BSA- exact algorithm computing LTS estimate

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Abstract

The main result of this paper is a new exact algorithm computing the estimate given by the Least Trimmed Squares (LTS). The algorithm works under very weak assumptions. To prove that, we study the respective objective function using basic techniques of analysis and linear algebra.

1 Introduction

In general, (linear) regression analysis is concerned with problems of the following type. One random variable Y called response variable is supposed to fit linear regression model $Y = x^T \beta^0 + e$, where $x \in \mathbb{R}^p$ is a vector of explanatory variables (random or not), $\beta^0 \in \mathbb{R}^p$ is a vector of regression coefficients and e is an error term. The aim of regression analysis is to estimate β^0 having n measurements of Y and X. These measurements will be denoted as vector $Y = (Y_1, \ldots, Y_n)$ and as a design matrix

$$X = \begin{pmatrix} x_1^1 & x_1^2 & \dots & x_1^p \\ x_2^1 & x_2^2 & \dots & x_2^p \\ \vdots & \vdots & & \vdots \\ x_n^2 & x_n^2 & \dots & x_n^p \end{pmatrix}, \tag{1}$$

vector x_i stands for a transposition of *i*-th row of the matrix X.

The best known estimate of β^0 is the estimate given by the (ordinary) least squares method (OLS estimate)

$$\hat{\beta}^{(OLS,n)} = (X^T X)^{-1} X^T Y, \tag{2}$$

which is in fact the projection of Y into the linear envelope of the columns of X. Unfortunately, the OLS estimate was shown to be very sensitive with respect to data contamination of many kinds (for more see [5]). Therefore, other estimates which are less sensitive or, in other words, more *robust* were introduced. One of such estimates is the estimate given by the *Least Trimmed Squares* method (LTS estimate) proposed by Rousseeuw in 1984 [4].

OLS estimate (2) is actually obtained as a minimum of the OLS objective function (OLS-OF) defined as a sum of squares of residuals $r_i(\beta) = Y_i - x_i^T \beta$, i.e., the OLS-OF reads

$$OF^{(OLS,X,Y)}(\beta) = \sum_{i=1}^{n} (Y_i - x_i^T \beta)^2.$$
 (3)

The basic idea of the LTS method is that the contaminating data points lay out of the main bulk of data and hence their residuals are bigger. It means that in order to obtain a more robust estimate of regression coefficients we ignore (trim) some portion of data points with

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¹All vectors in this text are treated as column vectors.

biggest residuals. Formally, the LTS estimate is defined as a minimum of the LTS objective function (LTS-OF)

$$OF^{(LTS,n,h)}(\beta) = \sum_{i=1}^{h} r_{(i)}^{2}(\beta),$$
 (4)

where h is a parameter which determines how many (n-h) data points is to be trimmed and $r_{(i)}(\beta)$ stands for the i-th smallest residuum at β . Since it is not reasonable to ignore more than a half of data points, h usually takes values between n/2 and n.

1.1 Algorithms

As we will see in the following section, there exists a straightforward algorithm always giving the exact value of the LTS estimate, but it requires $\binom{n}{h}$ computations of OLS estimates for h not trimmed data points. As this algorithm (and its modifications, see [1]) has been the only known exact algorithm, another faster ways how to obtain the LTS estimates were introduced. All these faster algorithms are probabilistic, i.e., it is not sure they return the exact value of the LTS estimate. There exist two kinds of probabilistic algorithms which may be described, using terminology from [2], as algorithms finding β satisfying the weak and strong necessary condition respectively. In fact, β satisfies the weak necessary condition if, and only if, it is a local minimum of the LTS-OF. Algorithms finding β 's satisfying the weak conditions have been proposed independently several times, first such algorithm is from [7], its modification by the same author can be found in [8], another algorithm of this type was introduced along with the notion of weak necessary condition in [2], and a version for large data sets is described in [6]. In the case of the strong condition the situation is simple as there is only one representative: Feasible Solution Algorithm [3].

Since we are going to study an algorithm solving the problem of minimizing of the LTS-OF, we can forget the complex statistical background and formulate it as follows:

Problem 1. Find the LTS estimate

$$\hat{\beta}^{(LTS,n,h)} = \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \sum_{i=1}^h r_{(i)}^2(\beta) = \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \sum_{i=1}^h (y_i - \beta x_i^T)^2, \tag{5}$$

where $n > p \ge 1$, $Y = (y_1, \dots, y_n)^T$, $X = (x_1, \dots, x_n)^T$ is a matrix from $\mathbb{R}^{n,p}$, and h is an integer such that $p \le h \le n$.

Further, let us denote the data for which the problem is defined by

$$\mathcal{D} = \{ (y_i, x_i^T) \mid i \in \hat{n} \}.$$

Prior to introduction of the new exact algorithm, we need to study the LTS-OF as the algorithm is based on some special properties of it. Having described these properties, we will first propose one-dimensional version of the algorithm which is easier to demonstrate, then the general case will be given.

2 Objective function

2.1 Discrete reformulation of LTS-OF

For every $\beta \in \mathbb{R}^p$ only h data with least squared residuals appear in (4). Every such h-element subset of all data \mathcal{D} can be unambiguously determined by 0-1 vector $w \in \mathbb{R}^n$, where $w^i = 1$ if (y_i, x_i^T) is an element of this subset and $w^i = 0$ otherwise – in this sense we will be speaking about a *subset* w. For any element of the set of all such vectors

$$Q^{(n,h)} = \{ w \in \mathbb{R}^n | \ w^i \in \{0,1\}, i \in \hat{n}, w^1 + \dots + w^n = h \},$$
(6)

we define two sets

$$I_w = \{k \in \hat{n} \mid w^k = 1\},$$
 (7)
 $O_w = \{k \in \hat{n} \mid w^k = 0\}.$

Clearly, for any β there exists at least one $w \in Q^{(n,h)}$ so that $\sum_{i=1}^h r_{(i)}^2(\beta) = \sum_{i=1}^n w^i r_i^2(\beta)$. Employing this fact we get:

$$\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^h r_{(i)}^2(\beta) = \min_{\beta \in \mathbb{R}^p, w \in Q^{(n,h)}} \sum_{i=1}^n w^i r_i^2(\beta)$$
 (8)

$$= \min_{w \in Q^{(n,h)}} \left(\min_{\beta \in \mathbb{R}^p} (WY - WX\beta)^T (WY - WX\beta) \right)$$
(9)
$$= \min_{w \in Q^{(n,h)}} ||WY - WX(X^TWX)^{-1}X^TWY)||^2,$$
(10)

$$= \min_{w \in Q^{(n,h)}} ||WY - WX(X^TWX)^{-1}X^TWY)||^2, \tag{10}$$

where $W = \operatorname{diag}(w)$. Having this equation, we can propose a new objective function of the LTS defined on $Q^{(n,h)}$

$$J(w) = \|W(Y - X(X^T W X)^{-1} X^T W Y))\|^2.$$
(11)

It is straightforward that J(w) is the minimum of the OLS-OF for the subset w, i.e. $\min_{\beta \in \mathbb{R}^p} \widetilde{OF}^{(OLS,WX,WY)}(\beta)$. Finally, we can also reformulate (5) to the following form

$$\hat{\beta}^{(LTS,n,h)} = (X^T W^* X)^{-1} X^T W^* Y, \tag{12}$$

where

$$w^* = \underset{w \in Q^{(n,h)}}{\min} J(w) \quad \text{and} \quad W^* = \text{diag}(w^*).$$
 (13)

2.2Domain of LTS-OF

The discrete version of the LTS-OF proposed in the previous paragraph is well known and has already been described in many articles dealing with the LTS, especially with computing the LTS estimate. In the present paragraph we shall discuss the non-discrete LTS-OF, i.e. $OF^{(LTS,n,h)}(\beta)$, where $\beta \in \mathbb{R}^p$. Several of the features, we are going to propose, were already mentioned in [9]. We will reprove them and broaden them somewhat.

Definition 1. We define a relation $Z \subset \mathbb{R}^p \times Q^{(n,h)}$ by

$$(\beta, w) \in Z \Leftrightarrow \sum_{i=1}^{h} r_{(i)}^2(\beta) = \sum_{i=1}^{n} w^i r_i^2(\beta).$$

Further, we define a set $\mathcal{U} \subset \mathbb{R}^p$ as the set where Z is a mapping from \mathbb{R}^p to $Q^{(n,h)}$. Complement of \mathcal{U} to \mathbb{R}^p is denoted by \mathcal{H} .

Assertion 1. For $\beta \in \mathbb{R}^p$ there exists only one $w \in Q^{(n,h)}$ so that $(\beta, w) \in Z$, i.e., $\beta \in \mathcal{U}$, if, and only if, $r_{(h)}^2(\beta) < r_{(h+1)}^2(\beta)$.

Indeed, if $r_i^2(\beta) = r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta) = r_j^2(\beta)$ and $(\beta, w) \in Z$ then also $(\beta, \hat{w}) \in Z$ where \hat{w} is created from w by swopping the i-th and j-th elements.

Corollary 2. The following holds:

$$\mathcal{H} = \{ \beta \in \mathbb{R}^p \, | \, r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta) \}.$$

For every $\beta \in \mathcal{H}$ there exist $i, j \in \hat{n}$ such that $r_{(h)}^2(\beta) = r_i^2(\beta) = r_j^2(\beta) = r_{(h+1)}^2(\beta)$, this equality is equivalent to $r_i(\beta) = \pm r_j(\beta) \Leftrightarrow y_i \mp y_j + (x_i^T \mp x_j^T)\beta = 0$.

Assumption 1. Let us assume that for Problem 1 that

- 1. $(\forall i, j \in \hat{n}, i \neq j)(x_i \neq \pm x_j),$
- 2. $(\forall i \in \hat{n})(||x_i|| \neq 0)$.

If Assumption 1 is fulfilled, then $y_i \mp y_j + (x_i \mp x_j)\beta = 0$ is represents a hyperplane, i.e., a closed set having Lebesgue measure 0. Since \mathcal{H} is a finite union of such sets, it is also closed and of Lebesgue measure 0.

Assertion 3. If Assumption 1 is fulfilled, we get

- 1. $\mu_L(\mathcal{H}) = 0$, i.e. the Lebesgue measure of \mathcal{H} is zero,
- 2. the set \mathcal{U} is open.

Assume that for two different $\beta_1, \beta_2 \in \mathcal{U}$

$$\{\beta \in \mathbb{R}^p \mid \beta = \beta_1 + t(\beta_2 - \beta_1), t \in [0, 1]\} \cap \mathcal{H} = \emptyset,$$

i.e., the line between β_1 and β_2 does not cross the set \mathcal{H} , then on this line we must have $r_{(h)}^2(\beta) < r_{(h+1)}^2(\beta)$ and so $Z(\beta_1) = Z(\beta_2)$. In words, the space \mathbb{R}^p is "divided" by the set \mathcal{H} into a finite number m of open disjoint subsets of \mathcal{U} .

Definition 2. For Problem 1 we define a sequence of $m \in \mathbb{N}$ sets $\mathcal{U}^{(seq)} = \{U_i\}_{i=1}^m$ such that

- 1. U_i is open and connected³, for all i = 1, ..., m,
- 2. $U_i \cap U_j = \emptyset$, for all $i, j, i \neq j$,
- 3. $\bigcup_{i=1}^{m} U_i = \mathcal{U}$,
- 4. $\bigcup_{i=1}^{m} \partial U_i = \mathcal{H}$.

We say that $U_i, U_j \in \mathcal{U}^{(seq)}$ are neighbours if $i \neq j$ and $\partial U_i \cap \partial U_j \neq \emptyset$. Further, we define a set $W^{(min)}$ of m vectors from $Q^{(n,h)}$

$$W^{(min)} = \{w_1, \dots, w_m \mid w_i = Z(\beta), \text{ where } \beta \in U_i, i \in \hat{m}\}.$$

The sequence $\mathcal{U}^{(\text{seq})}$ is uniquely determined by conditions 1, 2 and 3, condition 4 is implied by condition 3. The elements of $W^{(\text{min})}$ are correctly defined due to the fact that $Z(\beta) = Z(\hat{\beta})$ for all $\beta, \hat{\beta} \in U_i$ where $i \in \hat{m}$ arbitrary.

2.2.1 One-dimensional example

As the above introduced definitions and assertions are crucial for understanding all the results of the following paragraphs, we will demonstrate their meaning on an example. The simplest instance of Problem 1 is the case of p=1 when the argument of the LTS-OF is a real number $\beta \in \mathbb{R}^1$.

Let us assume that Assumption 1 is fulfilled. Then all residuals $r_i^2 = (y_i - x_i\beta)^2$, as well as an arbitrary sum of them, are sharply convex parabolas. Thus, for every subset $w \in Q^{(n,h)}$ the function $OF^{(OLS,WX,WY)}(\beta) = \sum_{i=1}^n w^i (y_i - x_i\beta)^2$ is also an sharply convex parabola and the value of the discrete function J(w) is a minimum of it.

²The finiteness of the number of the subsets will be proved later; we will prove that $m \leq \binom{n}{p+1} 2^p$.

 $^{^{3}}$ By definition, an open set A is connected if it cannot be represented as the disjoint union of two or more nonempty open sets.

Example 1. Find the LTS-estimate of Problem 1 for the following settings:

- n = 9, p = 1, h = 5,
- $Y = (-0.90, -0.80, 33.32, -27.23, 12.63, -14.18, -3.79, -8.66, -16.45)^T$
- $X = (1.39, -2.25, 6.10, -8.50, 8.26, -8.67, 10.87, 13.70, 13.05)^T$.

As β is a scalar, it is easy to draw the graph of $OF^{(LTS,n,h)}(\beta)$ for Example 1. What we need is to know is how to determine the value $OF^{(LTS,n,h)}(\beta)$ for a given β . It can be easily done by evaluating and ordering the squared residuals $r_i^2(\beta)$ for all i. Employing the definition of the relation Z, we can say that we need to find a subset w such that $(\beta,w)\in Z$ – in other words, we need to find the parabola $OF^{(OLS,WX,WY)}(\beta)$, corresponding to the subset w, such that $OF^{(OLS,WX,WY)}(\beta) = OF^{(LTS,n,h)}(\beta)$ for the given β . The sector of the graph of $OF^{(LTS,n,h)}(\beta)$ for Example 1 containing all the local minima is depicted in Figure 1.

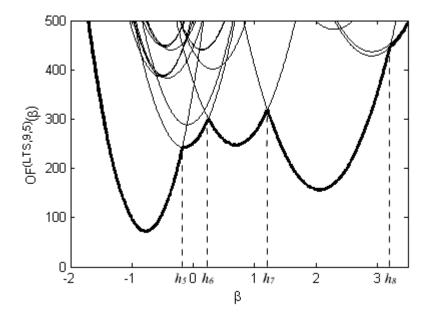


Figure 1: The bold line is the graph of the LTS-OF, the other parabolas (thin lines) are graphs of OLS-OF corresponding to various data subsets $w \in Q^{(9,5)}$.

It is clear that for data X Assumption 1 is fulfilled. The second part tells us that all squared residuals have parabolas as a graph and the first part that for any two parabolas the intersection of their graphs is a set of Lebesque measure 0 (a point, in the case of p=1). Using our notation, we can reformulate the last sentence in this way: the set \mathcal{H} is the set of β for which more than one parabola coincides with the graph of the LTS-OF or, equivalently, the set of β for which more than one subset w is in the relation Z with β . Denote $\mathcal{H} = \{h_1, \ldots, h_{m-1}\}$, where $m = \#\mathcal{U}^{(\text{seq})}$. For Example 1 $\mathcal{H} = \{-8.16, -7.44, -6.99, -3.92, -0.18, 0.25, 1.21, 3.20, 3.84\}$.

Regarding the set \mathcal{U} , we know that $\mathcal{U} = \mathbb{R}^1 \setminus \mathcal{H}$, thus the set \mathcal{U} is a union of m open intervals U_1, \ldots, U_m . It is obvious that the sequence $\mathcal{U}^{(\text{seq})}$ equals to the sequence of these intervals, i.e. $\mathcal{U}^{(\text{seq})} = \{U_i\}_{i=1}^m$. All the sets U_i and all the corresponding vectors $w_i \in W^{(\text{min})}$ are given in Table 1.

i	U_i	$\{j \in \hat{9} \mid w_i^j = 1\}$	i	U_i	$\{j \in \hat{9} \mid w_i^j = 1\}$
1	$(-\infty, -8.16)$	1,2,3,5,6	6	(-0.18, 0.25)	1,2,5,7,8
2	(-8.16, -7.45)	1,2,3,5,7	7	(0.25, 1.21)	1,2,5,6,7
3	(-7.45, -6.99)	1,2,5,6,7	8	(1.21, 3.20)	1,2,4,5,6
4	(-6,99,-3.92)	1,2,5,7,9	9	(3.20, 3.84)	1,2,3,4,6
5	(-3.92, -0.18)	1,2,7,8,9	10	$(3.84, +\infty)$	1,2,3,4,5

Table 1: The sets U_i and the corresponding w_i for Example 1.

Note that in general we have

$$i \neq j \Rightarrow w_i \neq w_i$$
.

For our example data $w_3 = w_7$. Note also that not all sets U_i must contain a local minimum. For us there are only 4 local minima: (1) in $\beta = -0.77$, value 71.96 (2) in $\beta = 0.14$, value 242.42 (3) in $\beta = 0.70$, value 246.87 and (4) in $\beta = 2.06$, value 156.15.

2.3 Local minima of LTS-OF

Now we will try to append to hitherto shown features and proposed notation some others, which will be useful from the point of view of the minimization of the objective function $OF^{(LTS,n,h)}(\beta)$. Without doubt it would be very useful to know if there exists a global minimum or if there could be more than one local minimum. Taking into account simultaneously the discrete form of the LTS-OF, (12) and (13) we have the proof of the existence of the global minimum and also the alternative formula for it.

As for the local minima, we have to employ the notation and facts from the previous paragraphs. We know that the domain of the LTS-OF can be written as a union of the open sets $\mathcal{U}^{(\text{seq})} = \{U_i\}_{i=1}^m$ and the set of measure zero \mathcal{H} . We also proved that for all $i = 1, \ldots, m$ and for all $\beta \in U_i$ we have $OF^{(LTS,n,h)}(\beta) = OF^{(OLS,W^iX,W^iY)}(\beta)$) where $W^i = \text{diag}(w_i)$ and $w_i \in W^{(\text{min})}$ (see Definition 2). This fact has an important consequence.

Definition 3. We say that a matrix $X \in \mathbb{R}^{n,p}$, $n \ge p$, has h-full rank if a rank of the matrix WX is p for all $w \in Q^{(n,h)}$, $W = \operatorname{diag}(w)$.

It is well known that the OLS estimate is unique if, and only if, the design matrix has rank p. As we compute the OLS estimate for h-element subset, we need the previous definition.

Assertion 4. The objective function of Problem 1 $OF^{(LTS,n,h)}(\beta)$ has a local minimum in $\beta_0 \in U_i \in \mathcal{U}^{(seq)}$ if, and only if, the function $OF^{(OLS,W^iX,W^iY)}(\beta)$) has a local minimum in $\beta_0 \in U_i$, $i \in \hat{m}$.

Moreover, if X has h-full rank, then

$$OF^{(LTS,n,h)}(\beta)$$
 has a local minimum at $\beta \in U_i \Leftrightarrow \beta = \hat{\beta}^{(OLS,W^iX,W^iY)}$,

where $W^i = diag(w_i)$.

How strong is the assumption that X has h-full rank? It depends on the values of parameters p and h which determine the dimensions of the sub-matrixes of X. If $h \gg p$ (usually true), then the assumption is very weak.

Assertion 4 tells us how to find local minima located in the open set \mathcal{U} . What if a local minimum is in the set \mathcal{H} ? In what follows, we will prove that even if a local minimum is in the set \mathcal{H} , it can be still found as the OLS estimate for some subset $w \in Q^{(n,h)}$.

- **Lemma 5.** 1. Function $f: \mathbb{R}^n \to \mathbb{R}^1$, $f \in C^{(2)}$ is a strictly convex function if and only if $\nabla^2 f(x)$ is a positive-definite matrix.
 - 2. A strictly convex function has maximally one strict minimum.

This Lemma is a classical result of mathematical analysis.

Lemma 6. For Problem 1 and every subset $w \in Q^{(n,h)}$ it holds that if the matrix WX has full rank (W = diag(w)), then the function $OF^{(OLS,WX,WY)}(\beta)$ is strictly convex.

The proof follows from from Lemma 5 and from the fact that $\nabla^2 OF^{(OLS,WX,WY)}(\beta) = X^TWX$, where X^TWX is positive-definite.

Lemma 7. Let functions f_1, \ldots, f_k be continuous having unique strict minimum, $f_i : \mathbb{R}^p \to \mathbb{R}^1$, and let $h(x) = \min\{f_1(x), \ldots, f_k(x)\}$ for all $x \in \mathbb{R}^p$. Define a set $S = \{x \in \mathbb{R}^p \mid f_1(x) = \cdots = f_k(x)\}$. If h has a local minimum at $x_0 \in S$, then f_i has the strict global minimum at x_0 for all $i = 1, \ldots, k$.

Prooj

If h has a local minimum at $x_0 \in S$, then there exists a neighbourhood $U_{(x_0)}$ of x_0 such that

$$(\forall x \in U_{(x_0)})(h(x) \ge h(x_0) = f_1(x_0) \dots = f_k(x_0)).$$

The same inequality is clearly true on $U_{(x_0)}$ for all $f_i(x)$ (note that $h(x) \leq f_i(x)$), and hence, due to the fact that all $f_i(x)$ have only one strict minimum, x_0 has to be also the point of the global minima of the functions f_i , i = 1, ..., k.

Q.E.D

Now we can propose the following not-surprising but important theorem.

Theorem 8. If the matrix X from Problem 1 has h-full rank, then for every local minimum at a point β_0 of the objective function $OF^{(LTS,n,h)}(\beta)$ there exists a vector $w \in W^{(min)}$ such that

$$\beta_0 = (X^T W X)^{-1} X^T W Y,$$

where W = diag(w).

Proof

If $\beta_0 \in \mathcal{U}$, the proof follows directly from Assertion 4.

Let us assume that $\beta_0 \in \mathcal{H}$. It means that there exist more than one subset being in relation Z with β_0 . Let us denote these subsets by $w_{i_1}, \ldots, w_{i_k}, k \geq 2$. Now employing the previous lemma – putting $f_j = OF^{(OLS,W_{i_j}X,W_{i_j}Y)}(\beta)$ for all j – and Lemma 6 we can be sure that β_0 is a point of global minima of all functions f_j . From Definition 2 we also know that $\mathcal{H} = \bigcup_{i \in \hat{m}} \partial U_i$. Thus, taking into account that $\beta_0 \in \mathcal{H}$, there are at least two of the subsets $w_{i_1}, \ldots, w_{i_k}, k \geq 2$ (corresponding to two neighbours from $\mathcal{U}^{(\text{seq})}$ – see Definition 2) which are elements of $W^{(\text{min})}$.

Q.E.D

3 Borders Scanning Algorithm – BSA

In the present section we shall propose a new algorithm for solving Problem 1. A principle of the algorithm is quite simple. It is based on the fact that

$$OF^{(LTS,n,h)}(\beta) = \min_{w \in Q^{(n,h)}} OF^{(OLS,WX,WY)}(\beta)$$
$$= \min_{i=1,\dots,m} OF^{(OLS,W^iX,W^iY)}(\beta),$$
(14)

where $W = \operatorname{diag}(w), W^i = \operatorname{diag}(w_i)$ and m and $w_i \in W^{(\min)}$ are defined in Definition 2. This equation claims that to get complete knowledge of the complicated function LTS-OF, it is enough to evaluate *only* m sharply convex objective functions of the OLS estimate $OF^{(OLS,W_iX,W_iY)}(\beta)$. Taking into account (10), (11) and Theorem 8, we can reformulate (14) as

$$\min_{\beta \in \mathbb{R}p} \sum_{i=1}^{h} r_{(i)}^{2}(\beta) = \min_{w \in Q^{(n,h)}} J(w) = \min_{i \in \hat{m}} J(w_i),$$

i.e. the LTS estimate for Problem 1 can be obtained by evaluating $J(w_i)$ for all $i \in \hat{m}$. More or less, most of algorithms take advantage of this fact. The question is how to determine (all) subsets $w_i \in W^{(\min)}$ most effectively. At first, we illustrate how the BSA does it in the one-dimensional case.

3.1 One-dimensional case

As written above, for Example 1 the set $W^{(\min)}$ consists of m = 10 elements and the set \mathcal{H} contains 9 points h_1, \ldots, h_9 . For each point $h_k, k \in \hat{9}$ there exist exactly two subsets w_{k_1} and w_{k_2} from $W^{(\min)}$ such that $(h_k, w_{k_1}) \in Z$ and $(h_k, w_{k_2}) \in Z$. These two subsets correspond to two parabolas whose intersection has a coordinate h_k and that can be easily determined for a given h_k using the following algorithm (which works for arbitrary p).

Program 1. How to find all $w \in Q^{(n,h)}$ such that $(\beta, w) \in Z$ for a given $\beta \in \mathbb{R}^p$:

- 1. For all $i \in \hat{n}$ evaluate squared residual $r_i^2(\beta)$ and order them. Define $i_k \in \{1, \ldots, n\}$ by $r_{i_k}^2(\beta) = r_{(k)}^2(\beta)$ for all $k = 1, \ldots, n$.
- 2. If $r_{(h)}^2(\beta) < r_{(h+1)}^2(\beta)$ then return the unique subset w such that $(w^i = 1 \Leftrightarrow r_i^2(\beta) \leq r_{(h)}^2(\beta))$.

If
$$r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$$
, let us suppose that

$$\begin{split} r_{i_1}^2(\beta) & \leq \dots \leq r_{i_l}^2(\beta) < r_{i_{l+1}}^2(\beta) = \dots = r_{i_h}^2(\beta) = \\ & = r_{i_{h+1}}^2(\beta) = \dots = r_{i_{l+t}}^2(\beta) < r_{i_{l+t+1}}^2(\beta) \leq \dots \leq r_{i_n}^2(\beta). \end{split}$$

Then return all the subsets wsuch that I_w (see (7)) contains l indices corresponding to i_1, \ldots, i_l and arbitrary (h-l)-element subset of indices corresponding to i_{l+1}, \ldots, i_{l+t} . Hence, there are $\binom{t}{h-l}$ subsets in relation Z with β .

In general, if Assumption 1 is fulfilled in the case of p=1, that for every $w_i \in W^{(\min)}$ there exists at least one point $h \in \mathcal{H}$ such that $(w_i, h) \in Z$. Taking this into account, we can state: if the set $\mathcal{H} = \{h_1, \ldots, h_{m-1}\}$ is known, all the subsets $W^{(\min)}$ can be obtained by performing Program 1 for each $h_k \in \mathcal{H}$.

Only remaining task is how to determine the set \mathcal{H} . According to its definition, a point β is an element of \mathcal{H} if and only if $r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$, i.e. this equality is sufficient and necessary condition. Since the condition contains ordered residuals, it can not be used directly – at first we need to find some candidates for which we will perform ordering of residuals. These candidates can be determined by the following necessary condition: if $\beta \in \mathcal{H}$, then there are two distinct indices i, j such that $r_i^2(\beta) = r_j^2(\beta)$. Let us denote the set containing all $\beta \in \mathbb{R}^1$ satisfying this necessary condition by H, i.e.

$$H = \{ \beta \in \mathbb{R}^1 \,|\, r_i^2(\beta) = r_i^2(\beta), i \neq j \}, \tag{15}$$

obviously $\#H \leq 2\binom{n}{p+1} = 2\binom{n}{2} = n(n-1)$ (note that the equation is quadratic, i.e., it has two solutions $\frac{y_i-y_j}{x_i-x_j}$ and $\frac{y_i+y_j}{x_i+x_j}$ – we still assume that Assumption 1 is fulfilled, hence $x_i \neq \pm x_j$).

Now we already have all necessary for proposing a one-dimensional version of BSA.

Program 2. BSA in the case of p = 1. Denote the elements of the set $Q^{(n,2)}$ by $\{v_1, \ldots, v_{\binom{n}{2}}\}$.

- 1. Set k = 1 and $J_{min} = +\infty$.
- 2. Denote the indices of two data points from subset v_k by i, j. Save solutions of equation $r_i^2(\beta) = r_j^2(\beta)$ as β_1 and β_2 , i.e., $\beta_1 = \frac{y_i - y_j}{x_i - x_j}$ and $\beta_2 = \frac{y_i + y_j}{x_i + x_j}$.
- 3. Evaluate and order residuals $r_l^2(\beta_1), l = 1, \ldots, n$.
- 4. If $r_{(h)}^2(\beta_1) = r_{(h+1)}^2(\beta_1)$ (i.e., $\beta_1 \in \mathcal{H}$), find subsets $w^{(1)}, w^{(2)} \in Q^{(n,h)}$ which are in relation Z with β_1 (use Program 1).
- 5. If $J(w^{(1)}) < J_{min}$, put $J_{min} = J(w^{(1)})$ and $\beta_{min} = \beta_1$. If $J(w^{(2)}) < J_{min}$, put $J_{min} = J(w^{(2)})$ and $\beta_{min} = \beta_2$.
- 6. If $\beta_1 \neq \beta_2$, repeat last two steps for β_2 (modify J_{min} and β_{min} accordingly).
- 7. If $k < \binom{n}{2}$, put k = k + 1 and go back to step 2.
- 8. Return β_{min} as the LTS estimate for Problem 1.

This algorithm works is finite if the \mathcal{H} contains only a finite number of points. Assumption 1 is a sufficient condition for this; it is not a necessary condition, but still it is very weak and easily verifiable.

3.2 Multidimensional case

In the case of p > 1, the situation is more complicated. The source of complication is the fact, that the set \mathcal{H} contains infinitely many points. In order to resolve this problem, we need to find some finite subset of \mathcal{H} , let us denote it \mathcal{H}_p , having the following property: for every $w \in W^{(\min)}$ there exists $\beta \in \mathcal{H}_p$ such that $(\beta, w) \in Z$.

Analogously to the case of p=1, we will be looking for candidates for being an element of \mathcal{H}_p in the set H, namely in some suitable finite subset H_p for which $\mathcal{H}_p \subset H_p \subset H$. In the case of p=1, the equality of the type $r_i^2(\beta) = r_j^2(\beta)$ can have at most two solutions, in the case of p>1, this equality only "decrements" the dimension by 1, i.e., the dimension of its solution is p-1. But we need the dimension to be zero, this can be reached by considering p "independent" equations of the type $r_i^2(\beta) = r_j^2(\beta)$, in other words, a system of p equations with p unknowns β^1, \ldots, β^p

$$r_{i_1}^2(\beta) = r_{i_2}^2(\beta)$$

 \vdots \vdots (16)
 $r_{i_p}^2(\beta) = r_{i_{p+1}}^2(\beta),$

where i_1, \ldots, i_{p+1} corresponds to one of (p+1)-element subsets of $\hat{n} = \{1, \ldots, n\}$, i.e., to one element of $Q^{(n,p+1)}$. Unfortunately, as we will prove later on, the system (16) is equivalent to 2^p linear systems of p equations. If all these systems are regular, then original system (16) can have up to 2^p solutions. Taking into account this number and the fact that there

are $\binom{n}{p+1} = \#Q^{(n,p+1)}$ different systems of type (16), the set H_p , which is to be defined as a set of solutions of all such systems, contains $\binom{n}{p+1}2^p$ points from $\mathbb{R}p$.

In the next section, all the sets introduced above (sets H, H_p and \mathcal{H}_p) will be redefined precisely and their mentioned (and some others) properties will be proved. In particular, we will propose assumptions which allow us to prove that the set \mathcal{H}_p is a suitable set for BSA.

3.3 Set \mathcal{H}_n

The goal of this section is to find a set \mathcal{H}_p for which it holds that for every $w \in W^{(\min)}$ there exists at least one $\beta \in \mathcal{H}_p$ such that $(\beta, w) \in Z$. We will define it as it was hinted above, therefor we will need to define the sets H and H_p containing candidates for being elements of \mathcal{H} and \mathcal{H}_p .

The set H is to be defined as in (15). The quadratic equation of the type $r_i^2(\beta) = r_i^2(\beta), i, j \in \hat{n}$ is equivalent to two linear ones

$$(x_i + x_j)^T \beta = y_i + y_j$$

$$(x_i - x_j)^T \beta = y_i - y_j,$$
(17)

each defining p-1-dimensional hyperplane.

Definition 4. Let us denote

$$H^{(i,j,\pm)} = \{\beta \in \mathbb{R}p \mid r_i^2(\beta) = r_j^2(\beta)\} = \{\beta \in \mathbb{R}p \mid y_i - x_i^T\beta = \pm (y_j - x_j^T\beta)\}$$

$$H^{(i,j,-)} = \{\beta \in \mathbb{R}p \mid y_i - x_i^T\beta = (y_j - x_j^T\beta)\} = \{\beta \in \mathbb{R}p \mid y_i - y_j = (x_i^T - x_j^T)\beta\}$$

$$H^{(i,j,+)} = \{\beta \in \mathbb{R}p \mid y_i - x_i^T\beta = -(y_j - x_i^T\beta)\} = \{\beta \in \mathbb{R}p \mid y_i + y_j = (x_j^T + x_j^T)\beta\}$$

for every $i, j \in \hat{n}, i \neq j$ and

$$H \quad = \quad \cup_{i,j \in \hat{n}, i \neq j} H^{(i,j,\pm)} = \left(\cup_{i,j \in \hat{n}, i \neq j} H^{(i,j,+)} \right) \cup \left(\cup_{i,j \in \hat{n}, i \neq j} H^{(i,j,-)} \right).$$

Obviously the set H from this definition is the same one as the definition given in (15). It is apparent that $\mathcal{H} \subset H$. We also know, that the sets $U_i, i \in \hat{m}$ from Definition 2 are separated by the set \mathcal{H} . The hyperplanes $H^{(i,j,+)}$ and $H^{(i,j,-)}$ divide $\mathbb{R}p$ into closed convex sets, so-called polytops, let us denote them $P_k, k \in \hat{K}$, where K is some finite number. We get that $\bigcup_{k \in \hat{K}} \partial P_k = H \supset \mathcal{H} = \bigcup_{i \in \hat{m}} \partial U_i$ and it implies that for every U_i there exist convex sets $P_{k_1}, \ldots P_{k_l}, l \geq 1$ such that $\bar{U}_i = \bigcup_{j \in \hat{l}} P_{k_j}$ and also that $m \leq K$.

It is well known fact, that a bounded polytop equals to a convex envelope of points which are intersections of the hyperplanes bordering the polytop. The smallest number of hyperplanes allowing their intersection to be a point (i.e. the set with dimension 0) equals the dimension of the space. For us the space is $\mathbb{R}p$ and the dimension is p. We propose some more notation to express what this fact means for our particular case.

Let $\circ \in \{+, -\}$ represent one of the arithmetical operations, either an addition or a subtraction, i.e. $x \circ y = x + y$ if $\circ = +$ and $x \circ y = x - y$ if $\circ = -$.

Let $\beta \in H$ be an intersection of q+1, $q \geq 1$, sets of the type $H^{(i,j,\pm)}$ such that $\beta \in H^{(i_1,i_2,\pm)} \cup H^{(i_2,i_3,\pm)} \cup \cdots \cup H^{(i_q,i_{q+1},\pm)}$ where $i_1,\ldots,i_{q+1} \in \hat{n}$ are distinct. It means that β is a solution of the following system

$$\begin{split} r_{i_1}^2(\beta) &= r_{i_2}^2(\beta) \\ &\vdots &\vdots \\ r_{i_q}^2(\beta) &= r_{i_{q+1}}^2(\beta). \end{split}$$

Note that if $r_{i_1}^2(\beta) = r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$ then, moreover, $\beta \in \mathcal{H}$. This system of equations is equivalent to the following 2^q systems

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_q} \circ_q x_{i_{q+1}})^T \beta = y_{i_q} \circ_q y_{i_{q+1}}.$$
(18)

where $(\circ_1, \dots, \circ_q)$ is an arbitrary element of the set product $\times_{i=1}^q \{+, -\}$

Definition 5. Let $\beta \in H$ be a solution of system (18) of q equations where $(\circ_1, \ldots, \circ_q) \in \times_{i=1}^q \{+,-\}$. Further, let us assume there exists no $i_{q+2} \in \hat{n} \setminus \{i_1, \ldots, i_{q+1}\}$ and $\circ_{q+1} \in \{+,-\}$ so that $(x_{i_{q+1}} \circ_{q+1} x_{i_{q+2}})^T \beta = y_{i_{q+1}} \circ_q y_{i_{q+2}}$ (i.e. $r_{i_1}^2(\beta) = \ldots = r_{i_{q+1}}^2(\beta) = r_{i_{q+2}}^2(\beta)$). Then we define an order of β

 $Ord(\beta) = number of linearly independent equations in (18).$

We also define a set of zero-dimensional intersections

$$H_p = \{ \beta \in H \mid \operatorname{Ord}(\beta) = p \}$$

and its subset \mathcal{H}_p of such $\beta \in \mathcal{H}_p$ for which $r_{i_1}^2(\beta) = \ldots = r_{i_{p+1}}^2(\beta) = r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$, where $i_1, \ldots, i_{p+1} \in \hat{n}$ are indices from the system of p linearly independent equations of type (18).

Now we have all the notation necessary for proposing the assertion which, despite being very simple and natural, will be used as the basis of BSA. At first, let us prove the following useful lemma – to be able to do it, we need this very weak assumption.

Assumption 2.

$$(\forall \beta \in \mathbb{R}p)(r_{(h)}^2(\beta) > 0),$$

i.e.,

$$OF^{(LTS,n,h)}(\hat{\beta}^{(LTS,n,h)}) > 0.$$

Lemma 9. Let us assume that Assumptions 1 and 2 are fulfilled and let $B \subset \mathbb{R}p$ be a set containing all solutions of the system of equations

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(x_{i_a} \circ_q x_{i_{a+1}})^T \beta = y_{i_a} \circ_q y_{i_{a+1}},$$
(19)

where $i_1, \ldots, i_{q+1} \in \hat{n}$ are distinct, $q \ge 1, \circ_1, \ldots, \circ_q \in \{+, -\}$ and $r_{i_1}^2(\beta) = \cdots = r_{i_{q+1}}^2(\beta) = r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$. Then for all $j, k \in \widehat{q+1}, j \ne k$ and $o \in \{+, -\}$ either

$$(\forall \beta \in B)((x_{i_j} \circ x_{i_k})^T \beta = y_{i_j} \circ y_{i_k}) \tag{20}$$

or

$$(\forall \beta \in B)((x_{i_j} \circ x_{i_k})^T \beta \neq y_{i_j} \circ y_{i_k})$$

is true.

Moreover, there exist $\circ_{l_1}, \ldots, \circ_{l_q} \in \{+, -\}$ such that system (19) is equivalent to the system

$$(x_{i_1} \circ_{l_1} x_{i_2})^T \beta = y_{i_1} \circ_{l_1} y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_1} \circ_{l_q} x_{i_{q+1}})^T \beta = y_{i_1} \circ_{l_q} y_{i_{q+1}}.$$
(21)

This holds even if $r_{i_1}^2(\beta) \neq r_{(h)}^2(\beta)$.

Proof

Assumption 2 implies that for all $s, t \in \hat{n}, s \neq t$ the system

$$(x_s + x_t)^T \beta = y_s + y_t$$

$$(x_s - x_t)^T \beta = y_s - y_t$$
(22)

has not any solution $\beta \in \mathbb{R}p$ such that $r_s^2(\beta) = r_t^2(\beta) = r_{(h)}^2(\beta)$. Indeed, if β_0 is a solution of both equations (22) and $r_s^2(\beta) = r_t^2(\beta) = r_{(h)}^2(\beta)$, then $r_s(\beta_0) = r_t(\beta_0)$ and simultaneously $r_s(\beta_0) = -r_t(\beta_0)$ and so $r_{(h)}^2(\beta_0) = 0$, i.e. Assumption 2 is not fulfilled (if $r_{(h)}(\beta_0) = 0$ then certainly $\beta_0 = \hat{\beta}^{(LTS,n,h)}$).

Now, let us assume, without loss to generality, that j < k holds for j, k. Then for any $\beta \in B$

$$(x_{i_j} \circ_j x_{i_{j+1}})^T \beta = y_{i_j} \circ_j y_{i_{j+1}}$$
$$(x_{i_{j+1}} \circ_{j+1} x_{i_{j+2}})^T \beta = y_{i_{j+1}} \circ_{j+1} y_{i_{j+2}}.$$

If j+1=k, then, according to Assumption 2 and the first paragraph of this proof, equation (20) holds for all $\beta \in B$ (when $\circ = \circ_j$) or it doesn't hold for any of them (when $\circ \neq \circ_j$). Further, let k be greater than j+1. By adding together the equations, in the case of $\circ_j = -$, or by subtracting them, in the case of $\circ_j = +$, we get the equivalent system

$$(x_{i_j} \circ_j x_{i_{j+1}})^T \beta = y_{i_j} \circ_j y_{i_{j+1}}$$
$$(x_{i_j} \circ'_{j+1} x_{i_{j+2}})^T \beta = y_{i_j} \circ'_{j+1} y_{i_{j+2}}$$
$$(x_{i_{j+2}} \circ_{j+2} x_{i_{j+3}})^T \beta = y_{i_{j+2}} \circ_{j+2} y_{i_{j+3}}.$$

We can repeat this step r-times, $r \leq q$, until j + r + 1 = k and finish the proof of the first part of the lemma by employing again assumption 2 for the system

$$(x_{i_j} \circ_j x_{i_{j+1}})^T \beta = y_{i_j} \circ_j y_{i_k} (x_{i_j} \circ'_{k-1} x_{i_k})^T \beta = y_{i_j} \circ'_{k-1} y_{i_k}.$$

The second part of the lemma can be proved by repeating the same steps for j=1 and k=q+1.

Q.E.L

The first part of the lemma tells us that each equation of the type $(x_{i_j} \circ x_{i_{j+2}})^T \beta = y_{i_j} \circ y_{i_k}, j, k \in \widehat{q+1}, \circ \in \{+, -\}$ is either equivalent with system (19) or set of solutions of this equation is disjoint with the set B (due to Assumption 2).

Now we can proceed to proving the most important assertion of this section – we shall prove that \mathcal{H}_p is suitable (in a sense of the previous section) set for BSA. Is it true in general or we have to assume that the data from Problem 1 satisfies some condition? The answer is that we have to propose some assumption which is, however, quite weak. The reason is that it could happen that the set \mathcal{H}_p is empty. For example, in the case of p=2, if all hyperplanes of the type $H^{(i,j,\circ)}$ are parallel, then there is no zero-dimensional intersection. It happens if and only if all vectors x_i from Problem 1 are parallel. Then all systems of equations of the type

$$(x_{i_1} \circ_1 x_{1_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$
$$(x_{i_1} \circ_2 x_{i_3})^T \beta = y_{i_1} \circ_2 y_{i_3}$$

are linearly dependent and so they have either no solution or the set of all solutions is onedimensional hyperplane. To avoid such a situation we propose the following assumption. **Assumption 3.** For all $(\circ_1, \ldots, \circ_{n-1}) \in \times_{i=1}^n \{+, -\}$ the matrix of the system of n-1 equations

$$(x_1 \circ_1 x_2)^T \beta = y_1 \circ_1 y_2$$

$$\vdots \qquad \vdots$$

$$(x_1 \circ_{n-1} x_n)^T \beta = y_1 \circ_{n-1} y_n$$

has rank p.

Obviously, it prevents the situation described above. This assumption has a consequence which will be crucial for the proof of the main assertion. Let us formulate it as a lemma.

Lemma 10. Let us assume that data from Problem 1 satisfies Assumption 3. If p > 1, then for

$$(\forall l \in \{2, \dots, p\})(\forall i_1, \dots, i_l \in \hat{n})(\forall \circ_1, \dots, \circ_{l-1} \in \{+, -\})$$

it holds that if the system of l-1 equations

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_1} \circ_{l-1} x_{i_l})^T \beta = y_{i_1} \circ_{l-1} y_{i_l}$$
(23)

has rank l-1, then there exist $i_{l+1} \in \hat{n} \setminus \{i_1, \dots, i_l\}$ and $o_l \in \{+, -\}$ such that the system of l equations

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_1} \circ_{l-1} x_{i_l})^T \beta = y_{i_1} \circ_{l-1} y_{i_l}$$

$$(x_{i_1} \circ_l x_{i_{l+1}})^T \beta = y_{i_1} \circ_l y_{i_{l+1}}$$

 $has \ rank \ l.$

Moreover, if $\beta \in \mathbb{R}p$ is a solution of (23) such that $r_{i_1}^2(\beta) = r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$ then i_{l+1} can be selected so that there exists β_1 such that $r_{i_1}^2(\beta_1) = \cdots = r_{i_l}^2(\beta_1) = r_{(h+1)}^2(\beta_1) = r_{(h+1)}^2(\beta)$

Proof

Let us denote the elements of the set of indices $\hat{n} \setminus \{i_1, \dots, i_l\}$ by $\{j_{l+1}, \dots, j_n\}$ and let us select signs $\circ_l, \dots, \circ_{n-1}$ arbitrarily. Then, due to Assumption 3, the system of equation

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_1} \circ_{l-1} x_{i_l})^T \beta = y_{i_1} \circ_{l-1} y_{i_l}$$

$$(x_{i_1} \circ_l x_{j_{l+1}})^T \beta = y_{i_1} \circ_1 y_{j_{l+1}}$$

$$\vdots \qquad \vdots$$

$$(x_{i_1} \circ_{n-1} x_{j_n})^T \beta = y_{i_1} \circ_{n-1} y_{j_n}$$

has rank p. We also know that l-1 vectors

$$((x_{i_1} \circ_1 x_{i_2}), \ldots, (x_{i_1} \circ_1 x_{i_l}))$$

are linearly independent. Then, due to Steinitz Theorem, there exist indices j_{l+1}, \ldots, j_{p+1} and k_l, \ldots, k_p (in other words, there exist p-l rows of the matrix of the system above) such that p vectors

$$((x_{i_1} \circ_1 x_{i_2}), \dots, (x_{i_1} \circ_{l-1} x_{i_l}), (x_{i_1} \circ_{k_l} x_{j_{l+1}}), \dots, (x_{i_1} \circ_{k_{l+1}} x_{j_p}))$$

form a base of the vector space $\mathbb{R}p$. Obviously, as the index i_{l+1} can be then taken each index from $\{j_{l+1},\ldots,j_{p+1}\}$.

The second part of the lemma is a consequence of the first one and the continuity of squared residuals. Let us denote the set of all solutions of the system (23) by $B \subset \mathbb{R}p$. We know that there exist $\beta, \beta_1 \in B$ and j_{l+1} such that $r_{i_1}^2(\beta) = \cdots = r_{i_l}^2(\beta) = r_{(h)}^2(\beta)$ and $r_{i_1}^2(\beta_1) = \cdots = r_{j_{l+1}}^2(\beta_1)$. Than, due to continuity of squared residuals, there must exists $\beta_2 \in B$ and i_{l+1} such that $r_{i_1}^2(\beta_2) = \cdots = r_{i_{l+1}}^2(\beta_2) = r_{(h)}^2(\beta_2) = r_{(h+1)}^2(\beta_2)$. Note that it could happen that $\beta_2 = \beta$ or $\beta_2 = \beta_1$.

Q.E.D

Assertion 11. Let us assume that Assumptions 1, 2 and 3 are fulfilled. If for the set $U_i \in \mathcal{U}^{(seq)}$, $i \in \hat{m}$ holds that $\partial U_i \neq \emptyset$, then

$$(\exists \beta \in \mathbb{R}p)(\beta \in \partial U_i \cap \mathcal{H}_p),$$

i.e. there exist $i_1, \ldots i_{p+1} \in \hat{n}$ and $o_1, \ldots, o_p \in \{+, -\}$ such that β is the only one solution of the system of p linearly independent equations

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_n} \circ_p x_{i_{n+1}})^T \beta = y_{i_n} \circ_p y_{i_{n+1}},$$

where moreover $r_{i_1}^2(\beta) = r_{(h)}^2(\beta) = r_{(h+1)}^2(\beta)$ is true.

Proof

Within the proof we shall partially use a syntax of computer programming -q will be treated as a variable which is a parameter of a loop, hence q = q + 1 means incrementing q of 1. The instruction go to \heartsuit means "go back to the line beginning with the sign \heartsuit ". Let us also suppose that p > 1, if p = 1, then the assertion is trivial.

Put q = 1. As $\partial U_i \neq \emptyset$, there are $\beta^{(1)} \in \partial U_i$, $i_1 \in I_{w_i}$ and $i_2 \in O_{w_i}$ (see (7) and for w_i see Definition 2) so that $r_{(h)}^2(\beta^{(1)}) = r_{(h+1)}^2(\beta^{(1)}) = r_{i_1}^2(\beta^{(1)}) = r_{i_2}^2(\beta^{(1)})$, i.e. there exists \circ_1 such that $\beta^{(1)} \in H^{(i_1, i_2, \circ_1)}$, i.e. $\beta^{(1)}$ is a solution of equation

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2}.$$

According to the previous lemma, there exist $i_3 \in \hat{n}$, $o_2 \in \{+, -\}$ and $\beta^{(2)} \in \mathbb{R}p$ such that $\beta^{(2)}$ is a solution of the system

$$(x_{i_1} \circ_1 x_{i_2})^T \beta = y_{i_1} \circ_1 y_{i_2} (x_{i_1} \circ_2 x_{i_3})^T \beta = y_{i_1} \circ_2 y_{i_3},$$

where moreover $r_{i_1}^2(\beta^{(2)}) = r_{(h)}^2(\beta^{(2)}) = r_{(h+1)}^2(\beta^{(2)})$.

 \heartsuit Put q=q+1. If q=p, the proof is completed. If it is not, the indices i_1,\ldots,i_q , signs $\circ_1,\ldots,\circ_{q-1}$ satisfy the assumptions of the previous lemma. Hence, there is an index

 i_{q+1} , a sign \circ_q and $\beta^{(q)}$ such that

$$(x_{i_1} \circ_1 x_{i_2})^T \beta^{(q)} = y_{i_1} \circ_1 y_{i_2}$$

$$\vdots \qquad \vdots$$

$$(x_{i_q} \circ_q x_{i_{q+1}})^T \beta^{(q)} = y_{i_q} \circ_q y_{i_{q+1}}$$
and $r_{i_1}^2(\beta^{(q)}) = r_{(h)}^2(\beta^{(q)}) = r_{(h+1)}^2(\beta^{(q)})$
Go to \heartsuit .
$$Q.E.D$$

The assertion ensures us that \mathcal{H}_p is a suitable for BSA and so we can propose also the multidimensional version of BSA. At the end of this section let us propose the following

Corollary 12. Let us assume that Assumptions 1, 2 and 3 are fulfilled. For the number $m = \#\mathcal{U}^{(seq)}$ (see Definition 2) we have

$$m \le \binom{n}{p+1} 2^p.$$

Proof

Due to Assertion 11 we know that $m \leq \#\mathcal{H}_p$. Then, the proof follows from the fact that $\mathcal{H}_p \subset H_p$, where $\#H_p \leq \binom{n}{p+1} 2^p$.

Q.E.D

BSA- new exact algorithm 4

4.1 Description of algorithm

The multidimensional version of BSA is a straightforward generalization of Program 2. Note that if Assumption 1 is satisfied, then in the case of p=1 it holds that $\mathcal{H}=\mathcal{H}_p$. If p>1then $\mathcal{H}_p \subseteq \mathcal{H}$. In short, BSA can be described as follows: find all $\beta \in \mathcal{H}_p$, by ordering the residuals verify whether β is also element of \mathcal{H}_p . If it is, find all $w \in Q^{(n,h)}$ such that $(\beta, w) \in \mathbb{Z}$ and evaluate J(w) (see (11)) for them. The minimal obtained value then corresponds to w^* (see (13)).

Program 3. BSA – generic definition for all dimensions. Denote all the elements of the set $Q^{(n,p+1)}$ by $\{v_1,\ldots,v_{\binom{n}{p+1}}\}$ and all the elements of the $cartesian \ product \times_{j=1}^{p} \{+,-\} \ by \ \{\circ^{(1)},\ldots,\circ^{(2^{p})}\}, \ where \ \circ^{(i)} = (\circ_{1}^{(i)},\ldots,\circ_{p}^{(i)}), i=1,\ldots 2^{p}.$

- 1. Set k = 1 and $J_{min} = +\infty$.
- 2. If $k > \binom{n}{n+1}$ go to step 9.
- 3. Denote the indices of data from subset v_k by i_1, \ldots, i_{p+1} , hence $i_1, \ldots, i_{p+1} \in \hat{n}$ are distinct and $v_k^{i_1} = \cdots = v_k^{i_{p+1}} = 1$. Put l = 1.
- 4. If $l > 2^p$, put k = k + 1 and go to step 2.
- 5. If the system of equations

$$(x_{i_1} \circ_1^{(l)} x_{i_2})^T \beta = y_{i_1} \circ_1^{(l)} y_{i_2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(x_{i_1} \circ_p^{(l)} x_{i_{p+1}})^T \beta = y_{i_1} \circ_p^{(l)} y_{i_{p+1}}$$
(24)

is regular, then denote its solution by β_0 , if it is not, put l = l + 1 and go to step 4.

- 6. Evaluate and order residuals $r^2(\beta_0)$.
- 7. If $r_{i_1}^2(\beta_0) = r_{(h)}^2(\beta_0) = r_{(h+1)}^2(\beta_0)$, find subsets $w^{(1)}, \ldots, w^{(g)} \in Q^{(n,h)}$ which are in relation Z with β_0 (use Program 1).
- 8. For j = 1, ..., g evaluate $J(w^{(j)})$. If $J(w^{(j)}) < J_{min}$, put $J_{min} = J(w^{(j)})$ and $w_{min} = w^{(j)}$.
- 9. Put l = l + 1 and go to step 4.
- 10. Return $\beta = \hat{\beta}^{(OLS,W_{min}X,W_{min}Y)}, W_{min} = diag(w_{min})$ as the LTS estimate for Problem 1.

Several of steps need to be commented on in details. Prior to that, let us prove that BSA always find the exact LTS estimate for Problem 1.

Theorem 13. If data of Problem 1 satisfy Assumptions 1, 2 and 3, then BSA returns the LTS estimate $\hat{\beta}^{(LTS,n,h)}$.

Proof

The proof follows straightforwardly from equation (14) and Assertion 11, which tells us that during BSA we evaluate J(w) for all $w \in W^{(\min)}$.

Q.E.D

Now, let us comment on the steps.

- steps 2-5 The goal of the algorithm is to find all points of the set \mathcal{H}_p and for every $\beta \in \mathcal{H}_p$ find all $U_i, i \in \hat{m}$ and the corresponding w_i such that $\beta \in \partial U_i$. In order to find all elements of \mathcal{H}_p , it is necessary to find all elements of H_p and it requires to resolve all possible systems of equations of type (24), i.e. it is necessary to go through all $\binom{n}{p+1} 2^p$ possibilities.
- step 5 Due to assertion 11 we know that "in most cases" we can omit non-regular systems without losing the assurance that we will find all the elements of \mathcal{H}_p (see the proof). Non-regularity would become a problem only if Assumption 3 is disrupted, i.e. at least one matrix $(n-1\times p+1)$ would not be regular. The assumptions will be discussed later on.
- **step 7** In this step β_0 is surely an element of H_p , to verify that $\beta_0 \in \mathcal{H}_p$ it is necessary to verify whether $r_{i_1}^2(\beta_0) = r_{(h)}^2(\beta_0) = r_{(h+1)}^2(\beta_0)$. If $r_{i_1}^2(\beta_0) \neq r_{(h)}^2(\beta_0) = r_{(h+1)}^2(\beta_0)$, then either $\beta_0 \notin \mathcal{H}_p$ or it will be found during the loop for another value of the parameter k.

If we assume that the equality $r_{i_1}^2(\beta_0)=r_{(k)}^2(\beta_0)$ has the same probability for all $k\in\hat{n}$ and for all $\beta_0\in H_p$, then the probability that the algorithm will go to step 8 from step 7 (i.e. that $r_{i_1}^2(\beta_0)=r_{(h)}^2(\beta_0)=r_{(h+1)}^2(\beta_0)$ holds) is p/(n-p+1).

step 7 and 8 How many w can be in relation Z with β_0 ? The number g depends on $l \in \{0, 1, \ldots, p-2\}$ for which $r_{i_1}^2(\beta_t) = r_{(h-l)}^2(\beta_t)$ and equals $\binom{p}{l+1}$. The worst case is $\binom{p}{\lfloor p/2 \rfloor}$ and the best one is p (see also Program 1).

To conclude the basic description of the algorithm, we will calculate the complexity of it.

In order to compute the exact LTS estimate $\hat{\beta}^{(LTS,n,h)}$ by BSA it is necessary to

- successively select all $\binom{n}{p+1}$ elements of the set $Q^{(n,p+1)}$,
- $\binom{n}{p+1} \cdot 2^p$ times resolve system (24) of p equations,
- $\binom{n}{n+1} \cdot 2^p$ times evaluate and order n residuals,

• $\binom{n}{p+1} \cdot 2^p \cdot \frac{p}{n-p+1} \cdot \binom{p}{[p/2]}$ times calculate the OLS estimate $\hat{\beta}^{(OLS,WX,WY)}$.

Of course, the complexity further depends on numerical methods used for ordering of the residuals and solving the systems of equation (in step 4 and also during calculating the OLS estimate in step 6) but such a discussion is beyond the scope of this work.

4.2 Assumptions – verification and disruption of them

As written above, Assumption 1 is quite weak and moreover easily verifiable. Assumption 2 is still weaker and we can rely on it without any doubt.

Concerning Assumption 3 the situation is a bit more complicated. To verify that all 2^{n-1} matrixes have full rank is too exhausting. On the other hand, an assumption that $(n-1\times p)$ matrix has rank p is quite weak (for n great enough) and we can rely on this is fulfilled.

However, if an intercept is considered, Assumption 3 is always disrupted for $o_1 = \cdots = o_{n-1} = -$ for the first column of the matrix contains only zeros and so the rank of the matrix is less or equal to p-1. Thus, if an intercept is considered, Assertion 11 (namely Lemma 10) is not proved and we lose the certainity that BSA always finds the exact LTS estimate. To resolve this problem, Assumption 3 has to be reformulated to the following form.

Assumption 4. For all $(\circ_1, \ldots, \circ_{n-1}) \in (\times_{i=1}^n \{+, -\}) \setminus \{(-, \ldots, -)\}$ the matrix of the system of equations

$$(x_1 \circ_1 x_2)^T \beta = y_1 \circ_1 y_2$$

$$\vdots \qquad \vdots$$

$$(x_1 \circ_{n-1} x_n)^T \beta = y_1 \circ_{n-1} y_n$$

has rank p.

Assuming that this Assumption 4 is fulfilled instead of Assumption 3 we can reprove Lemma 10 for models where an intercept is considered as follows.

Proof

The only difference between the proofs is selecting of the signs $\circ_l, \ldots, \circ_{n-1}$. In the original Lemma we can select them arbitrarily, here, if Assumption 4 is considered, we demand $(\exists k \in \{l, \ldots, n-1\}) (\circ_k \neq -)$. The rest of proof is completely the same.

Q.E.D

Conclusion

BSA proved to be quick enough to be usable for reasonably large data. Of course probabilistic algorithms are faster and they have found the exact solution of Problem 1 as well in all cases the author tested. BSA algorithm has been implemented in MATLAB and in C++ (by Roman Kápl) and is available by email.

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TBA

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