Cadabra and Python algorithms in General Relativity and Cosmology II: Gravitational Waves

Oscar Castillo-Felisola^a, Dominic T. Price^b, Mattia Scomparin^{c,*}

^aDepartamento de Física and Centro Científico y Tecnológico de Valparaíso (CCTVal),
 Universidad Técnica Federico Santa María
 Casilla 110-V, Valparaíso, Chile

 ^bDepartment of Mathematical Sciences, Durham University,
 South Road, Durham DH1 3LE, United Kingdom

 ^c Via del Grano 33,
 Mogliano Veneto, Italy

Abstract

Computer Algebra Systems (CASs) like Cadabra Software play a prominent role in a wide range of research activities in physics and related fields. We show how Cadabra language is easily implemented in the well established Python programming framework, gaining excellent flexibility and customization to address the issue of tensor perturbations in General Relativity. We obtain a performing algorithm to decompose tensorial quantities up to any perturbative order of the metric. The features of our code are tested by discussing some concrete computational issues in research activities related to first/higher-order gravitational waves.

Keywords: Computer Algebra System, Cadabra, Gravitation, Classical Field theory, Gravitational Waves, Perturbative Field Theory, Python, Cosmology, General Relativity.

1. Introduction

This article is the continuation of our previous work [1] describing the use of Cadabra algorithms within the framework of General Relativity and Cosmology. Cadabra is an open-source Computer Algebra System (CAS) designed for field-theory problems and specialized for both abstract and component computations. The Cadabra language is Python-oriented and works closely with the Python library for symbolic calculations SymPy, but uses its own Python class for storing expressions and uses LaTeX formatting for inputs and outputs.

In the last few years many introductory works about Cadabra [2–5] have appeared. Among all the applications described in this wide bibliography include problems dealing with field theory, differential equations and symbolic matrix algebra. Nevertheless, only a small number of them discuss the currently available spectrum of physical applications which development of Cadabra algorithms supported by Python coding allows. Thanks to its many language constructs and wide ranging standard library, Python is a very complete and flexible programming language employed in many high-performance scientific back-end and front-end applications. In addition, Python offers a wide range of external libraries such as *NumPy*, *SymPy*, *SciPy* and *Matplotlib* which make it a powerful and up-to-date ecosystem for dealing with mathematical, statistical, biological, human and social sciences. From this perspective, an in-depth investigation into how Cadabra algorithms could be consolidated and enhanced with Python is strongly motivated.

Of course there are many other CASs available and which are used in research on a daily basis and which contain specialist packages and functionality for dealing with a wide range of problems, however one of the main motivations for the writing of this paper is to display how even without dedicated functionality already available in the language

^{*}Corresponding author

Email addresses: o.castillo.felisola@protonmail.com (Oscar Castillo-Felisola), dominicprice@outlook.com (Dominic T. Price), mattia.scompa@gmail.com (Mattia Scomparin)

it is not difficult to create very powerful and generic programs by using Python with Cadabra. In particular, we wish to draw to the attention of the reader the following constructs which differ from other CASs and which are used throughout this paper when constructing algorithms and performing other manipulations:

- Expressions in Cadabra are mutable which provides powerful ways to interact with them, especially by combining this with Python iterator constructions which allows expressions to be visited and modified using for loops and other intuitive constructs.
- By allowing expressions to be queried for different property types, 'smart' algorithms can be written which respond differently to different types of input. Properties can also be dynamically attached to new symbols which allows the construction of functions which programmatically define new sets of objects.
- Python has many useful and performant inbuilt containers which have rich interfaces, and also provides implementations of many more useful container types such as defaultdict in the collections library. These make storing and accessing related expressions very easy and natural which improves the organisation of the code.
- As Cadabra uses a Late X format for inputting and outputting expressions, expressions can be created or manipulated by using Python's string and regex functions if a feature is not implemented in Cadabra. While this is not always an optimal solution, it is one of the great strengths of Python which makes it such a productive language that it is almost always possible to achieve some end.

In order to showcase as many of these features as possible, we have chosen to take a more in-depth look at tensor perturbation theory in General Relativity which is by no means the only field topic whose analysis is assisted by using the tools Python and Cadabra offer; but as it is a very well studied topic with a lot of literature dedicated to it we hope this makes the paper accessible to a large audience. Tensor perturbations, also known as Gravitational Waves, are disturbances in the curvature of the spacetime metric and represent one of the most definitive theoretical and phenomenological signatures of General Relativity as a standard theory of gravitation. Recently, gravitational waves have attracted a wide interest thanks to the detections made by the LIGO and Virgo interferometers sourced by compact binary coalescences [6–12].

Having taken gravitational waves as the most effective case study to examine, the main purpose of this work is therefore to put forth the first structured insight on the approach, design, and implementation of hybrid Cadabra and Python functions/commands and methods of defining a new coding environment within Cadabra. In particular, we use our multi-annual experience gained in both Cadabra and Python to provide a programming vision of the entire life-cycle of such a development process, which consists of five phases as displayed in Table 1.

Step	Description	Ref. Secs.
Theory	Analysis, modelling, formalism, reference objects	Sec. 2
Ref. libraries development	evelopment Supporting functions & environments, coding core paradigms	
Main algorithms development	Code development, optimization, new user-oriented commands	Sec. 4
Notebooks	Explore the new environment, upgrade Cadabra, communicate	Sec. 5, 6, 7
Testing & Discussion	Timing, performance, perspectives & generalizations	Sec. 8

Table 1: Steps of the development process with related reference sections of this article.

When talking about gravitational waves, methods related to *Perturbation Theory* are essential and Section 2 is dedicated to providing a general overview of tensorial perturbative expansions in General Relativity. In particular, the spacetime metric representing the gravitational field is treated as a series of successive, increasingly small tensor perturbations around a background, where an N-th order gravitational wave solution is obtained by (i) truncating the series by keeping only the first N terms, and (ii) forcing consistency with perturbed General Relativity equations and gauge conditions.

Operating tensorial expansions within Cadabra naturally leads to us introducing in Section 3 a set of new hybrid Cadabra and Python functions (see Table 2) collected inside a library which we call perturbations.cnb. These functions are united by the goal of defining the basic perturbative elements we will deal with. They fix the core

assumptions of our algorithms including the formalism, e.g. how perturbations will be represented, and the new user-oriented interface commands, e.g. how to invoke the new perturbative objects obtained in the code. This step is deeply influenced by the Pythonic aesthetic in favour of writing functions that are perfect for the immediate use-case, and which can be developed and tested separately before finally being put together.

Section 4 presents our first result, an algorithm called perturb() able to deal with tensor perturbations for every order of a generic tensor defined in terms of the spacetime metric. The following example illustrates the use of our machinery to perturb the Cristoffel connection $\Gamma^{\mu}_{\nu\tau}$, stored in the ch variable, up to the second order of metric tensor:

```
connection = perturb(ch,[gLow,gUpp],'pert',2)
```

where gLow and gUpp hold information about the metric tensor.

The code returns the Python dictionary connection which contains useful representations of the perturbation such as a symbolic decomposition in the 'sym' key:

$$\Gamma^{\mu}_{\ \nu\tau} = {}^{(\Gamma)}_{\mu}_{\ \nu\tau} + {}^{(1)}_{\ \Gamma}^{\mu}_{\ \nu\tau} + {}^{(2)}_{\ \Gamma}^{\mu}_{\ \nu\tau}$$

and an array of the perturbative orders in the 'ord' key, so in this case connection['ord'][2] is

$${}^{(2)}\Gamma^{\mu}_{\ \nu\tau} = \frac{1}{2}\partial_{\tau}{}^{(2)}h_{\nu}{}^{\mu} + \frac{1}{2}\partial_{\nu}{}^{(2)}h_{\tau}{}^{\mu} - \frac{1}{2}\partial^{\mu}{}^{(2)}h_{\nu\tau} - \frac{1}{2}{}^{(1)}h^{\mu\sigma}\partial_{\tau}{}^{(1)}h_{\nu\sigma} - \frac{1}{2}{}^{(1)}h^{\mu\sigma}\partial_{\nu}{}^{(1)}h_{\tau\sigma} + \frac{1}{2}{}^{(1)}h^{\mu\sigma}\partial_{\sigma}{}^{(1)}h_{\nu\tau}$$

More information about the data structure can be found in Table 4.

The algorithm is very performative and has been developed to be both flexible and intuitive, goals which the use of the timing analysis tools introduced in our fist paper [1] has helped us to achieve. The results of this analysis are reported in Section 8. The perturb() function takes full consideration of the symmetries and other fundamental properties of the perturbed object. This section also presents several examples of computations, focusing on the dependence of timings on properties of perturbed expressions.

The suite composed by the perturbations.cnb library and the perturb() algorithm defines a new Cadabra perturbative environment that must be explored and used alongside normal Cadabra notebooks. As toy models to provide a staging ground, Section 5 and Section 6 are completely dedicated to first order gravitational waves, obtained in the so-called harmonic gauge, with the respective sections calculating (i) the wave-equation which describes the propagation of first order gravitational waves, and (ii) the related wave-relations. In the last Section 7 we apply our machinery to reproduce some results obtained in Ref. [13], where a complete analytical analysis of vacuum high-order gravitational waves solutions is given. Hence, we provide a computational counterpart of such analysis, highlighting the natural predisposition of Cadabra Software and the newly developed perturbative environment to treat the heavy and onerous nature of high-order tensor calculations. In particular, we discuss how to obtain the numerical coefficients of higher-order gravitational wave solutions.

A final discussion is drawn in Section 8 where we analyse the timing, performance, and perspectives of our new perturbative environment, highlighting the full consistency of our results with the predicted exponential timing behaviour derived from the number of perturbed terms.

The experience gained through all phases of the development process, from designing to release, took us into a leading role in the design of Cadabra and Python algorithms. During the development process we needed to identify the algorithms' bottlenecks and boost their performance to satisfy notebooks' demands about practical manipulating, debugging, and programming issues not covered by Cadabra itself.

With reference to the process followed, we believe that there are some important takeaways of programming in Cadabra:

- Developing Cadabra algorithms supported by Python offers superior algorithm solutions. The well-established coding vision coming from Python suggests a more structured way of thinking about algorithms and provides a powerful factory of tools that can inspire many improvements of Cadabra.
- Cadabra architecture is completely predisposed to work with Python data-structures. Cadabra expression are
 one of the main paradigms of Cadabra's architecture and they are Python objects that can be treated as data
 inside more complex algorithm structures to be processed.

- Cadabra algorithms can be debugged and tested in a very smart way. We exhibit in the first part [1] for the first time some new approaches on Cadabra code debugging. A new approach to test timing performance has been introduced within the new timing.cnb library.
- The development of new computing environments induces an improvement of Cadabra's core. When working with our new perturbative environment, we needed to introduce new functions into Cadabra's core library, exhibiting a wide range of possible improvements for Cadabra. The fact that the introduction of a new environment has induced an improvement of the standard Cadabra environment shows that Cadabra has new undiscovered domains of improvements that approaches like ours can discover. From this perspective, Cadabra's code and community mutually support and inspire progress with each other.

Finally, it is important to notice that the results developed in this paper can be easily customized with respect to any physical model to which a perturbative approach must be applied, as is the case with statistical mechanics [14], radiative transfer [15], quantum mechanics [16], particle physics [17], fluid mechanics [18], chemistry [19], and many other contexts. Our algorithms are built in such a way that one can cut-and-paste expressions straight from this paper into a Cadabra notebook and customize the content according to specific needs. The main notebooks of this work will be completely available on the official GitLab repository.

Since new functions have been introduced in the Cadabra core libraries, our code requires a minimum of version 2.3.8 of Cadabra software, and as it is heavily based on the cooperation of Cadabra with Python it is completely incompatible with the old 1.x versions.

2. Overview of Tensor Perturbations in General Relativity

There is a vast, practically limitless, choice of works aimed to introducing the basics of General Relativity and its implications in Cosmology (see for example Refs. [20, 21]). In our analysis, we will follow the notations and conventions defined in the first part of our previous work [1].

As a starting point, let us consider spacetimes affected by small perturbations $h_{\mu\nu}$ about Minkowski spacetime $\eta_{\mu\nu}$. More precisely, the metric tensor $g_{\mu\nu}$ can be written

$$g_{\mu\nu} \equiv {}^{\scriptscriptstyle{(0)}}g_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \simeq \eta_{\mu\nu} + {}^{\scriptscriptstyle{(1)}}h_{\mu\nu}. \tag{1}$$

Note that the linear approximation has been employed and higher order terms than first have been neglected. Consequently, as $g_{\alpha\mu}g^{\mu\beta} = \delta^{\beta}_{\alpha}$, the contravariant metric can be written as $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, with $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. As a consequence, it is clear that the indices of all first-order tensorial quantities will be raised or lowered using the flat Minkowski metric. Adopting such perturbative decompositions, it is simple to exhibit that the first order Einstein's equations

$$^{(1)}R_{\mu\nu} = \kappa^{(1)}S_{\mu\nu},\tag{2}$$

can be rewritten as

$$\frac{1}{2}\partial_{\nu\rho}{}^{\scriptscriptstyle (1)}\!h_{\mu}{}^{\rho} - \frac{1}{2}\partial^{\rho}{}_{\rho}{}^{\scriptscriptstyle (1)}\!h_{\mu\nu} - \frac{1}{2}\partial_{\mu\nu}{}^{\scriptscriptstyle (1)}\!h_{\rho}{}^{\rho} + \frac{1}{2}\partial_{\mu}{}^{\rho}{}^{\scriptscriptstyle (1)}\!h_{\nu\rho} = \kappa^{\scriptscriptstyle (1)}\!S_{\mu\nu}, \tag{3}$$

with κ the constant gravitational coupling and $^{(1)}S_{\mu\nu}$ the first-order term of the source tensor, S, defined as

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T.$$

This equation does not possess unique solutions since, given a solution, it will always be possible to identify one another solution performing a coordinate transformation of the form

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\alpha}),\tag{4}$$

See https://gitlab.com/cdbgr/cadabra-gravity-II.

where $\partial_{\beta}\xi^{\alpha}$ is of the same order of $h_{\mu\nu}$. This property is known as *gauge invariance*. The redundancy can be removed by fixing a specific coordinate system. For our purposes, it is a good choice to work in the so-called *harmonic gauge*, defined by the condition

$$0 = \Gamma^{\sigma} = g^{\mu\nu} \Gamma^{\sigma}_{\mu\nu},\tag{5}$$

which perturbed up to first order and inserted in Eq. (3) in vacuum yields the first-order gravitational wave equation

$$\partial^{\alpha} \partial_{\alpha}^{(1)} h_{\mu\nu} = 0. \tag{6}$$

Without loss of generality, Eq. (6) is solved by tensor plane-wave parametrization solutions that can be parametrized as follows:

$${}^{\scriptscriptstyle (1)}h_{\mu\nu} = \mathbf{e}_{\mu\nu} \exp\left(ik_{\lambda}x^{\lambda}\right) + \bar{\mathbf{e}}_{\mu\nu} \exp\left(-ik_{\lambda}x^{\lambda}\right),\tag{7}$$

with

$$k_{\lambda}k^{\lambda} = 0$$
 and $\mathbf{e}^{\lambda}_{\nu}k_{\lambda} - \frac{1}{2}\mathbf{e}^{\mu}_{\mu}k_{\nu} = 0.$ (8)

Above, we have introduced the spacetime coordinate x^{μ} , the wave-number k_{λ} and the symmetric polarization tensor $\mathbf{e}_{\mu\nu}$ with its complex conjugate $\bar{\mathbf{e}}_{\mu\nu}$. Eqs. (8) are commonly known as the *first-order wave relations*.

The entire framework described in the present section can be extended to higher-order perturbations. As we will see in the following sections, the covariant metric can be decomposed as

$$g_{\mu\nu} \equiv \sum_{n=0}^{\bar{n}} {}^{(n)}g_{\mu\nu} = \sum_{n=0}^{\bar{n}} f_n\left(\eta, {}^{(1)}h, \cdots, {}^{(n)}h\right). \tag{9}$$

In our notation, ${}^{(n)}Y$ stands for the *n*-th perturbative order associated to the tensor *Y*, with \bar{n} the higher perturbative order we are interested in.

In such higher-order context the methods described for first-order perturbations and concept of gauge invariance hold. With reference to the main equations, Eqs. (5) and (6) become

$$0 = {}^{(n)}\Gamma^{\sigma}(\eta, {}^{(1)}h, \cdots, {}^{(n)}h) \text{ and } 0 = {}^{(n)}R_{\mu\nu}(\eta, {}^{(1)}h, \cdots, {}^{(n)}h) \text{ with } n \in \{0, ..., \bar{n}\}.$$
(10)

In particular, wave solutions propagating in vacuum satisfying Eqs. (10) admit a very general parametrization

$$^{(2n)}h_{\mu\nu} = \Phi^n \left\{^{(2n)}a \eta_{\mu\sigma}\alpha^{\sigma}_{ \nu} + iz\omega c^{-1}{}^{(2n)}b \eta_{\mu\sigma}\beta^{\sigma}_{ \nu}\right\} \exp\left(i2nk_{\rho}x^{\rho}\right) + c.c., \tag{11}$$

with, among all the objects introduced, $\Phi \equiv \mathbf{e}^{\mu}_{\nu} \mathbf{e}^{\nu}_{\mu}$ and α^{σ}_{ν} , β^{σ}_{ν} constant matrices. The numerical coefficients

$${\binom{2n}{a}, \binom{2n}{b}}$$
 for $n = 1, 2, \dots,$ (12)

characterize each perturbative order solution. Parametrization (11) means that the odd-order solutions do not contribute to the metric perturbation since one can see that all odd-order terms in the metric perturbation vanish identically. This feature may be related to the physical properties of the gravitational interaction [13].

3. Libraries

The code which accompanies this paper is organised into a number of modules, each of which contains functions and computations corresponding to a different section. See Figure 1 for the rooting. This section will introduce the content in the *libraries* directory, which contains general routines which are used throughout the remainder of the paper: (i) *header.cnb* setting up a common environment of property and object definitions, (ii) *perturbations.cnb* which defines general perturbative algorithms which are the building blocks of the discussion.

The remaining files in the *algorithms* directory are the notebooks which accompany the other sections in this paper; sec4_TensorPerturbationsGR.cnb defines an algorithm able to deal with tensor perturbations up to every order of a generic tensor defined in terms of the metric tensor (section 4), sec5_FirstOrderEqGW.cnb to get first-order gravitational waves equation (section 5), sec6_FirstOrderGWrel.cnb to get first-order gravitational wave relations (section 6), and sec7_HighOrderGWsol.cnb to study higher-order gravitational wave solutions (section 7).

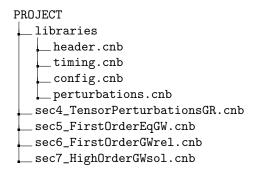


Figure 1: Rooting of the coding.

3.1. The header.cnb library

The header library is the same introduced in [1]. We refer to such paper for more information.

3.2. The config.cnb library: Defining global configuration variables

This package contains global configuration variables accessible from the notebooks. Its content simply defines the perturbation label and the maximum order in perturbation,

```
maxPertOrder = 2
pertLabel = 'pert'
```

It is particularly useful for setting the perturbation order of notebooks which differ from the default value by setting it before importing other notebooks, e.g. using

```
import libraries.config as config
config.maxPertOrder = 6
from sec4_TensorPerturbations import *
```

will now use maxPertOrder = 6, overriding the default value.

3.3. The perturbations.cnb library

This package contains various helper functions to define a standard approach to formally perturb expressions and equations in elementary ways, e.g. formally define standard perturbative formalism for Cadabra, inherit properties from unperturbed objects to perturbed ones, give perturbative decompositions of expressions and equations, substitute perturbative expansions. In order to use this package, we will import it with the standard Python import statement import libraries.perturbations as perturb.

Uploading the perturbations library, the following suite of functions will be available, of which we summarize the perspective of purpose in Table 2.

This subsection is dedicated to the definition of appropriate functions aimed at customizing the standard Cadabra commands with Python, adapting them to our purposes. Although the purpose is not immediately understood, the usefulness will be evident in the following

First of all we define a test environment, namely objects and properties to test our functions.

```
# Definitions for test cases
\partial{#}::PartialDerivative.
M{#}::LaTeXForm("\hat{\Psi}").
R{#}::LaTeXForm("\hat{\Phi}").
{M_{\mu\nu},R_{\mu\nu}}::Symmetric.
{T{#},M{#}}::Depends(\partial{#}).
```

Function	Input example	Output example
defPertSymbol	n, A	⁽ⁿ⁾ A
defPertList	\bar{n}, A	${}^{\scriptscriptstyle{(0)}}\!A,\ldots,{}^{\scriptscriptstyle{(\bar{n})}}\!A$
defPertSum	\bar{n}, A	$A = {}^{\scriptscriptstyle{(0)}}\!A + \ldots + {}^{\scriptscriptstyle{(\bar{n})}}\!A$
subsPertSums	$A = B$ $A = {}^{(0)}A + \dots + {}^{(\bar{n})}A$ $B = {}^{(0)}B + \dots + {}^{(\bar{n})}B$	$^{(0)}A + \dots + {^{(\bar{n})}}A = {^{(0)}}B + \dots + {^{(\bar{n})}}B$
getEquationPertOrder	$n, {}^{(0)}A + + {}^{(\bar{n})}A = {}^{(0)}B + + {}^{(\bar{n})}B$	$^{(n)}A = ^{(n)}B$

Table 2: The suite of functions defined in perturbations library with some examples. The ${}^{(n)}A$ and ${}^{(n)}B$ symbols denote perturbations, with associated weights and properties. The \bar{n} variable stands for the higher perturbative order of the decompositions. In particular, $n \le \bar{n}$.

```
defPertSymbol(ex:Ex, pertLabel:str, pertOrder:int) -> Ex

Returns a new object with the same structure as ex but with pertOrder appended to its name, and assigns the properties of ex to this new object
```

A really useful function to introduce is defPertSymbol, that, given a Cadabra object, a perturbative label, and a perturbative order, returns the associated the symbolic-perturbed Cadabra object. Formally defines standard perturbative formalism for Cadabra, inheriting properties from unperturbed objects to perturbed ones and fixing a perturbation label. Note: the function inherits only Symmetric, TableauSymmetry and Depends properties: other needed properties can be added by customizing the last block code at the end of the function, where such proprieties are queried to the Ex object.

```
def defPertSymbol (ex, pertLabel, pertOrder):
      # Create the perturbed version of the object by appending pertOrder to its name
      pert = ex.copy()
      pert.top().name += str(pertOrder)
       # Create anoter version where the indices are replaced by {#}
      pertGeneral = pert.copy()
       for index in pertGeneral.top().indices():
        index.erase()
       pertGeneral.top().append_child(${#}$)
10
       # Get the LaTeX form of ex with no indices
11
12
       exName = ex.copy()
      for index in exName.top().indices():
        index.erase()
14
15
       exLaTeX = exName._latex_()
       # Copy the LaTeX form of ex to pertGeneral, with the pertOrder written prefix and superscript
16
      LaTeXForm(pertGeneral, Ex(r'"\,^{^{(%s)}}%s"' % (pertOrder, exLaTeX)))
17
       # Assign properties of ex to pert and pertGeneral
18
       Weight(pertGeneral, Ex(f'label={pertLabel}, value={pertOrder}'))
19
20
      p_sym = Symmetric.get(ex)
       if p_sym is not None:
21
        p_sym.attach(pert)
22
      p_depends = Depends.get(ex)
23
       if p_depends is not None:
         p_depends.attach(pertGeneral)
25
       p_tab = TableauSymmetry.get(ex)
26
27
       if p_tab is not None:
         p_tab.attach(pert)
28
       return pert
```

Note that the function automatically recognizes some fundamental properties of the cadabra unperturbed object, and inheriths them inside the perturbed symbol. Additionally, let us highlight the use of tableaux symmetry. Some tests

```
defPertSymbol($M_{\mu\nu}$, 'pert',7);
```

```
{}^{\scriptscriptstyle{(7)}}\!\hat{\Psi}_{\mu\nu}
```

that takes the original Cadabra object $\hat{\Psi}_{\mu\nu}$ and gives the associated seventh-order perturbed symbol $\hat{\Psi}_{\mu\nu}$. In particular, the following tests hold:

```
# Testing Inherited Symmetry
x1 = canonicalise($M7_{\mu\nu}+M7_{\nu\mu}$);
# Testing Dependencies
x2 = unwrap($a \partial_{\rho}{M7_{\mu\nu}}$);
# Testing Inherited Weight
x3 = keep_weight($M7_{\mu\nu}$,$pert=7$);
x4 = keep_weight($M7_{\mu\nu}$,$pert=5$);
x5 = keep_weight($M7_{\mu}^{\nu}$,$pert=7$);
x6 = keep_weight($M7_{\mu}^{\nu}$,$pert=5$);
```

as can be easily seen in the Table 3:

Ex	Tested Property	Input	Output	Passed
x1	Symmetry	$\hat{\Psi}_{\mu\nu} + \hat{\Psi}_{\nu\mu}$	$2^{(7)}\hat{\Psi}_{\mu u}$	✓
x2	Dependencies	$\partial_{\rho}(a^{(7)}\hat{\Psi}_{\mu\nu})$	$a\partial_{ ho}{}^{^{(7)}}\hat{\Psi}_{\mu u}$	✓
хЗ	Weight	$\hat{\Psi}_{\mu u}$	$\hat{\Psi}_{\mu u}$	✓
x4	Weight	$^{(7)}\hat{\Psi}_{\mu u}$	0	√
х5	Weight	$\hat{\Psi}_{\mu}^{\ \nu}$	$\hat{\Psi}_{\mu}^{\ u}$	√
x6	Weight	$\hat{\Psi}_{\mu}^{\nu}$	0	√

Table 3: Test results for the defPertSymbol function.

Note that after the use of defPertSymbol($a_{\mu\nu}$, 'x',2), the new symbol y:=a2_{\mu\nu} results defined as $a_{\mu\nu}$. Also note that such function, once used, fixes weights and dependencies independently from type, position and number of indices, i.e. the previous example defines also the perturbative quantities (and properties) for $a_{\mu\nu\rho}$, and $a_{\mu\nu}$, and the other mixtures of indices.

Another function that we will use frequently is defPertList().

```
defPertList(ex:Ex, pertLabel:str, maxPertOrder:int) -> List[Ex]

Calls defPertSymbol on exfor all values of pertOrder up to and including maxPertOrder, and returns a list of all generated objects
```

It could be extremely useful for our scope define a set of perturbed symbols. This is exactly the purpose of the getPertList function, which takes a Cadabra object and produces with respect to it a Python list containing for each position from 0 (background) to maxPertOrder (the higher perturbative order desired) the associated perturbative symbol generated by defPertSymbol(). Referring to the passed arguments, such symbol will be characterized by a weight proportional to the array position and labelled by pertLabel. Formally the function defines all the perturbative symbols of an object from background to a max perturbative order.

```
def defPertList (ex, pertLabel, maxPertOrder):
    pertList = []

for i in range(maxPertOrder+1):
    symbol_i = defPertSymbol(ex,pertLabel,i)
    pertList.append(symbol_i)

return pertList
```

We thought it might be useful to highlight the initialization of the Python array pertLs=[] and the use of the pertLs.append() to valorize each position of the array with a Cadabra object in succession. By way of example, the following command

```
x = defPertList($M_{\mu\nu}$,'pert',2)
x[1];
```

```
\hat{\Psi}_{\mu\nu}
```

generates the list $[{}^{(0)}\hat{\Psi}_{\mu\nu}, {}^{(1)}\hat{\Psi}_{\mu\nu}, {}^{(2)}\hat{\Psi}_{\mu\nu}]$, where the *n*-th element has weight equal to *n* and inherits the the same dependencies of the $M_{\mu\nu}$ operator.

```
defPertSum(ex:Ex, exPertList:List[Ex]) -> Ex
Return an equation with ex on the left hand side, and a sum of the terms in exPertList on the right hand side i.e. write ex as a sum of its perturbative components
```

Formally defines the decomposition of an Ex object as the sum of perturbative orders contained in exPertList.

```
def defPertSum(ex, exPertList):
    pertSum := 0:
    for x in exPertList : pertSum += x
    return $@(ex) = @(pertSum)$
```

As an example, consider the following case

```
y1 := M_{\mu}:
y2 = defPertList(y1,'pert',3)
y3 = defPertSum(y1,y2);
```

$$\hat{\Psi}_{\mu} = \hat{\Psi}_{\mu} + \hat{\Psi}_{\mu} + \hat{\Psi}_{\mu} + \hat{\Psi}_{\mu} + \hat{\Psi}_{\mu} + \hat{\Psi}_{\mu}$$

```
subsPertSums(ex:Ex, partLabel:str, maxPertOrder:int, *pertSums:Ex) -> Ex
Substitute pertSums into ex, distribute the result and drop any terms higher than maxPertOrder
```

Substitutes a set of Ex pertSums into a given Ex object. The function automatically drops all terms higher than a maxPertOrder with respect to a certain partLabel. The function automatically distinguishes if the expression is or not an equation (drop weight never works fine in equations, hence we need to separate the terms and then rebuild the equation).

```
def subsPertSums(ex, pertLabel, maxPertOrder, *pertSums):
40
       # Substitute and distribute over the result
       for pertSum in pertSums:
42
         substitute(ex, pertSum)
43
       distribute(ex)
       # Remove higher order terms
45
46
       if ex.top().name == r"\equals":
         # Equation, need to drop left and right hand side separately
47
         1, r = lhs(ex), rhs(ex)
48
         for i in range(maxPertOrder+1,2*maxPertOrder+1):
           drop_weight(1, Ex(f"{pertLabel}={i}"))
50
           drop_weight(r, Ex(f"{pertLabel}={i}"))
51
         return $0(1) = 0(r)$
52
53
       else:
         for i in range(maxPertOrder+1,2*maxPertOrder+1):
54
           drop_weight(ex, Ex(f"{pertLabel}={i}"))
55
         return ex
```

The Python * symbol allows to pass an arbitrary number of argument objects to the function and entrusts a good flexibility about the function. The validity of the substitutions lies in the correct declarations passed through *pertSums. The effective functioning of this function can be seen with the following example:

```
z1 := R_{\mu}:
z2 = defPertList(z1,'pert',3)
z3 = defPertSum(z1,z2)
w = subsPertSums($M_{\mu}R_{\nu},'pert',3,y3,z3);
```

This code replaces inside the expression M_{\mu}R_{\nu} the defPertSum of M_{\mu} and R_{\nu} separately:

$${}^{(0)}\hat{\Psi}_{\mu}{}^{(0)}\hat{\Phi}_{\nu} + {}^{(0)}\hat{\Psi}_{\mu}{}^{(1)}\hat{\Phi}_{\nu} + {}^{(0)}\hat{\Psi}_{\mu}{}^{(2)}\hat{\Phi}_{\nu} + {}^{(0)}\hat{\Psi}_{\mu}{}^{(3)}\hat{\Phi}_{\nu} + {}^{(1)}\hat{\Psi}_{\mu}{}^{(0)}\hat{\Phi}_{\nu} + {}^{(1)}\hat{\Psi}_{\mu}{}^{(1)}\hat{\Phi}_{\nu} + {}^{(1)}\hat{\Psi}_{\mu}{}^{(2)}\hat{\Phi}_{\nu} + {}^{(2)}\hat{\Psi}_{\mu}{}^{(0)}\hat{\Phi}_{\nu} + {}^{(2)}\hat{\Psi}_{\mu}{}^{(1)}\hat{\Phi}_{\nu} + {}^{(2)}\hat{\Psi}_{\mu}{}^{(2)}\hat{\Phi}_{\nu} + {}^{(2)}\hat{\Psi}_{$$

Note that terms with an higher order than maxPertOrder equal to 3 are removed! Very useful for long computations to avoid terms that will be neglected!

```
getEquationPertOrder(ex:Ex, pertLabel:str, pertOrder:int) -> Ex

Return ex where all terms of an order different to pertOrder are discarded
```

The function getEquationPertOrder() gets the specified perturbative order from an equation. We introduce this function because the standard command keep weight doesn't work with equations but only with expressions. Given an expanded equation, gets an estabilished perturbative order

```
def getEquationPertOrder (ex, pertLabel, pertOrder):
    1 = keep_weight(lhs(ex), Ex(f"{pertLabel}={pertOrder}"))
    r = keep_weight(rhs(ex), Ex(f"{pertLabel}={pertOrder}"))
    return $@(1) = @(r)$
```

As an example, the action of the function is as follows

```
m = subsPertSums($M_{\mu}=R_{\mu}$,'pert',3,y3,z3);
getEquationPertOrder(m,'pert',2);
```

$${}^{(0)}\hat{\Psi}_{\mu} + {}^{(1)}\hat{\Psi}_{\mu} + {}^{(2)}\hat{\Psi}_{\mu} + {}^{(3)}\hat{\Psi}_{\mu} = {}^{(0)}\hat{\Phi}_{\mu} + {}^{(1)}\hat{\Phi}_{\mu} + {}^{(2)}\hat{\Phi}_{\mu} + {}^{(3)}\hat{\Phi}_{\mu}$$

$${}^{(2)}\hat{\Psi}_{\mu} = {}^{(2)}\hat{\Phi}_{\mu}$$

where only second order terms are extracted from the initial equation.

4. Tensor perturbations in General Relativity

The following section aims to introduce and implement a set of useful algorithms designed to make Cadabra a powerful environment to deal with the fundamental symbolical computations related to gravitational waves field. In particular, our goal is to introduce a standard package of functions within Cadabra software to deal with tensor perturbations up to every maxPertOrder order of every metric tensor object.

In particular, in this notebook we derive the covariant expansion of the metric tensor in terms of tensor perturbations. We extend this result to fundamental tensor objects in General Relativity (such as the Riemann tensor, Ricci tensor and others).

4.1. Setting the perturbative environment

For this example we shall import the header, config and perturbation modules provided with the bundle of notebooks. Other standard Cadabra modules are imported by the header itself.

```
from libraries.header import *
from libraries.perturbations import *
from libraries.config import maxPertOrder, pertLabel
```

See Section 3 for more information.

To characterize our perturbative calculations must define the higher perturbative order we will deal with. Hence we finish setting up the common environment by calling init_properties and defining the highest order to which we will be performing our calculations.

```
init_properties(coordinates=$t,x,y,z$, metrics=[$g_{\mu\nu}$, $\eta_{\mu\nu}$])
```

We finish the initialisation by assigning weights to the objects that will appear in our perturbative expansion

```
5 \eta{#}::Weight(label=pert,type=multiplicative,value=0).
6 \delta{#}::WeightInherit(label=pert,type=multiplicative,value=1).
7 \partial{#}::WeightInherit(label=pert,type=multiplicative,value=1).
```

As can be easily seen, such operators are weighted as background quantities.

4.2. Covariant Metric Perturbations gLow []

Let us consider a gravitational wave sourced at great distances from the Earth, e.g by two black holes mergers or two neuron star mergers. Due to the distance, the amplitude of the gravitational wave will be very small when it arrives on Earth and this assumption justifies us to put the following decomposition of the covariant metric tensor:

$$g_{\mu\nu} = \eta_{\mu\nu} + {}^{(1)}h_{\mu\nu} + {}^{(2)}h_{\mu\nu}, \qquad (13)$$

where $\eta_{\mu\nu}$ is the flat covariant metric tensor and ${}^{(i)}h_{\mu\nu}$ are small successive higher order tensor perturbations. In our notation, maxPertOrder stands for the maximum perturbative order with witch we want to deal with. Computationally, definition (13) can be implemented in a very smart way. First of all, considering the perturbative unit $h_{\mu\nu}$, its related properties can be defined as follows:

```
8 tensorPert:=h_{\mu\nu}:
9 # Properties
10 {h_{\mu\nu},h^{\mu\nu}::Symmetric.
11 h{#}::Depends(\partial{#}).
```

To begin, let us obtain a Python list of Cadabra perturbation symbols $[^{(0)}g_{\mu\nu},^{(1)}g_{\mu\nu},^{(2)}g_{\mu\nu}]$ by using our defPertList function

```
gPertList = defPertList($g_{\mu\nu}$,pertLabel,maxPertOrder)
```

The defPertList function defines, without showing them, the objects g0, g1 g2 that we can recall every time we need. Finally notice that the defPertList function assigns to gLow['ord'] a Python list, so no initialization gLow['ord']=[] as an empty list is needed.

Let us also initialize an empty Python dictionary:

```
13 gLow = {}
```

A dictionary is a collection of objects or variables which can be saved inside it unordered, changeable and indexed through a key item. In Python dictionaries, you can access the items of a dictionary by referring to its key name. For the sake of clarity on our approach, in this work each dictionary will be the key unit containing all the information for each perturbed tensor. As a matter of fact, we will also have a specific dictionary for the connection, a specific dictionary for the Ricci tensor and so on. All these dictionaries will have in common, a part trivial cases, the key names with which access to their values. Key names have a specific meaning referring to the perturbative information we want to use.

On that note, our model of dictionary is desribed in Table 4, where we use the gLow as a prototype, the same rules will be extended to all the other dictionaries

Key	Code	Output	Description
unp	gLow['unp']	$g_{\mu u}$	Unperturbed expression
ord	gLow['ord'][1]	$^{^{(1)}}\!h_{\mu u}$	Perturbative orders of the expression into a list
sym	gLow['sym']	$g_{\mu\nu} = {}^{\scriptscriptstyle{(0)}}g_{\mu\nu} + {}^{\scriptscriptstyle{(1)}}g_{\mu\nu} + {}^{\scriptscriptstyle{(2)}}g_{\mu\nu}$	Symbolic decomposition of the expression

Table 4: Model key names and the specific perturbative information we want to use in our dictionary structure.

Anyway, focusing back up for gLow, to valorize the dictionary structure described in Table 4, we can use the following code

```
gLow['unp'] = $g_{\mu\nu}$
gLow['sym'] = defPertSum(gLow['unp'],gPertList) # Symbolic decomposition
gLow['ord'] = defPertList($h_{\mu\nu}$,pertLabel,maxPertOrder) # Initialize (momentarily) ord as a pertList of h symbols
gLow['ord'][0] = $\eta_{\mu\nu}$
for i in range(maxPertOrder+1):
    1 = gPertList[i]
    r = gLow['ord'][i]
gLow['ord'][i] = $0(1) = 0(r)$
```

We highlight the use of out function getPertList, which assigns to gLow['ord'] a alist with all the perturbative orders characterized by a weight proportional to the array position and labelled by pert. All the properties of $h_{\mu\nu}$ are inherited inside the ⁽ⁱ⁾h by the same function. The function defPertSum sums all the perturbative orders into gLow['sym'].

In order to test the effectiveness of the code, we try to print the contents of the dictionary

```
gLow['unp'];
gLow['sym'];
gLow['ord'][1];
```

getting respectively

$$g_{\mu\nu}$$
 $g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + {}^{(1)}g_{\mu\nu} + {}^{(2)}g_{\mu\nu}$
 ${}^{(1)}h_{\mu\nu}$

Note that our algorithm for gLow inherits properties of $g_{\mu\nu}$ on $^{(i)}h_{\mu\nu}$ independentely to the position of their indices position. Consequentely, part of the command used can be already employed to calculate the contravariant metric perturbations.

4.3. Contravariant Metric Perturbations gUpp []

Similarly to above we can define the gUpp object with entries unp, sym and ord. Unlike before, the definition of the gUpp [] dictionary for covariant metric perturbations is a slightly more complicated, as they must defined in order to satisfy the following identity at each perturbative order:

$$g_{\mu\alpha}g^{\alpha\nu} = \delta^{\mu}_{\nu}, \tag{14}$$

with δ_{ν}^{μ} the Krönecker delta function that has a background WeightInherit propriety defined in Section 4.1. We begin with the same approach of the previous case and load temporarily gUpp[] as:

```
23  gUpp = {}
24  gUpp['unp'] = $g^{\mu\nu}$
25  gUpp['sym'] = defPertSum(gUpp['unp'],defPertList(gUpp['unp'],pertLabel,maxPertOrder))
26  gUpp['ord'] = [None]*(maxPertOrder+1) # Initialize (momentarily) ord as empty list of fixed positions
27  gUpp['ord'][0] = $g0^{\mu\nu}=\eta^{\mu\nu}$
```

Notice that the defPertList() function is called inside defPertSum. We initialized the ord as a list of None values, except the background to eta. Note the use of g0, defined and available thanks to the use of the defPertList command used for $g_{\mu\nu}$ in previous code. Note that in this case we directly deal with equalities for ord.

So far, we have that gUpp[] has the same content of gLow[] with raised indices. Some of these values have only been provisionally uploaded and they will be soon replaced. Now, let us introduce the identity (14) written in a slightly modified form

```
\label{eq:condition} $$ ident := 0 = \left( \right^{\nu}-g_{\rho}alpha\right); $$
```

to calculate the entries.

This identity can be written in terms of the perturbative expansions held in gLow['sym'] and gUpp['sym']. Let us substitute inside it all the obtained decompositions gLow['dec'] and gUpp['dec'] with our subsPertSums() function of Subsection 3.3,

subsPertSums(ident,pertLabel,maxPertOrder,gLow['sym'],gUpp['sym']);

$$0 = \delta_{\rho}^{\ \nu} - {}^{(0)}g_{\rho\alpha}^{\ (0)}g^{\alpha\nu} - {}^{(0)}g_{\rho\alpha}^{\ (1)}g^{\alpha\nu} - {}^{(0)}g_{\rho\alpha}^{\ (2)}g^{\alpha\nu} - {}^{(1)}g_{\rho\alpha}^{\ (0)}g^{\alpha\nu} - {}^{(1)}g_{\rho\alpha}^{\ (0)}g^{\alpha\nu} - {}^{(2)}g_{\rho\alpha}^{\ (1)}g^{\alpha\nu} - {}^{(2)}g_{\rho\alpha}^{\ (0)}g^{\alpha\nu}$$
(15)

Note that unnecessary high order terms like $^{^{(2)}}g_{\rho\alpha}^{}g^{\alpha\nu}$ are removed, since maxPertOrder=2.

As ${}^{(0)}g_{\mu\nu} = \eta_{\mu\nu}$ it is clear that ${}^{(0)}g_{\rho\alpha}{}^{\alpha\nu}g^{\alpha\nu} = \delta_{\rho}{}^{\nu}$ and so we can find ${}^{(0)}g^{\mu\nu}$ by looking at the *i*-th term in the expansion, which is what the function getEquationpertOrder from the libraries perturbations nortebook does. Examining only terms of order *i*, the quantity ${}^{(0)}g^{\alpha\nu}$ can only be accompanied by the factor ${}^{(0)}g_{\rho\alpha}$ which is the flat metric. By contracting with $\eta^{\mu\rho}$ we can isolate this quantity and then write it in terms of the lower order perturbations by substituting in for the values already calculated in gLow['ord'] and gUpp['ord']. Following such procedure, using Eq. (15) we can assign the correct values to gUpp['ord'] as follows:

```
for i in range(1,maxPertOrder+1):
       # Select the ith order perturbation
       order_i = getEquationPertOrder(ident,pertLabel,i)
32
33
       # Multiply through by g0^{\mathrm{hu}\ } and move gi^{\mathrm{hu}\ } to the other side
       manip.multiply_through(order_i, $g0^{\mu\rho}$)
34
35
       distribute(order_i)
       substitute(order_i, $g0^{\mu\rho}g0_{\rho\alpha}->\delta^{\mu}_{\alpha}$)
       manip.add_through(order_i, Ex(r'\delta^{\mu}_{\alpha}'+r'g'+str(i)+r'^{\alpha\nu}'))
37
       # Substitute definitions from gLow['ord'] and gUpp['ord']
38
       for j in range(0,i+1):
        substitute(order_i,gLow['ord'][j])
40
       for j in range(0,i):
41
        substitute(order_i,gUpp['ord'][j])
42
       # Simplify result and assign to gUpp['ord']
43
       distribute(order_i)
       eliminate kronecker(order i.repeat=True)
45
46
       eliminate_metric(order_i,repeat=True)
       canonicalise(order_i)
       gUpp['ord'][i] = order_i
```

In order to test the effectiveness of the code, we print the content of the dictionary

```
gUpp['unp'];
gUpp['sym'];
gUpp['ord'][1];
```

$$g^{\mu\nu} = {}^{(0)}g^{\mu\nu} + {}^{(1)}g^{\mu\nu} + {}^{(2)}g^{\mu\nu}$$

$$g^{\mu\nu} = -{}^{(1)}h^{\mu\nu}$$

4.4. Generalized Perturbative Function

As we have discussed in the previous sections, working with metric tensor perturbations has in a naturally way introduced a dictionary-model approach in order to collect all the obtained perturbative informations. Following the same philosophy, we modularized our coding in order to highlight the main logical building-blocks operations to decompose a generic tensor quantity. In fact, up to some little modifications, all operations previously discussed can be chained automatically to define a generic decomposition function that we call perturb.

```
perturb(ex:Ex, pertDicts:Dict[Ex], pertLabel:str, maxPertOrder:str) -> Dict[Ex]

Take an Ex object and produces with respect to it a Python dictionary containing the valorized perturbative structure defined in Table 4.
```

The perturb function below takes an Ex object and produces dictionaries similar to gUpp and gLow. Note that the ord entry is only calculated when the input ex is an equation.

```
def perturb(ex, pertDicts, pertLabel, maxPertOrder):
49
50
       # Initialize the pertDict object which will be returned
       pertDict = {}
51
       pertDict['ord'] = [None]*(maxPertOrder+1)
52
       if ex.top().name != r"\equals":
53
         # Not an equation, 'unp' is just ex and 'sym' the result of defPertSum
54
         pertDict['unp'] = ex
         pertDict['sym'] = defPertSum(ex,defPertList(ex,pertLabel,maxPertOrder))
56
57
       else:
         # 'unp' and 'sym' are as above
58
         pertDict['unp'] = ex[0]
59
         pertDict['sym'] = defPertSum(ex[0],defPertList(ex[0],pertLabel,maxPertOrder))
60
         # If pertDicts are provided, use them to
61
         if pertDicts is not None:
62
63
           # Decompose pertDicts into two components for 'sym' and 'ord'
           symPertList=[]
64
           ordPertList=[]
           for dic in pertDicts:
66
             symPertList.append(dic['sym'])
67
             ordPertList.append(dic['ord'])
           # Substitute symbolic decompositions
69
           subbed_ex = subsPertSums($@(ex)$, pertLabel, maxPertOrder, *symPertList, pertDict['sym'])
70
           # Iterate over all orders substituting order decompositions
           for i in range(0,maxPertOrder+1):
72
73
             cur_order = getEquationPertOrder(subbed_ex, pertLabel, i)
74
             for dec in ordPertList: # All objects to substitute
               if dec[0] is not None: # Leave symbolic the expressions
75
76
                 for j in range(i+1): # All orders up to i
                   substitute(cur_order, dec[j], repeat=True)
77
             # Simplifications
78
             distribute(cur_order)
             unwrap(cur_order,repeat=True)
80
             product_rule(cur_order)
             eliminate_metric(cur_order,repeat=True)
82
             eliminate_kronecker(cur_order,repeat=True)
83
             # Assign result of calculation to 'ord'
84
             pertDict['ord'][i] = cur_order
85
       return pertDict
```

In addition to the ex object to be perturbed and the standard setting arguments pertLabel and maxPertOrder, the perturb() function requires the pertDicts variable, namely the list perturbative units needed to perturb ex organized in terms of dictionaries whose structure is described in Table 4. Such perturbative units can be obtained by applying the perturb() function itself to the lower levels of the perturbative chain conducting to ex. As an example, as shown in Subsection 4.4.1, the perturbative expansion of the Einstein tensor $G_{\mu\nu}(g_{\mu\nu},R_{\mu\nu},R)$ can be obtained from the perturbative chain $g_{\mu\nu} \to g^{\mu\nu} \to \Gamma^{\sigma}_{\mu\nu} \to R_{\mu\nu\rho\sigma} \to R_{\mu\nu} \to R \to G_{\mu\nu}$. Then, the passed pertDicts are decomposed into the two lists symPertList and ordPertList, collecting all 'sym' and 'ord' contributes separately. Such lists are then employed as substitutions into cur_order, the symbolic decomposition of ex for each perturbative order. The last part of the function is dedicated to simplifications using standard Cadabra commands. In particular, for optimization reasons, the use of the burdensome canonicalise algorithm was deliberately avoided.

4.4.1. Tensor Perturbations: illustrative calculations

We are now ready to concretely test the tools developed in the previous sections, in particular the use of the perturb function. The following code creates in few line commands the tensor decomposition of the main metric objects of General Relativity passed from the *header.cnb* file, giving rise to a powerful approach to deal with higher-order perturbations within the Cadabra and Python framework:

```
connection = perturb(ch(), [gLow, gUpp], pertLabel, maxPertOrder)
riemann = perturb(rm(), [connection], pertLabel, maxPertOrder)
ricciTensor = perturb(rc(), [riemann], pertLabel, maxPertOrder)
ricciScalar = perturb(rs(), [gUpp,ricciTensor], pertLabel, maxPertOrder)
```

```
einstein = perturb(ei(), [gLow,ricciTensor,ricciScalar], pertLabel, maxPertOrder)
```

To verify if the code works correctly, we try to print the ricciTensor content:

```
ricciTensor['unp'];
ricciTensor['sym'];
ricciTensor['ord'][1];
```

which finally gives, as expected:

$$\begin{split} R_{\sigma\nu} &= {}^{(0)}\!R_{\sigma\nu} + {}^{(1)}\!R_{\sigma\nu} + {}^{(2)}\!R_{\sigma\nu} \\ {}^{(1)}\!R_{\nu\sigma} &= \frac{1}{2} \partial_{\sigma\tau} {}^{(1)}\!h_{\nu} {}^{\tau} - \frac{1}{2} \partial^{\tau} {}_{\tau} {}^{(1)}\!h_{\nu\sigma} - \frac{1}{2} \partial_{\nu\sigma} {}^{(1)}\!h_{\tau} {}^{\tau} + \frac{1}{2} \partial_{\nu} {}^{\tau} {}^{(1)}\!h_{\sigma\tau} \end{split}$$

A further tensor quantity that will prove extremely important in the following sections is the so-called harmonic gauge condition. The harmonic gauge condition is defined as

$$\Gamma^{\sigma} = g^{\mu\nu}\Gamma^{\sigma}_{\ \mu\nu} \,. \tag{16}$$

Using Cadabra language we can write:

```
92 hg{#}::LaTeXForm("\Gamma").
93 hg := hg^{\sigma} = g^{\mu\nu} ch^{\sigma}_{\mu\nu};
```

Following the same approach of the previous decompositions, we compute

harmonicGauge = perturb(hg,[gUpp,connection],pertLabel,maxPertOrder)

yielding

```
harmonicGauge['unp'];
harmonicGauge['sym'];
harmonicGauge['ord'][2];
```

$$\begin{split} \Gamma^{\sigma} \\ \Gamma^{\sigma} &= {}^{(0)}\!\Gamma^{\sigma} + {}^{(1)}\!\Gamma^{\sigma} + {}^{(2)}\!\Gamma^{\sigma} \\ {}^{(2)}\!\Gamma^{\sigma} &= \frac{1}{2}\partial_{\nu}{}^{(2)}\!h^{\nu\sigma} + \frac{1}{2}\partial^{\nu(2)}\!h_{\nu}{}^{\sigma} - \frac{1}{2}\partial^{\sigma(2)}\!h^{\nu}{}_{\nu} - \frac{1}{2}{}^{(1)}\!h^{\sigma\nu}\partial_{\rho}{}^{(1)}\!h^{\rho}{}_{\nu} - \frac{1}{2}{}^{(1)}\!h^{\sigma\nu}\partial^{\rho(1)}\!h_{\nu\rho} \\ &\quad + \frac{1}{2}{}^{(1)}\!h^{\sigma\nu}\partial_{\nu}{}^{(1)}\!h^{\rho}{}_{\rho} - \frac{1}{2}{}^{(1)}\!h^{\mu\nu}\partial_{\nu}{}^{(1)}\!h_{\mu}{}^{\sigma} - \frac{1}{2}{}^{(1)}\!h^{\mu\nu}\partial_{\mu}{}^{(1)}\!h_{\nu}{}^{\sigma} + \frac{1}{2}{}^{(1)}\!h^{\mu\nu}\partial^{\sigma(1)}\!h_{\mu\nu} \end{split}$$

We can also deal with matter sources like energy-momentum tensor and its trace:

```
93  mt := T_{\mu\nu}:
94  tr := T = g^{\mu\nu}T_{\mu\nu}:
95  #Properties
96  {T_{\mu\nu},T_{\mu\nu}::Symmetric.
97  T{#}::Depends(\partial{#}).
```

Using the perturb() function we calculate

```
98 matter = perturb(mt,None,pertLabel,maxPertOrder)
to get
```

```
matter['unp'];
matter['sym'];
matter['ord'][2];
```

$$T_{\mu
u}$$
 $T_{\mu
u} = {}^{\tiny (0)}T_{\mu
u} + {}^{\tiny (1)}T_{\mu
u} + {}^{\tiny (2)}T_{\mu
u}$
None

Note that if we never pass the list of perturbative dictionaries, a None is returned for ord.

An additional useful example is the following, where a mixture of sym and ord is passed to perturb the trace of the energy-momentum tensor:

It is clear that the tools developed in this Section can be completely customized according to need and used for obtaining higher perturbative orders of the objects already analyzed, just changing the value of maxPertOrder, or for the decomposition of any other tensor.

5. First-order gravitational-waves equation in presence of matter

At this point, we have the fundamental elements for the perturbative decomposition of relevant physical equations. In this Section 5 we linearize the Einstein equations in order to explicitly derive, within the harmonic gauge (16), the wave-equation which describes the propagation of first-order gravitational waves in presence of matter.

For this example we require the results from the previous tutorial sec4_TensorPerturbationsGR, and the header module is imported by inheritance.

```
from sec4_TensorPerturbationsGR import *
```

With the previous tutorials we have the fundamental elements for the perturbative decomposition of the Einstein's equations. More details on how obtaining these equations can be found in the tutorial *Einstein equations from a variational principle*.²

```
einEq := ei_{\mu\nu}=\kappa T_{\mu\nu}; substitute(_, ei()); G_{\mu\nu}=\kappa T_{\mu\nu} R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}
```

We remember that κ is a constant associated to the gravitational coupling. Einstein's equations can be reformulated by taking the trace of the equations:

```
# Multiply through by the metric
trEinEq := @(einEq);
manip.multiply_through(_, $g^{\mu\nu}$)

distribute(_);
# Replace contracted indices with scalar quantities
substitute(_, $g^{\mu\nu}T_{\mu\nu}->T$)
substitute(_, manip.eq_to_subrule(rs()))
```

 $^{^2}$ The tutorial is available in the web page https://cadabra.science/notebooks/einstein_equations.html

```
eliminate_metric(_)
eliminate_kronecker(_)
# Clean up
manip.multiply_through(_,$-1$);
```

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$$

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = g^{\mu\nu}\kappa T_{\mu\nu}$$

$$R = -T\kappa$$

We can now use the expressions einEq and trEinEq to write out the source term $S_{\mu\nu}$ which we define here using the perturb function from the pervious section

```
15 S_{\mu\nu}::Symmetric.
16 sm := S_{\mu\nu} = T_{\mu\nu} - 1/2 g_{\mu\nu} T;
17 source = perturb(_,[gLow,matter,matterTr],pertLabel,maxPertOrder)
```

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$$

It should be noted that for the source tensor $S_{\mu\nu}$ we decide to use only the symbolic decomposition, labelled by the key name 'sum' in the dictionary source. Printing the result we get

```
18     source['sym'];
19     source['ord'][0];
```

$$\begin{split} S_{\mu\nu} &= {}^{^{(0)}}\!S_{\mu\nu} + {}^{^{(1)}}\!S_{\mu\nu} + {}^{^{(2)}}\!S_{\mu\nu} \\ {}^{^{(0)}}\!S_{\mu\nu} &= {}^{^{(0)}}\!T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}{}^{^{(0)}}\!T^{\rho}{}_{\rho} \end{split}$$

In terms of this quantity the Einstein equations become

```
20    einEq2 := @(einEq):
21    manip.isolate(_, $rc_{\mu\nu}$);
22    # Insert expressions for source term and scalar Einstein equation
23    substitute(_, manip.isolate(sm, $T_{\mu\nu}$))
24    substitute(_, trEinEq);
25    # Clean up
26    distribute(_)
27    sort_product(_);
```

$$R_{\mu\nu} = \kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$$

$$R_{\mu\nu} = \kappa \left(\frac{1}{2} g_{\mu\nu} T + S_{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} T \kappa$$

$$R_{\mu\nu} = S_{\mu\nu} \kappa \tag{17}$$

Equation (17) is the same one obtained in our previous work [1] for $\Lambda = 0$ and the definition of $S_{\mu\nu}$. The first-order perturbed component of the above expression is given by

```
einEq2 = subsPertSums(einEq2,pertLabel,maxPertOrder,source['sym'],ricciTensor['sym']);
einEq2 = getEquationPertOrder(einEq2,pertLabel,1);
substitute(_,ricciTensor['ord'][1]);
```

$$\begin{split} R_{\mu\nu} + {}^{\text{\tiny (1)}} R_{\mu\nu} + {}^{\text{\tiny (2)}} R_{\mu\nu} &= {}^{\text{\tiny (0)}} S_{\mu\nu} \kappa + {}^{\text{\tiny (1)}} S_{\mu\nu} \kappa + {}^{\text{\tiny (2)}} S_{\mu\nu} \kappa \\ {}^{\text{\tiny (1)}} R_{\mu\nu} &= {}^{\text{\tiny (1)}} S_{\mu\nu} \kappa \\ \frac{1}{2} \partial_{\nu\tau} {}^{\text{\tiny (1)}} h_{\mu} {}^{\tau} - \frac{1}{2} \partial^{\tau} {}_{\tau} {}^{\text{\tiny (1)}} h_{\mu\nu} - \frac{1}{2} \partial_{\mu\nu} {}^{\text{\tiny (1)}} h_{\tau} {}^{\tau} + \frac{1}{2} \partial_{\mu} {}^{\tau} {}^{\text{\tiny (1)}} h_{\nu\tau} &= {}^{\text{\tiny (1)}} S_{\mu\nu} \kappa \end{split}$$

This equation does not permit a unique solution since given any solution it will always be possible to identify another solution by performing a coordinate transformation. This property is known as as *gauge invariance*. This redundancy can be removed by fixing a specific coordinate system: for our purposes it is a good choice to work in the *harmonic gauge*, defined by the condition:

```
31 hgCond := 0 = hg^{\sigma};
```

$$0 = \Gamma^{\sigma}$$

The first-order perturbed component of the harmonic gauge condition is

```
hgCond1 := @(hgCond):
hgCond1 = subsPertSums(_,pertLabel,maxPertOrder,harmonicGauge['sym']);
hgCond1 = getEquationPertOrder(_,pertLabel,1);
substitute(_,harmonicGauge['ord'][1]);
# Combine terms which only differ in height on contracted dummy indices
hgCond1 = substitute(_,$\partial_{\nu}{\nu}_{\nu}_{\nu}^{\sigma}}->\partial^{\nu}_{\nu}_{\nu}_{\nu}^{\sigma}});

0 = {}^{(0)}\Gamma^{\sigma} + {}^{(1)}\Gamma^{\sigma} + {}^{(2)}\Gamma^{\sigma}
0 = {}^{(1)}\Gamma^{\sigma}
0 = {}^{(1)}\Gamma^{\sigma}
0 = {}^{(1)}D^{\sigma}
```

This allows us to obtain the following constraint

```
manip.multiply_through(hgCond1, $2*\eta_{\alpha\sigma}$)
38
    manip.apply_through(_, $\partial_{\beta}{#}$);
    # Calculate derivatives
40
    distribute(_)
41
    product_rule(_)
42
    unwrap(_);
43
    # Eliminate metric and canonicalise
44
    eliminate_metric(_)
    canonicalise(_)
46
47
    eliminate_kronecker(_)
    rename_dummies(_);
```

$$0 = 2\partial_{\beta} \left[\eta_{\alpha\sigma} \left(\partial^{\nu^{(1)}} h_{\nu}{}^{\sigma} - \frac{1}{2} \partial^{\sigma^{(1)}} h^{\nu}{}_{\nu} \right) \right]$$

$$0 = 2\eta_{\alpha\sigma} \partial_{\beta}{}^{\nu^{(1)}} h_{\nu}{}^{\sigma} - \eta_{\alpha\sigma} \partial_{\beta}{}^{\sigma^{(1)}} h^{\nu}{}_{\nu}$$

$$0 = 2\partial_{\beta}{}^{\mu^{(1)}} h_{\alpha\mu} - \partial_{\alpha\beta}{}^{(1)} h^{\mu}{}_{\mu}$$

In the following code, the substitution $h^{\mu}_{\mu} \to h$ is made to allow the operator manip.to_lhs the automatically recognise that the addend should be moved to the left hand side of the equation.

$$0 = 2\partial_{\beta}^{\mu^{(1)}} h_{\alpha\mu} - \partial_{\alpha\beta}^{\quad (1)} h$$
$$\partial_{\alpha\beta}^{\quad (1)} h = 2\partial_{\beta}^{\mu^{(1)}} h_{\alpha\mu}$$
$$\partial_{\alpha\beta}^{\quad (1)} h^{\mu}_{\mu} = 2\partial_{\beta}^{\mu^{(1)}} h_{\alpha\mu}$$

As the partial derivatives commute, we can also deduce that

```
52 \partial_{\beta}^{\mu}(h1_{\alpha \mu})::Symmetric.
```

Introducing the gauge conditions just obtained in the perturbed Einstein equations gw_pert, some suitable algebraic manipulations provide the well-known wave equation in the presence of sources

```
einEqGauge = manip.multiply_through(einEq2, $-2$)
substitute(einEqGauge, $h1_{\mu}^{\mu}_->h1^{\mu}_{\mu}$);
substitute(_, hgCond1);
substitute(_, $\partial_{\mu\rho}(h1_{\mu}^{\rho}) -> \partial_{\mu\nu}(h1^{\rho}_{\rho})$)
canonicalise(_);
```

$$-\partial_{\nu\tau}{}^{(1)}h_{\mu}{}^{\tau} + \partial^{\tau}{}_{\tau}{}^{(1)}h_{\mu\nu} + \partial_{\mu\nu}{}^{(1)}h^{\tau}{}_{\tau} - \partial_{\mu}{}^{\tau}{}^{(1)}h_{\nu\tau} = -2^{(1)}S_{\mu\nu}\kappa$$

$$-\partial_{\nu\tau}{}^{(1)}h_{\mu}{}^{\tau} + \partial^{\tau}{}_{\tau}{}^{(1)}h_{\mu\nu} + 2\partial_{\nu}{}^{\tau}{}^{(1)}h_{\mu\tau} - \partial_{\mu}{}^{\tau}{}^{(1)}h_{\nu\tau} = -2^{(1)}S_{\mu\nu}\kappa$$

$$\boxed{\partial^{\tau}{}_{\tau}{}^{(1)}h_{\mu\nu} = -2^{(1)}S_{\mu\nu}\kappa} \qquad (Wave Equation)$$
(18)

In absence of matter fields we get

```
waveEq = substitute(_, $S1_{\mu\nu}->0$); \partial^{\tau}_{\tau}{}^{(1)}h_{\mu\nu} = 0 \tag{19}
```

The above equation shows that every component of the metric perturbation satisfy the wave equation.

6. First-order gravitational wave-relations

Working in the so-called harmonic gauge (see Eq. (16)), we consider plane-wave solutions of the wave equation (19) which describes the propagation of gravitational waves in absence of matter fields. We show that such solutions must satisfy the well-known wave-relations. This section is useful for showing the versatility of cadabra in managing tensor expressions that belong to the argument of exponential functions.

As with the notebook from the previous section we begin by importing the results of section 4, on top of which we import two functions from the cadabra standard library: replace_index which renames all indices in a subexpression, and get_basis_component which extracts the coefficient of a term.

```
from sec4_TensorPerturbationsGR import *
from cdb.core.manip import get_basis_component
from cdb.utils.indices import replace_index
```

6.1. Definitions

We introduce the spacetime coordinate x^{μ} , the polarization tensor $\mathbf{e}_{\mu\nu}$ and its complex conjugate $\bar{\mathbf{e}}_{\mu\nu}$

```
4 x::Coordinate.
5 x{#}::Depends(\partial{#}).
6 \pol1{#}::LaTeXForm("\mathbf{e}").
7 \pol2{#}::LaTeXForm("\bar{\mathbf{e}}").
```

Now we can consider the plane-wave solution, parameterised by the wave-number k_{λ} ,

```
 sol:=h1_{\mu\mu\mu}= pol1_{\mu\mu\mu}(i k_\lambda x^{\lambda}) + pol2_{\mu\mu\mu}(-i k_\lambda x^{\lambda}) : lambda ) :
```

$${}^{\scriptscriptstyle{(1)}}h_{\mu\nu} = \mathbf{e}_{\mu\nu} \exp\left(ik_{\lambda}x^{\lambda}\right) + \bar{\mathbf{e}}_{\mu\nu} \exp\left(-ik_{\lambda}x^{\lambda}\right)$$

where i is the imaginary unit. Here $h_{\mu\nu}$ is symmetric and inherits its dependence on the derivative ∂_{μ} through $\exp(\pm ik_{\lambda}x^{\lambda})$

```
9 \exp(i k_\lambda x{#})::Depends(\partial{#}).
10 \exp(-i k_\lambda x{#})::Depends(\partial{#}).
```

For the purposes of our analysis, we restrict to the vacuum wave-equation

```
wave_eq := \partial_{\sigma}{\partial^{\sigma}{h1_{\mu \nu}}} = 0;
```

$$\partial_{\sigma}^{\sigma^{(1)}} h_{\mu\nu} = 0$$

and by the harmonic gauge condition

 $\label{local_gauge_eq} $$ gauge_eq := \hat{\mu}_{\mu}_{\mu}_{\mu} - 1/2*partial_{\mu}_{\mu} = 0;$

$$\partial_{\mu}{}^{(1)}h^{\mu}{}_{\nu} - \frac{1}{2}\partial_{\nu}{}^{(1)}h^{\mu}{}_{\mu} = 0$$

More details on obtaining the wave-equation and the harmonic gauge first-order decomposition can be found in the previous tutorials. In order to handle the derivatives of the exponential function $\exp(\pm ik_{\lambda}x^{\lambda})$, the following replacement rules are required:

```
rule_1:=\partial_{\sigma}^{\sigma}(\exp(A??))->\partial_{\sigma}_{\exp(A??)*\partial^{\sigma}_{A??}}; rule_2:=\partial_{\mu}_{\exp(A??)}->\exp(A??)*\partial^{\mu}_{A??}; rule_3:=\partial_{\mu}_{\exp(A??)}->\exp(A??)*\partial_{\mu}_{A??};  \partial_{\sigma}^{\sigma}(\exp(A??)) \rightarrow \partial_{\sigma}(\exp(A??)) \partial^{\sigma}(A??)) \\ \partial^{\mu}(\exp(A??)) \rightarrow \exp(A??) \partial^{\mu}(A??) \\ \partial_{\mu}(\exp(A??)) \rightarrow \exp(A??) \partial_{\mu}(A??)
```

6.2. First wave-relation

We begin by imposing that the plane-wave $h_{\mu\nu}$ is a solution of the vacuum wave equation

```
wave_eq1 := @(wave_eq);
substitute(_, sol);
```

$$\partial_{\sigma} \,^{\sigma^{(1)}} h_{\mu\nu} = 0$$

$$\partial_{\sigma} \,^{\sigma} \left(\mathbf{e}_{\mu\nu} \exp\left(ik_{\lambda}x^{\lambda}\right) + \bar{\mathbf{e}}_{\mu\nu} \exp\left(-ik_{\lambda}x^{\lambda}\right) \right) = 0$$

To simplify the expression, we need to use the Cadabra specific converge construction, which is essentially a while loop which runs until the predicate expression no longer changes

```
converge(wave_eq1):
distribute(_)
product_rule(_)
unwrap(_)
wave_eq1;
```

$$\mathbf{e}_{\mu\nu}\partial_{\sigma}^{\ \sigma}\left(\exp\left(ik_{\lambda}x^{\lambda}\right)\right)+\bar{\mathbf{e}}_{\mu\nu}\partial_{\sigma}^{\ \sigma}\left(\exp\left(-ik_{\lambda}x^{\lambda}\right)\right)=0$$

Note that $\partial_{\sigma}^{\sigma} = \partial_{\sigma}\partial^{\sigma}$. In order to calculate the derivative ∂^{σ} we use substitute in rule_1 from above and use the replace_index texttrithm imported at the beginning to ensure that all the dummy pairs have unique names

```
23 substitute(_, rule_1)
24 replace_index(_, r'\exp', r'\lambda', r'\psi');
```

$$\mathbf{e}_{\mu\nu}\partial_{\sigma}\left(\exp\left(ik_{\psi}x^{\psi}\right)\partial^{\sigma}\left(ik_{\lambda}x^{\lambda}\right)\right) - \bar{\mathbf{e}}_{\mu\nu}\partial_{\sigma}\left(\exp\left(-ik_{\psi}x^{\psi}\right)\partial^{\sigma}\left(ik_{\lambda}x^{\lambda}\right)\right) = 0$$

We now use the unwrap texttrithm to take the constants out of the derivative, substitute in the metric where there are derivatives of the coordinate (i.e. $\partial^{\mu}x^{\alpha} \to \eta^{\mu\alpha}$) and then perform another unwrap.

```
unwrap(_)
substitute(_, $\partial^{\mu}{x^{\alpha}} -> \eta^{\mu\alpha}$);
unwrap(_);
```

$$\begin{aligned} &\mathbf{e}_{\mu\nu}ik_{\lambda}\partial_{\sigma}\left(\exp\left(ik_{\psi}x^{\psi}\right)\eta^{\sigma\lambda}\right) - \bar{\mathbf{e}}_{\mu\nu}ik_{\lambda}\partial_{\sigma}\left(\exp\left(-ik_{\psi}x^{\psi}\right)\eta^{\sigma\lambda}\right) = 0 \\ &\mathbf{e}_{\mu\nu}ik_{\lambda}\eta^{\sigma\lambda}\partial_{\sigma}\left(\exp\left(ik_{\psi}x^{\psi}\right)\right) - \bar{\mathbf{e}}_{\mu\nu}ik_{\lambda}\eta^{\sigma\lambda}\partial_{\sigma}\left(\exp\left(-ik_{\psi}x^{\psi}\right)\right) = 0 \end{aligned}$$

We now calculate the derivative of the exponentional using rule_3 and do another dummy index substitution as above

```
substitute(_, rule_3)
replace_index(_, r'\exp', r'\psi', r'\pi');
\mathbf{e}_{\mu\nu}ik_{\lambda}\eta^{\sigma\lambda}\exp\left(ik_{\pi}x^{\pi}\right)\partial_{\sigma}\left(ik_{\psi}x^{\psi}\right) + \bar{\mathbf{e}}_{\mu\nu}ik_{\lambda}\eta^{\sigma\lambda}\exp\left(-ik_{\pi}x^{\pi}\right)\partial_{\sigma}\left(ik_{\psi}x^{\psi}\right) = 0
```

To finish off this calculation we once more use unwrap to take out the constants before performing the substitution $\partial_{\mu}x^{\alpha} \to \delta_{\mu}^{\ \alpha}$ and cleaning up the result

```
unwrap(_); substitute(_, $\partial_{\mu}{x^{(\alpha)}} -> \delta_{\mu}^{(\alpha)}) # Can also do \partial_{\mu}^{(\mu)}{x^{(\alpha)}}: KroneckerDelta  

2    eliminate_kronecker(_); eliminate_metric(_)  

33    sort_product(_)  

34    sort_product(_)  

35    manip.multiply_through(wave_eq1, $-1$);  

-\mathbf{e}_{\mu\nu}k_{\lambda}\eta^{\sigma\lambda}\exp(ik_{\pi}x^{\pi})k_{\psi}\partial_{\sigma}x^{\psi} - \bar{\mathbf{e}}_{\mu\nu}k_{\lambda}\eta^{\sigma\lambda}\exp(-ik_{\pi}x^{\pi})k_{\psi}\partial_{\sigma}x^{\psi} = 0  

-\mathbf{e}_{\mu\nu}k_{\lambda}\eta^{\psi\lambda}\exp(ik_{\pi}x^{\pi})k_{\psi} - \bar{\mathbf{e}}_{\mu\nu}k_{\lambda}\eta^{\psi\lambda}\exp(-ik_{\pi}x^{\pi})k_{\psi} = 0  

-\exp(ik_{\pi}x^{\pi})\mathbf{e}_{\mu\nu}k_{\lambda}k^{\lambda} - \exp(-ik_{\pi}x^{\pi})\bar{\mathbf{e}}_{\mu\nu}k_{\lambda}k^{\lambda} = 0 
\exp(ik_{\pi}x^{\pi})\mathbf{e}_{\mu\nu}k_{\lambda}k^{\lambda} + \exp(-ik_{\pi}x^{\pi})\bar{\mathbf{e}}_{\mu\nu}k_{\lambda}k^{\lambda} = 0 
\exp(ik_{\pi}x^{\pi})\mathbf{e}_{\mu\nu}k_{\lambda}k^{\lambda} + \exp(-ik_{\pi}x^{\pi})\bar{\mathbf{e}}_{\mu\nu}k_{\lambda}k^{\lambda} = 0
```

This equation must be true for every value of $\mathbf{e}_{\mu\nu}$ and $\bar{\mathbf{e}}_{\mu\nu}$. Considering for example $\mathbf{e}_{\mu\nu}$, the first wave relation can be extracted using the get_basis_component

```
wave_eq1 = get_basis_component(wave_eq1, $\pol1{#}\exp(A??)$);
```

$$k_{\lambda}k^{\lambda} = 0$$
 (First Wave Relation)

Clearly, we would have achieved the same result using \pol2{#}\exp(A??) as the basis.

6.3. Second wave-relation

As $h_{\mu\nu}$ is a small perturbation around the flat spacetime, its indices can been raised or lowered using the Minkowski metric $\eta_{\mu\nu}$

```
37  sol2 := @(sol);
38  manip.multiply_through(_, $\eta^{\gamma}\)
39  distribute(_)
40  eliminate_metric(_);
```

$${}^{(1)}h_{\mu\nu} = \mathbf{e}_{\mu\nu} \exp\left(ik_{\lambda}x^{\lambda}\right) + \bar{\mathbf{e}}_{\mu\nu} \exp\left(-ik_{\lambda}x^{\lambda}\right)$$

$${}^{(1)}h^{\gamma}_{\nu} = \mathbf{e}^{\gamma}_{\nu} \exp\left(ik_{\lambda}x^{\lambda}\right) + \bar{\mathbf{e}}^{\gamma}_{\nu} \exp\left(-ik_{\lambda}x^{\lambda}\right)$$

Let us impose that the plane-wave h_{ν}^{γ} satisfies the harmonic-gauge condition and use distribute and unwrap to move the constants out of the derivative

```
41 wave_eq2 := @(gauge_eq);

42 substitute(_, so12)

43 distribute(_);

\partial_{\mu}{}^{(1)}h^{\mu}{}_{\nu} - \frac{1}{2}\partial_{\nu}{}^{(1)}h^{\mu}{}_{\mu} = 0
\partial_{\mu}\left(\mathbf{e}^{\mu}{}_{\nu}\exp\left(ik_{\lambda}x^{\lambda}\right)\right) + \partial_{\mu}\left(\bar{\mathbf{e}}^{\mu}{}_{\nu}\exp\left(-ik_{\lambda}x^{\lambda}\right)\right) - \frac{1}{2}\partial_{\nu}\left(\mathbf{e}^{\mu}{}_{\mu}\exp\left(ik_{\lambda}x^{\lambda}\right)\right) - \frac{1}{2}\partial_{\nu}\left(\bar{\mathbf{e}}^{\mu}{}_{\mu}\exp\left(-ik_{\lambda}x^{\lambda}\right)\right) = 0
```

As with the first wave-relation, we use unwrap to move the constants outside of the derivative, substitute in rule_3 to calculate the derivative before cleaning up the indices with replace_index and calling unwrap again

```
unwrap(_)
substitute(_, rule_3)
replace_index(_, r'\exp', r'\lambda', r'\psi')
unwrap(_);
```

$$\mathbf{e}^{\mu}_{\nu}\exp\left(ik_{\psi}x^{\psi}\right)ik_{\lambda}\partial_{\mu}x^{\lambda} - \bar{\mathbf{e}}^{\mu}_{\nu}\exp\left(-ik_{\psi}x^{\psi}\right)ik_{\lambda}\partial_{\mu}x^{\lambda} - \frac{1}{2}\mathbf{e}^{\mu}_{\mu}\exp\left(ik_{\psi}x^{\psi}\right)ik_{\lambda}\partial_{\nu}x^{\lambda} + \frac{1}{2}\bar{\mathbf{e}}^{\mu}_{\mu}\exp\left(-ik_{\psi}x^{\psi}\right)ik_{\lambda}\partial_{\nu}x^{\lambda} = 0$$

As before we replace $\partial_{\mu}x^{\lambda}$ with the Kronecker delta which we then resolve with eliminate_kronecker. We then separate out the coefficients of the two different exponential functions using factor_out

```
48    substitute(wave_eq2, $\partial_{\mu}{x^{\lambda}} -> \delta_{\mu}^{\lambda});
49    eliminate_kronecker(wave_eq2);
50    factor_out(wave_eq2,$\exp(i k_{\psi} x^{\psi}), \exp(-i k_{\psi} x^{\psi})$);
```

$$\mathbf{e}^{\mu}_{\nu} \exp\left(ik_{\psi}x^{\psi}\right) ik_{\lambda}\delta_{\mu}^{\lambda} - \bar{\mathbf{e}}^{\mu}_{\nu} \exp\left(-ik_{\psi}x^{\psi}\right) ik_{\lambda}\delta_{\mu}^{\lambda} - \frac{1}{2}\mathbf{e}^{\mu}_{\mu} \exp\left(ik_{\psi}x^{\psi}\right) ik_{\lambda}\delta_{\nu}^{\lambda} + \frac{1}{2}\bar{\mathbf{e}}^{\mu}_{\mu} \exp\left(-ik_{\psi}x^{\psi}\right) ik_{\lambda}\delta_{\nu}^{\lambda} = 0$$

$$\mathbf{e}^{\lambda}_{\nu} \exp\left(ik_{\psi}x^{\psi}\right) ik_{\lambda} - \bar{\mathbf{e}}^{\lambda}_{\nu} \exp\left(-ik_{\psi}x^{\psi}\right) ik_{\lambda} - \frac{1}{2}\mathbf{e}^{\mu}_{\mu} \exp\left(ik_{\psi}x^{\psi}\right) ik_{\nu} + \frac{1}{2}\bar{\mathbf{e}}^{\mu}_{\mu} \exp\left(-ik_{\psi}x^{\psi}\right) ik_{\nu} = 0$$

$$\exp\left(ik_{\psi}x^{\psi}\right) \left(\mathbf{e}^{\lambda}_{\nu} ik_{\lambda} - \frac{1}{2}\mathbf{e}^{\mu}_{\mu} ik_{\nu}\right) + \exp\left(-ik_{\psi}x^{\psi}\right) \left(-\bar{\mathbf{e}}^{\lambda}_{\nu} ik_{\lambda} + \frac{1}{2}\bar{\mathbf{e}}^{\mu}_{\mu} ik_{\nu}\right) = 0$$

This equation must be true for every value of $\mathbf{e}_{\mu\nu}$ and $\mathbf{\bar{e}}_{\mu\nu}$. Considering for example $\mathbf{e}_{\mu\nu}$, the second wave relation can be extracted using the get_basis_component function.

wave_eq2 = get_basis_component(wave_eq2,\$\exp(i k_{\psi} x^{\psi})\$);

$$e^{\lambda}_{\nu}ik_{\lambda} - \frac{1}{2}e^{\mu}_{\mu}ik_{\nu} = 0$$
 (Second Wave Relation) (20)

7. Higher-order gravitational-waves solutions in vacuum

In the previous Sections 5 and 6 we only exhibited first-order and second-order perturbative tensor examples. It is clear that the tools developed in Section 4 can be also managed to deal with higher-order problems. For reasons of clarity, we now apply our machinery to reproduce some results obtained in Ref. [13], where a complete analytical analysis of vacuum high-order gravitational waves solutions is given. The scope of the following Section 7 is to provide a computational counterpart of such analysis, highlighting the natural predisposition of Cadabra Software to treat the heavy and onerous nature of high-order tensor calculations. The formalism is introduced in Section 2.

Following the same method exhibited in Ref. [13] (see Section 2.6, Eqs. (63) and (64) of the reference paper), let us consider a general parametrization for background, first-order, and higher-order gravitational wave solutions that travel in the z-coordinate direction

$${}^{(0)}h_{\mu\nu} = 0$$
 ${}^{(1)}h_{\mu\nu} = \eta_{\mu\alpha}A^{\alpha}_{\ \nu} \exp(ik_{\rho}x^{\rho}),$ (21)

$${}^{(0)}h_{\mu\nu} = 0 \qquad {}^{(1)}h_{\mu\nu} = \eta_{\mu\alpha}A^{\alpha}_{\ \nu} \exp\left(ik_{\rho}x^{\rho}\right), \tag{21}$$

$${}^{(2n)}h_{\mu\nu} = \Phi^{n}\left({}^{(2n)}a\eta_{\mu\alpha}\alpha^{\alpha}_{\ \nu} + iz\omega c^{-1}{}^{(2n)}b\eta_{\mu\alpha}\beta^{\alpha}_{\ \nu}\right) \exp\left(i2nk_{\rho}x^{\rho}\right), \tag{22}$$

where 2n is the perturbative order, with $n = 1, 2, 3, \dots, N$. Clearly, odd-order solutions are null. In definition (22) we introduced the following quantities

$$\Phi \equiv {}^{(1)}A^{\mu}{}_{\nu}{}^{(1)}A^{\nu}{}_{\mu} = {}^{(1)}A^{\mu}{}_{\nu}{}^{(1)}A_{\mu}{}^{\nu}, \tag{23}$$

$$\alpha^{\mu}_{\nu} = \delta^{\mu}_{\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta^{\mu}_{\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad k^{\mu} \equiv \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ \omega/c \end{pmatrix}, \tag{24}$$

with ${}^{(1)}A^{\mu}_{\nu}$, the symmetric first-order polarization tensor expressed in terms of cross $({}^{^{(x)}}h)$ and plus $({}^{^{(+)}}h)$ states

In Eq. (22) the numerical coefficients $\{{}^{(2n)}a,{}^{(2n)}b\}$ will be the result of our computations, and must be found using Einstein's equations in vacuum and the harmonic coordinate conditions at each perturbative order.

$$R_{\mu\nu} = 0 \qquad g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} = 0, \tag{26}$$

Due to gravitational wave relations, at each perturbative order the following gauge conditions are satisfied:

$$k^{\mu}k_{\mu} = 0$$
 $k^{\mu^{(1)}}A_{\mu\nu} = 0$ $k_{\mu}\beta^{\mu\nu} = 0$ (27)

7.1. Definitions and modules

As with the notebooks from previous sections we begin by importing the results of Section 4 and functionality from the Cadabra standard library.

- import libraries.config as config
- config.maxPertOrder=4
- from sec4_TensorPerturbationsGR import *
- from cdb.utils.indices import replace_index

The header.cnb module has been automatically imported, whereas sec4_TensorPerturbationsGR imports our dictionaries gUpp, connection, ricciTensor, etc. containing the perturbative decompositions of the main tensorial quantities we will employ. The maximum perturbative order we will deal with is fixed by the maxPertOrder variable, which is valorized into the sec4_TensorPerturbationsGR module. This value must be set before executing the calculations. We suggest to use as a starting point low perturbative orders to do not face honerous computational times during the first tests. In our case, we set the value maxPertOrder=10.

7.2. Objects and components

The scope of this section is to reproduce the method obtained in Ref. [13, sec. 2.6], in particular the parameterised solutions presented in Eqs. (21) and (22) (with related tensorial parameters), using a *translation* into Cadabra objects. We start making the following property and equation definitions

```
# Normalization factor
     n1 := A^{\mathbb{L}}_{nu} A^{\mathbb{L}}_{nu} = \Phi.
     n2 := A^{\mathbb{L}}_{nu} A_{\mathbb{L}} = \Phi.
     # Properties of parametrizing objects
     A_{\mu\nu}::Symmetric.
     B {\mu\nu}::Symmetric.
10
     \exp{#}::Depends(\partial{#}).
11
     Z::Depends(\partial{#}).
12
     for i in range(maxPertOrder+1):
13
       LaTeXForm(Ex(f"a{i}"), Ex(r'"\,^{^{((s))}a"'} \% i))
       LaTeXForm(Ex(f"b{i}"), Ex(r'"\setminus,^{\{(%s)\}}b"' \% i))
15
```

Note that we use Z in place of the coordinate z in order to avoid the sympy_bridge problem (as Cadabra can not currently export expression which use the Depends property to Sympy) when we evaluate() the expressions. We do the algebraic manipulations using Z and then make the substitution $Z \to z$ when doing evaluations.

We now construct a list sol with parameterized solutions up to maxPertOrder. In order to avoid introduce extra metric contractions later, we define these for all the necessary positions of indices using the auxiliary function all_index_positions:

```
def all_index_positions(ex):
16
       tmp := @(ex).
17
       res = Ex(r"\comma")
18
       res.top().append_child($@(ex)$)
19
       for contraction, index in [($\eta^{\mu \lambda}$, r"\mu"), ($\eta^{\lambda \nu}$, r"\nu"), ($\eta_{\}
20
            mu \lambda}$, r"\mu")]:
         manip.multiply_through(ex, contraction)
2.1
         distribute(ex)
22
         eliminate_metric(ex)
23
         eliminate_kronecker(ex)
24
         replace_index(ex, r"\equals", r"\lambda", index)
25
         res.top().append_child($@(ex)$)
26
       return res
27
    sol = []
28
     # Zero and First order by hand
29
     sol.append($h0^{\mu nu} = 0$)
30
     sol.append( h1_{\mu\nu} = \epsilon_{\mu\lambda}A^{\alpha}_{\mu\lambda}_{\nu\nu} = \frac{\mu^{\mu}}{\mu^{\mu}}_{\nu\nu} 
31
     # Higher order solutions
32
     for i in range(2, maxPertOrder+1):
33
       # Construct symbols based on current pert order
34
       iex, ai, bi, hi = Ex(i), Ex(f'a\{i\}'), Ex(f'b\{i\}'), Ex(f'h\{i\}')
35
       # Define solutions for each order and append to `sol`
       if i%2 == 0:
37
38
         # Define the solution for even orders
         ex1 := Q(ai) \beta_{\mu} \ \delta^{\alpha}_{\nu}.
39
         ex2 := i O(bi) Z \omega/c \eta_{\mu\alpha} B^{\alpha}_{\nu}.
40
41
         ex3 := \exp(@(iex) i k_{\rho}x^{\rho}).
42
         ex := @(hi)_{\mu \in \mathbb{Z}} = \Phi(iex)/2 (@(ex1) + @(ex2)) @(ex3).
         # Append the solution to the sol list
43
         expand_power(ex)
44
         sol.append(ex)
45
46
       else:
         # Odd orders are all zero
         sol.append($@(hi)_{\mu = 0})
```

We will print out a couple of the elements in this list to ensure that the construction is correct

```
sol[-2][0];
sol[-1][0];
```

```
h_{\mu\nu} = 0
h_{\mu\nu} = \Phi \Phi (^{^{(4)}} a \eta_{\mu\alpha} \delta^{\alpha}_{\ \nu} + i^{^{(4)}} b Z \omega c^{-1} \eta_{\mu\alpha} B^{\alpha}_{\ \nu}) \exp(4ik_{\rho} x^{\rho})
```

At each perturbative order the gauge conditions $k^{\mu}k_{\mu} = 0$, $k^{\mu}A_{\mu\nu} = 0$ and $k_{\mu}B^{\mu\nu} = 0$ exposed in (27) are satisfied which we express for all index positions here:

7.3. Evaluation of the parameters

With reference to [13], we will now calculate the components of the parameters k_{μ} , $A_{\mu\nu}$ and $B_{\mu\nu}$.

We first define a function evaluate_and_complete which evaluates the components of an equation and uses these to complete a set of substitution rules.

```
def evaluate_and_complete(ex, rule, other_components):
    tmp = evaluate(rule, join(ex, other_components), rhsonly=True)
    subrules = comp.components_to_subrule(tmp)
    for subrule in subrules[r'\arrow']:
        subrule.name = r'\equals'
        ex.top().append_child(subrule)
    return ex
```

After definining the components of the Minkowski metric we then use this to write out the components of the parameters k_u , A_{uv} , B_{uv} and Φ ,

```
# Minkowski flat metric
66
                                  mink := { \text{t} = 1, \text{x} = -1, \text{ta}_{y} = -1, \text{ta}_{z} = -1 }.
                                 complete(mink, $\eta^{\mu\nu}$);
69
                                  # Position
                                 x := \{x_{t} = t, x_{x} = -x, x_{y} = -y, x_{z} = -z\};
                                  # Wavenumber (by Eq. 24)
71
                                  k := \{k^{t} = \omega/c, k^{z} = \omega/c\}.
 72
                                   evaluate\_and\_complete(k, $k_{\mathbb q} = \epsilon_{\mathbb q}^{\mu \rho}k^{\rho}, mink);
                                     # Solution's parameters (from eqns (65) and (25))
74
                                   A := \{A^{x}_{x}=-p, A^{x}_{y}=-m, A^{y}_{x}=-m, A^{y}_{y}=-m, A^{y}_{
 75
                                     evaluate\_and\_complete(A, $A^{\mu\nu} = \epsilon^{\mu\nu}A^{\mu\nu}_{\pi^0}, mink)
                                   evaluate\_and\_complete(A, $A_{\mu\nu} = \epsilon_{\mu\nu}A^{\rho}_{nu}, mink)
 77
 78
                                   evaluate_and_complete(A, $A_{\mu}^{\nu} = \eta^{\nu\sigma}A_{\mu\sigma}$, mink);
 79
                                  B := \{B^{t}_{t}=1, B^{t}_{z}=-1, B^{z}_{t}=1, B^{z}_{z}=-1\}.
                                  evaluate\_and\_complete(B, $B^{\mu\nu} = \text{nu}rho}B^{\mu}_{\nu}_{\nu} = \hat{\mu}_{\nu}_{\nu}
 80
                                   evaluate\_and\_complete(B, $B_{\mu\nu} = \epsilon_{\mu\nu}B^{\langle \mu\nu}B^{\langle \mu\nu}B^{
                                  evaluate\_and\_complete(B, $B_{\mu}^{nu} = \hat{\mu}_{\mu}^{nu}, mink);
82
                                   # Normalisation
 83
                                 norm = manip.swap_sides($@(n1)$)
                                  evaluate(norm, A);
```

to get the following evaluated components

```
\begin{split} & \left[ \eta_{tt} = 1, \eta_{xx} = -1, \eta_{yy} = -1, \eta_{zz} = -1, \eta^{tt} = 1, \eta^{xx} = -1, \eta^{yy} = -1, \eta^{zz} = -1 \right] \\ & \left[ x_t = t, x_x = -x, x_y = -y, x_z = -z \right] \\ & \left[ k^t = \omega c^{-1}, k^z = \omega c^{-1}, k_t = \omega c^{-1}, k_z = -\omega c^{-1} \right] \\ & \left[ A^x_{x} = -p, A^x_{y} = -m, A^y_{x} = -m, A^y_{y} = p, A^{xx} = p, A^{xy} = m, A^{yx} = m, A^{yy} = -p, A_{xx} = p, A_{xy} = m, A_{yx} = m, A_{yy} = -p, A_{x}^{x} = -p, A_{x}^{y} = -m, A_{y}^{x} = -m, A_{y}^{y} = p \right] \\ & \left[ B^t_{t} = 1, B^t_{z} = -1, B^z_{t} = 1, B^z_{z} = -1, B^{tt} = 1, B^{tz} = 1, B^{zt} = 1, B^{zz} = 1, B_{tt} = 1, B_{tz} = -1, B_{zt} = -1, B_{zt} = -1, B_{zt} = -1, B_{z}^{z} = -1 \right] \\ & \Phi = 2m^2 + 2p^2 \end{split}
```

7.4. Einstein Equation and Harmonic Gauge

To evaluate the numerical values of the parameters ${}^{\tiny (i)}a$ and ${}^{\tiny (i)}b$ we will need to calucalate the components of the Einstein equations and harmonic gauge condition at different perturbative orders. We begin by calculating the symbolic decomposition of these equations with subsPertSums function defined in Subsection 3.3

```
# Einstein Equation: symbolic decomposition
ein := 0 = rc_{\mu\nu};
ein = subsPertSums(ein,pertLabel,maxPertOrder,ricciTensor['sym']);
# Harmonic Gauge condition: symbolic decomposition
gau := 0 = g^{\mu\nu}_{\mu\nu} ch^{\sigma}_{\sigma}_{\mu\nu};
gau = subsPertSums(gau,pertLabel,maxPertOrder,gUpp['sym'],connection['sym']);

0 = R_{\mu\nu}
0 = R_{\mu\nu}
0 = {}^{(0)}_{R_{\mu\nu}} + {}^{(1)}_{R_{\mu\nu}} + {}^{(2)}_{R_{\mu\nu}} + {}^{(3)}_{R_{\mu\nu}} + {}^{(4)}_{R_{\mu\nu}}
0 = g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu}
0 = {}^{(0)}_{g^{\mu\nu}}{}^{(0)}\Gamma^{\sigma}_{\mu\nu} + {}^{(0)}_{g^{\mu\nu}}{}^{(1)}\Gamma^{\sigma}_{\mu\nu} + {}^{(0)}_{g^{\mu\nu}}{}^{(2)}\Gamma^{\sigma}_{\mu\nu} + {}^{(0)}_{g^{\mu\nu}}{}^{(1)}\Gamma^{\sigma}_{\mu\nu} + {}^{(1)}_{g^{\mu\nu}}{}^{(1)}\Gamma^{\sigma}_{\mu\nu} + {}
```

We now define the function subsParamSol which takes a symbolic decomposition ex of order pertOrder and substitutes in the parameterizations in the sol list.

To begin, we loop over all the metric perturbations ${}^{(i)}h$ up to maxPertOrder and substitute in the parameterized solutions from sol

```
92    def subsParamSol(ex,pertOrder):
93    # Substitute in the parameterized solutions
94    for j in range(1,pertOrder+1):
95         substitute(ex, sol[j])
```

We then unnest the resulting expression so that derivatives of products of exponentials are replaced with products of derivatives.

```
96  # Get derivatives of \exp only. Maximum 2nd derivatives in expression
97  converge(ex):
98   distribute(ex)
99   product_rule(ex)
100  unwrap(ex)
```

We also need to lower the indices of the derivatives,

```
# Calculate derivatives of \exp: (a) lower indices on derivatives

substitute(ex, $\partial^{\mu}{Q??} -> \eta^{\mu\rho}\partial_{\rho\nu}^{Q??}$)

substitute(ex, $\partial^{\mu}_{\nu}^{Q??} -> \eta^{\mu\rho}\partial_{\rho\nu}^{Q??}$)

substitute(ex, $\partial_{\mu}^{\nu}^{Q??} -> \eta^{\nu\rho}\partial_{\mu\rho}^{Q??}$)
```

We are now ready to perform the differentiation by the substituting in the definitions $\partial_{\nu} \exp(a i k_{\rho} x^{\rho}) \rightarrow a i k_{\nu}$ and $\partial_{\mu\nu} \exp(a i k_{\rho} x^{\rho}) \rightarrow -a^2 k_{\mu} k_{\nu}$,

```
# Calculate derivatives of \exp: (b) execute derivatives with substitutions

for j in range(1,pertOrder+1):
    jex = Ex(j)

substitute(ex, $\partial_{\nu}{\exp(@(jex) i k_{\rho}x^{\rho})} -> @(jex) i k_{\nu}$)

substitute(ex, $\partial_{\mu\nu}{\exp(@(jex) i k_{\rho}x^{\rho})} -> -@(jex) @(jex) k_{\mu}k_{\nu}$)

substitute(ex, $\partial_{\mu\nu}{\exp(@(jex) i k_{\rho}x^{\rho})} -> 1$)
```

Now that we have substituted in these expressions we can insert the gauge symmetries

```
# Insert gauge symmetries
substitute(ex,gau1)
substitute(ex,gau2)
substitute(ex,gau3)
```

To finish off, we perform some final simplifications by setting the second derivatives of the coordinates to 0, removing trace terms of *B* and eliminating any contracted metrics and Kronecker deltas:

```
115
      eliminate_metric(ex, repeat=True)
      eliminate_kronecker(ex, repeat=True)
116
      \# Set second derivatives of coordinates to 0
      substitute(ex, $\partial_{\mu\nu}{Z} -> 0$)
118
      119
      substitute(ex, \pi^{\infty}_{\infty} -> 0$)
120
      121
      # Trace of B matrix is zero
122
      substitute(ex, $B^{\rho}_{\rho} -> 0$)
123
      substitute(ex, $B_{\rho}^{\rho} -> 0$)
124
      return ex
```

We use this to substitute in the parameterized solutions to $g^{\mu\nu}$, $\Gamma^{\rho}_{\mu\nu}$ and $R_{\mu\nu}$ which will be needed to fully expand the solutions

```
# Copy the the lists calculated in section 4
126
     evalG = gUpp['ord'].copy()
127
     evalC = connection['ord'].copy()
     evalR = ricciTensor['ord'].copy()
129
130
     # Substitute in the orders of the parameterized solutions
     for i in range(1,maxPertOrder+1):
131
       evalG[i] = subsParamSol(evalG[i].i)
132
       evalC[i] = subsParamSol(evalC[i],i)
133
       evalR[i] = subsParamSol(evalR[i],i)
134
```

We define a final function evalParam which evaluates ex over the given components and simplifies the result

```
135
   def evalParam(ex, components):
     ex = comp.get_component(ex, components)
136
     # Lower indices
137
     substitute(ex, $Z->z$)
138
139
     distribute(ex)
     eliminate_kronecker(ex, repeat=True)
140
     eliminate_metric(ex, repeat=True)
141
     # Extract component and enter values
142
     expand_dummies(ex, join(A, join(B, join(k, mink))))
143
     substitute(ex, join(k, mink))
     145
         0$)
     -> 0$)
     substitute(ex, norm)
147
     # Clean up
148
```

```
149     collect_factors(ex)
150     distribute(ex)
151     eliminate_kronecker(ex)
152     return ex
```

7.5. Calculation of coefficients

We are now ready to calculate the coefficients a and b. We will keep them in two lists in a dictionary such that the *i*th element contains the value of ${}^{(i)}a$ or ${}^{(i)}b$ We define and initialise these here with the first two orders which are trivial:

```
# Define the list of coefficients

coeffs = {
    'a': [None] * (maxPertOrder+1),
    'b': [None] * (maxPertOrder+1)

for }

coeffs['a'][0], coeffs['b'][0] = $a0 -> 0$, $b0 -> 0$,

coeffs['a'][1], coeffs['b'][1] = $a1 -> 0$, $b1 -> 0$
```

We now iterate through the remaining orders calculating the values of these coefficients at these orders by examining the Einstein and gauge equations at this order and substituting in the values calculated for the lower orders.

We begin by defining expressions for the variables ${}^{(i)}a$ and ${}^{(i)}b$ so that we can pull them in to our equations later,

```
160     for i in range(2, maxPertOrder+1):
161         ai, bi = Ex(f'a{i}'), Ex(f'b{i}')
```

For odd orders the values are always 0 and so we make a special case for these values,

```
# Check if the order is odd (and thus = 0) or even (and requires calculation)
if i%2 == 1:
coeffs['a'][i] = $@(ai) -> 0$
coeffs['b'][i] = $@(bi) -> 0$
```

For even orders we begin by getting the *i*-th order of the equation and substituting in the decompositions of the Einstein equation and gauge conditions which we calculated earlier

```
166
        else:
          # Get the perturbative order of each equation
167
         eq1 = getEquationPertOrder(ein,pertLabel,i)
168
         eq2 = getEquationPertOrder(gau,pertLabel,i)
169
170
          # Substitute in the perturbed orders
         for j in range(i+1):
171
            substitute(eq1, evalR[j]) #ein
172
            substitute(eq2, evalG[j]) #gau
173
            substitute(eq2, evalC[j]) #gau
```

We now simplify the gauge equation in the usual style by eliminating metric tensors which can raise and lower indices, substituting in the gauge conditions and enforcing the tracelessness of B,

Then, we go through the previous perturbative orders which we have already calculated numerical values for and substitute these in.

```
# Substitute in the previously calculated coefficients
for j in range(1,j):
substitute(eq1, coeffs['a'][j])
```

In order to calculate the numerical value at this order, we extract the t components of the equations and use the linsolve algorithm from the cdb.core.solve library to solve for ${}^{(i)}a$ and ${}^{(i)}b$ which we then append to the coeffs arrays

```
188
          # Evaluate equations
          eq1 = evalParam(eq1, $t, t$)
189
          eq2 = evalParam(eq2, $t$)
190
          # Solve the component equations as a linear system
191
          sols = solv.linsolve($@(eq1), @(eq2)$, $@(ai), @(bi)$)
192
193
          # Append our newly calculated solutions
          coeffs['a'][i] = sols[0][0]
194
          coeffs['b'][i] = sols[0][1]
195
```

We can now display the numerical values of $^{(i)}a$ and $^{(i)}b$:

$$\begin{bmatrix} {}^{\scriptscriptstyle (1)}a \to 0, {}^{\scriptscriptstyle (1)}b \to 0 \end{bmatrix} \quad \begin{bmatrix} {}^{\scriptscriptstyle (2)}a \to \frac{3}{16}, {}^{\scriptscriptstyle (2)}b \to -\frac{1}{8} \end{bmatrix} \quad \begin{bmatrix} {}^{\scriptscriptstyle (3)}a \to 0, {}^{\scriptscriptstyle (3)}b \to 0 \end{bmatrix} \quad \begin{bmatrix} {}^{\scriptscriptstyle (4)}a \to -\frac{1}{64}, {}^{\scriptscriptstyle (4)}b \to -\frac{1}{32} \end{bmatrix}$$

Although exemplified the code for maxPertOrder=4, we manage to calculate the terms up to maxPertOrder=10. With respect to the reference paper, we have found from our calculations that the ⁽⁴⁾ a coefficient is $-\frac{1}{64}$ instead of $\frac{1}{64}$. As the use of this value has led to calculating higher order coefficient which agree with the reference paper, we believe that this is a typographical error in the paper. For the sake of completeness, we report the remaining higher order coefficients,

$$\begin{bmatrix} {}^{\text{(5)}}a \rightarrow 0, {}^{\text{(5)}}b \rightarrow 0 \end{bmatrix} \quad \begin{bmatrix} {}^{\text{(6)}}a \rightarrow \frac{13}{4608}, {}^{\text{(6)}}b \rightarrow -\frac{1}{1536} \end{bmatrix} \quad \begin{bmatrix} {}^{\text{(7)}}a \rightarrow 0, {}^{\text{(7)}}b \rightarrow 0 \end{bmatrix} \\ \begin{bmatrix} {}^{\text{(8)}}a \rightarrow -\frac{83}{131072}, {}^{\text{(8)}}b \rightarrow -\frac{65}{147456} \end{bmatrix} \quad \begin{bmatrix} {}^{\text{(9)}}a \rightarrow 0, {}^{\text{(9)}}b \rightarrow 0 \end{bmatrix} \quad \begin{bmatrix} {}^{\text{(10)}}a \rightarrow \frac{5023}{31457280}, {}^{\text{(10)}}b \rightarrow \frac{163}{3145728} \end{bmatrix}$$

8. Discussion and conclusions

Computer algebra systems are essential to many research problems in physics and other related fields, and due to the wide variety of technical challenges which these problems demand these tools need to possess a large amount of flexibility and customization. One of the major advantages of Cadabra is its direct connection with Python which allows it to take advantage of the powerful constructs and the numerous libraries which Python has. In this article we have continued on from the discussion of the previous paper [1], where the applicability of Cadabra to calculations in general relativity was demonstrated, to show that more general algorithms can be constructed and exemplifying this by using tensor perturbations in general relativity as a case study. This allowed us to show Cadabra's capabilities in handling abstract tensor equations and performance when handling very large expressions, as well as providing the possibility of extension beyond general relativity into other fields where perturbation theory is used such as atomic physics, condensed matter theory and particle physics.

The central algorithm of the paper is the perturb function from section 4 which provides a general way of decomposing an object as a perturbative expansion in terms of other objects, which in this paper has allowed us to represent many different quantities in terms of metric perturbations. By employing the results from this we first derived the wave-equation in the harmonic gauge for the propagation of first order gravitational waves in the presence of matter and their wave-relations in a vacuum. Having calculated this, we then show that Cadabra is capable of handling high-order tensor calculation by calculating coefficients for wave equations up to the 10th order.

We present here some comments on the complexity of the perturb function and the calculation of the higher order coefficients to illustrate the limits of how far can be explored using this technique before the computational

power required becomes excessive. We next look at Fig. 2 which shows the amount of time spent performing various routines inside the perturb function for several different objects.

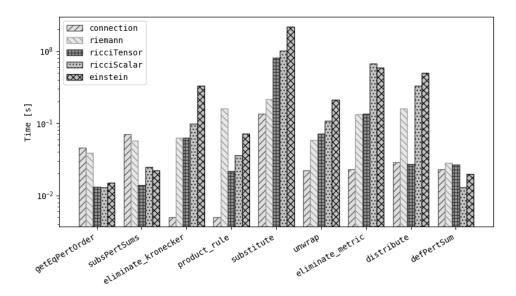


Figure 2: Breakdown of time spent in the subroutines of perturb

The non-uniformity of this graph is a result of various inherent properties of the different expressions which mean that the various algorithms must do different amounts of work on each object. This is most clearly shown in the product_rule algorithm, where the connection which relies only on a single derivative of the metric spends almost no time but the Riemann tensor which is composed of more complicated derivatives of Christoffel symbols spends the most time out of any algorithm. However, the overall behaviour is defined by the most expensive operation which is in all cases substitute; as this is a property which does not depend on any property of the expression other than its size, this shows that the dominant property which governs the complexity of the perturb algorithm is size of the expression which is directly related to the number of terms in the expression

We next concentrate on the graph in Fig. 3 which shows the complexity of the perturb function over different perturbative orders.

As can be seen, the complexity for each object is well described by an exponential relation with the perturbative order. This general pattern is a consequence of the number of terms at each perturbative order. To give a theoretical justification that this grows exponentially, we can consider the connection given by

$$\Gamma^{\mu}_{\ \nu\tau} = \frac{1}{2} g^{\mu\sigma} (\partial_{\tau} g_{\nu\sigma} + \partial_{\nu} g_{\tau\sigma} - \partial_{\sigma} g_{\nu\tau}) \tag{28}$$

Terms in the perturbative expansion of ${}^{(n)}\Gamma^{\mu}_{\nu\tau}$ consist of a product of metric perturbations ${}^{(n)}h_{\mu\nu}$ and exactly one derivative $\partial_{\rho}{}^{(n)}h_{\mu\nu}$. As the sum of the individual perturbative orders of ${}^{(n)}h_{\mu\nu}$ must equal n, the total number of possible terms is given by the number of ordered partitions of n which can be shown to be given by $P(n) = 2^{n-1}$. As ${}^{(n)}h_{\mu\nu}$ is symmetric there are three unique combinations of indices for the derivative term and therefore the total number of terms at each order is given by $3 \cdot 2^{n-1}$, which agrees very well with the growth factor shown in the graph.

Finally we make a quick comment on the behaviour shown in Fig. 4 which shows the comparative amounts of time it takes to run the sec4_TensorPerturbations.cnb notebook and the sec7_HigherOrderWaves.cnb notebook.

The red and blue lines closely follow each other until the fifth order where the growth rate of the time it takes for sec7_HigherOrderWaves.cnb to run diverges. This is a very good illustration of the impossibility of extrapolating such graphs to predict the runtime of higher order calculations. At the lower orders, running sec4_TensorPerturbations.cnb

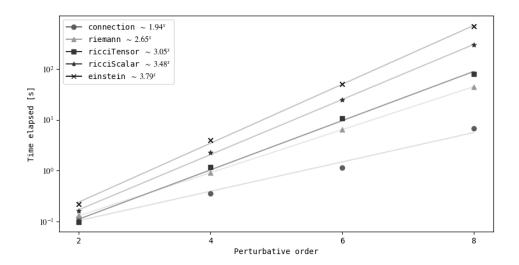


Figure 3: Total computational time of perturb over a range of perturbative orders fitted to exponential functions

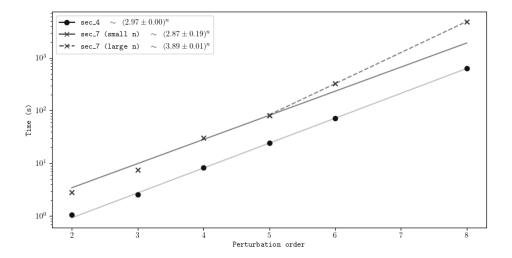


Figure 4: Total time elapsed in the computation of the perturbed objects, for different orders of perturbations. The marks denote the computational timing, while the solid line is the fit to the scatter plot.

to initialise the perturbed quantities is most expensive operation and is the dominant behaviour which causes the two lines to be approximately parallel; however as higher orders are probed the complexity of the computation performed in sec7_HigherOrderWaves.cnb which has smaller overall constants but a higher growth factor starts to dominate causing the divergence. It is possible that at even higher orders there may be other parts of the calculation which scale even worse but which are hidden at the orders we have explored.

The perturb function is specifically intended to be as general as possible so that it is readily customizable to suit the requirements of fields outside of general relativity. We encourage readers to modify the function to explore different possibilities for the use of the algorithm. We also leave possible improvements in efficiency and other optimisations, especially in order to minimise the amount of memory required for large perturbative terms, to a future

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