A Simple and Fast Approach for Computing the Fusion Reactivities with Arbitrary Ion Velocity Distributions

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Calculating fusion reactivity involves a complex six-dimensional integral of the fusion cross section and ion velocity distributions of two reactants. We demonstrate a simple Monte Carlo approach that efficiently computes this integral for arbitrary ion velocity distributions with a time complexity of O(N), where N is the number of samples. This approach generates random numbers that satisfy the reactant velocity distributions. In cases where these numbers are not readily available, we propose using Gaussian random numbers with weighted factors. For cases where only a small number of N samples are available, a $O(N^2)$ method can be used. We benchmarked this approach against analytical results for drift bi-Maxwellian distributions and provided examples of drift ring beam and slowing down distributions. Our results show that the error can be less than 1% with $N \sim 10^4$ samples for our standard approach.

Keywords: Fusion Reactivity, Monte-Carlo, Arbitrary Velocity Distributions

1. Introduction

Fusion reactivity $\langle \sigma v \rangle$ is the integral of fusion cross section and the reactants' velocity distribution functions

$$\langle \sigma v \rangle = \int \int dv_1 dv_2 \sigma(|v_1 - v_2|) |v_1 - v_2| f_1(v_1) f_2(v_2), \quad (1)$$

where f_1 and f_2 are the normalized velocity distribution functions of two ions, i.e., $\int f_j(\mathbf{v}_j)d\mathbf{v}_j = 1$ with j = 1, 2, and $d\mathbf{v}_j = d\mathbf{v}_{xj}d\mathbf{v}_{yj}d\mathbf{v}_{zj}$. Here, $\sigma = \sigma(E)$ or $\sigma = \sigma(v)$ is the fusion cross section, with *E* being the energy in the center-of-mass frame

$$E = \frac{1}{2}m_r v^2, \quad v = |\mathbf{v}| = |\mathbf{v}_1 - \mathbf{v}_2|, \quad m_r = \frac{m_1 m_2}{m_1 + m_2}, \quad (2)$$

where m_1 and m_2 are the mass of the two reactants, and m_r is the reduced mass of the system.

Equation (1) is not only important for calculating the fusion yield in laboratory [1] or stellar [2] plasmas, but it is also useful for obtaining spectrum information of the distribution functions $f_{1,2}$ from a diagnostic perspective [3]. However, calculating $\langle \sigma v \rangle$ for arbitrary f_1 and f_2 is difficult since it involves a six-dimensional (6D)

velocity integral, which is usually computed numerically [4, 5]. Kolmes et al. [6] used a mix of quadrature and Monte Carlo algorithm to study the fusion yield of plasma with velocity-space anisotropy at constant energy. Nath et al. [7] reduced the 6D integral to a 3D integral for drift tri-Maxwellian distributions, which is numerically tractable. Several analytical 1D integral results are summarized in Ref. [8], with the drift bi-Maxwellian distribution being the most general one, which can be reduced to Maxwellian, bi-Maxwellian, and beam-Maxwellian cases.

Numerically integrating Eq.(1) for arbitrary ion velocity distributions is generally considered to be complicated in the literature (see e.g. [7]). Although a fast orthogonal polynomial expansion method was proposed in Ref. [5], it is limited to velocity distributions that are independent of the azimuthal angle ϕ in spherical coordinates and therefore not generally applicable. Similarly, Ref. [9] used a similar approach for energy spectra diagnostic of unscattered neutrons produced by deuterium-deuterium and deuterium-tritium fusion reactions. While Monte Carlo high-dimensional integral methods (see e.g. Ref. [4]) can be applied to arbitrary ion velocity distributions, their computation efficiency, i.e., computation speed and accuracy, highly depends on the sampling method used.

In this work, we propose a simple and effective Monte

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Carlo approach for computing the 6D integral in Eq.(1). Unlike general Monte Carlo integral methods such as [4], our approach is specifically designed for this problem, which enables us to achieve maximum computation efficiency. Moreover, we found that the approaches used in first-principle particle simulation codes [10, 11] to calculate the fusion yield are valid for arbitrary velocity distributions and can be used to calculate Eq.(1). However, our approach is more flexible when we are only interested in calculating the fusion reactivity integral in Eq.(1). The proposed Monte Carlo approach has a time complexity of O(N), where N is the number of samples. In this paper, we also compare three types of this approach, which can be used for different situations.

Section 2 describes the approach used in this work. In Section 3, we benchmark our results against analytical results for drift bi-Maxwellian distributions, and apply our approach to drift ring beam and slowing down distributions. Finally, in Section 4, we summarize our findings.

2. Monte-Carlo Approach

The fusion reaction rate per unit volume and per unit time can be calculated as [1, 2]

$$R_{12} = \frac{n_1 n_2}{1 + \delta_{12}} \langle \sigma v \rangle, \tag{3}$$

where n_1 and n_2 are the number densities of the two reactants, respectively, and δ_{12} is equal to 0 for different reactants and 1 for the same reactants.

Eq.(3) implies a physical meaning, namely, that the fusion reactivity $\langle \sigma v \rangle$ represents the probability of a fusion reaction occurring. Thus, we select one particle from species 1 and one particle from species 2, and calculate $\sigma(|v_1 - v_2|)|v_1 - v_2|$ for these two particles. We repeat this process *N* times, and as *N* approaches infinity, the average value of each $\sigma(|v_1 - v_2|)|v_1 - v_2|$ will be the integral value of Eq.(1). This yields a simple Monte-Carlo approach (Method 1, standard approach) to compute Eq.(1):

- Step 1: Generate a random particle with velocity v₁ = (v_{1x}, v_{1y}, v_{1z}) that satisfies the velocity distribution f₁(v₁), and a random particle with velocity v₂ = (v_{2x}, v_{2y}, v_{2z}) that satisfies the velocity distribution f₂(v₂).
- Step 2: Calculate $\sigma(|v_1 v_2|)|v_1 v_2|$ for these two particles.
- Step 3: Repeat Steps 1 and 2 for N times.

• Step 4: Obtain the average value of each $\sigma(|v_1 - v_2|)|v_1 - v_2|$, which is the integral value of Eq.(1).

This approach has a time complexity of O(N).

In some situations, such as when using experimental diagnostic data, the number of samples N may be small. In these cases, we can use the following approach (Method 2) to compute Eq.(1):

- Step 1: Generate N_1 particles randomly with velocities $v_1 = (v_{1x}, v_{1y}, v_{1z})$ that satisfy the velocity distribution $f_1(v_1)$, and N_2 particles with velocities $v_2 = (v_{2x}, v_{2y}, v_{2z})$ that satisfy the velocity distribution $f_2(v_2)$.
- Step 2: Calculate $\sigma(|v_1 v_2|)|v_1 v_2|$ for each pair of particles, resulting in a total of $N_1 \times N_2$ pairs.
- Step 3: Obtain the average value of each $\sigma(|v_1 v_2|)|v_1 v_2|$, which is the integral value of Eq.(1).

Usually, $N = N_1 \simeq N_2$. This approach has a time cost of $O(N_1N_2) \simeq O(N^2)$.

Both Method 1 and Method 2 require generating random numbers that satisfy the reactant velocity distributions. In cases where these numbers are not readily available, we can modify Method 1 to obtain Method 3, which uses weighted factors and the following equation

$$\langle \sigma v \rangle = \int \int dv_1 dv_2 \sigma(|v_1 - v_2|) |v_1 - v_2| w(v_1, v_2) f_{1g}(v_1) f_{2g}(v_2).$$
(4)

Here, the weight function is defined as

$$w(\mathbf{v}_1, \mathbf{v}_2) = \frac{f_1(\mathbf{v}_1) f_2(\mathbf{v}_2)}{f_{1g}(\mathbf{v}_1) f_{2g}(\mathbf{v}_2)}.$$

We can compute Eq. (4) using Method 3, which involves the following steps:

- Step 1: Generate a random particle with velocity $v_1 = (v_{1x}, v_{1y}, v_{1z})$ that satisfies the velocity distribution $f_{1g}(v_1)$, and another random particle with velocity $v_2 = (v_{2x}, v_{2y}, v_{2z})$ that satisfies the velocity distribution $f_{2g}(v_2)$.
- Step 2: Calculate $\sigma(|v_1 v_2|)|v_1 v_2|w(v_1, v_2)$ for these two particles.
- Step 3: Repeat Steps 1 and 2 for *N* times.
- Step 4: Obtain the average value of each $\sigma(|v_1 v_2|)|v_1 v_2|w(v_1, v_2)$, which is the integral value of Eq. (1).

	Method 1, O(N)	Method 2, O(N^2)	Method 3, O(N)
Step 1	Initialize (vt1x,vt1y,vt1z,vd1x,vd1y,vd1z,vt2x,vt2y,vt2z,vd2x,vd2y,vd2z,mr,N)		
Step 2 generate N particles of 1 and 2	<pre>% generate N random numbers with the velocity distributions of f1 and f2 v1x=vt1x*randn(N,1)+vd1x; % randn -> exp(-x^2/2) v1y=vt1y*randn(N,1)+vd1y; v1z=vt1z*randn(N,1)+vd1z; v2x=vt2x*randn(N,1)+vd2x; v2y=vt2y*randn(N,1)+vd2y; v2z=vt2z*randn(N,1)+vd2z;</pre>		<pre>% generate N random numbers with Gaussian velocity distributions f1g and f2g for f1 and f2 vt=sqrt(vt1x^2/6+vt1y^2/6+vt1z^2/3+vt2x^2/6+vt2y^2/6+vt2z^2/3+vd 1x^2+vd1y^2+vd1z^2+vd2x^2+vd2y^2+vd2z^2); % choose a vt v1x=vt+randn(N,1); % randn -> exp(-x^2/2) v1y=vt+randn(N,1); v1z=vt+randn(N,1); v2z=vt+randn(N,1); v2z=vt+randn(N,1); v2z=vt+randn(N,1); % weight of Monte-Carlo integral, f1*f2/(f1g*f2g) wgt=vt^6/(vt1x+vt1y+vt1z+vt2x+vt2y+vt2z)*exp(-(v1x-vd1x).^2/(2+vt1x^2) -(v1y-vd1y).^2/(2+vt1y^2) -(v1z-vd1z).^2/(2+vt1x^2) (2*vt2x^2) -(v2y-vd2y).^2/(2+vt2y^2) -(v2z-vd2z).^2/(2+vt2z^2) +(v1x.^2+v1y.^2+v1z.^2+v2x.^2+v2y.^2+v2z.^2) /(2*vt2z));</pre>
Step 3 obtain v= v1-v2	v=sqrt((v1x- v2x).^2+(v1y- v2y).^2+(v1z-v2z).^2); wgt=1+0.*v;	$ \begin{array}{l} \mbox{for } i=1:N \ \ \ \ \ v=zeros(N*N,1) \\ \mbox{for } j=1:N \\ v((i-1)*N+j)=sqrt((v1x(i)-v2x(j)).^2 \\ +(v1y(i)-v2y(j)).^2+(v1z(i)-v2z(j)).^2); \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ $	v=sqrt((v1x-v2x).^2+(v1y-v2y).^2+(v1z-v2z).^2);
Step 4 obtain <sigma*v></sigma*v>	EkeV=0.5*mr*v.^2/(ge*1e3); % J -> keV sgm=fsgmdt(EkeV); % DT fusion cross section sgmv=mean(sgm.*v.*wgt); % take average to obtain the final <sigma*v></sigma*v>		

Figure 1: Sample code demonstrating three Monte-Carlo methods for computing the 6D fusion reactivity integral for drift tri-Maxwellian velocity distributions.

Method 3 is actually an important sampling Monte Carlo approach[4]. A good choice of f_{1g} and f_{2g} can reduce the requirement of N. In this work, we use Gaussian distributions for f_{1g} and f_{2g} . The time cost of Method 3 is also O(N).

Figure 1 provides sample code programs to demonstrate the above three Monte Carlo methods used to calculate the 6D fusion reactivity integral for drift tri-Maxwellian velocity distributions given by

$$f_{j}(\mathbf{v}_{j}) = \left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{v_{txj}v_{tyj}v_{tzj}}\right) \exp\left[-\frac{(v_{xj}-v_{dxj})^{2}}{2v_{txj}^{2}} - \frac{(v_{yj}-v_{dyj})^{2}}{2v_{tyj}^{2}} - \frac{(v_{zj}-v_{dzj})^{2}}{2v_{tzj}^{2}}\right].$$
(5)

Here, v_{txj} , v_{tyj} , and v_{tzj} are the thermal velocities in each direction, and v_{dxj} , v_{dyj} , and v_{dzj} are the drift velocities in each direction, with j = 1, 2. These three simple codes can quickly compute all the results in Nath et al [7], with Method 1 being the most effective (see Sec. 3).

3. Benchmarks and Applications

To demonstrate the methods presented in Section 2, we compare the results with analytical solutions for drift bi-Maxwellian distributions[8]. Additionally, we compare the three methods for drift ring beam[12, 13] and slowing down[13] distributions and use the D-T fusion reaction cross-section data from Ref.[14].

3.1. Drift bi-Maxwellian distribution

The distribution functions are given by

$$f_{j}(\mathbf{v}_{j}) = \frac{1}{T_{\parallel j}^{1/2} T_{\perp j}} \left(\frac{m_{j}}{2\pi k_{B}}\right)^{3/2} \cdot \exp\left[-\frac{m_{j} v_{\perp j}^{2}}{2k_{B} T_{\perp j}} - \frac{m_{j} (v_{\parallel j} - v_{dj})^{2}}{2k_{B} T_{\parallel j}}\right], \quad (6)$$

where j = 1, 2, and k_B is the Boltzmann constant. Here, $\int f_j(\mathbf{v}_j) d\mathbf{v}_j = 1$, $v_{\perp j}^2 = v_{xj}^2 + v_{yj}^2$, and $v_{\parallel j} = v_{zj}$. The drift tri-Maxwellian distribution Eq.(5) reduces to the drift bi-Maxwellian distribution Eq.(6) by taking $v_{txj} =$ $v_{tyj} = \sqrt{k_B T_{\perp j}/m_j}$, $v_{tzj} = \sqrt{k_B T_{\parallel j}/m_j}$, and $v_{dxj} = v_{dyj} =$ 0 in Eq.(5). With this drift bi-Maxwellian distribution, the 6D integral Eq.(1) reduces to a 1D integral[8], which is a function of only T_r , R_t , and E_d , where

$$T_r = \frac{(2T_{\perp r} + T_{\parallel r})}{3}, \ R_t = \frac{T_{\perp r}}{T_{\parallel r}}, \ E_d = k_B T_d = \frac{m_r v_d^2}{2},$$

where $v_d = v_{dz2} - v_{dz1}$. Additionally, we have

$$T_{\parallel r} = \frac{m_1 T_{\parallel 2} + m_2 T_{\parallel 1}}{m_1 + m_2}, \ T_{\perp r} = \frac{m_1 T_{\perp 2} + m_2 T_{\perp 1}}{m_1 + m_2}.$$

Figure 2 shows the benchmark results of the 6D Monte-Carlo approach to the analytical 1D integral[8] for drift bi-Maxwellian distributions, which exhibit good agreement with $R_t = 2$, $E_d = 20$ keV and $N_1 =$



Figure 2: Comparison between the results obtained using the 6D Monte-Carlo approach and the analytical 1D integral method[8] for drift bi-Maxwellian distributions.



Figure 3: Comparison of computation time and error using 6D Monte-Carlo approach Method 1 for drift bi-Maxwellian distributions with different values of *N*.

 10^4 . To obtain the error of each method, the results are repeated three times for each case. The total computation time of each method is also the computer time taken. We observe that the total computer cost for computing the 6D Monte-Carlo results in Fig.2 using Method 1 for 20 points of T_r with $N = 10^4$ and repeat 3 times is 0.11 seconds, with an error less than 1%. To achieve a similar level of accuracy, Methods 2 and 3 require around 50 times more computation time. Method 2 requires the smallest value of N among these three methods.

Figure 3 compares the computation time and error with different values of N, using 6D Monte-Carlo approach Method 1 for drift bi-Maxwellian distributions. We find that $N = 10^5$ is sufficient for these parameters ($R_t = 0.5$, $E_d = 20$ keV). The computation time is not accurately proportional to O(N) due to the fact that for high values of N, the vector program scheme can save some computation costs.

To make an accurate comparison of the performance of the three methods, it is necessary to use the same level of computational precision. However, since precision is influenced by many parameters and is difficult to control, this work can only provide a rough comparison. When the same level of computational precision is achieved, Method 1 requires a smaller number of samples, $N_1 < N_2^2$, compared to Method 2. This is because Method 1 has a more accurate sampling of the distribution function than Method 2, while the calculation of the reactivity sum is similar. On the other hand, Method 3 requires a larger number of samples, $N_3 > N_1$, compared to Method 1, which is understandable because Method 1 has a much simpler integral weight $\sigma(|\mathbf{v}_1 - \mathbf{v}_1|)$ $|v_2||v_1 - v_2|$ than the weight $\sigma(|v_1 - v_2|)|v_1 - v_2|w(v_1, v_2)|$ in Method 3.

3.2. Drift ring beam distribution

The drift ring beam distribution, which includes both parallel and perpendicular drifts as well as temperature anisotropy, is given by[12]

$$f_{j}(\mathbf{v}_{j}) = f_{zj} \cdot f_{\perp j} = \frac{1}{\sqrt{\pi}v_{tzj}} \exp\left[-\frac{(v_{zj}-v_{dzj})^{2}}{v_{tzj}^{2}}\right] \cdot \frac{1}{\pi A_{j}v_{t\perp j}^{2}} \exp\left[-\frac{(\sqrt{(v_{xj}-v_{dxj})^{2}+(v_{yj}-v_{dyj})^{2}}-v_{drj})^{2}}{v_{t\perp j}^{2}}\right] (7)$$

where $A_j = \exp(-\frac{v_{drj}^2}{v_{t\perp j}^2}) + \sqrt{\pi}(\frac{v_{drj}}{v_{t\perp j}})\operatorname{erfc}(-\frac{v_{drj}}{v_{t\perp j}})$, and $\int f_j(\mathbf{v}_j)d\mathbf{v}_j = 1$, for j = 1, 2. The error function $\operatorname{erfc}(-x) = 1 + \operatorname{erf}(x)$, and $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. The 1D analytical form of Eq.(1) for this distribution is not

yet available. Note also that there exists a $\sqrt{2}$ difference between the definition of thermal velocity v_t here and Eq.(5).

Appendix A provides instructions on how to generate random numbers with this distribution. Figure 4 compares the drift ring beam using the three methods in Section 2. Once again, we observe that Method 1 is the most efficient among them, and Methods 2 requires smallest N.

3.3. Slowing down distribution

The isotropic slowing down distribution is given by [13]

$$f_j(\mathbf{v}_j) = \frac{3}{4\pi \ln[1 + v_{bj}^3/v_{cj}^3]} \frac{H(v_{bj} - v)}{v^3 + v_{cj}^3},\tag{8}$$

where $\int f_j(\mathbf{v}_j)d\mathbf{v}_j = 1$ for j = 1, 2, and H(x) is the Heaviside function, defined as H(x < 0) = 0, H(x > 0) = 1, and H(0) = 1/2. The 1D analytical form of Eq.(1) for this distribution is not yet available.

Instructions for generating random numbers with this distribution are provided in Appendix A. Figure 5 compares the slowing down distribution using the three methods described in Sec.2. Once again, we see that Method 1 is the most effective. For Method 3, we also compared two types of random numbers: Gaussian $f_{1g,2g}$, and uniform $f_{1g,2g}$ in $v_{xj,yj,zj} \in [-v_{bj}, v_{bj}]$ for j = 1, 2. Both methods yielded similar results, indicating the robustness of this approach.

4. Summary and Discussion

We have developed a simple Monte-Carlo approach to compute the 6D fusion reactivity integral Eq.(1) for arbitrary ion velocity distributions. We compared three types of this approach for several typical distributions, such as drift bi-Maxwellian, drift ring beam, and slowing down distributions. Our results show that this approach is both robust and effective.

The second method is suitable for situations when N is small, with a time cost of $O(N^2)$. The first method is found to be the most effective one among them, with a time cost of O(N). However, it still requires a routine to generate the corresponding random numbers of the given distributions, as in the second method. The third method uses a weight function to remove the requirement of generating corresponding random numbers, with a time cost of O(N). For these three methods, the typical requirement for $N_{1,2,3}$ is $N_1 \approx 10^4 - 10^5$, $N_2 \approx 5 \sqrt{N_1} \approx 10^3$, and $N_3 \approx 50N_1 \approx 10^6 - 10^7$.



Figure 4: Comparison of three Monte-Carlo methods for drift ring beam distributions, where $v_{dj} = [v_{djx}, v_{djy}, v_{djz}, v_{djr}]$.



Figure 5: Comparison of the three Monte-Carlo methods for slowing down distributions, where $v_t = \sqrt{2k_BT_r/m_r}$.

Overall, our Monte-Carlo approach provides a practical and efficient tool for computing the fusion reactivity integral. Although the basic ideas behind our Monte Carlo approach for computing the fusion reactivity integral may not be new, the approach presented in this work is still noteworthy for its simplicity and efficiency. Similar Monte Carlo pairwise treatments have been used in particle simulation codes, such as those in Refs. [10, 11], to calculate the fusion yield for arbitrary velocity distributions. Furthermore, the Fokker-Planck binary collision model for plasma particle simulation [15] can also be retrospectively related to this approach. However, our approach, as demonstrated with three methods, is more flexible and applicable to a wider range of situations where only the fusion reactivity integral, Eq.(1), needs to be calculated. Thus, we believe that our approach is valuable and worth summarizing to the community. In future work, further optimization of the algorithms and exploring new applications of this approach in related fields can be pursued. The computation source codes used in this work are avaiable at https://github.com/hsxie/fusionreactivity.

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Appendix A. Random numbers for drift ring beam and slowing down distributions

To generate a velocity v with distribution f(v) from a uniform $u \in [0, 1)$ random number using a monotonic function transformation v = v(u), we use the relation

$$v(u + \Delta u) = v + \Delta v, \Delta u = f(v)\Delta v,$$
 (A.1)

which can be written as

$$f(v)dv = du. \tag{A.2}$$

Solving for *u* gives

$$u = u(v) = \int f(v')dv'.$$
(A.3)

We can then calculate the transformation v = v(u) from the inverse function of u = u(v).

To model the distributions of drift ring beams, we use the product of two distributions: $f(\mathbf{v}) = f_z(v_z) \cdot f_\perp(v_x, v_y)$, where $f_z(v_z)$ can be generated using a standard Gaussian random number function. The distribution $f_\perp(v_x, v_y)$ is given by

$$f_{\perp} = \frac{1}{\pi A v_{t\perp}^2} \exp\left[-\frac{(\sqrt{(v_x - v_{dx})^2 + (v_y - v_{dy})^2} - v_{dr})^2}{v_{t\perp}^2}\right],$$

where $v_{\perp} = \sqrt{(v_x - v_{dx})^2 + (v_y - v_{dy})^2} \in [0, \infty)$ and ϕ is the angle between the *x*-axis and the velocity vector \mathbf{v}_{\perp} in the *xy*-plane. The quantity *A* is defined as $A = \exp(-v_{dr}^2/v_{t\perp}^2) + \sqrt{\pi}(v_{dr}/v_{t\perp})\operatorname{erfc}(-v_{dr}/v_{t\perp})$, and $\int f_j(\mathbf{v}_j)d\mathbf{v}_j = 1$. In the (v_{\perp}, ϕ) space, we have $f(v_{\perp}, \phi) = f(v_{\perp})f(\phi)$, where

$$f(v_{\perp}) = \frac{2v_{\perp}}{Av_{t\perp}^2} \exp\left[-\frac{(v_{\perp} - v_{dr})^2}{v_{t\perp}^2}\right], \ 0 \le v_{\perp} < \infty$$
$$f(\phi) = \frac{1}{2\pi}, \ 0 \le \phi < 2\pi.$$
(A.4)

The coefficients are normalized such that $\int_0^{\infty} f(v_{\perp}) dv_{\perp} = 1$ and $\int_0^{2\pi} f(\phi) d\phi = 1$. To generate ϕ , we use a uniform random number $u \in [0, 1)$ and set $\phi = 2\pi u$.

The relationship between v_{\perp} and the uniform random number *u* is given by the following equation

$$u = \int f(v_{\perp}) dv_{\perp} = \frac{1}{A} \left\{ \sqrt{\pi} \frac{v_{dr}}{v_{t\perp}} \left[\operatorname{erf} \left(\frac{v_{dr}}{v_{t\perp}} \right) - \operatorname{erf} \left(\frac{v_{dr} - v_{\perp}}{v_{t\perp}} \right) \right] + \exp \left(- \frac{v_{dr}^2}{v_{t\perp}^2} \right) - \exp \left(- \frac{(v_{\perp} - v_{dr})^2}{v_{t\perp}^2} \right) \right\}, \quad (A.5)$$

which satisfies the requirements u(0) = 0 and $u(\infty) = 1$. In the case of a usual Maxwellian/Gaussian distribution with $v_{dr} = 0$ and A = 1, we have

$$u = -\exp(-v_{\perp}^2/v_{t\perp}^2) + 1,$$

so that

$$v_{\perp} = v_{t\perp} \sqrt{-\ln(1-u)},$$

which is one of the standard ways to generate a Gaussian random distribution. When $v_{dr} \neq 0$, we can obtain the inverse function $v_{\perp}(u) = u^{-1}(v_{\perp})$ numerically using 1D interpolation, since $u(v_{\perp})$ is known and monotonically increasing. Then, we can obtain the velocity components (v_x, v_y) using the following equations:

$$v_x = v_{\perp} \cos \phi + v_{dx}, \ v_y = v_{\perp} \sin \phi + v_{dy}.$$

Note that ϕ and v_{\perp} should use independent random numbers *u*.

Similarly, for the slowing-down distribution in (v, ϕ, θ) space, we have

$$f(\mathbf{v}) = f(v)f(\theta)f(\phi), \quad f(\theta) = \frac{1}{\pi}, \quad f(\phi) = \frac{1}{2\pi}, \\ f(v) = \frac{3v^2}{\ln[1+v_b^3/v_c^3]} \frac{H(v_b - v)}{v^3 + v_c^3}, \quad (A.6)$$

which means $0 \le \theta < \pi$ and $0 \le \phi < 2\pi$ are uniformly distributed. We have

$$u = \int f(v)dv = \ln[1 + v^3/v_c^3] / \ln[1 + v_b^3/v_c^3],$$

with $v \in [0, v_b)$, $u \in [0, 1)$, i.e.,

$$v = v_c \left\{ \exp\left[u \ln(1 + v_b^3 / v_c^3) \right] - 1 \right\}^{1/3}$$

After generating random numbers of (v, θ, ϕ) , we can obtain (v_x, v_y, v_z) via

 $v_x = v \sin \theta \cos \phi$, $v_y = v \sin \theta \sin \phi$, $v_z = v \cos \theta$.

For arbitrary distributions, generating random numbers is not always straightforward. However, there are numerical libraries available, such as UNURAN [16].

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