf{5}}+4 \times \mathbf{1} \\
& \rightarrow \mathbf{5}+9 \times \mathbf{1} \tag{9}
\end{align*}
$$

These are the only possibilities, factoring out the trivial variations discussed earlier. Note that the $\mathbf{1 4}$ cannot branch into 14 singlets of $S U(5)$ as that would imply that the algebra generators $T^{a}$ of the subgroup are all null matrices.

Furthermore, the four possibilities in expression (9) correspond to valid embeddings. The reason is simple: the set of 24 matrices with dimensions $14 \times 14$ forming the indicated representations of $S U(5)$ (which happen to be reducible) are certainly unitary and all have unit determinant, therefore in all four cases they are a subset of the $14^{2}-1$ matrices which form the fundamental representation of $S U(14)$.

In this way, it is not hard to handle even bigger rank differences: for example, there are 104 inequivalent ways of embeddings $S O(10)$ in $S U(100)$. Furthermore, from the branching rules of the fundamental representation of $S U(n)$ it is straightforward to extract a projection matrix which can then be used to decompose other representations. The number of $U(1)$ 's which commute with the subgroup $H$ is given by the number irreps into which the fundamental representation of $S U(n)$ decomposes, minus 1 (see [25]). This means that

[^9]for the cases shown in expression (9), the first embedding admits 4 extra $U(1)$ 's - i.e. $H=S U(5) \times U(1)^{4}$ would be possible - while the second/third/forth embeddings can accommodate up to 5/5/9 U(1)'s. Finally, with some adaptations this technique can also be used on the other classical groups: $G=S O(n)$ and $S p(2 n)$.

There is one more function to be discussed in relation to symmetry breaking, namely SubgroupCoefficients. What it does, its input, as well as its output are rather elaborate, so let us proceed in steps.

Consider an $S U(3)$ triplet $\left(t_{1}, t_{2}, t_{3}\right)^{T}$ and and anti-triplet $\left(\bar{t}_{1}, \bar{t}_{2}, \bar{t}_{3}\right)^{T}$ such that the combination $t_{i} \bar{t}_{i}$ is invariant under this group. As mention above, it is possible to embed $S U(2)$ in $S U(3)$ such that a triplet branches into a doublet plus a singlet. To be explicit,

$$
\left(\begin{array}{c}
t_{1}  \tag{10}\\
t_{2} \\
t_{3}
\end{array}\right)=B_{1}\left(\begin{array}{c}
d_{1} \\
d_{2} \\
s
\end{array}\right) \quad \text { and }\left(\begin{array}{c}
\bar{t}_{1} \\
\bar{t}_{2} \\
\bar{t}_{3}
\end{array}\right)=B_{2}\left(\begin{array}{c}
d_{1}^{\prime} \\
d_{2}^{\prime} \\
s^{\prime}
\end{array}\right),
$$

where the $d_{i}$ and $d_{i}^{\prime}$ are the components of doublets, while $s$ and $s^{\prime}$ are $S U(2)$ singlets; $B_{1}$ and $B_{2}$ are 3 by 3 unitary matrices. We may re-write the original $S U(3)$ invariant in terms of the new components, and from $S U(2)$ invariance alone we know that

$$
\begin{equation*}
t_{1} \bar{t}_{1}+t_{2} \bar{t}_{2}+t_{3} \bar{t}_{3}=\alpha_{1}\left(d_{1} d_{2}^{\prime}-d_{2} d_{1}^{\prime}\right)+\alpha_{2} s s^{\prime} \tag{11}
\end{equation*}
$$

and all that is left is to determine is $\alpha_{1}$ and $\alpha_{2}$. These two numbers would be free if we were to ask only for invariance under the $S U(2)$ subgroup; however, invariance under the full $S U(3)$ group fixes them (to be precise, it fixes their ratio). Note also that the $\alpha_{i}$ depend on how we choose to normalize the $S U(2)$ and $S U(3)$ invariants, as well as on the matrices $B_{1}$ and $B_{2}$. For example, a change $t_{i} \bar{t}_{i} \rightarrow 2 t_{i} \bar{t}_{i}$ would imply doubling the $\alpha_{i}$, while insertion of a minus sign in the third column of $B_{1}$ would lead to a change of sign for $\alpha_{2}$.

With these caveats in mind, SubgroupCoefficients calculates the $\alpha_{i}$ coefficients and the $B_{i}$ matrices for a product of representations, assuming the explicit representation matrices returned by RepMatrices and the invariants as normalized by Invariants. It works only for regular embeddings.

```
SubgroupCoefficients[<group>,<list of representations>,
<projection matrix>,<subgroup information>]
```

Details on how to use this function can be found in the program's built-in documentation files. For the example at hand, we would write

```
In[\rho]:= breakInfo = {{1, {1}}};
    result = SubgroupCoefficients[SU3, {3, -3}, SU2, breakInfo];
ln[0]:= (* The Bi matrices *)
    MatrixForm /@ result[[1, 2]]
```



```
In[v]:= (* The \alpha ( coefficients *)
    result[[2, 5, 1]]
Out[0]= {-1, 1}
```

From the rest of the data saved in the variable result - which is not fully printed above - one can tell that the first number in the last output is the coefficient of the contraction of the two doublets $\left(\alpha_{1}\right)$ and the second number corresponds to the singlets $\left(\alpha_{2}\right)$. We can test this result with Invariants. The relevant expressions are

```
In[-]:= invSU3 = Invariants[SU3, {3, -3}][[1]] /.
    {a[1] -> t1, a[2] ->t2, a[3] ->t3, b[1] ->tb1, b[2] ->tb2, b[3] ->tb3}
Out[0]= t1 tb1 + t2 tb2 + t3 tb3
In[-]:= inv1SU2 = Invariants[SU2, {2, 2}][[1]] /. {a[1] -> d1, a[2] -> d2,
        b[1] }->\textrm{dp1,b}\textrm{b}[2]->\textrm{dp}2
    inv2SU2 = Invariants[SU2, {1, 1}][[1]] /. {a[1] > s, b[1] > sp}
Out[0]= d2 dp1 - d1 dp2
Out[0]= S Sp
```

From the $B_{i}$ matrices printed above we get the dictionary \{t1->d1, t2->d2, t3->s, tb1->dp2, tb2->-dp1, tb3->sp\} and with it we can indeed make a correspondence between the $S U(3)$ and $S U(2)$ invariants, using the coefficients $\left(\alpha_{1}, \alpha_{2}\right)=(-1,1)$ :

```
In[~]:= invSU3 /. {t1 }->\textrm{d}1,\textrm{t}2->\textrm{d}2,\textrm{t}3->\textrm{s},\textrm{tb}1->\textrm{dp}2,\textrm{tb}2->-\textrm{dp}1,\textrm{tb}3->\textrm{sp}
    -inv1SU2 + inv2SU2 == %
Out[-]= - d2 dp1 + d1 dp2 + s sp
Out[0]= True
```

This is a simple example which can be worked out by hand; however, more elaborate cases often need to be considered (see for example the Georgi-Jarlskog relations [26]). SubgroupCoefficients can handle products of an arbitrary number of representations, but
note that in all cases the interpretation of its output requires an understanding of the corresponding output of the RepMatrices and Invariants functions.

## 4. Permutation groups

The permutations of $m$ objects form a group which is usually called $S_{m}$. The family of finite groups $S_{1}, S_{2}, \ldots$ is important in various contexts, including in the analysis of the decomposition of direct products of Lie group representations (this is to be discussed in subsection 4.3). Mostly because of this last connection, GroupMath contains several functions related to the permutation groups which are described in this section. The following is a quick summary of the most important features of these groups.

The number of elements of the $S_{m}$ finite group (i.e. its order) is $m$ !. Each of its irreducible representations can be labeled with a partition $\lambda$ of $m$, which is a list of non-increasing natural numbers $\lambda_{i}$ adding up to $m$. For example, $\{4\},\{3,1\},\{2,2\},\{2,1,1\}$ and $\{1,1,1,1\}$ are the possible partitions of 4 , therefore $S_{4}$ has 5 irreducible representations. A partition $\lambda=\left\{\lambda_{1}, \lambda_{2}, \cdots\right\}$ is very often represented by a Young diagram which contains $\lambda_{i}$ boxes in row $i$, with all rows left-aligned. For instance,

$$
\begin{equation*}
\{5,4,1,1\}=\stackrel{\square}{\square} . \tag{12}
\end{equation*}
$$

Partitions also play another role. As with any finite group, the elements of $S_{m}$ can be organized in conjugacy classes and, in the case of the permutation groups, the conjugacy classes themselves can be labeled with partitions. Take $S_{4}$ : using the cycle notation, the 5 classes consist on the permutations of the form $(\cdots),(\cdots)(\cdot),(\cdot)(\cdot \cdot),(\cdot \cdot)(\cdot)(\cdot)$ and $(\cdot)(\cdot)(\cdot)(\cdot) .{ }^{10}$

A Young tableau with shape $\lambda$ is obtained by filling the Young diagram of $\lambda$ with natural numbers. A standard Young tableau must be filled with the values $1,2, \ldots, m$ (assuming $\lambda$ is a partition of $m$ ), and these numbers must be arranged in such a way that they increase along each row (from left to right) and along each column (from top to bottom). For example

$$
\begin{array}{|l|l|}
\hline 1 & 3  \tag{13}\\
\hline 2 & 4 \\
\hline
\end{array}
$$

Less stringent conditions apply to a semi-standard Young tableau. ${ }^{11}$ It is filled with the natural numbers up to some value $c$ which does not need to be related to the shape $\lambda$, and repetitions are allowed. Numbers must increase along each column and they cannot decrease along rows. For instance, there are three such tableaux for $c=2$ and $\lambda=\{3,1\}$ :

$$
\begin{array}{|l|l|l|l|l|l|}
\hline 1 & 1 & 1  \tag{14}\\
\hline 2 & & \\
\hline
\end{array}, .
$$

[^10]It turns out that the number of standard Young tableaux with shape $\lambda$ is the same as the dimension of the irreducible representation (of some permutation group) associated to this partition. On the other hand, the number of semi-standard Young tableau of shape $\lambda$ and using the natural numbers up to $c$ is given by the so-called Hook content formula (see [27]). ${ }^{12}$

The group $S_{m_{1}} \times S_{m_{2}} \times \cdots$ is contained in the larger $S_{m}$ group with $m=\sum_{i} m_{i}$, and it might be important to know how do the representations of these two groups relate. For example the irreducible representation of $S_{5}$ associated to $\lambda=\{4,1\}$ is reducible when restricted to the subgroup $S_{2} \times S_{2} \times S_{1}$ : it decomposes as two copies of $(\{2\},\{2\},\{1\})$ plus one copy of $(\{1,1\},\{2\},\{1\})$ and $(\{2\},\{1,1\},\{1\})$. We can also ask what are the representations of $S_{5}$ which contain the irreducible representation $(\{2\},\{2\},\{1\})$ of $S_{2} \times$ $S_{2} \times S_{1}$ (for example). The generic answer to this type of question is provided by the well known Littlewood-Richardson rule [28]. In this particular case, it provides the following answer involving the functions $s_{\lambda}$ (these are the Schur functions):

$$
\begin{equation*}
s_{\{2\}} s_{\{2\}} s_{\{1\}}=s_{\{5\}}+2 s_{\{4,1\}}+2 s_{\{3,2\}}+s_{\{2,2,1\}}+s_{\{3,1,1\}} . \tag{15}
\end{equation*}
$$

The definition and importance of the functions appearing here will not be discussed further. Instead, I will simply mention that the equation above can be interpreted as saying that $(\{2\},\{2\},\{1\})$ is contained once in the irreducible representations $\{5\},\{2,2,1\}$ and $\{3,1,1\}$ of $S_{5}$, and twice in both $\{4,1\}$ and $\{3,2\}$.

After this quick review of the main features of the permutation group $S_{m}$, we move on to a description of the GroupMath functions connected to it.

## Partitions, Young diagrams and Young tableaux

It is possible to draw the Young diagram associated to a partition $\lambda$ with the function YoungDiagram.

## YoungDiagram[<partition>]



The standard Young diagrams associated with a given partition can be computed by calling GenerateStandardTableaux. By default, each tableau is given as a list of lists, following the format $\{<$ numbers in row 1>, <numbers in row 2>, ...\}. With the option Draw set to True, they are instead printed as Young tableaux.

[^11]```
GenerateStandardTableaux[<partition>]
```

$$
\begin{aligned}
& \operatorname{In}[\rho]:=\text { GenerateStandardTableaux [\{3, 3, 1\}, Draw } \rightarrow \text { True] }
\end{aligned}
$$

$$
\begin{aligned}
& , \\
& ,, \\
& \\
&
\end{aligned}
$$

One can transpose Young diagrams and tableaux by converting their rows into columns and vice-versa. The functions TransposePartition and TransposeTableaux perform such operation.

TransposePartition[<partition>]

## TransposeTableaux[<tableaux>]

```
In[v]:= TransposePartition[{3, 3, 1}]
Out[-]={3, 2, 2}
```

$\operatorname{In}[-]:=\operatorname{TransposeTableaux}[\{\{1,2,3\},\{4,5,6\},\{7\}\}]$
Out[ $[=\{\{1,4,7\},\{2,5\},\{3,6\}\}$

The number of semi-standard Young tableau with some shape $\lambda$ filled with the natural numbers $1,2, \cdots, c$ can be computed with the function HookContentFormula.

HookContentFormula[<partition>, <c>]

```
In[-]:= HookContentFormula[{3, 1}, 2]
Out[0]= 3
```

Note that the number of tableaux shown in (14) is indeed 3.

## $S_{m}$ representations

We will now consider functions involving the representations (and classes) of the $S_{m}$ group. The dimension of an irreducible representation given by a partition $\lambda$ can be calculated with SnIrrepDim; SnClassOrder computes the order (the number of elements) of a class given by a partition $\mu$ and SnCl assCharacter calculates the character of a class $\mu$ in an irreducible representation $\lambda$.

## SnIrrepDim[<irrep partition>]

```
In[ø]:= SnIrrepDim[{3, 3, 1}]
Out[0]= 21
```


## SnClassOrder[<class partition>]

## SnClassOrder [\{2, 2\}]

Out[ $[=3$

## SnClassCharacter[<irrep partition>,<class partition>]

```
In[v]:= SnClassCharacter[{11, 1}, {6, 3, 3}]
Out[[]= - 1
```

Here is a short code which prints the character table of $S_{4}$ using SnClassCharacter:

```
\(\ln [\cdot]:=\mathbf{m}=\mathbf{4}\);
    classesSm = irrepsSm = IntegerPartitions [m];
    tbl = Table[SnClassCharacter[irrep, class], \{irrep, irrepsSm\},
        \{class, classesSm\}];
    tbl = Transpose[Prepend [Transpose[tbl], Style[\#, \{Bold, Red\}] \& /@irrepsSm]];
    PrependTo[tbl, Style[\#, \{Bold, Red\}] \& /@ Join[\{"Irrep\\Class"\}, classesSm]];
    Grid[tbl, Frame -> All]
```

Out $[0]=$| Irrep\Class | $\{4\}$ | $\{3,1\}$ | $\{2,2\}$ | $\{2,1,1\}$ | $\{1,1,1,1\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{4\}$ | 1 | 1 | 1 | 1 | 1 |
|  | $\{3,1\}$ | -1 | 0 | -1 | 1 |
| 3 |  |  |  |  |  |
| $\{2,2\}$ | 0 | -1 | 2 | 0 | 2 |
| $\{2,1,1\}$ | 1 | 0 | -1 | -1 | 3 |
| $\{1,1,1,1\}$ | -1 | 1 | 1 | -1 | 1 |

The matrices associated with an irreducible representation $\lambda$ can be calculated explicitly in a particular basis - with SnIrrepGenerators. However, note that the number of elements
of $S_{m}(=m!)$ grows rapidly with $m$, which means that even for small $m$ 's it would be unwise to provide the representation matrices for each element. On the other hand, it is known that the full group can be generated by just the transposition (12) (exchange of the first two objects) and the cyclic permutation $(12 \cdots m) .{ }^{13}$ Therefore SnIrrepGenerators provides only the matrices for these two elements of the group. They are always real and orthogonal (if this last property is not required, the option OrthogonalizeGenerators->False can be used to significantly speed up the calculation).

## SnIrrepGenerators[<partition>]

$$
\begin{aligned}
& \ln [\sigma]:=\text { MatrixForm /@ SnIrrepGenerators [\{3, 1\} ] } \\
& \text { Out }[\cdot]=\left\{\left(\begin{array}{ccc}
\frac{1}{2} & -\frac{1}{2 \sqrt{3}} & -\sqrt{\frac{2}{3}} \\
-\frac{1}{2 \sqrt{3}} & \frac{5}{6} & -\frac{\sqrt{2}}{3} \\
-\sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{3}
\end{array}\right),\left(\begin{array}{ccc}
-\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\
-\frac{1}{2 \sqrt{3}} & -\frac{1}{6} & \frac{2 \sqrt{2}}{3} \\
-\sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{3}
\end{array}\right)\right\}
\end{aligned}
$$

```
In[\rho]:= MatrixForm /@ SnIrrepGenerators[{3, 1},OrthogonalizeGenerators }->\mathrm{ False]
```

Out[ $]=\left\{\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1\end{array}\right),\left(\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1\end{array}\right)\right\}$

The product of various irreducible representations $\lambda_{i}$ of a permutation group can be decomposed into irreducible parts with DecomposeSnProduct. The output is a list of the irreducible representations contained in the product, and their multiplicity. There is no limit to the number of representations being multiplied.

## DecomposeSnProduct[<list of partitions>]

```
In[o]:= DecomposeSnProduct[{{4, 2}, {3, 2, 1}}]
Out[f]= {{{5, 1}, 1}, {{4, 2}, 2}, {{4, 1, 1}, 2}, {{3, 3}, 1}, {{3, 2, 1}, 3},
    {{3,1,1, 1}, 2}, {{2, 2, 2}, 1}, {{2, 2, 1, 1}, 2}, {{2, 1, 1, 1, 1}, 1}}
```

In this particular example, the output affirms that the $S_{6}$ product $\{4,2\} \times\{3,2,1\}$ decomposes as $1\{5,1\}+2\{4,2\}+\cdots+1\{2,1,1,1,1\}$.

As mentioned in the brief summary of the features of permutations groups, we may analyze what happens to $S_{m}$ 's representations under the subgroup $S_{m_{1}} \times S_{m_{2}} \times \cdots$ with $m_{1}+m_{2}+\cdots=m$. The function SnBranchingRules calculates the decomposition of an

[^12]irreducible representation of $S_{m}$, while LittlewoodRichardsonCoefficients computes all irreps of $S_{m}$ which contain a given representation of the subgroup.

## SnBranchingRules[<Sm irrep partition>,<\{m1,m2,...\}>]

$\ln [\cdot]=\operatorname{SnBranchingRules}[\{4,1\},\{3,2\}]$
outf $-=\{\{\{\{3\},\{2\}\}, 1\},\{\{\{2,1\},\{2\}\}, 1\},\{\{\{3\},\{1,1\}\}, 1\}\}$
The first argument of this function $(\{4,1\})$ must be a partition which identifies an irreducible representation of some $S_{m}$ group. The second argument ( $\{3,2\}$ ) is simply a list of the indices $m_{i}$ of the subgroup $S_{m_{1}} \times S_{m_{2}} \times \cdots$. The output for the example shown here implies that the irreducible representation $\{4,1\}$ of $S_{5}$ decomposes as $1(\{3\},\{2\})+1(\{2,1\},\{2\})+$ $1(\{3\},\{1,1\})$ under $S_{3} \times S_{2}$.

## LittlewoodRichardsonCoefficients[<list of partitions>]

```
In[f]:= LittlewoodRichardsonCoefficients[{{2, 1}, {2}}]
Out[0]={{{4, 1}, 1}, {{3, 2}, 1}, {{3, 1, 1}, 1}, {{2, 2, 1}, 1}}
```

In the code above, the irreducible representation $(\{2,1\},\{2\})$ of $S_{3} \times S_{2}$ was passed to the function LittlewoodRichardsonCoefficients. The output indicates that it can be found once in each of the following $S_{5}$ irreps: $\{4,1\}$ (in agreement with the previous example where the SnBranchingRules function was used), $\{3,2\},\{3,1,1\}$ and $\{2,2,1\}$.

## Permutation symmetry of products of Lie group representations

The decomposition of products of Lie group representations can be performed with the function ReduceRepProduct described earlier. However, when there are repeated representations in these products, it is possible to say more about their decomposition in irreducible representations of the Lie group.

Consider these two examples:

$$
\begin{align*}
\mathbf{2} \times \mathbf{2} & =\mathbf{1}_{A}+\mathbf{3}_{S} \quad[S U(2)]  \tag{16}\\
\mathbf{1 0} \times \mathbf{1 0} & =\mathbf{1}_{S}+\mathbf{4 5}_{A}+\mathbf{5 4} \mathbf{4}_{S} \quad[S O(10)] \tag{17}
\end{align*}
$$

In both cases the subscripts indicate what happens to an irrep under permutations of the representations on the left. More generally, we may think of the direct product

$$
V^{m}=\underbrace{V \times V \times \cdots V}_{m}
$$

of a vector space $V$ with dimension $n$. We take $|1\rangle,|2\rangle, \cdots,|n\rangle$ to be basis vectors of $V$, in which case the set of all $\left|i_{1} i_{2} \cdots i_{m}\right\rangle \equiv\left|i_{1}\right\rangle\left|i_{2}\right\rangle \cdots\left|i_{m}\right\rangle$ form a basis of $V^{m}$. Under
permutations $\pi$ of the $V$ factors,

$$
\begin{equation*}
\left|i_{1} i_{2} \cdots i_{m}\right\rangle \xrightarrow{\pi}\left|i_{\pi(1)} i_{\pi(2)} \cdots i_{\pi(m)}\right\rangle \tag{18}
\end{equation*}
$$

the $n^{m}$-dimensional vector space $V^{m}$ decomposes into irreducible subspaces $V_{\lambda}^{m}$ labeled by partitions $\lambda$ of $m$. This means that under permutations the elements of $V_{\lambda}^{m}$ do not mix with those of another $V_{\lambda^{\prime} \neq \lambda}^{m}$. For example, in the case of $m=2$ and $n=3$ we have a 9 -dimensional vector space $V^{2}$ which decomposes into $V_{\{2\}}^{2} \equiv V_{S}^{2}$ and $V_{\{1,1\}}^{2} \equiv V_{A}^{2}$ (the symmetric and the anti-symmetric subspaces). The first space, $V_{\{2\}}^{2}$, is generated by the six vectors

$$
\begin{equation*}
\{|11\rangle,(|12\rangle+|21\rangle) / \sqrt{2},(|13\rangle+|31\rangle) / \sqrt{2},|22\rangle,(|23\rangle+|32\rangle) / \sqrt{2},|33\rangle\} \tag{19}
\end{equation*}
$$

while

$$
\begin{equation*}
\{(|12\rangle-|21\rangle) / \sqrt{2},(|13\rangle-|31\rangle) / \sqrt{2},(|23\rangle-|32\rangle) / \sqrt{2}\} \tag{20}
\end{equation*}
$$

forms a basis of $V_{\{1,1\}}^{2}$. Permutations do not mix the two subspaces.
On the other hand, any linear transformation on $V$ induces a linear transformation $T$ on $V^{m}$ and crucially the action of $T$ commutes with the operation $\pi$ shown in (18). Hence $V^{m}$ is decomposable into irreducible subspaces under the action of $T \times \pi$. Since this is true for any transformation $T$ induced by a linear transformation on $V$, it is certainly true if we consider those forming a Lie group.

In the case of $m=3$ and $n=6$, the relevant subspaces are $V_{\{3\}}^{3}$ (of dimension 56), $V_{\{2,1\}}^{3}$ (of dimension 140) and $V_{\{1,1,1\}}^{3}$ (of dimension 20). Now take for example the 6 dimensional irreducible representation of $S U(3)$ acting on $V: V_{\{3\}}^{3}$ decomposes under $S U(3)$ as $\mathbf{1}+\mathbf{2 7}+\overline{\mathbf{2 8}}, V_{\{2,1\}}^{3}$ decomposes as $2(\mathbf{8}+\mathbf{2 7}+\overline{\mathbf{3 5}})$ and $V_{\{1,1,1\}}^{3}$ decomposes as $\mathbf{1 0}+\overline{\mathbf{1 0}}$. It is no coincidence that $V_{\{2,1\}}^{3}$ contains two copies of the $S U(3)$ representations $\mathbf{8}, \mathbf{2 7}$ and $\overline{35}$ : it is a consequence of the fact that under $S_{3}$ permutations the entire subspace $V_{\{2,1\}}^{3}$ transforms as the 2-dimensional irreducible representation $\{2,1\}$. In other words, $V_{\{2,1\}}^{3}$ transforms under the irreducible representations $(\mathbf{8},\{2,1\}),(\mathbf{2 7},\{2,1\})$ and $(\overline{\mathbf{3 5}},\{2,1\})$ of $S U(3) \times S_{3}$. Using subscripts for the $S_{3}$ irreps, the full decomposition reads

$$
\begin{equation*}
\mathbf{6} \times \mathbf{6} \times \mathbf{6}=\mathbf{1}_{\{3\}}+\mathbf{2} \mathbf{7}_{\{3\}}+\overline{\mathbf{2 8}}_{\{3\}}+\mathbf{8}_{\{2,1\}}+\mathbf{2} \mathbf{7}_{\{2,1\}}+\overline{\mathbf{3 5}}_{\{2,1\}}+\mathbf{1 0}_{\{1,1,1\}}+\overline{\mathbf{1 0}}_{\{1,1,1\}} . \tag{21}
\end{equation*}
$$

It is often important to have this kind of data (see for instance [6]). In those cases, the function ReduceRepProduct is insufficient. However, at least in principle it is easy to recover the permutation group information. Consider some representation $R$ of $G$ with weights $\omega_{i}$, $i=1, \cdots, n$. The $n^{2}$-dimensional reducible representation $R^{2}$ has weights $\omega_{i}+\omega_{j}$, with $i$ and $j$ taking all values from 1 to $n$. However, $R^{2}$ splits in a symmetric subspace $R_{\{2\}}^{2}$ and an anti-symmmetric one $R_{\{1,1\}}^{2}$. The weights associated with the former are $\omega_{i}+\omega_{j}$ with $1 \leq i \leq j \leq n$, while the weights associated with the latter are $\omega_{i}+\omega_{j}$ with $1 \leq i<j \leq n$. More generally, the rule is as follows. Weights $\omega_{i}$ are a reference to the entries of the diagonal elements of a Lie algebra $G$ in some representation $R$, hence the character (i.e. trace) of the actual group transformation matrices are the sum $\sum_{j} \exp \left(i \omega_{j}\right) \equiv \sum_{j} x_{j}$. Then, the character of $R_{\lambda}^{m}$, where $\lambda$ is a partition of $m$, are given by the Schur polynomial $s_{\lambda}\left(x_{1}, x_{2}, \cdots, x_{n}\right)$.

For example, $s_{\{2\}}\left(x_{1}, x_{2}, \cdots, x_{n}\right)=\sum_{1 \leq i \leq j \leq n} x_{i} x_{j}$ hence from each monomial we extract the weights $\omega_{i}+\omega_{j}$ with $1 \leq i \leq j \leq n$, which matches what was expected for $R_{\{2\}}^{2}$. On the other hand, $s_{\{1,1\}}\left(x_{1}, x_{2}, \cdots, x_{n}\right)=\sum_{1 \leq i<j \leq n} x_{i} x_{j}$ which yield the weights $\omega_{i}+\omega_{j}$ with $1 \leq i<j \leq n$ for $R_{\{1,1\}}^{2}$.

Since Schur polynomials can be readily calculated, is quite easy to calculate the decomposition of $R^{m}$ in irreducible spaces under the joint action of the Lie group and the permutation group, somewhat in analogy with what can be done to decompose $R^{m}$ disregarding the permutation group. However a brute force approach will only work as long as the dimension of $R^{m}$ in not too large. Fortunately, the authors of the computer code LIE describe in the program's manual [7] an algorithm do to the computation efficiently. Following the nomenclature used in LIE, this algorithm was implemented in a function called Plethysms.

## Plethysms[<simple Lie group>,<representation>,<partition>]

```
In[\mp@code{]:= Plethysms[SU3, 6, {3}, UseName }->\mathrm{ True]}
Out[f]={{\overline{\mathbf{28}},\mathbf{1}},{\mathbf{1},\mathbf{1}},{\mathbf{27},\mathbf{1}}}
In[\sigma]:= Plethysms[SU3, 6, {2, 1}, UseName }->\mathrm{ True]
Out[\sigma]={{\overline{\mathbf{35}},\mathbf{1}},{8, 1},{27, 1}}
ln[`]:= Plethysms[SU3, 6, {1, 1, 1}, UseName }->\mathrm{ True]
Out[0]={{\overline{10},1},{10,1}}
```

The user needs to provide a simple Lie group $G$, a representation $R$ of that group, and a partition $\lambda$ of some number $m$, and the output is a list of the representations of $G$ (with multiplicity) contained in $R_{\lambda}^{m}$. The option UseName -> True can be used to convert Dynkin coefficients into names of representations in the output.

In practical situations, it might be necessary to consider (a) products of different representations, with some of them repeated, and (b) non-simple Lie groups. Relying on Plethysms, the function PermutationSymmetry handles these more complicated cases, being comparable in this respect to ReduceRepProduct. The difference between the two is that PermutationSymmetry also calculates the relevant permutation symmetries; in other words, it provides the decomposition of products of representations of a semi-simple Lie group $G$ in irreducible representations of the group $G \times S_{m} \times S_{m^{\prime}} \times \cdots .{ }^{14}$ The value of the integers $m$, $m^{\prime}$ depends on the product considered. For example, in the $S U(2)$ product of $\mathbf{2} \times \mathbf{2} \times \mathbf{3} \times \mathbf{3}$ the relevant permutation group is $S_{2} \times S_{2}$, while in $2 \times 2 \times 2 \times 3$ it is $S_{3} \times S_{1}$.

## PermutationSymmetry[<Lie group>, <list of representations>]

[^13]```
In[f]:= PermutationSymmetry[SU2, {2, 2, 3, 3}, UseName }->\mathrm{ True]
Out[o]={{{1, 2}, {3, 4}}, {{{7,{\square\square, \square\square}},1}, {{3, {\square\square, \square\square}}, 2},
```



```
        {{5,{\boxminus,\square\square}},1},{{1,{\square,\square}},1},{{1,{\boxminus,\square\square}},1}}}
```

```
In[v]:= PermutationSymmetry[SU2, {2, 2, 2, 3}, UseName }->\mathrm{ True]
Out[ ]={{{1, 2, 3}, {4}},{{{6,{\square\square\square,\square}},1}, {{2, {\square\square\square,\square}}, 1},
    {{2,{\square,\square}},1},{{4,{\square\square,\square}},1},{{4,{\square,\square}},1}}}
```

```
\(\ln [\rho]:=\operatorname{PermutationSymmetry[\{ E6,~SU5\} ,~\{ \{ 27,5\} ,~\{ 27,5\} \} ,~UseName~} \rightarrow\) True]
Out[ \([=\{\{\{1,2\}\},\{\{\{\overline{\mathbf{3 5 1}} \otimes 15,\{\square\}\}, 1\},\{\{\overline{\mathbf{3 5 1}} \otimes 10,\{\square\}\}, 1\},\{\{\overline{\mathbf{2 7}} \otimes 15,\{\square\}\}, 1\}\),
    \(\{\{\overline{\mathbf{2 7}} \otimes 10,\{\boxminus\}\}, 1\},\{\{\overline{\mathbf{3 5 1}} \otimes 15,\{\square\}\}, 1\},\{\{\overline{\mathbf{3 5 1}} \otimes 10,\{\square\}\}, 1\}\}\}\)
```

The output of the function consists of a list with two parts: \{<part1>, <part2>\}. The content of the first part is rather trivial: it identifies the position of repeated representations. For example, in the product $\mathbf{2 \times 2 \times 3 \times 3}$ the first and second representations are the same, and so are the last two, hence the first part reads $\{\{1,2\},\{3,4\}\}$. The second part of the output is the non-trivial one: it consists on the list of the irreducible representations of the group $G \times S_{m} \times S_{m^{\prime}} \times \cdots$ in the format \{<representation of G>,\{<partition of $m>,<p a r t i t i o n ~ o f ~ m ’>, \ldots\},<m u l t i p l i c i t y>\}$. It is possible to use the option UseName -> True to convert the representation of $G$ to a name and at the same time draw Young diagrams for all the partitions.

The function PermutationSymmetryOfInvariants operates in a similar fashion, except that it only displays the invariants under $G$ in a product. Therefore the second part of the output of this function does not contain the piece of information <representation of G> mentioned earlier.

## PermutationSymmetryOfInvariants[<Lie group>,<list of representations>]

```
In[\varnothing]:= PermutationSymmetryOfInvariants[SU5, {5, 10, 10, 5, 10, 10}, UseName }->\mathrm{ True]
Out[ ] = {{{1, 4}, {2, 3, 5, 6}},
```



## The special unitary groups and $S_{m}$

There is a particularly close relation between the special unitary groups and the permutation groups. Indeed, it is possible to specify a representation of $S U(n)$ with a Young tableaux, or equivalently a partition $\lambda$ (which in turn can be associated to an irreducible representation of a permutation group). That is because (a) the product of $m$ copies of the fundamental representation $S U(n)$ with a permutation symmetry $\lambda$ transforms irreducibly under $S U(n)$ and (b) all irreducible representations of $S U(n)$ can be obtained in this way.

As such, the irreducible representations of special unitary groups can be labeled with either Dynkin coefficients or partitions, and the conversion between the two notations can be performed with the functions ConvertToPartitionNotation and ConvertPartitionToDynkinCoef. It is worth noting that to get Dynkin coefficients from a partition it is necessary to know the value of the $n$ in $S U(n)$.

```
ConvertToPartitionNotation[<SU(n) representation>]
```

```
In[`]:= ConvertToPartitionNotation[{1, 0}] (* triplet of SU(3) *)
    DrawYoungDiagram[%]
Out[0]= {1}
Out[0]=
In[~]:= ConvertToPartitionNotation[{0, 1}] (* anti-triplet of SU(3) *)
    DrawYoungDiagram[%]
Out[0]= {1, 1}
Out[-]=}
```

ConvertPartitionToDynkinCoef[<n>, <partition>]
$\ln [\cdot]:=$ ConvertPartitionToDynkinCoef [3, \{1\}]
ConvertPartitionToDynkinCoef[3, \{1, 1\}]
Out[ $\left[=\left\{\begin{array}{ll}\mathbf{1}, 0\end{array}\right\}\right.$
Out $[0]=\{\boldsymbol{0}, \mathbf{1}\}$

## 5. Concluding remarks

The various functions described here will hopefully constitute a useful aid for performing computations with Lie and permutation groups. GroupMath's built-in documentation files include information on some details and lesser important functions which were not mentioned in this text.

Future versions of the program are likely to include code for systematically expressing $S U(n)$ invariants in the tensor formalism, using the Levi-Civita $\epsilon$ and the Kronecker $\delta$ tensors, since such a notation is simple and convenient. Another possibility under consideration is the addition of a limited amount of code related to finite groups beyond the ones describing permutations.

## Acknowledgments

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## A. Brief review of Lie algebras

## Simple, semi-simple and reductive Lie algebras

A Lie algebra $L$ is a vector space endowed with a binary operation usually called the Lie bracket (or commutator) $[\cdot, \cdot]$ which is (i) linear, (ii) anti-symmetric and (iii) obeys the Jacobi identity. The dimension $d$ of the algebra is simply the dimension of the vector space, so that all elements of $L$ can be written as a linear combination of $d$ generators.

Consider now the 2-dimensional Lie algebra $L$ generated by $x$ and $y$ and such that $[x, y]=x$. There is something peculiar about this example: by taking the commutator among all elements of $L$ we do not get back the full algebra $L$ but rather only the elements proportional to $x$, so $[L, L] \neq L$. This algebra is said to be solvable because, starting with $L^{(1)} \equiv L$, the sequence of algebras $L^{(i)} \equiv\left[L^{(i-1)}, L^{(i-1)}\right]$ become equal to the trivial algebra $\{0\}$ for a large enough value of $i$. In contrast to these cases, the semi-simple Lie algebras are those for which $[L, L]=L$. This definition excludes elements $x \in L$ which commute with all the algebra (i.e. $[x, L]=\{0\}$ ). Lie algebras $A$ exclusively composed of such elements, $[A, A]=\{0\}$, are called commutative and they are associated to $U(1)^{d}$ abelian groups (with $d$ being the dimension of the algebra $A$ ). The set of reductive Lie algebras consist of the semi-simple Lie algebras and their direct sum with abelian algebras.

For any Lie algebra $L$ we must have $[L, L] \subset L$, and any subspace $S$ of $L$ which also closes under the Lie bracket, $[S, S] \subset S$, is a subalgebra. Furthermore, a subspace $I$ of $L$ with the property $[I, L] \subset I$ is called an ideal (which is also a subalgebra). A non-abelian Lie algebra $L$ is called simple if it does not have any ideals apart from the obvious ones: $L$ itself and $\{0\}$. Semi-simple Lie algebras can be decomposed into a direct sum of simple ones, hence they are the algebra of Lie groups $G_{1} \times G_{2} \times \cdots$ where the $G_{i}$ are called simple groups.

If is therefore enough to study simple Lie algebras in order to understand the semi-simple and reductive ones. The rest of this appendix discusses some properties of semi-simple Lie algebras which, without loss of generality, can be seen as being simple.

## Cartan subalgebra, root vectors and roots.

A maximum of $r$ of the algebra generators commute among themselves. The vector space generated by them is called a Cartan subalgebra and the number $r$ is called the rank of the algebra.

The remaining generators $e_{\alpha}$ (root vectors) can be organized in such a way that ad (h) acts diagonally on them for any $h$ in the Cartan subalgebra, meaning that

$$
\begin{equation*}
\left[h, e_{\alpha}\right]=\alpha(h) e_{\alpha} \tag{22}
\end{equation*}
$$

for some number $\alpha(h)$. The function $\alpha$ itself is called a root. Since this function is linear, it is fully determined by the values $\alpha\left(h_{i}\right)$ for some basis $\left\{h_{1}, h_{2}, \cdots, h_{r}\right\}$ of the Cartan
subalgebra. Therefore pragmatically one can identify a root with a list of $r$ numbers. The set of all roots forms the root system $\Delta$.

## Positive and simple roots

If $\alpha$ is a root so is $-\alpha$, therefore the root vectors come in pairs $\left\{e_{\alpha}, e_{-\alpha}\right\}$. It is then appropriate to distinguish positive from negative roots. This can be done by selecting $r$ independent roots $\alpha_{i}$ and expressing each root as a linear combination $c_{i} \alpha_{i}$; the positive roots are those for which the first non-zero $c_{i}$ coefficient is positive. Simple roots are positive roots which cannot be expressed as the sum of two other positive roots, and there are always as many of them as the rank $r$ of the algebra.

## Inner product of roots, the Cartan matrix and Dynkin diagrams

Any root $\alpha$ can be associated to an element $h_{\alpha}$ of the Cartan subalgebra through the relation

$$
\begin{equation*}
\alpha\left(h^{\prime}\right) \equiv \operatorname{Tr}\left[\operatorname{ad}\left(h^{\prime}\right) \operatorname{ad}\left(h_{\alpha}\right)\right]=\sum_{\gamma \in \Delta} \gamma\left(h^{\prime}\right) \gamma\left(h_{\alpha}\right) \tag{23}
\end{equation*}
$$

which is valid for any element $h^{\prime}$ of the Cartan subalgebra. An inner product $\langle\cdot, \cdot\rangle$ among roots is often defined with this $\alpha \leftrightarrow h_{\alpha}$ correspondence:

$$
\begin{equation*}
\langle\alpha, \beta\rangle \equiv \beta\left(h_{\alpha}\right) . \tag{24}
\end{equation*}
$$

For a set of simple roots $\alpha_{i}$, the Cartan matrix $A$ has entries

$$
\begin{equation*}
A_{i j}=\frac{2\left\langle\alpha_{i}, \alpha_{j}\right\rangle}{\left\langle\alpha_{j}, \alpha_{j}\right\rangle} \tag{25}
\end{equation*}
$$

and it fully characterizes a complex simple Lie algebra $L$. It can be represented graphically as a Dynkin diagram. It turns out that there is a limited number of possibilities for $A$ and consequently for $L$ : $S U(n)$ with $n \geq 2, S O(n)$ with $n=5$ or $\geq 7, S p(2 n)$ with $n \geq 3$, plus the exceptional Lie algebras $G_{2}, F_{4}, E_{6}, E_{7}$ and $E_{8}$.

## Chevalley-Serre relations

With the simple roots $\alpha_{i}$ we may define a particularly interesting set of $3 r$ algebra elements:

$$
\begin{equation*}
e_{i} \propto e_{\alpha_{i}}, f_{i} \propto e_{-\alpha_{i}} \text { and } h_{i} \equiv \frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle} h_{\alpha_{i}} \tag{26}
\end{equation*}
$$

such that the following Chevalley-Serre relations hold:

$$
\begin{align*}
{\left[e_{i}, f_{j}\right] } & =\delta_{i j} h_{j}  \tag{27}\\
{\left[h_{i}, e_{j}\right] } & =A_{j i} e_{j}  \tag{28}\\
{\left[h_{i}, f_{j}\right] } & =-A_{j i} f_{j} \tag{29}
\end{align*}
$$

Except for $S U(2)$, these $3 r$ elements do not generate the full algebra. However, the missing generators can easily be obtained by repeatedly computing the commutator of the various $e$ 's and $f$ 's above: $\left[e_{i}, e_{j}\right],\left[\left[e_{i}, e_{j}\right], e_{k}\right], \ldots$, and the same for the $f$ 's.

## Representations, weights and Dynkin coefficients

A representation $R$ is a linear and structure-preserving map which associates a linear transformation over some vector space $\Phi$ to each Lie algebra element. The vector space is usually $\mathbb{C}^{n}$ and linear transformations over $\mathbb{C}^{n}$ are just $n \times n$ matrices, so it is often convenience to think of a representation as a set of matrices (equal in number to the dimension $d$ of the algebra). The integer $n$ is the dimension of the representation (assumed here to be finite).

If it is not possible to simultaneously block-diagonalize all matrices of $R$ with a basis change, this representation is said to be irreducible (an irrep for short); otherwise it is reducible. The latter are direct sums of irreps.

There is a particularly convenient way of labeling the $n$ components of $\Phi$. All elements of the Cartan subalgebra can be represented as diagonal matrices (in the same basis); for example the $h_{i}$ of the Chevalley-Serre relations act on some component $\omega$ of $\Phi$ as follows:

$$
\begin{equation*}
R\left(h_{i}\right) \Phi^{\omega}=\omega\left(h_{i}\right) \Phi^{\omega} . \tag{30}
\end{equation*}
$$

Just like with roots, $\omega$ (called a weight) can be viewed as either a linear function or an $r$-dimensional vector. It is then possible to assign a Cartan subalgebra element $h_{\omega}$ to each weight $\omega$ in analogy to the procedure used for roots, and in this way the inner product $\langle\cdot, \cdot\rangle$ mentioned earlier can be extended to weights as well. The last equation becomes

$$
\begin{equation*}
R\left(h_{i}\right) \Phi^{\omega}=2 \frac{\left\langle\omega, \alpha_{i}\right\rangle}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle} \Phi^{\omega} \tag{31}
\end{equation*}
$$

It should be noted that two or more components of $\Phi$ might be associated to the same weight.

Weights can be sorted as follows. Each weight is written as a linear combination of the simple roots $\alpha_{i}$ and then the coefficients are compared: $\omega=c_{i} \alpha_{i}$ is considered to be higher/larger than $\omega^{\prime}=c_{i}^{\prime} \alpha_{i}$ if the first non-zero value of $c_{i}-c_{i}^{\prime}$ for $i=1, \cdots, r$ is positive; otherwise the weight $\omega$ is lower/smaller than $\omega^{\prime}$. With this understanding, there will always be a weight $\Lambda$ which is the highest in every irreducible representation, and it can be used to unambiguously label it. In particular, irreps are often specified in terms of their Dynkin coefficients:

$$
\begin{equation*}
\Lambda_{i}=2 \frac{\left\langle\Lambda, \alpha_{i}\right\rangle}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle} . \tag{32}
\end{equation*}
$$

## The Weyl group, orbits, and dominant weights

For any representation, if $\omega$ is a weight so is

$$
\begin{equation*}
s_{\alpha}(\omega) \equiv \omega-2 \frac{\langle\omega, \alpha\rangle}{\langle\alpha, \alpha\rangle} \tag{33}
\end{equation*}
$$

for any root $\alpha$. The $s_{\alpha}$ transformations, which can be seen as reflecting the weight vectors $\omega$, generate a group called the Weyl group W . The orbit of $\omega$ in W consists of the set of
weights which can be obtained from $\omega$ by performing on it one or more reflections. Weights in the same orbit are said to be conjugate to one-another. It turns out that in all Weyl group orbits there is a single dominant weight with non-negative Dynkin coefficients.

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[^0]:    ${ }^{1}$ Repeated indices are assumed to be summed over.
    ${ }^{2}$ More generally, the convention is that the biggest root $\alpha_{\max }$ of a simple algebra has norm 1: $\left\langle\alpha_{\max }, \alpha_{\max }\right\rangle=1$.

[^1]:    ${ }^{3}$ The algorithms used by DominantWeights, DominantConjugate, ReflectWeight and WeylOrbit are essentially the ones in [19, 20].

[^2]:    RepMatrices[<group>,<representation>]

[^3]:    RepMinimalMatrices[<group>,<representation>]

[^4]:    ${ }^{4}$ Furthermore，when there are repeated representations being multiplied，the Invariants function ensures that the output decomposes in a very specific way into irreducible representations of the relevant permutation group．More information on this feature can be found in the built－in documentation of the function．

[^5]:    ${ }^{5}$ It should be pointed out that projection matrices are not unique: several of them can be associated to the same branching rules.

[^6]:    DecomposeRep[<group>,<representation of group>,<subgroup>,
    <projection matrix>]

[^7]:    ${ }^{6}$ Just by looking at the Dynkin diagram of $G$ and following simple rules one can find all such subalgebras.

[^8]:    ${ }^{7}$ In practice, it is not necessary to check the branching rules for all representations $\rho$ of $G$ : it suffices to consider the defining representation $(1,0,0, \cdots, 0)$, except for $S O(2 n)$ where it is also necessary to consider the branching rules of the spinor representation $(0, \cdots, 0,0,1)$.
    ${ }^{8}$ The symmetries discussed here are given by the outer automorphism group.

[^9]:    ${ }^{9}$ Previously called FindAllEmbeddings.

[^10]:    ${ }^{10}$ Consider four objects $a, b, c$ and $d$. The elements (1423) and (13) (24) correspond to the permutations $\{a, b, c, d\} \rightarrow\{c, d, b, a\}$ and $\{a, b, c, d\} \rightarrow\{c, d, a, b\}$.
    ${ }^{11}$ It is important to note that there is no consensus in the literature on the definition of this type of tableaux.

[^11]:    ${ }^{12}$ This number also corresponds to the size of the irreducible representation of $S U(c)$ associated to the partition or Young diagram $\lambda$ : for example $\lambda=\{3,1\}$ is associated to the triplet representation of $S U(2)$ if $c=2$. The relation between partitions and representations of the special unitary groups is explored in section 4.3.

[^12]:    ${ }^{13}$ The remaining elements of $S_{m}$ can be obtained by repeated multiplication of just these two.

[^13]:    ${ }^{14}$ The cost of this extra information is a longer computation time.

