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Noise Space Decomposition Method for two dimensional sinusoidal model

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Abstract

In this paper, we address the estimation of the parameters of the two dimensional sinusoidal signal model. The proposed method is the two dimensional extension of the one dimensional noise space decomposition method. The proposed methods provide consistent estimators of the unknown parameters and they are non-iterative in nature. We propose a pairing algorithm also, which helps in identifying the frequency pair. It is observed that the mean squares errors of the proposed estimators are quite close to the asymptotic variance of the least squares estimators.

Key Words and Phrases: Sinusoidal model, Prony's algorithm, Monte Carlo Simulation; Strong Consistency.

1 Introduction

In this paper, we consider the following two-dimensional sinusoidal model

$$y(s,t) \;=\; \sum_{k=1}^{p}\Big(A_k^0\cos(s\lambda_k^0+t\mu_k^0)+B_k^0\sin(s\lambda_k^0+t\mu_k^0)\Big)+e(s,t) \qquad (1)$$
$$s=1,\ldots,M;\quad t=1,\ldots,N,$$

where A_k^0s and B_k^0s are the unknown amplitudes, λ_k^0s and μ_k^0s are the unknown frequencies and $\lambda_k^0,\mu_k^0\in(0,\pi)$. The additive component $\{e(s,t)\}$ is from a independent and identically distributed (*i.i.d.*) random field, and the number of components p is assumed to be known. Given a sample $\{y(s,t), s=1,\ldots,M;\ t=1,\ldots,N\}$, the problem is to estimate A_k^0, B_k^0, λ_k^0 and μ_k^0, $k=1,\ldots,p$.

The first term on the right hand side of (1) is known as the signal component. The detection and estimation of the signal component in presence of additive noise is an important and

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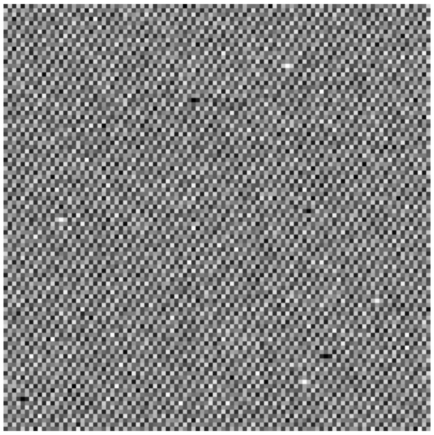

Figure 1: The image plot of a simulated dataset.

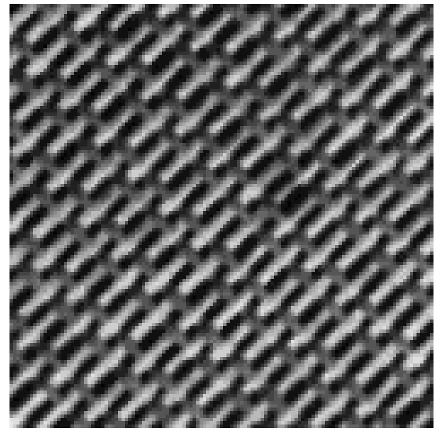

Figure 2: The image Plot of a real dataset.

classical problem in Statistical Signal Processing. It is observed in Zhang and Mandrekar (2001) and in Yuan and Subba Rao (1993) that such models can be used in modeling black and white (BW) regular textures. Figure 1 represents the 2-D image plot of simulated $y(s,t)$ whose grey levels at (s,t) is proportional to the value of $y(s,t)$. Our problem is to extract the regular texture from the contaminated one. Figure 2 represents the image plot of a real texture. The data given in Figure 2 have been analyzed by Nandi, Prasad and Kundu (2010) quite effectively using model (1). The best estimator, as expected, is the least squares estimator (LSE). The LSEs are strongly consistent and asymptotically normally distributed. But it is well known that finding the least square estimators of the frequencies is a numerically difficult problem and the procedure tends to be computationally intensive. The function required to be optimized is highly non-linear in its parameters even in case of one dimensional model and one needs to use an iterative procedure. Due to the presence of several local minima, convergence might be a tricky problem. Recently, Nandi, Prasad and Kundu (2010) proposed an efficient algorithm to estimate the unknown parameters of (1) which provides estimators asymptotically equivalent to the least squares estimators. In this paper, we develop a non iterative procedure to estimate the unknown frequencies of model (1) extending the one-dimensional (1-D) noise space decomposition (NSD) method proposed by Kundu and Mitra (1995). The proposed two dimensional (2-D) NSD method provides consistent estimators of the unknown frequencies.

Zhang and Mandrekar (2001), Kundu and Gupta (1998), Nandi and Kundu (1999) and Bansal, Hamedani and Zhang (1999) considered model (1) with $B_k^0 = 0$, whereas Zhang (1991) considered the model with sine term only *i.e.* with $A_k^0 = 0$, and with *i.i.d.* errors. This is a basic model in many fields, such as antennae array processing, geophysical perception, biomedical spectral analysis etc.

Kundu and Gupta (1998) and Bansal, Hamedani and Zhang (1999) assumed the errors to be *i.i.d.* and obtained the strong consistency and the asymptotic normality of the LSEs. Bansal, Hamedani and Zhang (1999) obtained some sufficient conditions on the non-linear function (deterministic part) under which the LSEs are strongly consistent and asymptotically normally distributed. These sufficient conditions are satisfied by harmonic type functions, which are also of interest in one dimensional models where the sufficient conditions provided by Wu (1981) and Jennrich (1969) are not satisfied. Kundu and Gupta (1998) obtained the theoretical results when $\min(M,N) \to \infty$, whereas Bansal, Hamedani and Zhang (1999) proved the same results when $MN \to \infty$. Zhang (1991) showed that a 2-D ARMA process is an appropriate model of 2-D sinusoids in white noise and presented a time domain analysis technique for resolving several closely spaced 2-D sinusoids in white noise. In Nandi and Kundu (1999), the errors are finite order 2-D moving average process and the error variance is finite and they mainly considered the LSEs of the different parameters and study their large sample properties.

The organization of this paper is as follows. Some preliminary ideas are given in section 2. Here we provide a different formulation of model (1) and briefly discuss the Prony's estimators and its extension in two dimension. The 2-D NSD method for model (1) is discussed in section 3. Two pairing algorithms are proposed in section 4. The consistency results of the proposed estimators are provided in section 5. Numerical results are provided in section 6 and finally we conclude the paper in section 7.

2 Preliminaries

In this section, we first provide an equivalent formulation of the signal component of model (1) using complex exponentials. Then, we briefly discuss the Prony's method, which was proposed in 1795, to find the non-linear parameters on a similar one-dimensional model in noiseless situation.

We write model (1) as $y(s,t) = m(s,t) + e(s,t)$, where $m(s,t)$ is the signal component. Then we observe that using complex exponentials, $m(s,t)$ can be written as

$$m(s,t) = \sum_{k=1}^{2p} c_k^0 e^{i(s\gamma_k^0 + t\delta_k^0)} \tag{2}$$

with

$$\begin{array}{lllll} i = \sqrt{-1}, & C_{2k} = \dfrac{A_k^0 + iB_k^0}{2}, & C_{2k-1} = \dfrac{A_k^0 - iB_k^0}{2}, & & k = 1, \ldots, p; \\ \gamma_{2k} = -\lambda_k, & \gamma_{2k-1} = \lambda_k, & \delta_{2k} = -\mu_k, & \delta_{2k-1} = \mu_k, & k = 1, \ldots, p. \end{array}$$

Now $\gamma_k^0, \delta_k^0 \in (-\pi, \pi)\backslash\{0\}$ and c_k^0, $k = 1, \ldots, 2p$ are complex-valued. This form is quite useful in tackling the technical details.

2.1 Different Other Estimators

The LSE of the unknown parameters are obtained by minimizing the following residual sum of squares with respect to unknown parameters A_k, B_k λ_k and μ_k, $k = 1, \ldots, p$.

$$\sum_{s=1}^{M}\sum_{t=1}^{N}\left(y(s,t) - \sum_{k=1}^{p}\left[A_k \cos(s\lambda_k + t\mu_k) + B_k \sin(s\lambda_k + t\mu_k)\right]\right)^2 .$$

In section 6, we compare the LSE and another estimator, known as approximate LSE (ALSE) with the proposed NSD estimators. The ALSE of λ_k and μ_k, $k = 1, \ldots, p$ are obtained by maximizing the 2-D periodogram function defined as follows;

$$I(\lambda,\mu) = \frac{1}{MN}\left|\sum_{s=1}^{M}\sum_{t=1}^{N} y(s,t)e^{-j(s\lambda+t\mu)}\right|^2 .$$

The maximization is done locally and sequentially under the constraints

$$|A_1^0|^2 + |B_1^0|^2 \geq |A_2^0|^2 + |B_2^0|^2 \geq \cdots \geq |A_p^0|^2 + |B_p^0|^2.$$

required to resolve the identifiability issues. Once the non-linear frequencies are obtained, corresponding linear parameters A_ks and B_ks are estimated as

$$\tilde{A}_k = \frac{2}{MN}\sum_{s=1}^{M}\sum_{t=1}^{N} y(s,t)\cos(s\tilde{\lambda}_k + t\tilde{\mu}_k),$$

$$\tilde{B}_k = \frac{2}{MN}\sum_{s=1}^{M}\sum_{t=1}^{N} y(s,t)\sin(s\tilde{\lambda}_k + t\tilde{\mu}_k),$$

where $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ are the ALSEs of λ_k and μ_k respectively. The ALSE's are asymptotically equivalent to the LSE's with the same rate of convergence (Kundu and Nandi; 2003).

2.2 Prony's Method

Prony's (1795) idea of fitting the sum of exponentials to the data has been extensively used in Signal Processing and Numerical Analysis. The method is described in several text books (Barrodale and Oleski; 1981, Kay; 1988) in details. The proposed 2-D NSD method is based on 1-D NSD method and the later uses some concepts of Prony's method. So we describe it here briefly. Suppose $m_1(1), \ldots, m_1(N)$ are N data points from

$$m_1(t) = \sum_{k=1}^{q}\alpha_k e^{j\beta_k t}, \quad t = 1, \ldots, N \quad \text{where} \quad \beta_i \neq \beta_k, \quad i \neq k,$$

as Prony's method was proposed for noiseless data. Prony observed that there exists $q+1$ constants, say, $g_1, \ldots, g_{q+1}$ such that it satisfies

$$\begin{array}{ccccccccc} g_1 m_1(1) & + & g_2 m_1(2) & + & \cdots & + & g_{q+1} m_1(q+1) & = & 0 \\ g_1 m_1(2) & + & g_2 m_1(3) & + & \cdots & + & g_{q+1} m_1(q+2) & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \\ g_1 m_1(N-q) & + & g_2 m_1(N-q+1) & + & \cdots & + & g_{q+1} m_1(N) & = & 0 \end{array}$$

and there is a one - one correspondence between $\mathbf{g} = (g_1, \ldots, g_{q+1})$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)$ subject to the condition $\sum_{i=1}^{q+1} g_i^2 = 1$ and $g_1 > 0$. Then the following q-degree polynomial

$$g_1 + g_2 x^2 + \cdots + g_{q+1} x^q = 0$$

has roots $e^{i\beta_1}, e^{i\beta_2}, \ldots, e^{i\beta_q}$. Thus $\beta_1, \ldots, \beta_q$ can be estimated once $g_1, \ldots, g_{q+1}$ are estimated. It is also observed that g_ks are independent of $\alpha_1, \ldots, \alpha_q$, $k = 1, \ldots, q$. Since Prony's algorithm is applicable to noiseless data, several problem specific adoptions have been considered and the method can be used to estimate the starting values of the nonlinear parameters of any iterative scheme.

The above idea can be extended for the 2-D case also. We concentrate on the form (2) of $m(s,t)$. We write the signal component $\{m(s,t), s = 1, \ldots, M; t = 1, \ldots, N\}$ of the model (1) in the following matrix form

$$\begin{bmatrix} m(1,1) & \cdots & m(1,N) \\ \vdots & \vdots & \cdots \\ m(M,1) & \cdots & m(M,N) \end{bmatrix} = \mathbf{M_S}, \quad \text{(say)}. \tag{3}$$

Note that there exists a vector $\mathbf{a} = (a_1, \ldots, a_{2p+1})$, with $|\mathbf{a}|^2 = 1$ and $a_1 > 0$, such that

$$\begin{bmatrix} m(1,1) & \cdots & m(1,N) \\ \vdots & \vdots & \cdots \\ m(M,1) & \cdots & m(M,N) \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & a_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{2p+1} & \vdots & \cdots & 0 \\ 0 & a_{2p+1} & \cdots & a_1 \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & a_{2p+1} \end{bmatrix} = \mathbf{0} \tag{4}$$

$$= \mathbf{M_S A}, \quad \text{(say)}.$$

Similarly there exists a vector $\mathbf{b} = (b_1, \ldots, b_{2p+1})$, with $|\mathbf{b}|^2 = 1$ and $b_1 > 0$, such that

$$\begin{bmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & b_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ b_{2p+1} & \vdots & \cdots & 0 \\ 0 & b_{2p+1} & \cdots & b_1 \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & b_{2p+1} \end{bmatrix} \begin{bmatrix} m(1,1) & \cdots & m(1,N) \\ \vdots & \vdots & \cdots \\ m(M,1) & \cdots & m(M,N) \end{bmatrix} = \mathbf{0}$$

$$= \mathbf{B}\mathbf{M_S}, \quad \text{(say)}.$$

Consider the two polynomial equations

$$B_1(z) = a_1 + a_2 z + \cdots + a_{2p+1} z^{2p} = 0, \tag{5}$$

$$B_2(z) = b_1 + b_2 z + \cdots + b_{2p+1} z^{2p} = 0, \tag{6}$$

then the equation (5) has the roots $e^{i\gamma_1}, \ldots, e^{i\gamma_{2p}}$ and the equation (6) has the roots $e^{i\delta_1}, \ldots, e^{i\delta_{2p}}$. Thus the roots are basically in the form $\exp(\pm i\mu_k)$.

3 Two-Dimensional NSD Method

In this section first we propose the 2-D NSD method to estimate the non-linear frequencies of the 2-D sinusoidal model (1). Like the Prony's method presented in section 2, we will use the form (2) for developing the algorithm. The proposed method which is basically an extension of 1-D NSD method to two-dimension, is as follows.

From the s^{th} row of the data matrix $\mathbf{Y}$, described in equation (3), construct the matrix $\mathbf{A_s}$ for any $N - 2p \geq L \geq 2p$ as follows,

$$\mathbf{A_s} = \begin{bmatrix} y(s,1) & \cdots & y(s,L+1) \\ \vdots & \vdots & \vdots \\ y(s,N-L) & \cdots & y(s,N) \end{bmatrix}.$$

Obtain the $(L+1) \times (L+1)$ matrix $\mathbf{B}$ as

$$\mathbf{B} = \frac{1}{(N-L)M} \sum_{s=1}^{M} \mathbf{A_s}^H \mathbf{A_s}.$$

Suppose the singular value decomposition of $\mathbf{B}$ is

$$\mathbf{B} = \sum_{i=1}^{L+1} \lambda_i \mathbf{u_i} \mathbf{u_i}^H,$$

where $\lambda_1 \geq \cdots \geq \lambda_{L+1}$ are the ordered eigen values of $\mathbf{B}$ and $\mathbf{u}_i$ is the normalized eigen vector corresponding to λ_i.

Now using the same idea as Kundu and Mitra (1995), construct the estimated signal subspace $\mathcal{S}$ and the estimated noise subspace $\mathcal{N}$ as follows:

$$\mathcal{S} = \{\mathbf{u_1} : \cdots : \mathbf{u_{2p}}\} \quad \text{and} \quad \mathcal{N} = \{\mathbf{u_{2p+1}} : \cdots : \mathbf{u_{L+1}}\}.$$

We use the estimated noise space $\mathcal{N}$ to estimate $\mathbf{a} = (a_1, \ldots, a_{2p+1})$, the constants to construct the polynomial equation. Consider $(L+1) \times (L+1-p)$ matrix $\mathbf{B_1}$ as follows:

$$\mathbf{B_1} = [\mathbf{u_{2p+1}} : \cdots : \mathbf{u_{L+1}}] = \begin{bmatrix} b_{1,1} & \cdots & b_{1,L+1-2p} \\ \vdots & \vdots & \vdots \\ b_{L+1,1} & \cdots & b_{L+1,L+1-2p} \end{bmatrix}.$$

Now the aim is to obtain an basis of $\mathbf{B_1}$, which is of the same form as matrix $\mathbf{A}$ in equation (4). Partition the matrix $\mathbf{B_1}$ as follows:

$$\mathbf{B_1^T} = [\,\mathbf{B_{1k}^T} \quad : \quad \mathbf{B_{2k}^T} \quad : \quad \mathbf{B_{3k}^T}\,]$$

for $k = 0, 1, \ldots, L-2p$, where $\mathbf{B_{1k}^T}$, $\mathbf{B_{2k}^T}$ and $\mathbf{B_{3k}^T}$ are of the orders $(L+1-2p) \times k$, $(L+1-2p) \times (2p+1)$ and $(L+1-2p) \times (L-k-2p)$ respectively. Consider the matrix

$$[\,\mathbf{B_{1k}^T} \quad : \quad \mathbf{B_{3k}^T}\,].$$

Since it is a random matrix, it is of rank $(L-2p)$ (full rank) almost surely. Therefore, there exists an $L-2p+1$ column vector $\mathbf{X_{k+1}} \neq \mathbf{0}$, such that

$$\begin{bmatrix} \mathbf{B_{1k}} \\ \mathbf{B_{3k}} \end{bmatrix} \mathbf{X_{k+1}} = \mathbf{0}.$$

Consider the $(2p+1)$ vector $\hat{\mathbf{a}}^{k+1} = (\hat{a}_{k+1,1}, \ldots, \hat{a}_{k+1,2p+1})$, where

$$\left(\hat{a}^{k+1}\right)^T = \mathbf{B_{12}} \mathbf{X_{k+1}}.$$

By proper normalization, we can make $\hat{a}_{k+1,1} > 0$ and $||\hat{\mathbf{a}}^{k+1}||^2 = 1$ for $k = 0, 1, \ldots, L-2p$. Therefore, there exist vectors $\mathbf{X_1}, \ldots, \mathbf{X_{L-2p+1}}$ such that

$$\mathbf{B_1}\,[\,\mathbf{X_1} \quad : \quad \ldots \quad : \quad \mathbf{X_{L-2p+1}}\,] = \begin{bmatrix} \hat{a}_{1,1} & 0 & \cdots & 0 \\ \vdots & \hat{a}_{2,1} & \cdots & \vdots \\ \hat{a}_{1,2p+1} & \vdots & \cdots & \hat{a}_{L-2p+1,1} \\ 0 & \hat{a}_{2,2p+1} & \cdots & \vdots \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & \hat{a}_{L-2p+1,2p+1} \end{bmatrix}.$$

In the noiseless situation $\hat{\mathbf{a}}^1 = \ldots = \hat{\mathbf{a}}^{2p+1} = \hat{\mathbf{a}}$. So it is reasonable to use any one of $\hat{\mathbf{a}}^k,\ k = 0, 1, \ldots, L-2p$ or all of them to estimate $\nu_1, \ldots, \nu_{2p}$. One can consider the average of all the $\hat{\mathbf{a}}_k$s and use it as an estimate of $\mathbf{a}$. We have considered that $\hat{\mathbf{a}}$ for which the prediction error is minimum. To obtain the prediction errors we consider the following method:

From model (1), we obtain

$$\sum_{s=1}^{M} y(s,t) = \sum_{k=1}^{p}\sum_{m=1}^{M}\left\{A_k^0\cos(s\lambda_k^0 + t\mu_k^0) + B_k^0\sin(s\lambda_k^0 + t\mu_k^0)\right\} + \sum_{s=1}^{M} e(s,t), \tag{7}$$

and we also obtain

$$y_1(t) = \sum_{k=1}^{p}\left\{a_k^0\cos(t\mu_k^0) + b_k^0\sin(t\mu_k^0)\right\} + e_1(t) = m_2(t) + e_1(t),\ (say) \tag{8}$$

where

$$a_k^0 = A_k^0\sum_s\cos(s\lambda_k^0) + B_k^0\sum_s\sin(s\lambda_k^0),\ b_k^0 = -A_k^0\sum_s\sin(s\lambda_k^0) + B_k^0\sum_s\cos(s\lambda_k^0),$$

and $e_1(t) = \sum_s e(s,t)$. Similarly as the form in equation (2), $m_2(t)$ is also written as

$$m_2(n) = \sum_{k=1}^{2p} d_k^0 e^{it\delta_k^0},\ \ d_{2k}^0 = \frac{a_k^0 + ib_k^0}{2},\ \ d_{2k-1}^0 = \frac{a_k^0 - ib_k^0}{2}. \tag{9}$$

Now for all $i = 1, \ldots, 2p+1$, consider $\hat{\mathbf{a}}_i$ and solving the polynomial equation (5) obtain the corresponding $\delta_1, \ldots, \delta_{2p}$. Then the linear parameters of the corresponding one dimensional model (8) are obtained and finally we obtain the prediction error of this model (8).

Exactly in the same way $\mathbf{b}$ can be estimated using the columns of the data matrix $\mathbf{Y}$ and from the roots of the polynomial equation (6), we obtain the estimates of $\beta_1, \ldots, \beta_p$.

4 Pairing Algorithm

In this section we propose two pairing algorithms to estimate the pairs $\{(\lambda_k, \mu_k); k = 1, \ldots, p\}$ for the model (1). One algorithm is based on $p!$ search. It is computationally efficient for small values of p, say $p = 2, 3$ and the other is based on p^2-search, so it is efficient for large values of p, *i.e.* when p is greater than 3. Suppose the estimates obtained using the method in section 3 are $\{\hat{\lambda}_{(1)}, \ldots, \hat{\lambda}_{(p)}\}$ and $\{\hat{\mu}_{(1)}, \ldots, \hat{\mu}_{(p)}\}$.

4.1 Algorithm 1

Consider all possible $p!$ combination of pairs $\{(\hat{\lambda}_{(i)}, \hat{\mu}_{(i)}) : i = 1, \ldots, p\}$ and calculate the sum of the periodogram function for each combination as

$$I(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1}^{p}\frac{1}{MN}\left|\sum_{m=1}^{M}\sum_{n=1}^{N} y(m,n)e^{-j(m\lambda_k + n\mu_k)}\right|^2.$$

Consider that combination as the paired estimates of $\{(\lambda_i, \mu_i) : i = 1, \ldots, p\}$ for which this $I(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is maximum.

4.2 Algorithm 2

Consider the periodogram function $I_1(\lambda, \mu)$ of each pair $(\lambda_i, \mu_k),\ i = 1, \ldots, p,\ k = 1, \ldots, p.$

$$I_1(\lambda, \mu) = \frac{1}{MN} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} y(m,n) e^{-j(m\lambda + n\mu)} \right|^2 .$$

Compute $I_1(\lambda, \mu)$ over $\{(\hat{\lambda}_{(i)}, \hat{\mu}_{(k)}), i, k = 1, \ldots, p\}$. Choose the largest p values of $I(\hat{\lambda}_{(i)}, \hat{\mu}_{(k)})$ and the corresponding $\{(\hat{\lambda}_{(k)}, \hat{\mu}_{(k)}), k = 1, \ldots, p\}$ are the paired estimates of $\{(\lambda_k, \mu_k), k = 1, \ldots, p\}$.

5 Consistency Results

In this section we establish the strong consistency of the frequency estimators obtained by 2-D NSD method. We prove the results by using the form (2). To prove the strong consistency we need the following assumptions as in the line of Rao, Zhao and Zhou (1994) or Kundu and Mitra (1995) on the parameters of the model (1).

Assumption 1 $\{e(s,t)\}$ *is an array of independent and identically distributed real valued random variables with mean zero and finite variance* σ^2.

Assumption 2 $\lambda_1, \ldots, \lambda_p$ *are distinct and so also are* $\mu_1, \ldots, \mu_p$.

Assumption 3 $A_1^0, \ldots, A_p^0$ *and* $B_1^0, \ldots, B_p^0$ *are arbitrary real numbers not identically equal to zero.*

THEOREM 1 *Under the Assumptions 1 and 2, the estimators* $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$ *and* $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_p)$ *obtained by the method described in section 3 are strongly consistent estimators of* $\boldsymbol{\lambda}^0 = (\lambda_1^0, \ldots, \lambda_p^0)$ *and* $\boldsymbol{\mu}^0 = (\mu_1^0, \ldots, \mu_p^0)$ *respectively.*

To prove Theorem 1, we need the following lemmas.

Lemma 1 *Let* $\mathbf{Q} = ((Q_{ik}))$ *and* $\mathbf{W} = ((W_{ik}))$ *be two* $r \times r$ *Hermitian matrices with spectral decomposition*

$$\mathbf{Q} = \sum_{i=1}^{r} \gamma_i \mathbf{q}_i \mathbf{q}_i^H, \qquad \gamma_1 \geq \ldots \geq \gamma_r,$$

$$\mathbf{W} = \sum_{i=1}^{r} \delta_i \mathbf{w}_i \mathbf{w}_i^H, \qquad \delta_1 \geq \ldots \geq \delta_r,$$

where γ_is and δ_is are eigen values of $\mathbf{Q}$ and $\mathbf{W}$ respectively and $\mathbf{q}_i$ and $\mathbf{w}_i$ are the orthonormal eigenvectors of $\mathbf{Q}$ and $\mathbf{W}$ associated with γ_i and δ_i respectively, for $i = 1, \ldots, r$. Further assume that

$$\delta_{n_{h-1}+1} = \cdots = \delta_{n_h} = \tilde{\delta}_h; \qquad 0 = n_0 \; < \; n_1 \; < \; \cdots \; < \; n_s = p; \quad h = 1, \ldots, s;$$

$$\tilde{\delta}_1 \; > \; \tilde{\delta}_2 \; > \; \cdots \tilde{\delta}_s$$

and that $|q_{ik} - w_{ik}| \; < \; \alpha$, $i, k = 1, \ldots, r$, then there exists a constant K independent of α such that

$$\begin{aligned} &(1) \quad |\gamma_i - \delta_i| \; < \; K\alpha, \qquad\qquad i = 1, \ldots, r, \\ &(2) \sum_{i=n_{h-1}+1}^{n_h} \mathbf{q}_i \mathbf{q}_i^H \; = \sum_{i=n_{h-1}+1}^{n_h} \mathbf{w}_i \mathbf{w}_i^H + \mathbf{C^{(h)}}, \end{aligned}$$

with

$$\mathbf{C^{(h)}} = ((C_{lk}^{(h)})), \qquad |C_{lk}^{(h)}| \leq K\alpha.$$

Proof of Lemma 1 The proof mainly follows from von Neumann's (1937) inequality. For details see Bai, Miao and Rao (1990).

Lemma 2 *Let $g_n(x)$ be a sequence of polynomials of degree l, with roots $x_1^{(n)}, \ldots, x_l^{(n)}$ for each n. Let $g(x)$ be a polynomial of degree l, with roots $x_1, \ldots, x_l$. If $g_n(x) \to g(x)$ as $n \to \infty$ for all x, then with proper rearrangement the roots of $g_n(x)$, $x_k^{(n)}$ converge to the roots of $g(x)$, i.e. to x_k, $k = 1, \ldots, l$.*

Proof of Lemma 2 See Bai (1986).

Proof of Theorem 1 We note that using form (2)

$$\begin{aligned} y(s,t) &= \sum_{k=1}^{2p} c_k^0 e^{j(s\gamma_k^0 + t\delta_k^0)} + e(s,t) \\ &= \sum_{k=1}^{2p} g_{ks}^0 e^{jt\delta_k^0} + e(s,t), \quad t = 1, \ldots, N, \end{aligned}$$

where $g_{ks}^0 = c_k^0 e^{js\gamma_k^0}$, $k = 1, \ldots, 2p$ and $s = 1, \ldots, M$. Now consider the $(u,v)^{th}$ element of the matrix $\frac{1}{N-L}\mathbf{A_s^H A_s}$ for any s. The $y(s,t)$ and $e(s,t)$ being real-valued, we have

$$\begin{aligned} &\left(\left(\frac{1}{N-L}\mathbf{A_s^H A_s}\right)\right)_{u,v} \\ = \; &\frac{1}{N-L} \sum_{w=1}^{N-L+1} \bar{y}(s, u+w) y(s, v+w) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N-L}\sum_{w=1}^{N-L+1}\left(\sum_{k=1}^{2p}\bar{g}^0_{ks}e^{-j(u+w)\delta^0_k}+\bar{e}(s,u+w)\right)\left(\sum_{k=1}^{2p}g^0_{ks}e^{j(v+w)\delta^0_k}+e(s,v+w)\right)\\
&= \frac{1}{N-L}\sum_{w=1}^{N-L+1}\left(\sum_{k=1}^{2p}\bar{g}^0_{ks}e^{-j(u+w)\delta^0_k}\right)\left(\sum_{k=1}^{2p}g^0_{ks}e^{j(v+w)\delta^0_k}\right)\\
&\qquad +\frac{1}{N-L}\sum_{w=1}^{N-L+1}\bar{e}(s,u+w)\sum_{k=1}^{2p}g^0_{ks}e^{j(v+w)\delta^0_k}\\
&\qquad +\frac{1}{N-L}\sum_{w=1}^{N-L+1}e(s,v+w)\sum_{k=1}^{2p}\bar{g}^0_{ks}e^{-j(u+w)\delta^0_k}\\
&\qquad +\frac{1}{N-L}\sum_{w=1}^{N-L+1}\bar{e}(s,u+w)e(s,v+w),\\
&= T_1(s)+T_2(s)+T_3(s)+T_4(s)
\end{aligned}
$$

Now we observe that

$$
\begin{aligned}
T_1(s) &= \frac{1}{N-L}\left\{\sum_{w=1}^{N-L+1}\left(\sum_{k=1}^{2p}|g^0_{ks}|^2e^{j\delta^0_k(v-u)}\right)\right.\\
&\qquad\left.+\sum_{w=1}^{N-L+1}\left(\sum_{k\neq l}^{2p}\bar{g}^0_{ks}g^0_{lm}e^{j\delta^0_l(v+w)-j\delta^0_k(u+w)}\right)\right\}\\
&= \sum_{k=1}^{2p}|g^0_{ks}|^2e^{j\delta^0_k(v-u)}+O\left(\frac{1}{N-L}\right)\\
&\qquad\qquad\text{(For fixed } L \text{ and large } N\text{)}
\end{aligned}
$$

Now by the law of iterated logarithm (Chung; 1974) of M-dependent sequence, we say that

$$
\begin{aligned}
T_2(s) &= O\left(\frac{\log\log(N-L)}{N-L}\right)^{1/2} = T_3(s),\\
T_4(s) &= \begin{cases}\sigma^2 & \text{if } u=v\\ 0 & \text{if } u\neq v,\end{cases} \qquad\qquad s=1,\ldots,M.
\end{aligned}
$$

Hence

$$
\lim_{N\to\infty}\frac{1}{N-L}\mathbf{A_s^H}\mathbf{A_s}=\sigma^2\mathbf{I}_{L+1}+\mathbf{\Omega^{(L)}}^{\mathbf{H}}\mathbf{D_s}\mathbf{\Omega^{(L)}} \qquad \text{a.s.},
$$

where

$$
\mathbf{\Omega^{(L)}}=\begin{bmatrix} e^{-j\delta^0_1} & \cdots & e^{-j(L+1)\delta^0_1}\\ \vdots & \vdots & \vdots\\ e^{-j\delta^0_{2p}} & \cdots & e^{-j(L+1)\delta^0_{2p}}\end{bmatrix}
$$

and

$$
\mathbf{D_s}=\mathrm{diag}\{|g_{1s}|^2,|g_{2s}|^2,\cdots|g_{2ps}|^2\}.
$$

$\mathbf{I}_{L+1}$ is the identity matrix of order $L+1$. Therefore, averaging over all rows,

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{M(N-L)}\sum_{s=1}^{M}\mathbf{A_s^H A_s} &= \sigma^2\mathbf{I}_{L+1}+\mathbf{\Omega^{(L)}}^{\mathbf{H}}\mathbf{D}\mathbf{\Omega^{(L)}} \\ &= \mathbf{S} \quad \text{(say)},\end{aligned}$$

where $\mathbf{D}=\sum_{s=1}^{M}\mathbf{D_m}$. Note that due to Assumption 2, the rank of the matrix $\mathbf{\Omega^{(L)}}$ is $2p$ and due to Assumption 3, the rank of $\mathbf{D}$ is also $2p$. Let the ordered eigen values of $\mathbf{S}$ be

$$\lambda_{(1)}\geq\lambda_{(2)}\geq\cdots\lambda_{(p)}>\lambda_{(p+1)}=\cdots\lambda_{(L+1)}=\sigma^2$$

and suppose the singular value decomposition of $\mathbf{S}$ is

$$\mathbf{S}=\sum_{k=1}^{L+1}\lambda_{(k)}^2\mathbf{s_k s_k^H}.$$

Here $\mathbf{s_k}$ is the orthonormal eigen vector corresponding to the eigen value $\lambda_{(k)}$. Therefore using Lemma 1, we have

$$\sum_{k=2p+1}^{L+1}\mathbf{\hat{u}_i\hat{u}_i^H}\longrightarrow\sum_{k=2p+1}^{L+1}\mathbf{s_i s_i^H}$$

i.e. the vector space generated by $(\mathbf{\hat{u}_{2p+1}},\ldots,\mathbf{\hat{u}_{L+1}})$ converges to the vector space generated by $(\mathbf{s_{2p+1}},\ldots,\mathbf{s_{L+1}})$. Now similarly as Kundu and Mitra (1995), it can be shown that the vector space generated by $(\mathbf{\hat{u}_{2p+1}},\ldots,\mathbf{\hat{u}_{L+1}})$ has a unique basis of the form

$$\begin{bmatrix}\hat{a}_{1,1} & 0 & \cdots & 0\\ \vdots & \hat{a}_{2,1} & \cdots & \vdots\\ \hat{a}_{1,2p+1} & \vdots & \cdots & \hat{a}_{L-2p+1,1}\\ 0 & \hat{a}_{2,2p+1} & \cdots & \vdots\\ \vdots & 0 & \cdots & \vdots\\ 0 & 0 & \cdots & \hat{a}_{L-2p+1,2p+1}\end{bmatrix},$$

with $\hat{a}_{k,1}\ >\ 0$ and $||\mathbf{\hat{a}}_k||=1$, where $\mathbf{\hat{a}}_k=(\hat{a}_{k,1},\ldots,\hat{a}_{k,2p+1})$ for $k=1,\ldots,L-2p+1$. Similarly the vector space generated by $(\mathbf{s_{2p+1}},\ldots,\mathbf{s_{L+1}})$ has a unique basis of the form

$$\begin{bmatrix}a_1 & 0 & \cdots & 0\\ \vdots & a_1 & \cdots & \vdots\\ a_{2p+1} & \vdots & \cdots & a_1\\ 0 & a_{2p+1} & \cdots & \vdots\\ \vdots & 0 & \cdots & \vdots\\ 0 & 0 & \cdots & a_{2p+1}\end{bmatrix},$$

with $a_1 > 0$ and $||\mathbf{a}_k|| = 1$, $\mathbf{a}_k = (a_1, \ldots, a_{2p+1})$. This implies that

$$\hat{\mathbf{a}}_k \to \mathbf{a} \quad \text{a.s.}, \quad k = 1, \ldots, L - p + 1.$$

Therefore, using Lemma 2, we can conclude that the roots obtained from the polynomial equation (5) with $\hat{\mathbf{a}}_k$, are consistent estimators of $\delta_1^0, \ldots, \delta_{2p}^0$ for $k = 1, \ldots, L - 2p + 1$.

To prove the strong consistency of $\hat{\gamma}_1, \ldots, \hat{\gamma}_{2p}$, consider the t^{th} column in place of s^{th} row. Then using the same technique as above, it can be shown that $\hat{\gamma}_1, \ldots, \hat{\gamma}_{2p}$ are strongly consistent estimators of $\gamma_1^0, \ldots, \gamma_{2p}^0$.

6 Numerical Experiments

In section 3 we have developed a method to estimate the frequencies of the 2-D sinusoidal models. The large sample properties of the estimators are examined in the previous section. In this section, we study the small sample properties of the estimators using simulated data. All the computations are performed at the Indian Institute of Technology, Kanpur on Sun Workstation using the random number generator of Press *et al.*(1993). NAG subroutines are used for eigen decomposition and to obtain the roots of different polynomial equations. The numerical experiments have been conducted for different values of σ^2, the error variance.

We consider the following model with two components for the simulation study:

$$\begin{aligned} y(s,t) \;=\; & A_1^0 \cos(s\lambda_1^0 + t\mu_1^0) + B_1^0 \sin(s\lambda_1^0 + t\mu_1^0) \\ & + A_2^0 \cos(s\lambda_2^0 + t\mu_2^0) + B_2^0 \sin(s\lambda_2^0 + t\mu_2^0) + e(s,t), \\ & \qquad s = 1, \ldots, 30; \quad t = 1, \ldots, 30; \end{aligned} \tag{10}$$

with

$$A_1^0 = 4.0, B_1^0 = 5.0, \lambda_1^0 = 2.0, \mu_1^0 = 1.0,$$
$$A_2^0 = 3.5, B_2^0 = 5.5, \lambda_2^0 = 2.5, \mu_2^0 = 1.5.$$

The error random variables $e(s,t)$s are $i.i.d.$ normal random variables with mean zero and variance σ^2. We consider $\sigma = 0.01, 0.03, 0.05, 0.1, 0.3, 0.5, 1.0$ and 2.0. For each σ we generate 1000 different data sets using different sequences of $e(s,t)$ and the frequencies are estimated using the method described in section 3. We obtain the average estimates and the mean squared errors over one thousand replication. We have also calculated the ALSEs and the LSEs for the frequencies of the model (10) maximizing locally the 2-D periodogram function and minimizing the residual sum of squares respectively. The mean squared errors (MSEs) of the LSEs and the ALSEs and the asymptotic variances (ASYVs) of the least squares estimators are also reported for comparison. The asymptotic variances (ASYV) are obtained using the distribution obtained for the model (1) in Kundu and Gupta (1998) using the true parameter values. All these results are reported in Table 1. For different σ, we have reported the ALSEs, the NSD estimators, the LSEs and the ASYVs in the columns. The mean squared errors of the corresponding estimators are given in the brackets in the next row. As the model (10) has two components we have used Algorithm 2 described in section 4.2 to obtain the final estimates of the frequencies.

From the results of simulation of the model (10), it is observed that the proposed 2-D NSD method works quite well for different values of σ, the error variance. For small values of σ, the NSD estimators work better than the ALSEs and for large σ both the ALSEs and the NSD estimators perform almost identically. In case of one dimensional model the performances of the NSD estimators depends on L, the extended order. But it has been observed in simulation study that the performances of the 2-D NSD estimators do not depend much on the choice of L. It works better for small values of L ($\leq M/3$ or $\leq N/3$). For all L, less than equal to 10, the NSD estimators work almost in a similar way in terms of MSEs. As L increases, its performances deteriorate. Also the computational cost is very high if L is large as compared to small L. Though the ALSEs are computed as the local maxima of the 2-D periodogram function, the computational cost is still very high as compared to the NSD estimators. For L less than 20 *i.e.* $2M/3$, the ALSEs are more computationally expensive than the NSD estimators.

7 Conclusions

In this chapter, we have considered the estimation of the frequencies of the the 2-D sinusoidal model under the assumption of *i.i.d.* errors. Though the LSE is the most reasonable estimator, it is well known that obtaining the LSEs even in 1-D, is a difficult problem. We have developed an consistent non-iterative procedure to estimate the unknown parameters of the 2-D sinusoidal model (1) which is an extension of the 1-D NSD method. We have proposed two pairing algorithms to estimate the final set of frequencies. It has been observed that the proposed method provides consistent estimators. Numerical results indicate that the 2-D NSD estimators can be used as the starting values to obtain the LSEs for the sinusoidal model (1) in most of the cases. Also for the 2-D sinusoidal model the proposed estimators work better than the ALSEs.

Recently, Prasad and Kundu (2009) used three dimensional superimposed sinusoidal model to analyze colored textures. It seems the proposed 2-D NSD method can be extended to three dimension also. Work is in progress, it will be reported later.

Table 1: The ALSEs, the NSD estimators, the LSEs, the corresponding MSEs and the asymptotic variances of the different parameters of the 2-D sinusoidal model.

σ	Parameter	ALSE	NSD estimator	LSE	ASYV
.01	λ_1	1.99699	2.00001	1.99999	
		(9.08479e-06)	(3.54279e-09)	(5.39972e-11)	7.2267387e-11
	μ_1	1.00029	1.00000	0.999994	
		(8.61867e-08)	(1.61219e-09)	(5.52318e-11)	7.2267387e-11
	λ_2	2.49917	2.50001	2.50002	
		(6.75802e-07)	(2.63496e-09)	(2.40655e-11)	6.9716774e-11
	μ_2	1.49932	1.50000	1.50001	
		(4.57080e-07)	(8.79792e-10)	(2.45539e-11)	6.9716774e-11
.03	λ_1	1.99698	2.00001	1.99999	
		(9.12686e-06)	(3.13326e-08)	(4.87039e-10)	6.5040645e-10
	μ_1	1.00029	1.00000	0.999999	
		(8.61407e-08)	(1.48746e-08)	(4.84410e-10)	6.5040645e-10
	λ_2	2.49917	2.50001	2.50001	
		(6.75478e-07)	(2.26150e-08)	(2.33181e-10)	6.2745098e-10
	μ_2	1.49932	1.50000	1.50000	
		(4.57967e-07)	(7.75861e-09)	(2.28176e-10)	6.2745098e-10
.05	λ_1	1.99698	2.00002	2.00000	
		(9.14752e-06)	(8.77772e-08)	(1.40601e-09)	1.8066845e-09
	μ_1	1.00029	1.00000	1.00000	
		(8.68718e-08)	(4.05905e-08)	(1.41209e-09)	1.8066845e-09
	λ_2	2.49918	2.50001	2.50001	
		(6.75214e-07)	(5.56449e-08)	(6.58749e-10)	1.7429194e-09
	μ_2	1.49933	1.50000	1.50000	
		(4.54628e-07)	(2.21050e-08)	(6.51090e-10)	1.7429194e-09
0.1	λ_1	1.99697	2.00002	2.00000	
		(9.19954e-06)	(3.31397e-07)	(5.46929e-09)	7.2267388e-09
	μ_1	1.00029	1.00000	0.999999	
		(9.84439e-08)	(1.59495e-07)	(5.61064e-09)	7.2267388e-09
	λ_2	2.49918	2.50001	2.50001	
		(6.83887e-07)	(2.26631e-07)	(2.43253e-09)	6.9716783e-09
	μ_2	1.49932	1.50000	1.50000	
		(4.64469e-07)	(9.38960e-08)	(2.43484e-09)	6.9716783e-09

(continued...)

σ	Parameter	ALSE	NSD estimator	LSE	ASYV
.3	λ_1	1.99696	1.99999	2.00000	
		(9.38446e-06)	(3.16364e-06)	(4.90887e-08)	6.5040652e-08
	μ_1	1.00029	0.99999	0.999999	
		(2.11111e-07)	(1.48538e-06)	(4.81352e-08)	6.5040652e-08
	λ_2	2.49918	2.50000	2.50001	
		(7.34819e-07)	(2.25009e-06)	(2.35643e-08)	6.2745102e-08
	μ_2	1.49932	1.49998	1.49999	
		(5.08247e-07)	(7.75322e-07)	(2.29973e-08)	6.2745102e-08
0.5	λ_1	1.99696	2.00000	1.99998	
		(9.68857e-06)	(9.18530e-06)	(1.41363e-07)	1.8066847e-07
	μ_1	1.00029	1.00000	1.00002	
		(4.36193e-07)	(4.08143e-06)	(1.41096e-07)	1.8066847e-07
	λ_2	2.49918	2.50001	2.50000	
		(8.28642e-07)	(5.46881e -06)	(6.58413e-08)	1.7429194e-07
	μ_2	1.49932	1.50003	1.50000	
		(6.02746e-07)	(2.19510e-06)	(6.57252e-08)	1.7429194e-07
1.0	λ_1	1.99694	1.99969	1.99999	
		(1.11344e-05)	(3.84807e-05)	(5.33798e-07)	7.2267403e-07
	μ_1	1.00030	0.99991	1.00001	
		(1.49700e-06)	(1.78891e-05)	(5.34425e-07)	7.2267403e-07
	λ_2	2.49918	2.50009	2.49999	
		(1.28682e-06)	(2.20381e-05)	(2.81330e-07)	6.9716793e-07
	μ_2	1.49931	1.49990	1.50000	
		(1.02375e-06)	(8.93373e-06)	(2.82741e-07)	6.9716793e-07
2.0	λ_1	1.99690	1.99979	1.99993	
		(1.66416e-05)	(1.64697e-04)	(2.14585e-06)	2.8906954e-06
	μ_1	1.00033	0.99981	1.00005	
		(5.70992e-06)	(7.68182e-05)	(2.15722e-06)	2.8906954e-06
	λ_2	2.49917	2.49967	2.50001	
		(3.10050e-06)	(9.20901e-05)	(1.04708e-06)	2.7886711e-06
	μ_2	1.49928	1.50013	1.49998	
		(2.68293e-06)	(4.02589e-05)	(1.03805E-06)	2.7886711E-06

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