# Robust and Parallel Bayesian Model Selection 

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#### Abstract

Effective and accurate model selection is an important problem in modern data analysis. One of the major challenges is the computational burden required to handle large data sets that cannot be stored or processed on one machine. Another challenge one may encounter is the presence of outliers and contaminations that damage the inference quality. The parallel "divide and conquer" model selection strategy divides the observations of the full data set into roughly equal subsets and perform inference and model selection independently on each subset. After local subset inference, this method aggregates the posterior model probabilities or other model/variable selection criteria to obtain a final model by using the notion of geometric median. This approach leads to improved concentration in finding the "correct" model and model parameters and also is provably robust to outliers and data contamination.


Keywords: Machine learning, Bayesian statistics, model selection, scalable inference.

## 1. INTRODUCTION

In many data modeling scenarios, many plausible models are available to fit to the data, each of which may result in drastically different predictions and

[^0]conclusions. Being able to select the right model for inference is a crucial task. As our main example, we consider model selection for a normal linear model:
\[

$$
\begin{equation*}
Y=X \beta+\epsilon, \epsilon \sim N\left(0, \sigma^{2} I\right) \tag{1}
\end{equation*}
$$

\]

where $Y$ is an $N$ dimensional response vector, $X$ is an $N \times D$ dimensional design matrix and $\beta$ is a $D$ dimensional vector of regression parameters. Here the candidate models to be selected could refer to the sets of significant variables. 5 In a Bayesian setting, we have a natural probabilistic evaluation of models through posterior model probabilities. Depending on the objectives of the data analysis, we may be interested in assessing the belief on which is the "best" model or obtaining predictions with minimum error.

Existing procedures to accomplish the aforementioned goals, however, will perform poorly under the presence of outliers and contaminations. In addition, Markov chain Monte Carlo (MCMC) algorithms for these methods do not scale to big data situations. The goal of this paper is to investigate a "divide-andconquer" method that integrates with existing Bayesian model selection techniques, in a way that is robust to outliers and, moreover, allows us to perform Bayesian model selection in parallel.

Our "divide-and-conquer" strategy is based on the ideas for robust inference using the notion of the geometric median [1], especially the median posterior in the Bayesian context [2, 3]. Previous work in this area has focused on the performance in parametric inference. Our contribution in this paper is to demonstrate ${ }_{20}$ the effectiveness of these ideas in selecting the correct class of models on top of the parameters. In particular, we show that the model aggregated across different subsets (the "divide") has improved concentration to the true model class compared to the one using the full data set. This concentration is in terms of the posterior model probabilities to the point mass assigned to the true model.
${ }_{25}$ The result also holds jointly with the concentration of the parameter estimates, and under the presence of outliers and hence demonstrates robustness. We carry out extensive numerical studies on simulation data and a real data example to demonstrate the performance of our proposed approach.

## 2. BAYESIAN MODEL SELECTION

In Bayesian model selection, we define the prior model probability $\operatorname{Pr}\left(M_{k}\right)$ for each of the model $M_{k}(k=1, \ldots, K)$ under consideration. For model $M_{k}$, we additionally have parameters $\left(\beta_{k}, \sigma_{k}^{2}\right)$ with prior $\operatorname{Pr}\left(\beta_{k}, \sigma_{k}^{2} \mid M_{k}\right)$, which leads to a likelihood $\operatorname{Pr}\left(Y \mid \beta_{k}, \sigma_{k}^{2}, M_{k}\right)$. Thus, the posterior model probability for model $M_{k}, \operatorname{Pr}\left(M_{k} \mid-\right)$, is proportional to

$$
\operatorname{Pr}\left(M_{k}\right) \int \operatorname{Pr}\left(Y \mid \beta_{k}, \sigma_{k}^{2}, M_{k}\right) \operatorname{Pr}\left(\beta_{k}, \sigma_{k}^{2} \mid M_{k}\right) \mathrm{d} \beta_{k} \mathrm{~d} \sigma_{k}^{2}
$$

However, as noted in [4, choosing the model with the highest posterior model probability is not always the best option nor should one neglect the risk of model uncertainty. Instead of resorting to a single model for predicted values $\tilde{Y}$ (or some quantity of interest in general), [5] proposes to average over the model uncertainty with Bayesian model averaging (BMA) to obtain a posterior mean and variance of $\tilde{Y}$ at a covariate level $\tilde{X}$ :

$$
\begin{aligned}
E[\tilde{Y} \mid \tilde{X}, Y]= & \sum_{k=1}^{K} E\left[\tilde{Y} \mid \tilde{X}, Y, M_{k}\right] \operatorname{Pr}\left(M_{k} \mid \tilde{X}, Y\right) \\
\operatorname{Var}(\tilde{Y} \mid \tilde{X}, Y)= & \sum_{k=1}^{K} \operatorname{Pr}\left(M_{k} \mid X, Y\right)\left(\operatorname{Var}\left(\tilde{Y} \mid \tilde{X}, Y, M_{k}\right)+\right. \\
& \left.E\left[\tilde{Y} \mid \tilde{X}, Y, M_{k}\right]^{2}\right)-E[\tilde{Y} \mid X, Y]^{2}
\end{aligned}
$$

30 We will focus on BMA in our theoretical developments in this paper. Our numerical experiments, however, will show that our divide-and-conquer strategy is also effective in applying on other model selection methods.

The first alternative to BMA is the median probability model, which can be shown to be optimal if we must choose one model for prediction 4. In this approach, we define the posterior inclusion probability of each predictor $x_{d}(d=1, \ldots, D)$ as the sum of posterior model probabilities of the models that include predictor $x_{d}$, namely $p_{d}=\sum_{k: x_{d} \in M_{k}} \operatorname{Pr}\left(M_{k} \mid X, Y\right)$. The median probability model is the model that includes the predictors $x_{d}$ if $p_{d} \geq 1 / 2$.

Second, using the maximum value of the likelihood for each model $\operatorname{Pr}\left(Y \mid \hat{\beta}_{k}, \hat{\sigma}_{k}^{2}, M_{k}\right)$, where $\left(\hat{\beta}_{k}, \hat{\sigma}_{k}^{2}\right)$ is the maximum likelihood estimate of $\left(\beta_{k}, \sigma_{k}^{2}\right)$, we can perform
penalized model selection through the Akaike information criterion (AIC) 6] or the Bayesian information criterion (BIC) [7] by selecting the model with the lowest information criterion:

$$
\begin{array}{r}
\mathrm{AIC}=-2 \log \operatorname{Pr}\left(Y \mid \hat{\beta}_{k}, \hat{\sigma}_{k}^{2}, M_{k}\right)+2(D+1), \\
\mathrm{BIC}=-2 \log \operatorname{Pr}\left(Y \mid \hat{\beta}_{k}, \hat{\sigma}_{k}^{2}, M_{k}\right)+(D+1) \log N .
\end{array}
$$

The final model selection technique we will consider is stochastic variable selection through the spike and slab model [8], which allows for variable shrinkage under high-dimensional models. For the purposes of this paper, we will use the rescaled spike and slab model [9. To perform posterior inference in this model, we first define $Y^{\prime}=\sqrt{\frac{N}{\hat{\sigma}^{2}}} Y$ where $\hat{\sigma}^{2}$ is the unbiased estimate of $\sigma^{2}$ under the full model and let $\nu_{0}>0$ be some small number. The model is defined to be the following mixture model:

$$
\begin{aligned}
Y^{\prime} & \sim N\left(X \beta, N \sigma^{2} I\right), \beta_{S S_{d}} \sim N\left(0, J_{d} \tau_{d}^{2}\right) \\
\sigma_{s s}^{-2} & \sim \operatorname{Gamma}(a, b), J_{d} \sim(1-w) \delta_{J_{d}}\left(\nu_{0}\right)+w \delta_{J_{d}}(1) \\
\tau_{d}^{-2} & \sim \operatorname{Gamma}\left(a_{\tau}, b_{\tau}\right), w \sim \operatorname{Uniform}(0,1)
\end{aligned}
$$

## 3. DIVIDE-AND-CONQUER AND ROBUST BAYESIAN MODEL SELECTION

In our robust model selection strategy, we divide $N$ observations into $R$ subsets of roughly equal sample size. Then inference, model selection and prediction is performed for the linear model $Y_{(j)}=X_{(j)} \beta+\epsilon_{(j)}$ independently across $j=1, \ldots, R$ subsets using the existing Bayesian model selection proce${ }_{45}$ dures, which are then combined to form a final model or a combined prediction value.

Given linear model (1), we first define the following priors on a normal likelihood with response variable $Y$ and $D$-dimensional predictor $X$. The $N$ observations are divided into $R$ subsets with $s$ observations within each subset.

One has,

$$
\begin{aligned}
\operatorname{Pr}\left(\sigma_{(j)}^{-2}\right) & =\operatorname{Gamma}(a, b) \\
\operatorname{Pr}\left(\beta_{(j)} \mid \sigma_{(j)}^{2}\right) & =N\left(\beta_{0}, \sigma_{(j)}^{2} \Sigma_{0}\right)
\end{aligned}
$$

To compensate for the data division, we raise the likelihood of the divided data $\operatorname{Pr}\left(Y_{(j)} \mid X_{(j)}, \beta, \sigma^{2}\right)$ to the $R$-th power and adjust the normalizing constant accordingly so that the likelihood for $Y_{j}$ is:

$$
\left(\frac{R}{2 \pi \sigma_{(j)}^{2}}\right)^{N / 2} \exp \left\{\frac{-R}{2 \sigma^{2}}\left(Y_{(j)}-X_{(j)} \beta_{(j)}\right)^{T}\left(Y_{(j)}-X_{(j)} \beta_{(j)}\right)\right\}
$$

The intuition and motivation for raising the subset likelihood to $R$-th power is to adjust the potentially inflated variance of the subset posterior distribution. Exploiting conjugacy, we obtain the full conditionals for data subset $j=1, \ldots, R$ :

$$
\begin{aligned}
\operatorname{Pr}\left(\beta_{(j)} \mid-\right) & =N\left(\mu_{\beta}, \sigma^{2} \Sigma_{\beta}\right) \\
\mu_{\beta} & =\Sigma_{\beta}\left(\beta_{0} \Sigma_{0}^{-1}+R X_{(j)}^{T} Y_{(j)}\right) \\
\Sigma_{\beta} & =\left(\Sigma_{0}^{-1}+R X_{(j)}^{T} X_{(j)}\right)^{-1} \\
\operatorname{Pr}\left(\sigma_{(j)}^{-2} \mid-\right) & =\operatorname{Gamma}\left(a^{\prime}, b^{\prime}\right) \\
a^{\prime} & =a+\frac{N+D}{2} \\
b^{\prime} & =b+\frac{R}{2} \epsilon^{T} \epsilon+\frac{1}{2}\left(\beta_{(j)}-\beta_{0}\right)^{T} \Sigma_{0}^{-1}\left(\beta_{(j)}-\beta_{0}\right) \\
\epsilon & =\left(Y_{(j)}-X_{(j)} \beta_{(j)}\right)
\end{aligned}
$$

Let $\Sigma_{X}=I+R X_{(j)} \Sigma_{0} X_{(j)}^{T}$, then integrating out the parameters gives us the following marginal distribution $\operatorname{Pr}\left(Y_{(j)} \mid X_{(j)}\right)$ :

$$
\frac{\left(\frac{R}{2 \pi}\right)^{\frac{N}{2}} b^{a} \Gamma\left(a+\frac{N}{2}\right)\left|\Sigma_{X}\right|^{-\frac{1}{2}} / \Gamma(a)}{\left(b+\frac{R}{2}\left(Y_{(j)}-X_{(j)} \beta_{0}\right)^{T} \Sigma_{X}^{-1}\left(Y_{(j)}-X_{(j)} \beta_{0}\right)\right)^{a+\frac{N}{2}}}
$$

For distributed AIC and BIC model evaluation, we raise the likelihood term of the AIC and BIC formula to the power of $R$ :

$$
\begin{array}{r}
\mathrm{AIC}_{R}=-2 R \log \operatorname{Pr}\left(Y_{(j)} \mid \hat{\beta}_{k}, \hat{\sigma}_{k}^{2}, M_{k}\right)+2(D+1) \\
\mathrm{BIC}_{R}=-2 R \log \operatorname{Pr}\left(Y_{(j)} \mid \hat{\beta}_{k}, \hat{\sigma}_{k}^{2}, M_{k}\right)+(D+1) \log N
\end{array}
$$

In applying our procedure with the spike and slab prior, we derived the full Gibbs sampler for our procedure. For posterior inference in the spike and slab model, let $\Delta=\operatorname{diag}\left\{J_{1} \tau_{1}^{2}, \ldots, J_{D} \tau_{D}^{2}\right\}$, we can perform Gibbs sampling by drawing from the following posteriors:

$$
\begin{aligned}
\operatorname{Pr}\left(\beta_{S S(j)} \mid-\right) & =N\left(\mu_{\beta_{S S}}, \Sigma_{\beta_{S S}}\right) \\
\Sigma_{\beta_{S S}} & =\left(\Delta^{-1}+\frac{R}{N \sigma_{S S(j)}^{-2}} X_{(j)}^{T} X_{(j)}\right)^{-1}, \\
\mu_{\beta_{S S}} & =\Sigma_{\beta_{S S}}\left(\frac{R}{N \sigma_{S S(j)}^{-2}} X_{(j)}^{T} Y_{(j)}\right), \\
\operatorname{Pr}\left(\sigma_{S S(j)}^{-2} \mid-\right) & =\operatorname{Gamma}\left(a_{S S}^{\prime}, b_{S S}^{\prime}\right), \\
a_{S S}^{\prime} & =a+\frac{N}{2}, \\
b_{S S}^{\prime} & =b+\frac{R}{2 N}\left(Y_{(j)}-X_{(j)} \beta_{S S(j)}\right)^{T}\left(Y_{(j)}-X_{(j)} \beta_{S S(j)}\right), \\
w_{d 1} & =(1-w) \nu_{0}^{-1 / 2} \exp \left\{-\frac{\beta_{S S(j) d}^{2}}{2 \nu_{0} \tau_{d}^{2}}\right\}, \\
w_{d 2} & =w \exp \left\{-\frac{\beta_{S S(j) d}^{2}}{2 \tau_{d}^{2}}\right\}, \\
\operatorname{Pr}\left(\tau_{d}^{-2} \mid-\right) & \propto w_{d 1} \delta_{J_{d}}\left(\nu_{0}\right)+w_{d 2} \delta_{J_{d}}(1), \\
\operatorname{Pr}(w \mid-) & =\operatorname{Beta}\left(1+\left|\left\{d: J_{d}=1\right\}\right|, 1+\left|\left\{d: J_{d}=\nu_{0}\right\}\right|\right)
\end{aligned}
$$

Once inference is built on each subset, the key step is to aggregate the subset models (or estimates) together into a final model (or estimate). To aggregate our results, we collect the $R$ number of subset models or estimates and find the geometric median between these $R$ elements. The geometric median for a set of elements $\left\{x_{1}, \ldots, x_{R}\right\}$ valued on a Hilbert space $\mathbb{H}$, is defined as

$$
\begin{equation*}
x_{*}=\operatorname{med}_{g}\left(x_{1}, \ldots, x_{R}\right)=\operatorname{argmin}_{y \in \mathbb{H}} \sum_{j=1}^{R}\left\|y-x_{j}\right\|, \tag{2}
\end{equation*}
$$

where $\|\cdot\|$ is the norm associated with the inner product in $\mathbb{H}$ [3]. The solution can generally be effectively approximated using the Weiszfeld algorithm 10.

For instance, in the case of aggregating the posterior model probabilities across $R$ subsets of data, the geometric median operates on the space of posterior distributions and the geometric median posterior model probability, $\operatorname{Pr}_{*}\left(M_{k} \mid X, Y\right)$, is defined as:

$$
\begin{equation*}
\underset{P \in \Pi_{K}}{\operatorname{argmin}} \sum_{j=1}^{R}\left\|P-\operatorname{Pr}\left(M_{k} \mid X_{(j)}, Y_{(j)}\right)\right\|, \tag{3}
\end{equation*}
$$

where $\operatorname{Pr}\left(M_{k} \mid X_{(j)}, Y_{(j)}\right)$ is the posterior model probabilities for subset $j$, and $\Pi_{K}$ denotes the space of distributions on $K$ support points. The metric $\|\cdot\|$ here can be taken as the Euclidean metric, or an integral probability metric (IPM) defined as $\|P-Q\|=\sup _{f \in \mathcal{F}}\left|\int f(x) \mathrm{d}(P-Q)(x)\right|$ for some class of functions $\mathcal{F}$ [11, 12].

For the model selection techniques discussed earlier (AIC, BIC, and the median model selection), we can choose a final model in two ways: One, we can select the best model locally on each subset, use it for prediction, and then aggregate the results (estimate combination). Or two, we can take the median of the model selection criteria and choose that particular model on each subset and then aggregate the results to get a final model (model combination).

However, in Bayesian model averaging and spike and slab modeling we do not choose a final model. We can still perform model or estimate combination by aggregating the posterior model probabilities. We consider both model and estimate combinations in our experiments and show that they yield similar results in our experimental settings.

## 4. IMPROVED CONCENTRATION AND ROBUSTNESS

In this section we provide theoretical justification on the robustness in the divide-and-conquer strategy. In particular, we focus on BMA. Additionally, we show that the aggregated model class from our strategy concentrates faster, in terms of posterior model probabilities, to the correct class compared to using the whole data set at once. This concentration result can be joint with parameter estimation, and also applies in a way that exhibits robustness against outliers.

```
Algorithm 1: Algorithm for robust model selection in the case of BMA.
    for \(j \in\{1, \ldots, R\}\) do
        Raise likelihood to \(R\)-th power
        Compute inference for \(P\left(\theta \mid M_{k}, X_{(j)}, Y_{(j)}\right)\) for \(k=1, \ldots, K\)
        Draw predictive values from predictive posterior \(P\left(\tilde{Y} \mid M_{k}, X_{(j)}, Y_{(j)}\right)\)
        for \(k=1, \ldots, K\)
        Calculate posterior model probabilities \(\left\{P\left(M_{k} \mid X_{(j)}, Y_{(j)}\right)\right\}_{k=1, \ldots, K}\)
    Calculate geometric median of posterior model probabilities over the
        subsets using (3).
Approximate geometric medians of posterior parameter probabilities or predictive values given individual models over the subsets using (2).
Obtain BMA estimate: \(E[\tilde{Y} \mid Y, X]=\sum_{k=1}^{K} E_{*}\left[\tilde{Y} \mid X, Y, M_{k}\right] P r_{*}\left(M_{k} \mid X, Y\right)\)
```

Note that we do not raise the subset likelihood to $R$-th power in our current theoretical analysis, but the results can be generalized by imposing slightly stronger entropy conditions on the model.

Let $\mathcal{S}$ be the domain of $\theta=\left(M_{k}, \beta, \sigma^{2}\right)$, our set of model indices and parameters. Let $\theta_{0}$ be the true data generating parameter, and let ( $X_{1}, Y_{1}$ ) be a generic data point. Let $p_{0}(y \mid x):=p\left(y \mid x, \theta_{0}\right)$ be the true conditional density of $Y_{1}$ given $X_{1}$, and $p_{0}(x)$ be the true density of the covariates $X_{1}$. We denote $p_{\theta}(y \mid x):=p(y \mid x, \theta)$. Let $P_{\theta}$ be the distribution defined by $p_{0}(x) \times p_{\theta}(y \mid x)$ and $P_{0}$ is the true distribution $p_{0}(x) \times p_{0}(y \mid x)$. For convenience, we denote $P_{0} f=P_{0} f\left(X_{1}, Y_{1}\right)=E_{p_{0}}\left[f\left(X_{1}, Y_{1}\right)\right]$ where $E_{p_{0}}[\cdot]$ is the expectation under $p_{0}(y \mid x) \times p_{0}(x)$. We denote $P_{0}^{N}$ as the true probability measure taken on the data $(X, Y)$ of size $N$ and $P_{0}^{N} f=E_{P_{0}^{N}}[f(X, Y)]$. Lastly, we denote $\mathcal{D}(\epsilon, \mathcal{P}, d)$ as the $\epsilon$-packing number of a set of probability measures $\mathcal{P}$ under the metric $d$, which is the maximal number of points in $\mathcal{P}$ such that the distance between any pair is at least $\epsilon$. We implicitly assume here that $\mathcal{P}$ is separable. The following Theorem 1 follows from a modification of Theorem 2.1 in [13]:

Theorem 1. Assume that there is a sequence $\varepsilon_{N}$ such that $\varepsilon_{N} \rightarrow 0$ and $N \varepsilon_{N}^{2} \rightarrow$
$\infty$ as $N \rightarrow \infty$, a constant $C$, and a set $\mathcal{S}_{N} \in \mathcal{S}$ so that

90
2. $\operatorname{Pr}\left(\mathcal{S} \backslash \mathcal{S}_{N}\right) \leq e^{-N \varepsilon_{N}^{2}(C+4)}$.
3. $\operatorname{Pr}\left(\theta:-P_{0} \log \frac{p_{\theta}\left(Y_{1} \mid X_{1}\right)}{p_{0}\left(Y_{1} \mid X_{1}\right)} \leq \varepsilon_{N}^{2}\right.$,

$$
\left.P_{0}\left(\frac{p_{\theta}\left(Y_{1} \mid X_{1}\right)}{p_{0}\left(Y_{1} \mid X_{1}\right)}\right)^{2} \leq \varepsilon_{N}^{2}\right) \geq e^{-N \varepsilon_{N}^{2} C}
$$

where $\mathcal{P}_{\mathcal{S}_{N}}=\left\{p_{0}(x) \times p_{\theta}(y \mid x): \theta \in \mathcal{S}_{N}\right\}$ and $d_{H}$ is the Hellinger distance. Then we have

$$
\begin{gather*}
P_{0}^{N}\left(\operatorname{Pr}\left(\theta: d_{H}\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}^{2} \mid X, Y\right)>\delta\right) \leq \\
\frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta} \tag{4}
\end{gather*}
$$

for any $0<\delta<1$ and sufficiently large $T>0$ such that $L T^{2} \geq C+4$ and $L T^{2}-1>L$, where $L$ is a universal constant.

The proof of Theorem 1 is in the Appendix. As noted by [13], the important assumptions are Assumptions 1 and 3. Essentially, Assumption 1 constrains the size of the parameter domain $\mathcal{S}$ to be not too big, whereas Assumption 3 ensures sufficient mass of the prior on a neighborhood of the true parameter. to the true $\theta_{0}$ with high probability, where the closeness is measured in terms of the Hellinger distance between the likelihoods. Note that the RHS of 4 consists of three terms. The dominant term is the power-law decay in $N \varepsilon_{N}^{2}$. The other two exponential decay terms result from technical arguments in the existence of tests that sufficiently distinguish between distributions [14, 15.

Next we describe the concentration behavior of BMA. We focus on the situations where all the candidate models are non-nested, i.e. only one model contains distributions that are arbitrarily close to the truth. Without loss of generality, we let $M_{1}$ be the true model.

Theorem 2 (BMA of Non-Nested Models). Suppose the assumptions in Theorem 1 hold. Also assume that, for sufficiently small $\epsilon>0, d\left(P_{\theta}, P_{0}\right)>\epsilon$
for any $\theta \in \mathcal{S}_{-1}:=\left\{\left(M_{k}, \beta, \sigma^{2}\right): k \neq 1\right\}$. Let $L$ be the same universal constant arising in Theorem 1. We have

1. For any given $0<\delta<1$,

$$
\begin{align*}
& P_{0}^{N}\left(\operatorname{Pr}\left(M_{1} \mid X, Y\right)<1-\delta\right) \leq \\
& \quad \frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta} \tag{5}
\end{align*}
$$

for sufficiently large $N$.
2. For any given $0<\delta<1$,

$$
\begin{align*}
& P_{0}^{N}\left(d_{E}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)>\delta\right) \leq \\
& \quad \frac{\sqrt{2}}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 \sqrt{2} e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 \sqrt{2} e^{-2 N \varepsilon_{N}^{2}}}{\delta} \tag{6}
\end{align*}
$$

for sufficiently large $N$, where $d_{E}$ is the Euclidean distance, and $\mathbf{e}_{1}$ is the point mass on $M_{1}$.
3. For any $0<\delta<\sqrt{(\sqrt{2}-1)^{2}+1} / 2$,

$$
\begin{align*}
& P_{0}^{N}\left(d_{H}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)>\delta\right) \leq \\
& \quad \frac{(\sqrt{2}-1)^{2}+1}{\sqrt{2} C^{2} N \varepsilon_{N}^{2} \delta^{2}}+\frac{\left((\sqrt{2}-1)^{2}+1\right) e^{-L N \varepsilon_{N}^{2}}}{\delta^{2}}+  \tag{7}\\
& \quad \frac{\left((\sqrt{2}-1)^{2}+1\right) e^{-2 N \varepsilon_{N}^{2}}}{\delta^{2}}
\end{align*}
$$

for sufficiently large $N$.

## Proof of Theorem 2,

Proof of 1 . Consider large enough $N$ and fix a sufficiently large $T>0$. We 120 have

$$
\begin{align*}
& \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right) \leq T \varepsilon_{N}^{2} \mid X, Y\right) \\
= & E_{\operatorname{Pr}}\left[\operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right) \leq T \varepsilon_{N}^{2} \mid M_{k}, X, Y\right) \mid X, Y\right] \tag{8}
\end{align*}
$$

where $E_{P r}[\cdot \mid X, Y]$ denotes the posterior expectation
and $\operatorname{Pr}\left(\cdot \mid M_{k}, X, Y\right)$ denotes the posterior distribution given model $M_{k}$
$=\operatorname{Pr}\left(M_{1} \mid X, Y\right) \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right) \leq T \varepsilon_{N}^{2} \mid M_{1}, X, Y\right)$,
by the condition that $d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}^{2}$ for any $\theta \in \mathcal{S}_{-1}$ and any $T>0$ eventually. Hence

$$
\begin{equation*}
\operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right) \leq T \varepsilon_{N}^{2} \mid X, Y\right) \geq 1-\delta \tag{10}
\end{equation*}
$$

implies

$$
\begin{equation*}
\operatorname{Pr}\left(M_{1} \mid X, Y\right) \geq 1-\delta \tag{11}
\end{equation*}
$$

The result then follows from Theorem 1. which implies that 10 occurs with probability at least

$$
1-\left(\frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta}\right)
$$

Proof of 2. Note that 11) implies

$$
\begin{equation*}
d_{E}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)=\sqrt{\left(1-\operatorname{Pr}\left(M_{1} \mid X, Y\right)\right)^{2}+\sum_{k \neq 1} \operatorname{Pr}\left(M_{k} \mid X, Y\right)^{2}} \leq \sqrt{2} \delta \tag{12}
\end{equation*}
$$

since $\left(1-\operatorname{Pr}\left(M_{1} \mid X, Y\right)\right)^{2} \leq \delta^{2}$ and $(\delta, 0, \ldots, 0)$ is an optimizer of the optimization

$$
\max \sum_{i=2}^{K} x_{i}^{2} \text { subject to } \sum_{i=2}^{K} x_{i} \leq \delta
$$

Hence (5) and 12 together imply

$$
P_{0}^{N}\left(d_{E}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right) \geq \sqrt{2} \delta\right) \leq \frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta}
$$

By redefining $\tilde{\delta}=\sqrt{2} \delta$, we get (6).
Proof of 3. Note that 11) implies

$$
\begin{align*}
d_{H}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right) & =\sqrt{\frac{1}{2}\left(\left(\sqrt{1-\operatorname{Pr}\left(M_{1} \mid X, Y\right)}\right)^{2}+\sum_{k \neq 1} \operatorname{Pr}\left(M_{k} \mid X, Y\right)\right)} \\
& \leq \sqrt{\frac{1}{2}\left((1-\sqrt{1-\delta})^{2}+\delta\right)} \tag{13}
\end{align*}
$$

since $x_{i}=\delta /(k-1)$ for all $i \neq 0$ gives the optimizer of the optimization

$$
\max \sum_{i \neq 0} \sqrt{x_{i}} \text { subject to } \sum_{i \neq 0} x_{i} \leq \delta
$$

Hence (5) and (13) together imply

$$
\begin{array}{r}
P_{0}^{N}\left(d_{H}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)>\sqrt{\frac{1}{2}\left((1-\sqrt{1-\delta})^{2}+\delta\right)}\right) \leq  \tag{14}\\
\frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta}
\end{array}
$$

Note that $(1-\sqrt{1-\delta})^{2}$ is a convex function in $\delta$ for $0<\delta<1$ and is equal to 0 at $\delta=0$. Thus $(1-\sqrt{1-\delta})^{2} \leq(\sqrt{2}-1)^{2} \delta$ for $0<\delta<1 / 2$, where $(\sqrt{2}-1)^{2}$ is the slope of the line between $(0,0)$ and $\left(1 / 2,(1-\sqrt{1-1 / 2})^{2}\right.$. Hence, for $0<\delta<1 / 2$, we have

$$
\sqrt{\frac{1}{2}\left((1-\sqrt{1-\delta})^{2}+\delta\right)} \leq \sqrt{\left((\sqrt{2}-1)^{2}+1\right) \frac{\delta}{2}}
$$

Combining with 14), we have

$$
\begin{array}{r}
P_{0}^{N}\left(d_{H}\left(\operatorname{Pr}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)>\sqrt{\left((\sqrt{2}-1)^{2}+1\right) \frac{\delta}{2}}\right) \leq  \tag{15}\\
\frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta} .
\end{array}
$$

By redefining $\tilde{\delta}=\sqrt{\left((\sqrt{2}-1)^{2}+1\right) \delta / 2}$, we get (7).
Note that the assumption $d\left(P_{\theta}, P_{0}\right)>\epsilon$ for any $\theta \in \mathcal{S}_{-1}$ and sufficiently small $\epsilon$ is a manifestation of the non-nested model situation, asserting that only one model is "correct". Result 1 is a concentration on the posterior probability of picking the correct model to be close to 1 .

Result 2 translates this in terms of the Euclidean distance between the model posterior probability and the point mass on the correct model. Result 3 is an alternative using the Hellinger distance. Note that the concentration ${ }_{30}$ bound for Hellinger distance (7) is inferior to that for Euclidean distance (6) for small $\delta$ since $\delta^{2}$ instead of $\delta$ shows up in the RHS of (7). This is because in our proof, the function $\sqrt{(1-\sqrt{1-\delta})^{2}+\delta}$ that appears in (14) has derivative $1 /\left(2 \sqrt{(1-\delta)\left((1-\sqrt{1-\delta})^{2}+\delta\right)}\right)$ which is $\infty$ at $\delta=0$, and thus no linearization is available when $\delta$ is close to 0 .

Theorem 2 can be modified to handle the case where multiple models contain
the truth. In particular, the expression inside the probability in (5) becomes

$$
\sum_{r \in \mathcal{M}} \operatorname{Pr}\left(M_{r} \mid X, Y\right)<1-\delta
$$

where $\mathcal{M}$ is the collection of all $r$ such that $M_{r}$ contains the true model. In (6) and (7), the use of $\mathbf{e}_{1}$ is replaced by an existence of some probability vector (dependent on $N$ ) supported on the indices in $\mathcal{M}_{r}$. In other words, one now allows comparing with an arbitrary allocation of probability masses to all true models in the concentration bound. These modifications can be seen by following the arguments in the proof of Theorem 2. Specifically, (9) would be modified as

$$
\sum_{r \in \mathcal{M}} \operatorname{Pr}\left(M_{r} \mid X, Y\right) \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right) \leq T \varepsilon_{N}^{2} \mid M_{r}, X, Y\right)
$$

Then (10) would imply a modified version of 11, namely

$$
\sum_{r \in \mathcal{M}} \operatorname{Pr}\left(M_{r} \mid X, Y\right) \geq 1-\delta
$$

giving the claimed modification for (5). Then, following (12), we could find a probability vector to make all $\left(1-\operatorname{Pr}\left(M_{r} \mid X, Y\right)\right)^{2}$ terms vanish except one, which is in turn bounded by $\delta^{2}$. This gives the claimed modifications for (6) and (7).

The following result states how a divide-and-conquer strategy can improve the concentration rate of the posterior model probabilities towards the correct model:

Theorem 3 (Concentration Improvement). Suppose the assumptions in Theorem 22 hold. Let $s=N / R$, and $q=\frac{\sqrt{2}}{C^{2} s \varepsilon_{s}^{2} \delta}+\frac{2 \sqrt{2} e^{-L s \varepsilon_{s}^{2}}}{\delta}+\frac{2 \sqrt{2} e^{-2 s \varepsilon_{s}^{2}}}{\delta}$. For sufficiently large s, letting $\alpha, \nu$ be constants such that $0<q<\alpha<1 / 2$ and
$0 \leq \nu<(\alpha-q) /(1-q)$, we have:

1. $\operatorname{Pr} r_{*}\left(M_{k} \mid X, Y\right)$, the geometric median under $d_{E}$ of $\left\{\operatorname{Pr}\left(M_{k} \mid\left(X_{(j)}, Y_{(j)}\right)\right)\right\}_{j=1, \ldots, R}$, satisfies

$$
\begin{align*}
& P_{0}^{N}\left(d_{E}\left(\operatorname{Pr}_{*}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)>C_{\alpha} \delta\right) \leq \\
& \quad\left(e^{(1-\nu) \psi\left(\frac{\alpha-\nu}{1-\nu}, q\right)}\right)^{-R}, \tag{16}
\end{align*}
$$

where $C_{\alpha}=(1-\alpha) \sqrt{1 /(1-2 \alpha)}$, and $\psi(\alpha, q)=(1-\alpha) \log \frac{1-\alpha}{1-q}+\alpha \log \frac{\alpha}{q}$.
2. Let $K$ be the number of model classes, then:

$$
\begin{align*}
P_{0}^{N} & \left(\operatorname{Pr}_{*}\left(M_{1} \mid X, Y\right)<1-C_{\alpha} \delta \sqrt{\frac{K-1}{K}}\right) \leq  \tag{17}\\
& \left(e^{(1-\nu) \psi\left(\frac{\alpha-\nu}{1-\nu}, q\right)}\right)^{-R}
\end{align*}
$$

3. Suppose in addition that, for any $P_{\theta^{1}}, P_{\theta^{2}}$ such that $\theta^{i}=\left(M_{1}, \beta^{i},\left(\sigma^{2}\right)^{i}\right)$ for $i=1,2$, we have

$$
\begin{equation*}
d_{H}\left(P_{\theta^{1}}, P_{\theta^{2}}\right) \geq \tilde{C} \rho_{k}\left(\theta^{1}, \theta^{2}\right)^{\gamma} \tag{18}
\end{equation*}
$$

where $\rho_{k}\left(\theta^{1}, \theta^{2}\right)=\left\|k\left(\cdot, \theta^{1}\right)-k\left(\cdot, \theta^{2}\right)\right\|_{\mathbb{H}}$, with $k$ being a characteristic kernel defined on the space $\left\{\theta=\left(M_{1}, \cdot, \cdot\right)\right\}$ and $\mathbb{H}$ is the corresponding reproducing kernel Hilbert space ( RKHS ), and $\tilde{C}>0$ and $\gamma>0$ are constants. Moreover, assume that there is a universal constant $\tilde{K}$ such that $e^{-\tilde{K} s \varepsilon_{s}^{2} / 2} \leq \varepsilon_{s}$ for all s, and we choose $\varepsilon_{s}$ such that $\tilde{q}=\frac{1}{C s \varepsilon_{s}^{2}}+4 e^{-\tilde{K} s \varepsilon_{s}^{2} / 2}<1 / 2$. Then

$$
\begin{aligned}
& P_{0}^{N}\left(\operatorname{Pr}_{*}\left(M_{1} \mid X, Y\right)>1-C_{\alpha} \delta \sqrt{\frac{K-1}{K}}\right. \\
& \left.\quad\left\|P r_{*}\left(\theta \mid M_{1}, X, Y\right)-\delta_{0}\right\|_{\mathcal{F}_{k}} \leq C_{\alpha} \tilde{T} \epsilon_{s}^{1 / \gamma}\right) \\
& \geq 1-\left(e^{(1-\nu) \psi\left(\frac{\alpha-\nu}{1-\nu}, q\right)}\right)^{-R}-\left(e^{\psi(\alpha, q)}\right)^{-R}
\end{aligned}
$$

where $\|\cdot\|_{\mathcal{F}_{k}}$ is defined as $\|P-Q\|_{\mathcal{F}_{k}}=\left\|\int k(x, \cdot) d(P-Q)(x)\right\|_{\mathbb{H}}, \tilde{T}>0$ is a sufficiently large constant, $\operatorname{Pr}_{*}\left(\theta \mid M_{1}, X, Y\right)$ is the geometric median of $\left\{\operatorname{Pr}\left(\theta \mid M_{1},\left(X_{(j)}, Y_{(j)}\right)\right)\right\}_{j=1, \ldots, R}$ under the $\|\cdot\|_{\mathcal{F}_{k}}$-norm, and $\delta_{0}$ is the delta measure at the true parameter.

The significance of Theorem 3 is the improvement of the concentration from power-law decay in Theorem 2 to exponential decay, as the number of subsets grows. Such type of results is known in the case of parameter estimation (e.g., [2, 3]). Theorem 3 generalizes to the case of model selection. Results 1 and 2 describe the exponential concentration for the model posteriors to the correct model, while Result 3 states the joint concentration in both the model posterior and the parameter posterior given the correct model, when one adopts a second
layer of divide-and-conquer on the parameter posterior conditional on each inhas been used in existing work like [2, 3,

Finally, the following theorem highlights that the concentration improvement still holds even if the data are contaminated to a certain extent:

Theorem 4 (Robustness to Outliers). Using the notation in Theorem 3, but assume instead that, for $j$ where $1 \leq j \leq\lfloor(1-\nu) R\rfloor+1$,

$$
\begin{aligned}
& P_{0}^{s}\left(d_{E}\left(\operatorname{Pr}\left(M_{k} \mid X_{(j)}, Y_{(j)}\right), \mathbf{e}_{1}\right)>\delta\right) \leq \\
& \quad \frac{\sqrt{2}}{C^{2} s \varepsilon_{s}^{2} \delta}+\frac{2 \sqrt{2} e^{-L s \varepsilon_{s}^{2}}}{\delta}+\frac{2 \sqrt{2} e^{-2 s \varepsilon_{s}^{2}}}{\delta},
\end{aligned}
$$

the conclusion of Theorem 3 still holds.

7 Theorem 4 stipulates that when a small number of subsets are contaminated by arbitrary nature, the geometric median approach still retains the same exponential concentration.

Proofs of Theorems 3 and 4 .
The proofs of both theorems rely on a key theorem on geometric median in [1], restated in the Appendix. We focus on Theorem 3, as the proof for Theorem 4 is a straightforward modification in light of Theorem 5

Proof of 1. Immediate by noting that

$$
P_{0}^{s}\left(d_{E}\left(\operatorname{Pr}\left(M_{k} \mid X_{(j)}, Y_{(j)}\right), \mathbf{e}_{1}\right)>\delta\right) \leq q,
$$

for all $j=1, \ldots, R$, and applying Theorem 5 .
Proof of 2. Note that

$$
\begin{equation*}
d_{E}\left(\operatorname{Pr}_{*}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right) \geq\left(1-\operatorname{Pr}_{*}\left(M_{1} \mid X, Y\right)\right) \sqrt{\frac{K}{K-1}} \tag{19}
\end{equation*}
$$

To see this, let $a=\operatorname{Pr}_{*}\left(M_{1} \mid X, Y\right)$. We have

$$
d_{E}\left(\operatorname{Pr}_{*}\left(M_{k} \mid X, Y\right), \mathbf{e}_{1}\right)=\sqrt{(1-a)^{2}+\sum_{i=2}^{K} x_{i}^{2}}
$$

where $x_{i}$ 's satisfy $\sum_{i=2}^{K} x_{i}=1-a$. Since $(1-a) /(K-1)$ is the optimizer of the optimization

$$
\min \sum_{i=2}^{K} x_{i}^{2} \text { subject to } \sum_{i=2}^{K} x_{i}=1-a
$$

we get $\sqrt{(1-a)^{2}+\sum_{i=2}^{K} x_{i}^{2}} \geq(1-a) \sqrt{K /(K-1)}$.
Hence (16) and (19) together give

$$
P_{0}^{N}\left(\operatorname{Pr}_{*}\left(M_{1} \mid X, Y\right)<1-C_{\alpha} \delta \sqrt{\frac{K-1}{K}}\right) \leq\left(e^{(1-\nu) \psi\left(\frac{\alpha-\nu}{1-\nu}, q\right)}\right)^{-R}
$$

Proof of 3. Under the additional assumptions, we can invoke Corollary 3.5 in [3] to obtain that

$$
P_{0}^{N}\left(\left\|P r_{*}\left(\theta \mid M_{1}, X, Y\right)-\delta_{0}\right\|_{\mathcal{F}_{k}}>C_{\alpha} \tilde{T} \epsilon_{s}^{1 / \gamma}\right) \leq\left(e^{\psi(\alpha, q)}\right)^{-R}
$$

The result follows from applying a union bound and together with 17.

## 5. SIMULATIONS AND DATA ANALYSIS

For the BMA, AIC, BIC and median probability model tests, we generate data from a model $Y=X \beta+\epsilon$, where $X$ is a $5000 \times 10$ matrix and $\beta$ is a 10 dimensional vector with 3 true predictors. We assess the aforementioned model selection techniques with four tests, over 10 trials for the contamination and magnitude tests and over 20 trials for the coverage test on 1 and 10 subsets for the magnitude and coverage tests and 1 and 50 subsets for the contamination tests with 1,000 iterations on each MCMC chain and a burn-in period of the initial 500 iterations.

The first test is the contamination test which examines the root mean square error (RMSE) of held-out test data $\tilde{Y}$ of size 50 against the number of outliers
present (as many as 5 in our experiments) in the training data, $Y$. We generate outliers by taking the maximum of the absolute value of the data and add a given magnitude value. Each outlier has a relative magnitude of 10,000 meaning that we find the largest output, $Y_{i^{*}}$ such that $i^{*}=\operatorname{argmax}_{i}\left\{\left|Y_{i}\right|: i=1, \ldots, N\right\}$, so that the value of the outlier is $Y_{i^{*}}+\left(\operatorname{sgn}\left(Y_{i^{*}}\right) \times 10000\right)$. For the contamination test, we expect to see superior performance with regards to RMSE of the 50 subset median posterior as long as the number of outliers per subset does not exceed 1. Figure 1 demonstrates the robustness of our technique to the number of outliers when we divide the data into subsets. We can see that the empirical $95 \%$ distribution of the RMSE over 10 trials for 50 subsets (green dashed line) falls dramatically below that of the RMSE distribution of 1 subset for each model selection technique when outliers are present except in the case when 50 outliers are present for Bayesian model averaging which approaches the point where the theoretical guarantees of our method are violated.

The second test assesses the RMSE of the held-out test data of size 50 against the increasing relative magnitude of one outlier present in the training data. We expect to see nearly constant RMSE on the 10 subset run as the relative magnitude of a single outlier increases, thus the procedure is robust. We can see in Figure 2 that the RMSE of distributed variants of the model selection techniques are lower than the single processor variants as the number of outliers increases. In the magnitude test, we can categorically observe that 10 subset RMSE is invariant to the relative magnitude of one outlier present in the data whereas the RMSE grows rapidly on one subset.

The next test assesses the $95 \%$ frequentist posterior coverage of the true heldout predictive value of size $1, \tilde{Y}$, against the increasing relative magnitude of one outlier in the training data. To calculate coverage we generate 50 independent MCMC chains at each level of outlier magnitude and calculate the proportions of chains which include the true predictive value within the $2.5 \%$ and $97.5 \%$ percentiles of the posterior predictive draws. For the coverage test we see that the empirical coverage of a single predictive value for the distributed subsets is, on average, $95 \%$ regardless of the magnitude of the outlier as opposed to the


Figure 1: Contamination test.
empirical coverage for the single subset. In the 1 subset case, we can see that the empirical coverage degrades almost to zero as the magnitude of the outlier grows. (see Figure 3).

Our last evaluation is the coverage of the regression coefficients and the ability for our model selection techniques to choose the correct model under the distributed setting with a single outlier of magnitude 10,000 . We compare the posterior credible interval of the regression coefficients for 1 and 10 subsets. Note that we do not include nested models in our evaluations or models larger ${ }_{230}$ than the true model (i.e models with more than 3 covariates included). Furthermore, we perform this evaluation under two settings: One, where we combine the optimal local model seleceted on each subset ("Model Combination") or if


Figure 2: Magnitude of outlier test.
we combine the subposterior estimates and select the optimal model globally ("Estimate Combination") As seen in Figure 4, the parallel technique is able to select the correct model 1 subset test, the outlier leads to the incorrect model being selected. Additionally, Figure 5 demonstrates that model and estimate combination yield similar results with the regression coefficient coverage test.

Also, we would like to see if the results still hold between model and estimate combination for the other simulation studies performed. Figs. 6, 7, and 8 show selection in each of the tests evaluated.

Furthermore, we wish to evaluate our method a large synthetic dataset with the same synthetic generating process as above, but with one million observa-


Figure 3: Testing empirical coverage of predictive value.
tions divided over 50 processors. Here, we examine the behavior of our method when we increase the magnitude of one outlier in the dataset and when we increase the number of outliers with fixed magnitude. In Figure 9, we can see that our performance is robust when the number of outliers per subset fulfills Theorem 4. When the number of outliers reaches 40 and 50 , we see start to see a noticeable degradation of our method's predictive ability. However, this degradation is still small relative to what we might observe in the case where we do not divide the data into subsets.

Additionally, we would like to see the computational gain of dividing the data for this situation in terms of CPU time for running the model selection and inference procedure. For one subset the average computation time is $91,829.15$ seconds with a standard error of 190.80 seconds. For ten subsets, the average computation time is $10,301.60$ seconds with a standard error of 81.28 seconds. And for fifty subsets, the average computation time is $29,49.74$ seconds with a standard error of 16.61 seconds which signifies that we obtain critical computa-

## Estimate Combination



Figure 4: Posterior regression parameter coverage test results, estimate combination.
tional performance when dividing our method across multiple processors.
Lastly, we evaluate our parallel model selection method on the diabetes data set used in [16]. The diabetes data consists of a $442 \times 10$ dimension design matrix scaled with unit norm and zero mean and a single response vector. We held out 45 observations for test evaluation and plotted the posterior $95 \%$ credible intervals for the predictive values centered at zero after subtracting the true predictive value. We can see in Fig. 10 that, after dividing the data across 5 subsets, we can attain a tighter credible interval over the true value for each model selection technique.


Figure 5: Posterior regression parameter coverage test results, model combination.

## 6. CONCLUSION

While a substantial body of work exists for fast and scalable Bayesian infer- ence methods, few research methods are available on robust and scalable model selection. We have studied in this paper a divide-and-conquer strategy that contributes to filling this gap. This strategy operates by taking the geometric median of posterior model probabilities or other selection criteria that extends previous results focusing on parametric inference. We show theoretically how the strategy, particularly in the setting of BMA, can be robust to outliers and, moreover, exhibits faster concentration to the true model in terms of posterior





- 1 Subset, Med. 450 Subsets, Med. - 1 Subset, $95 \%$ Dist. --- 50 Subsets, $95 \%$ Dist.

BIC, Estimate Combination

BIC, Model Combination


- 1 Subset, Med. 450 Subsets, Med. - 1 Subset, $95 \%$ Dist. --- 50 Subsets, $95 \%$ Dist.


- 1 Subset, Med. $\quad 50$ Subsets, Med. $\qquad$ 1 Subset, 95\% Dist. --- 50 Subsets, $95 \%$ Dist.

Figure 6: Contamination test


Figure 7: Coverage test.


Figure 8: Magnitude test.


Figure 9: Synthetic big data results.


Figure 10: Diabetes test data results.
model probabilities. The concentration result also applies to the joint setting of model selection and parameter estimation. We illustrate with both simulation data and a real data example how a variety of our strategy leads to more robust inference compared to standard approach that does not divide data into subsets. The strategy we present is simple to execute and is foreseen to have good practical value.

## 7. APPENDIX

## Proof of Theorem 1.

The proof is a modification of that for Theorem 2.1 in [13. Take any $\epsilon>2 \varepsilon_{N}$, we have, by Assumption 1.

$$
\log \mathcal{D}\left(\frac{\epsilon}{2}, \mathcal{P}_{\mathcal{S}_{N}}, d\right) \leq \log \mathcal{D}\left(\varepsilon_{N}, \mathcal{P}_{\mathcal{S}_{N}}, d\right) \leq N \varepsilon_{N}^{2}
$$

Then, by Theorem 7.1 in [13], there exists tests $\phi_{N}$ and a large enough constant $T$ (chosen later) such that

$$
\begin{equation*}
P_{0}^{N} \phi_{N} \leq e^{N \varepsilon_{N}^{2}} e^{-L N T^{2} \varepsilon_{N}^{2}} \frac{1}{1-e^{-L N T^{2} \varepsilon_{N}^{2}}}, \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{\theta \in \mathcal{S}_{N}: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}} P_{\theta}^{N}\left(1-\phi_{N}\right) \leq e^{-L N T^{2} \varepsilon_{N}^{2}} \tag{21}
\end{equation*}
$$

for a universal constant $L>0$, any $N>0$, and $P_{\theta}^{N}$ denotes the probability measure on $(X, Y)$ under $\left(X_{1}, Y_{1}\right) \sim p_{0}(x) \times p_{\theta}(y \mid x)$.

By 20, we have

$$
\begin{equation*}
P_{0}^{N} \operatorname{Pr}\left(\theta: \theta: d\left(P_{\theta}, P_{0}\right)>L \varepsilon_{N}^{2} \mid X, Y\right) \phi_{N} \leq P_{0}^{N} \phi_{N} \leq 2 e^{-L N \varepsilon_{N}^{2}} \tag{22}
\end{equation*}
$$

as $N \rightarrow \infty$, if we choose $L T^{2}-1>L$. Now, since

$$
P_{0} \frac{p_{\theta}\left(Y_{1} \mid X_{1}\right)}{p_{0}(Y \mid X)}=\int \frac{p_{\theta}(y \mid x)}{p_{0}(y \mid x)} p_{0}(d y \mid x) p_{0}(d x)=\int p_{\theta}(d y \mid x) p_{0}(d x)=1
$$

by Fubini's theorem, we have

$$
P_{0}^{N} \int_{\mathcal{S} \backslash \mathcal{S}_{N}} \prod_{i=1}^{N} \frac{p_{\theta}\left(Y_{i} \mid X_{i}\right)}{p_{0}\left(Y_{i} \mid X_{i}\right)} \operatorname{Pr}(\mathrm{d} \theta) \leq \operatorname{Pr}\left(\mathcal{S} \backslash \mathcal{S}_{N}\right)
$$

Hence, by Fubini's theorem again,

$$
\begin{align*}
& P_{0}^{N} \int_{\theta \in \mathcal{S}: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}} \prod_{i=1}^{N} \frac{p_{\theta}\left(Y_{i} \mid X_{i}\right)}{p_{0}\left(Y_{i} \mid X_{i}\right)} \operatorname{Pr}(\mathrm{d} \theta)\left(1-\phi_{N}\right) \\
\leq & \Pi\left(\mathcal{S} \backslash \mathcal{S}_{N}\right)+\int_{\theta \in \mathcal{S}_{N}: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}} P_{\theta}^{N}\left(1-\phi_{N}\right) \operatorname{Pr}(\mathrm{d} \theta) \\
\leq & \Pi\left(\mathcal{S} \backslash \mathcal{S}_{N}\right)+e^{-L N T^{2} \varepsilon_{N}^{2}} \text { by } 21 \\
\leq & 2 e^{-N \varepsilon_{N}^{2}(C+4)} \tag{23}
\end{align*}
$$

if $K M^{2} \geq C+4$, by Assumption 2 .
By Lemma 1 (stated below) and Assumption 3, with probability at least $1-1 /\left(C^{2} N \varepsilon_{N}^{2}\right)$, we have

$$
\begin{equation*}
\int \prod_{i=1}^{N} \frac{p_{\theta}\left(Y_{i} \mid X_{i}\right)}{p_{0}\left(Y_{i} \mid X_{i}\right)} \operatorname{Pr}(\mathrm{d} \theta) \geq e^{-2 N \varepsilon_{N}^{2}} \operatorname{Pr}\left(B_{n}\right) \geq e^{-N \varepsilon_{N}^{2}(2+C)} \tag{24}
\end{equation*}
$$

where

$$
B_{n}=\left\{\theta:-P_{0} \log \frac{p_{\theta}\left(Y_{1} \mid X_{1}\right)}{p_{0}\left(Y_{1} \mid X_{1}\right)} \leq \varepsilon_{N}^{2}, P_{0}\left(\frac{p_{\theta}\left(Y_{1} \mid X_{1}\right)}{p_{0}\left(Y_{1} \mid X_{1}\right)}\right)^{2} \leq \varepsilon_{N}^{2}\right\}
$$

Let $A_{N}$ be the event that 24 holds. We have

$$
\begin{aligned}
& P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N} \mid X, Y\right)\left(1-\phi_{N}\right) \mathbf{1}_{A_{N}} \\
= & P_{0}^{N} \frac{\int_{\theta: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}} \prod_{i=1}^{N} \frac{p_{\theta}\left(Y_{i} \mid X_{i}\right)}{p_{0}\left(Y_{i} \mid X_{i}\right)} \operatorname{Pr}(\mathrm{d} \theta)}{\int \prod_{i=1}^{N} \frac{p_{\theta}\left(Y_{i} \mid X_{i}\right)}{p_{0}\left(Y_{i} \mid X_{i}\right)} \operatorname{Pr}(\mathrm{d} \theta)}\left(1-\phi_{N}\right) \mathbf{1}_{A_{N}} \\
\leq & \left.e^{N \varepsilon_{N}^{2}(2+C)} 2 e^{-N \varepsilon_{N}^{2}(C+4)} \text { by } 23\right) \text { and } 24 \\
= & 2 e^{-2 N \varepsilon_{N}^{2}} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N} \mid X, Y\right) \\
= & P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N} \mid X, Y\right) \phi_{N}+P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)\right. \\
& \left.>T \varepsilon_{N} \mid X, Y\right)\left(1-\phi_{N}\right) \mathbf{1}_{A_{N}}+P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)\right. \\
& \left.>T \varepsilon_{N} \mid X, Y\right)\left(1-\phi_{N}\right)\left(1-\mathbf{1}_{A_{N}}\right) \\
\leq & P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N} \mid X, Y\right) \phi_{N}+P_{0}^{N} \operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)\right. \\
& \left.>T \varepsilon_{N} \mid X, Y\right)\left(1-\phi_{N}\right) \mathbf{1}_{A_{N}}+P_{0}^{N}\left(A_{N}^{c}\right) \text { for sufficiently large } T \\
\leq & 2 e^{-L N \varepsilon_{N}^{2}}+2 e^{-2 N \varepsilon_{N}^{2}}+\frac{1}{C^{2} N \varepsilon_{N}^{2}},
\end{aligned}
$$

by 22 , (25) and the property of $A_{N}$. By Chebyshev's inequality, we have

$$
P_{0}^{N}\left(\operatorname{Pr}\left(\theta: d\left(P_{\theta}, P_{0}\right)>T \varepsilon_{N}^{2} \mid X, Y\right)>\delta\right) \leq \frac{1}{C^{2} N \varepsilon_{N}^{2} \delta}+\frac{2 e^{-L N \varepsilon_{N}^{2}}}{\delta}+\frac{2 e^{-2 N \varepsilon_{N}^{2}}}{\delta},
$$

which concludes the theorem.

Lemma 1. For any $\epsilon>0$ and probability distribution $\Pi$ defined on the set

$$
\begin{equation*}
\left\{\theta:-P_{0} \log \frac{p_{\theta}(Y \mid X)}{p_{0}\left(Y_{1} \mid X_{1}\right)} \leq \epsilon^{2}, P_{0}\left(\frac{p_{\theta}(Y \mid X)}{p_{0}\left(Y_{1} \mid X_{1}\right)}\right)^{2} \leq \epsilon^{2}\right\} \tag{25}
\end{equation*}
$$

we have, for every $C>0$,

$$
\begin{equation*}
P_{0}^{N}\left(\int \prod_{i=1}^{N} \frac{p_{\theta}\left(Y_{i} \mid X_{i}\right)}{p_{0}\left(Y_{i} \mid X_{i}\right)} \Pi(\mathrm{d} \theta) \leq e^{-(1+C) N \epsilon^{2}}\right) \leq \frac{1}{C^{2} N \epsilon^{2}} \tag{26}
\end{equation*}
$$

Theorem 5 (Adopted from [1]). Consider a Hilbert space $(\mathbb{H},\langle\cdot, \cdot\rangle)$ and $\xi_{0} \in$ $\mathbb{H}$. Let $\hat{\xi}_{1}, \ldots, \hat{\xi}_{R} \in \mathbb{H}$ be a collection of independent random $\mathbb{H}$-valued elements. Let $\alpha, q, \nu$ be constants such that $0<q<\alpha<1 / 2$ and $0 \leq \nu<(\alpha-q) /(1-q)$. Suppose that there exists $\epsilon>0$ such that for all $j$, where $1 \leq j \leq\lfloor(1-\nu) R\rfloor+1$,

$$
P\left(\left\|\hat{\xi}_{j}-\xi_{0}\right\|>\epsilon\right) \leq q
$$

Let $\hat{\xi}_{*}=\operatorname{med}_{g}\left(\hat{\xi}_{1}, \ldots, \hat{\xi}_{R}\right)$ be the geometric median of $\left\{\hat{\xi}_{1}, \ldots, \hat{\xi}_{R}\right\}$. Then

$$
P\left(\left\|\hat{\xi}_{*}-\xi_{0}\right\|>C_{\alpha} \epsilon\right) \leq\left(e^{(1-\nu) \psi\left(\frac{\alpha-\nu}{1-\nu}, q\right)}\right)^{-R}
$$

where $C_{\alpha}=(1-\alpha) \sqrt{1 /(1-2 \alpha)}$, and

$$
\psi(\alpha, q)=(1-\alpha) \log \frac{1-\alpha}{1-q}+\alpha \log \frac{\alpha}{q}
$$

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