Testing the parametric form of the conditional variance in regressions based on distance covariance

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Abstract

In this paper, we propose a new test for checking the parametric form of the conditional variance based on distance covariance in nonlinear and nonparametric regression models. Inherit from the nice properties of distance covariance, our test is very easy to implement in practice and less effected by the dimensionality of covariates. The asymptotic properties of the test statistic are investigated under the null and alternative hypotheses. We show that the proposed test is consistent against any alternative and can detect local alternatives converging to the null hypothesis at the parametric rate $1/\sqrt{n}$ in both the nonlinear and nonparametric settings. As the limiting null distribution of the test statistic is intractable, we propose a residual bootstrap to approximate the limiting null distribution. Simulation studies are presented to assess the finite sample performance of the proposed test. We also apply the proposed test to a real data set for illustration.

Key words: Distance covariance, heteroscedasticity, nonparametric regression models, residual bootstrap.

1 Introduction

Regression models usually assume homoscedastic error which usually makes the statistical inference substantially simplified in many scenarios. While real data from applications often admit some heteroscedastic structures. Efficient statistical inference in homoscedastic cases may fail to work for models with heteroscedastic structure. Thus it is importance to test heteroscedasticity in regression models. Consider the heteroscedastic regression model

$$Y = m(X) + \sigma(X)\varepsilon, \tag{1.1}$$

where (Y, X) is a random vector with real-valued response variable Y and p-dimensional predictor vector X, $m(x) = E(Y|X = x)$ is the regression function, the error term ε is independent of X with $E(\varepsilon) = 0$ and $Var(\varepsilon) = 1$, and $\sigma^2(X) = var(Y|X)$ is the unknown conditional variance function.

There exist tremendous works in the literature on testing heteroscedasticity for model (1.1). Early tests in this field usually utilized parametric methods to construct test statistics, see Bickel (1978), Breusch and Pagan (1979), White (1980), Cook and Weisberg (1983), among many others. All parametric tests are based on residuals obtained after fitting a model with a specified conditional variance function. Thus these tests may lose power when the parametric form of the variance function is misspecified. Later works for testing heteroscedasticity considered more robust nonparametric tests. Examples include Dette and Munk (1998), Zhu, Fujikoshi, and Naito (2001), Dette (2002), Zheng (2009), Su and Ullah (2013), Guo et al. (2020), Tan et al. (2021), and Xu and Cao (2021).

Some authors further considered a more general problem of checking the parametric form of the conditional variance function in regression models. Wang and Zhou (2007) proposed a nonparametric test based on kernel method for accessing the adequacy of a given parametric variance function. Samarakoon and Song (2011) developed a test for the parametric form of the variance function based on the minimized L_2 distance between a nonparametric variance function estimator and the parametric variance function estimator. Samarakoon and Song (2012) further considered a nonparametric empirical smoothing lack-of-fit test for checking the adequacy of a given parametric variance structure. These methods use local smoothing estimation to construct the test statistics which are usually called local smoothing tests. This type of tests can only detect the local alternatives that depart from the null at the rate $1/\sqrt{nh^{p/2}}$. This rate can be really slow for large p which causes the power of these tests dropped very quickly. Here n is the sample size and h is the bandwidth in the nonparametric estimation. Another method for testing the parametric form of the variance usually constructed the test statistics based on empirical processes. Dette et al.(2007) constructed a Kolmogorov-Smirnov and a Cramér-von Mises test based on the difference between the empirical distributions of residuals under the null and the alternatives. Koul and Song (2010) proposed a consistent test for checking the parametric form of the conditional variance based on the Khmaladze martingale transformation of a marked empirical process of calibrated squared residuals. This type of tests are functionals of the averages of empirical processes of residuals which is usually called global smoothing test as averaging is a globally smoothing step. They usually can detect the local alternatives departing from the null at the parametric rate $1/\sqrt{n}$. Although this convergence rate is not related to the dimension p, global smoothing tests also suffers severely from the dimensionality problem in practice due to the data sparseness in multidimensional spaces.

In this paper we proposed a new test for checking the adequacy of the parametric form of the conditional variance in the nonlinear and nonparametric regression models. Our method is based on a measure of dependence between the covariate X and the residual obtained after fitting a parametric variance function. One of the most popular measure of dependence in statistical community is the approach of distance covariance (dCov) proposed by Székely et al. (2007). The distance covariance admits some nice properties: (i) it is dimension free in the sense that the dimensions of random vectors can be arbitrary; (ii) it is nonnegative and is zero only if the random vectors are independent; (iii) it has a closed form expression and is very easy to implement in practice. Thus we utilize the distance covariance in this paper to construct the test statistic. Sen and Sen(2014) and Xu and He (2021) also adopted this method to construct goodness-of-fit tests for linear regression models. We investigate the asymptotic properties of the test statistic under the null and the alternative hypotheses. Interestingly, our test can detect the local alternatives distinct from the null at the parametric rate $1/\sqrt{n}$ in both the nonlinear and nonparametric regression models. Further, inherit from the nice properties of distance covariance, our test is easy to compute in practice and is less effected by the dimensionality of the covariate. Since the limiting null distribution of the proposed test is rather complicated, we propose a residual bootstrap to approximate the limiting null distribution. The validity of the residual bootstrap is also investigated.

The rest of this paper is organized as follows. In Section 2, we give a short review of the distance covariance and then construct the test statistic. In Section 3, we investigate the asymptotic properties of the test statistic under the null and alternative hypotheses in nonlinear and nonparametric regression models. In Section 4, a residual bootstrap is proposed to approximate the limiting null distribution and its validity is also established in this section. In Section 5, we study the finite-sample performance of our tests by simulations and a real data analysis. Section 6 contains a discussion. All technical proofs are included in the Appendix.

2 Test statistic construction

Let $\mathcal{M} = \{\sigma^2(\cdot, \theta) : \theta \in \Theta \subset \mathbb{R}^d\}$ be a given parametric family of functions. We are interested in testing whether the conditional variance $\sigma^2(X)$ in (1.1) belongs to $\mathcal{M}$ or not. Thus, the null hypothesis can be restated as

$$H_0 : \sigma^2(X) = \sigma^2(X, \theta_0), \quad \text{for some } \theta_0 \in \Theta \subset \mathbb{R}^d,$$

whereas the alternative hypothesis is

$$H_1 : \sigma^2(X) \neq \sigma^2(X, \theta), \quad \forall\, \theta \in \Theta.$$

To illustrate our methodology, consider a random variable $\eta = \frac{Y - m(X)}{\sigma(X, \tilde{\theta}_0)}$, where $\tilde{\theta}_0$ is defined as the same way in (2.3) of Dette et al. (2007), i.e.,

$$\tilde{\theta}_0 = arg \min_{\theta \in \Theta} E[(Y - m(X))^2 - \sigma^2(X, \theta)]^2 = arg \min_{\theta \in \Theta} E[\sigma^2(X) - \sigma^2(X, \theta)]^2 \tag{2.1}$$

Under the null H_0, it follows from Assumption 4(a) in Section 3 that $\tilde{\theta}_0 = \theta_0$ and $\eta = \varepsilon$. Note that H_0 is tantamount to $\eta \perp\!\!\!\perp X$, where $\perp\!\!\!\perp$ denotes the statistical independence. Then we can construct the test statistic by any criterion that measures the dependence of η and X. Székely et al. (2007) proposed the distance covariance and distance correlation to test and measure dependence between two random vectors. As the measure of distance covariance is dimension free in the sense that the dimensions of random vectors can be arbitrary and very easy to implement in practice, it becomes very popular in the statistical community. Thus we in this paper construct the test statistic based on the measure of distance covariance.

First we give a short review of the method of distance covariance. Let $Z \in \mathbb{R}^p$ and $W \in \mathbb{R}^q$ be two random vectors. According to Székely et al. (2007), if $E(\|Z\| + \|W\|) < \infty$, then $dCov^2(Z, W) = 0$ if and only if Z and W are independent, where $dCov^2(Z, W)$ is defined by

$$dCov^2(Z, W) = \frac{1}{c_p c_q} \int_{\mathbb{R}^{p+q}} \frac{|f_{Z,W}(t, s) - f_Z(t) f_W(s)|^2}{||t||^{1+p} ||s||^{1+q}} dtds, \tag{2.2}$$

$f_{Z,W}$ is the joint characteristic function of Z and W, $f_Z(t)$ and $f_W(s)$ are respectively the characteristic functions of Z and W, $c_p = \pi^{\frac{1+p}{2}} / \Gamma(\frac{1+p}{2})$, $\Gamma(\cdot)$ is the gamma function, and $\| \cdot \|$ is the Euclidean norm. Székely and Rizzo (2009) further obtained an analytic form of $dCov^2(Z, W)$:

$$dCov^2(Z, W) = E[U(Z, Z')V(W, W')], \tag{2.3}$$

where

$$U(Z, Z') = \|Z - Z'\| - E(\|Z - Z'\| | Z) - E(\|Z - Z'\| | Z') + E(\|Z - Z'\|),$$

$$V(W,W') = \|W-W'\| - E(\|W-W'\||W) - E(\|W-W'\||W') + E(\|W-W'\|)$$

and (Z',W') is an independent copy of (Z,W). Let $(Z_i,W_i), i=1,...,n$ be an independent and identically distributed sample from (Z,W). Székely and Rizzo (2014) proposed an unbiased estimator of the distance covariance $dCov^2(Z,W)$:

$$d\hat{C}ov_n^2(Z,W) = \frac{1}{n(n-3)} \sum_{1\le i\ne j\le n} Z_{ij}W_{ij}, \tag{2.4}$$

where

$$Z_{ij} = \|Z_i - Z_j\| - \tfrac{1}{n-2}\textstyle\sum_{k=1}^n \|Z_i - Z_k\| - \tfrac{1}{n-2}\sum_{l=1}^n \|Z_j - Z_l\| + \tfrac{1}{(n-1)(n-2)}\sum_{k,l=1}^n \|Z_k - Z_l\|$$
$$W_{ij} = \|W_i - W_j\| - \tfrac{1}{n-2}\textstyle\sum_{k=1}^n \|W_i - W_k\| - \tfrac{1}{n-2}\sum_{l=1}^n |W_j - W_l| + \tfrac{1}{(n-1)(n-2)}\sum_{k,l=1}^n \|W_k - W_l\|.$$

Now we give the construction of the test statistic. Suppose that $\{(X_i,\eta_i), i=1,...,n\}$ is an independent and identically distributed sample from the distribution of (X,η). Recall that the null hypothesis H_0 is equivalent to $dCov^2(X,\eta)=0$, then we can construct the test statistic based on $d\hat{C}ov_n^2(X,\eta)$. Set

$$A_{ij} = \|X_i - X_j\| - \frac{1}{n-2}\sum_{k=1}^n \|X_i - X_k\| - \frac{1}{n-2}\sum_{l=1}^n \|X_j - X_l\| + \frac{1}{(n-1)(n-2)}\sum_{k,l=1}^n \|X_k - X_l\|$$
$$B_{ij} = |\eta_i - \eta_j| - \frac{1}{n-2}\sum_{k=1}^n |\eta_i - \eta_k| - \frac{1}{n-2}\sum_{l=1}^n |\eta_j - \eta_l| + \frac{1}{(n-1)(n-2)}\sum_{k,l=1}^n |\eta_k - \eta_l|.$$

According to the assertion (2.4), an unbiased estimator of $dCov^2(X,\eta)$ is

$$d\hat{C}ov_n^2(X,\eta) = \frac{1}{n(n-3)} \sum_{1\le i\ne j\le n} A_{ij}B_{ij} \tag{2.5}$$

Note that $d\hat{C}ov_n^2(X,\eta)$ involves the unknown regression function $m(\cdot)$ and parameter $\tilde{\theta}_0$. They should be substituted by their empirical analogues. Thus, our final test statistic is

$$\hat{U}_n = \frac{1}{n(n-3)} \sum_{1\le i\ne j\le n} A_{ij}\hat{B}_{ij} \tag{2.6}$$

where $\hat{B}_{ij}$ is defined in the same way as B_{ij} except $\eta_i = \frac{Y_i - m(X_i)}{\sigma(X_i,\tilde{\theta}_0)}$ being replaced by its estimator $\hat{\eta}_i = \frac{Y_i - \hat{m}(X_i)}{\sigma(X_i,\hat{\theta}_n)}$, where $\hat{m}(\cdot)$ is a consistent estimator of the regression function $m(\cdot)$ and $\hat{\theta}_n$ is the nonlinear least squares estimation of $\tilde{\theta}_0$, that is,

$$\hat{\theta}_n = arg\min_{\theta\in\Theta} \sum_{i=1}^n [(Y_i - \hat{m}(X_i))^2 - \sigma^2(X_i,\theta)]^2. \tag{2.7}$$

3 Main results

3.1 Asymptotic properties in nonlinear regressions

In this subsection, we consider the nonlinear regression model $Y = m(X, \beta_0) + \sigma(X)\varepsilon$ in (1.1), where $m(X, \beta_0)$ is a given function with unknown parameter β_0. Let $\hat{\beta}_n$ be any $\sqrt{n}$-consistent estimator of β_0. Then our test statistic $\hat{U}_n$ is given in (2.6) with $\hat{\eta}_i$ and $\hat{\theta}_n$ respectively replaced by

$$\hat{\eta}_i = \frac{Y_i - m(X_i, \hat{\beta}_n)}{\sigma(X_i, \hat{\theta}_n)} \quad \text{and} \quad \hat{\theta}_n = arg \min_{\theta \in \Theta} \sum_{i=1}^{n} [(Y_i - m(X_i, \hat{\beta}_n))^2 - \sigma^2(X_i, \theta)]^2.$$

To derive the asymptotic properties of $\hat{U}_n$ in nonlinear settings, we assume the following regularity conditions. Let $F_\varepsilon(\cdot)$ and $f_\varepsilon(\cdot)$ be the cumulative function and the density function of ε, respectively.

Assumption 1. The true parameter β_0 lies in the interior of the compact subset of $\mathbb{R}^p$ and the estimator $\hat{\beta}_n$ of β_0 satisfies

$$\sqrt{n}(\hat{\beta}_n - \beta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(Y_i, X_i, \beta_0) + o_p(1),$$

where $l(\cdot)$ is a vector function such that $E[l(Y, X, \beta_0)] = 0$ and $E[l(Y, X, \beta_0) l(Y, X, \beta_0)^\top]$ exists and is positive definite.

Assumption 2.

(a). The function $m(x, \beta)$ is twice continuously differentiable in β. Let $\dot{m}(x, \beta) = \frac{\partial m(x,\beta)}{\partial \beta}$ and $\ddot{m}(x, \beta) = \frac{\partial^2 m(x,\beta)}{\partial \beta \partial \beta^\top}$. Moreover, $E\|\dot{m}(X, \beta_0)\|^4 < \infty$, $|m(x, \beta)| \leq \kappa(x)$, and $\|\ddot{m}(x, \beta)\| \leq \kappa(x)$ for all $\beta \in U(\beta_0)$, where $U(\beta_0)$ is some neighborhood of β_0 and $\kappa(x)$ is a measurable function such that $E\|\kappa(X)\|^{4+\gamma} < \infty$ for some $\gamma > 0$.

(b). $E(Y^4) < \infty$ and $E(\|X\|^{4+\gamma}) < \infty$.

Assumption 3. Let $\dot{f}_\varepsilon(t) = \frac{df_\varepsilon(t)}{dt}$ be the derivative of the density function $f_\varepsilon(t)$. $\dot{f}_\varepsilon(\cdot)$ satisfies the uniform Hölder continuity condition, i.e., there exist two positive constants M_0 and m_0 such that $|\dot{f}_\varepsilon(t_1) - \dot{f}_\varepsilon(t_2)| \leq M_0 |t_1 - t_2|^{m_0}$ for any t_1, t_2. Furthermore, $\int_{-\infty}^{\infty} f_\varepsilon(t) |\dot{f}_\varepsilon(t)| dt < \infty$.

Assumption 4.

(a). The vector $\tilde{\theta}_0$ lies in the interior of the compact subset Θ in $\mathbb{R}^d$ and is the unique minimizer of (2.1).

(b). $\inf_x \sigma^2(x, \theta) > 0$ for all $\theta \in \Theta$ and $\inf_{\|\theta - \tilde{\theta}_0\| > \delta} E[\sigma^2(X, \theta) - \sigma^2(X, \tilde{\theta}_0)]^2 > 0$ for any $\delta > 0$.

(c). The function $\sigma^2(x, \theta)$ is third order continuously differentiable with respect to θ and x. Set $\dot{\sigma}(x, \theta) = \frac{\partial \sigma(x,\theta)}{\partial \theta}$ and $\ddot{\sigma}(x, \theta) = \frac{\partial^2 \sigma(x,\theta)}{\partial \theta \partial \theta^\top}$, we have $E\|\dot{\sigma}(X, \tilde{\theta}_0)\|^4 < \infty$ and $E\|\sigma(X)\dot{\sigma}(X, \tilde{\theta}_0)\|^4 < \infty$.

(d). $\|\ddot{\sigma}(x, \theta)\| \leq \kappa(x)$ and $\|\sigma(x)\ddot{\sigma}(x, \theta)\| \leq \kappa(x)$ for all $\theta \in U(\tilde{\theta}_0)$, where $U(\tilde{\theta}_0)$ is some neighborhood of $\tilde{\theta}_0$ and $\kappa(x)$ is a measurable function such that $E\|\kappa(X)\|^{4+\gamma} < \infty$ for some $\gamma > 0$.

(e). The matrix $\Sigma = E[\dot{\sigma}^2(X,\tilde{\theta}_0)\dot{\sigma}^2(X,\tilde{\theta}_0)^\top] - E[(\sigma^2(X) - \sigma^2(X,\tilde{\theta}_0))\ddot{\sigma}^2(X,\tilde{\theta}_0)]$ is non-singular.

Assumptions 1, 2 and 4 are commonly used in the literature of testing heteroscedasticity, see Dette et al. (2007) and Zheng (2009) for instance. Assumption 3 is similar to the regularity condition 4 in Xu and Cao (2021) which is used to investigate the convergence rate of the empirical U-process in the decomposition of $\hat{U}_n$.

The next theorem gives the asymptotic properties of $\hat{U}_n$ under the null hypothesis H_0. Its proof will be given in the Appendix. To facilitate the statement, let $Z = (\varepsilon, X)$ and let $F_Z(\cdot)$ be the cumulative distribution function of Z.

Theorem 3.1. *Suppose that Assumptions 1-4 hold. Then under the null H_0, we have in distribution*

$$n\hat{U}_n \longrightarrow \sum_{k=1}^{\infty} \lambda_k(\mathcal{Z}_k^2 - 1) + 4\mathcal{N}^\top \mathcal{P}_1 + 4\mathcal{W}^\top \Sigma^{-1}\mathcal{P}_2 + 8A_\varepsilon \mathcal{W}^\top \Sigma^{-1}\mathcal{P}_3 + 2A_\varepsilon \mathcal{W}^\top \Sigma^{-1} M_2 \Sigma^{-1}\mathcal{W} + Q_\varepsilon \mathcal{N}^\top M_1 \mathcal{N}, \tag{3.1}$$

where $A_\varepsilon = E[\varepsilon F_\varepsilon(\varepsilon)]$, $\Sigma = E[\dot{\sigma}^2(X_i,\theta_0)\dot{\sigma}^2(X_i,\theta_0)^T]$, $Q_\varepsilon = E[f_\varepsilon(\varepsilon)]$, $\mathcal{Z}_1, \mathcal{Z}_2, \cdots$ *are independent standard normal random variables, the eigenvalues* $\{\lambda_k\}_{k=1}^{\infty}$ *are the solutions of the integral equation*

$$\int C_\varepsilon(\varepsilon_i, \varepsilon_j) C_x(X_i, X_j)\phi_k(Z_j) dF_Z(Z_j) = \lambda_k \phi_k(Z_i)$$

with $\{\phi_k(\cdot)\}_{k=1}^{\infty}$ *being orthonormal eigenfunctions and*

$$\begin{aligned} C_\varepsilon(\varepsilon_i, \varepsilon_j) &= |\varepsilon_i - \varepsilon_j| - E(|\varepsilon_i - \varepsilon_j||\varepsilon_i) - E(|\varepsilon_i - \varepsilon_j||\varepsilon_j) + E(|\varepsilon_i - \varepsilon_j|) \\ C_x(X_i, X_j) &= \|X_i - X_j\| - E(\|X_i - X_j\||X_i) - E(\|X_i - X_j\||X_j) + E(\|X_i - X_j\|), \end{aligned}$$

$M_1 = E[\{\frac{\dot{m}(X_1,\beta_0)}{\sigma(X_1,\theta_0)} - \frac{\dot{m}(X_2,\beta_0)}{\sigma(X_2,\theta_0)}\}\{\frac{\dot{m}(X_1,\beta_0)}{\sigma(X_1,\theta_0)} - \frac{\dot{m}(X_2,\beta_0)}{\sigma(X_2,\theta_0)}\}^\top C_x(X_1, X_2)]$, $M_2 = E[\{\frac{\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma(X_1,\tilde{\theta}_0)} + \frac{\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma(X_2,\tilde{\theta}_0)}\}\{\frac{\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma(X_1,\tilde{\theta}_0)} + \frac{\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma(X_2,\tilde{\theta}_0)}\}^T C_x(X_1, X_2)]$, *and* $(\mathcal{Z}_i, \mathcal{N}, \mathcal{W}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) \in \mathbb{R}^{5p+1}$ *are an zero-mean Gaussian random vector. The covariance matrix of* $(\mathcal{Z}_i, \mathcal{N}, \mathcal{W}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ *is rather complicated and postponed to the Appendix.*

Now we discuss the asymptotic properties of $\hat{U}_n$ under the global alternative and the local alternative hypotheses. Consider the local alternative hypotheses converging to null at the parametric rate $1/\sqrt{n}$:

$$H_{1n} : \sigma^2(X) = \sigma^2(X,\theta_0) + \frac{1}{\sqrt{n}} s(X)$$

for some function $s(\cdot)$ with $E[s^2(X)] < \infty$. To derive the asymptotic properties of $\hat{U}_n$ under the local alternatives H_{1n}, we require some further regularity conditions.

Assumption 5. $E\|s(X)\dot{\sigma}(X,\theta_0)\|^4 < \infty$ and $\|s(x)\ddot{\sigma}(x,\theta)\| \le \kappa(x)$ for all $\theta \in U(\theta_0)$, where $U(\theta_0)$ is some neighborhood of θ_0 and $\kappa(x)$ is a measurable function such that $E\|\kappa(X)\|^{4+\gamma} < \infty$ for some small $\gamma > 0$.

The next theorem states the asymptotic properties of $\hat{U}_n$ under various alternative hypotheses. Its proof will be given in the Appendix.

Theorem 3.2. *(1). Suppose that Assumptions 1-5 hold. Then under the local alternative H_{1n}, we have in distribution*

$$
\begin{aligned}
n\hat{U}_n \quad \longrightarrow \quad & \sum_{k=1}^{\infty} \lambda_k(\mathcal{Z}_k^2 - 1) + 4\mathcal{N}^\top \mathcal{P}_1 + 4\mathcal{W}^\top \Sigma^{-1}\mathcal{P}_2 + 8A_\varepsilon \mathcal{W}^\top \Sigma^{-1}\mathcal{P}_3 + 2A_\varepsilon \mathcal{W}^\top \Sigma^{-1} M_2 \Sigma^{-1}\mathcal{W} \\
& + Q_\varepsilon \mathcal{N}^\top M_1 \mathcal{N} + 4E[s(X)\dot{\sigma}^2(X_i,\theta_0)]^\top \Sigma^{-1}\mathcal{P}_2 + 4A_\varepsilon \mathcal{W}^T M_2 \Sigma^{-1} E[s(X)\dot{\sigma}^2(X_i,\theta_0)] \\
& + 8A_\varepsilon E[s(X)\dot{\sigma}^2(X_i,\theta_0)]^\top \Sigma^{-1}\mathcal{P}_3 + 2A_\varepsilon E[s(X)\dot{\sigma}^2(X_i,\theta_0)]^T \Sigma^{-1} M_2 \Sigma^{-1} E[s(X)\dot{\sigma}^2(X_i,\theta_0)]
\end{aligned}
$$

where $\Sigma = E[\dot{\sigma}^2(X_i,\theta_0)\dot{\sigma}^2(X_i,\theta_0)^T]$ and the quantities A_ε, Q_ε, $\lambda_i, \mathcal{Z}_i, \mathcal{N}, \mathcal{W}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and M are defined in Theorem 3.1.
(2). Suppose that Assumptions 1-4 hold. Then under the global alternative H_1, we have in distribution

$$\sqrt{n}[\hat{U}_n - dCov^2(\eta, X)] \longrightarrow N(0, \sigma_1^2),$$

where $\sigma_1^2 = 4var\{\mathcal{G}(\eta, X) + K_1^T l(Y, X, \beta_0) + [\sigma^2(X)\varepsilon^2 - \sigma^2(X,\tilde{\theta}_0)]K_2^T \Sigma^{-1}\dot{\sigma}^2(X,\tilde{\theta}_0)\}$ with

$$
\begin{aligned}
K_1 &= -E[(\frac{\dot{m}(X_1,\beta_0)}{\sigma(X_1,\tilde{\theta}_0)} - \frac{\dot{m}(X_2,\beta_0)}{\sigma(X_2,\tilde{\theta}_0)})I(\eta_1 > \eta_2)C_x(X_1,X_2)], \\
K_2 &= -E[(\frac{\eta_1\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma(X_1,\tilde{\theta}_0)} - \frac{\eta_2\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma(X_2,\tilde{\theta}_0)})I(\eta_1 > \eta_2)C_x(X_1,X_2)], \\
\mathcal{G}(\eta_i, X_i) &= E[C_\eta(\eta_i,\eta_j)C_x(X_i,X_j)|\eta_i, X_i] - dCov^2(\eta, X), \\
C_\eta(\eta_i,\eta_j) &= |\eta_i - \eta_j| - E(|\eta_i - \eta_j||\eta_i) - E(|\eta_i - \eta_j||\eta_j) + E(|\eta_i - \eta_j|),
\end{aligned}
$$

and $C_x(\cdot,\cdot)$ giving in Theorem 3.1.

It follows from Theorem 3.2 that under the global alternative H_1, our test statistic $n\hat{U}_n$ diverges to infinity in probability at the rate $\sqrt{n}$. Furthermore, our test statistic can detect the local alternative distinct from the null at the parametric rate $n^{-1/2}$ in nonlinear settings. This is the fastest convergence rate in hypothesis testing.

3.2 Asymptotic properties in nonparametric regressions

Now we consider a nonparametric regression model $Y = m(X) + \sigma(X)\varepsilon$ in (1.1), where $m(\cdot)$ is the unknown regression function. To construct the test statistic, we need to estimate $m(\cdot)$ by a nonparametric method such as the Nadaraya-Watson estimator. Set

$$\hat{m}(X_i) = \frac{\sum_{j=1,j\neq i}^n K_h(X_i - X_j)Y_j}{\sum_{j=1,j\neq i}^n K_h(X_i - X_j)},$$

where h is the bandwidth, $K_h(\cdot) = K(\cdot/h)/h^p$, and $K(\cdot)$ is a kernel function. Let $\hat{\eta}_i = \frac{Y_i - \hat{m}(X_i)}{\sigma(X_i,\hat{\theta}_n)}$, then the test statistic $\hat{U}_n$ is given in (2.6). To derive the asymptotic behaviors of $\hat{U}_n$ in the nonparametric regression model, we impose some extra regularity conditions. Let $f_X(\cdot)$ be the density function of X.

Assumption 6.

(a) $f_X(\cdot)$ has a compact support $\bar{\Theta}$ and is k-times continuously differentiable on $\bar{\Theta}$. Let $f_X^{(k)}(x)$ be the k-th derivatives of $f_X(x)$. There exists a neighborhood of 0, say $U(0)$, such that $|f_X^{(k)}(x+u) - f_X^{(k)}(x)| \leq c\|u\|$ for all $u \in U(0)$, where c is a positive constant.

(b) The regression function $m(\cdot)$ is k-times continuously differentiable and its k-th derivative is continuous and bounded. Let $m^{(k)}(x)$ be the k-th derivatives of $m(x)$. There exists a neighborhood of 0, say $U(0)$, such that $|m^{(k)}(x+u) - m^{(k)}(x)| \leq c\|u\|$ for all $u \in U(0)$, where c is a positive constant.

(c) The continuous kernel function $K(u)$ is bounded and satisfies $\int K(u)du = 1$, $K(u) = K(-u)$, $\int u_1^{l_1} u_2^{l_2} \cdots u_p^{l_p} K(u)du = 0$ for all $0 < l_1 + \cdots + l_p < k$, and $\int u_1^{l_1} u_2^{l_2} \cdots u_p^{l_p} K(u)du \neq 0$ for all $l_1 + \cdots + l_p = k$.

(d) The bandwidth satisfies that $h \to 0$, $nh^{2k} \to 0$ and $nh^{2p} \to \infty$ as $n \to \infty$.

Assumption 6(a) is typical in the nonparametric estimation which avoids the boundary effect problem. Assumptions 6(b)-(d) are commonly used in the literature of testing heteroscedasticity in nonparametric models, see Zhu et al. (2001) and Zheng (2009) for instance.

The following theorem presents the limit distribution of the test statistics $\hat{U}_n$ under both the null and alternative hypotheses. Its proof will be given in the Appendix.

Theorem 3.3. *(1). Suppose that Assumptions 2(b), 3, 4 and 6 hold. Then under the null H_0, we have in distribution*

$$n\hat{U}_n \to \sum_{k=1}^{\infty} \lambda_k(\mathcal{Z}_k^2 - 1) + 4\mathcal{W}^T\Sigma^{-1}\mathcal{P}_2 + 8A_\varepsilon \mathcal{W}^T\Sigma^{-1}\mathcal{P}_3 + 2A_\varepsilon \mathcal{W}^\top\Sigma^{-1}M_2\Sigma^{-1}\mathcal{W} + 4A_\varepsilon E\|X_1 - X_2\|$$

where $\mathcal{Z}_1, \mathcal{Z}_2, \cdots$ are independent standard normal random variables, the eigenvalues $\{\lambda_q\}_{q=1}^{\infty}$ are the solutions of the integral equation

$$\int H(Z_i, Z_j)\psi_q(Z_j)dF_Z(Z_j) = \lambda_q \psi_q(Z_i)$$

with $Z_i = (\varepsilon_i, X_i)$, and $(\mathcal{Z}_i, \mathcal{W}, \mathcal{P}_2, \mathcal{P}_3) \in \mathbb{R}^{3p+1}$ is a zero-mean Gaussian random vector. Here the kernel function $H(\cdot,\cdot)$ and the covariance matrix of $(\mathcal{Z}_i, \mathcal{W}, \mathcal{P}_2, \mathcal{P}_3)$ are complicated and are postponed in the Appendix.
(2). Suppose that Assumptions 2(b), 3, 4 and 6 hold. Then under the local alternative H_{1n}, we have in distribution

$$\begin{aligned} n\hat{U}_n \longrightarrow & \sum_{i=1}^{\infty} \lambda_i(\mathcal{Z}_i^2 - 1) + 4\mathcal{W}^T\Sigma^{-1}\mathcal{P}_2 + 8A_\varepsilon\mathcal{W}^T\Sigma^{-1}\mathcal{P}_3 + 4E[s(X)\dot{\sigma}^2(X_i,\theta_0)]^T\Sigma^{-1}\mathcal{P}_2 \\ & + 8E[s(X)\dot{\sigma}^2(X_i,\theta_0)]^T\Sigma^{-1}\mathcal{P}_3 + 4A_\varepsilon E(\|X_1 - X_2\|) + 4A_\varepsilon\mathcal{W}^T M_2\Sigma^{-1}E[s(X)\dot{\sigma}^2(X_i,\theta_0)] \\ & + 2A_\varepsilon E[s(X)\dot{\sigma}^2(X_i,\theta_0)]^T\Sigma^{-1}M_2\Sigma^{-1}E[s(X)\dot{\sigma}^2(X_i,\theta_0)] + 2A_\varepsilon\mathcal{W}^\top\Sigma^{-1}M_2\Sigma^{-1}\mathcal{W} \end{aligned}$$

where $\Sigma = E[\dot{\sigma}^2(X_i,\theta_0)\dot{\sigma}^2(X_i,\theta_0)^T]$ and the random vector $(\mathcal{Z}_i, \mathcal{W}, \mathcal{P}_2, \mathcal{P}_3)$ is the same as in (1).
(3). Suppose that Assumptions 2(b), 3, 4 and 6 hold. Then under the global alternative H_1, we have in distribution

$$\sqrt{n}[\hat{U}_n - dCov^2(\eta, X)] \longrightarrow N(0, \sigma_2^2),$$

where $\sigma_2^2 = 4var\{\mathcal{G}(\eta, X) + \mathcal{I}_1(\eta, X) + [\sigma^2(X)\varepsilon^2 - \sigma^2(X, \tilde{\theta}_0)]K_2^T \Sigma^{-1} \dot{\sigma}^2(X, \tilde{\theta}_0)\}$, K_2 and $\mathcal{G}(\eta, X)$ are defined in Theorem 3.2, and $\mathcal{I}_1(\eta, X)$ is rather complicated and will be given in the Appendix.

According to Theorem 3.3, it is readily seen that the test statistic $n\hat{U}_n$ is consistent with asymptotic power 1 under the global alternative H_1 and can detect the local alternative H_{1n} departing from the null at the parametric rate $n^{-1/2}$ in nonparametric regression settings, even though the nonparametric estimation is involved in our test statistic. This convergence rate is in line with the result in Dette et al. (2007) where they proposed a Kolmogorov-Smirnov and a Cramér-von Mises type of test statistic for the parametric form of the variance function based on residual empirical processes in nonparametric regressions. Note that $n\hat{U}_n$ is also a Cramér-von Mises test statistic based on an empirical process of characteristic functions. Thus our test can be also viewed as a global smoothing test.

4 Bootstrap approximation

It follows from Theorems 3.2 and 3.3 that the test statistic $\hat{U}_n$ is not asymptotically distribution-free as its limiting null distributions depend on the unknown data generating process. To determine the critical value for our test, we suggest a residual bootstrap to approximate the limiting null distribution of $n\hat{U}_n$. This method is also used by Wang and Zhou (2007), Sen and Sen (2014), Guo et al. (2020), and Tan et al. (2021). The algorithm of residual bootstrap is as follows.

1. Generate the bootstrap errors $\{\varepsilon_i^*\}_{i=1}^n$ by randomly resampling with replacement from the standardized variables $(\hat{\eta}_i - \bar{\hat{\eta}})/\{n^{-1}\sum_{i=1}^n(\hat{\eta}_i - \bar{\hat{\eta}})^2\}^{1/2}$, $1 \leq i \leq n$, where $\hat{\eta}_i = [Y_i - \hat{m}(X_i)]/\sigma(X_i, \hat{\theta}_n)$ and $\bar{\hat{\eta}} = n^{-1}\sum_{i=1}^n \hat{\eta}_i$.

2. Generate a bootstrap sample according to the model $Y_i^* = \hat{m}(X_i) + \sigma(X_i, \hat{\theta}_n)\varepsilon_i^*$. Let $\hat{m}^*(X_i)$ and $\sigma^2(X_i, \hat{\theta}_n^*)$ be the bootstrap estimators based on $\{(Y_i^*, X_i) : i = 1, \cdots, n\}$ and let $\hat{\varepsilon}_i^* = [Y_i^* - \hat{m}^*(X_i)]/\sigma(X_i, \hat{\theta}_n^*)$.

3. Define the bootstrap version of the test statistic U_n^* based on $(X_1, \hat{\varepsilon}_1^*), \cdots, (X_n, \hat{\varepsilon}_n^*)$:

$$\hat{U}_n^* = \frac{1}{n(n-3)} \sum_{1 \leq i \neq j \leq n} A_{ij} \hat{B}_{ij}^*,$$

where $\hat{B}_{ij}^*$ is defined in the same way as B_{ij} with η_i replacing by $\hat{\varepsilon}_i^*$.

4. Repeat the above steps a large number of times, say B times. The critical value for a given significant level α is determined by the upper α quantile of the bootstrap distribution $\{nU_{n,i}^* : i = 1, \cdots, B\}$ of the test statistic.

Here $\hat{m}(X_i) = m(X_i, \hat{\beta}_n)$ and $\hat{m}^*(X_i) = m(X_i, \hat{\beta}_n^*)$ in the nonlinear regression case and $\hat{m}(X_i) = \sum_{j \neq i}^n K_h(X_i - X_j)Y_j / \sum_{j \neq i}^n K_h(X_i - X_j)$ and $\hat{m}^*(X_i) = \sum_{j \neq i}^n K_h(X_i - X_j)Y_j^* / \sum_{j \neq i}^n K_h(X_i - X_j)$ in the nonparametric regression case. In the next theorem, we establish the asymptotic validity of the residual bootstrap approximation for our test statistic. Its proof will be given in the Appendix.

Theorem 4.1. *Suppose that Assumptions 1-6 hold.*
(1). Under the null H_0 and the local alternative H_{1n}, $n\hat{U}_n^*$ given $\{(Y_i, X_i) : i = 1, \cdots, n\}$ converges

in distribution to the limiting null distribution of $n\hat{U}_n$ in probability.
(2). Under the global alternative H_1, $n\hat{U}_n^*$ given $\{(Y_i, X_i) : i = 1, \cdots, n\}$ converges in distribution to the limiting null distribution of $n\hat{U}_n$ in probability, except for $Z = (\varepsilon, X)$ replacing by $Z = (\eta, X)$.

5 Numerical studies

5.1 Simulations

In this subsection, we conduct detailed simulation studies to investigate the finite sample performance of the proposed test statistics $\hat{U}_n$. We also make a comparison between our test, the Dette et al. (2007) test T_n^{CvM}, and the Wang and Zhou (2007) test T_n^{WZ} in both the nonlinear and nonparametric regression settings. The Cramér-von Mises (CvM) test statistic of Dette et al. (2007) is defined as

$$T_n^{CvM} = n \int [\hat{F}_{\hat{\varepsilon}}(y) - \hat{F}_{\hat{\eta}}(y)]^2 d\hat{F}_{\hat{\varepsilon}}(y),$$

where $\hat{F}_{\hat{\varepsilon}}(y) = \frac{1}{n}\sum_{i=1}^n I(\hat{\varepsilon}_i \leq y)$ with $\hat{\varepsilon}_i = [Y_i - \hat{m}(X_i)]/\hat{\sigma}(X_i)$, $\hat{F}_{\hat{\eta}}(y) = \frac{1}{n}\sum_{i=1}^n I(\hat{\eta}_i \leq y)$ with $\hat{\eta}_i = [Y_i - \hat{m}(X_i)]/\sigma(X_i, \hat{\theta}_n)$, and $\hat{\sigma}(\cdot)$ is a nonparametric estimator of $\sigma(\cdot)$. The critical value of T_n^{CvM} is determined by the smooth residual bootstrap as suggested by Dette et al. (2007). The test statistic of Wang and Zhou (2007) is

$$T_n^{WZ} = \frac{1}{n(n-1)h^p} \sum_{1 \leq i \neq j \leq n} K(\frac{X_i - X_j}{h})\{[Y_i - \hat{m}(X_i)]^2 - \sigma^2(X_i, \hat{\theta}_n)\}\{[Y_j - \hat{m}(X_j)]^2 - \sigma^2(X_j, \hat{\theta}_n)\},$$

where $K(\cdot)$ is the kernel functions and h is the bandwidth. For the kernel function and the bandwidth selection in the nonparametric estimations of Dette et al. (2007) and Wang and Zhou (2007), one can refer their papers for details.

In the following simulation examples, $a = 0$ and $a \neq 0$ correspond to the null hypothesis and the alternative hypotheses respectively. The significant level is set to be 0.05. The simulation results are based on the average of 1000 replications and the bootstrap approximation of $B = 500$ samples. The sample sizes are 50 and 100 in each model. The dimension p of covariates is set to be $2, 4, 8$ to see how the performance of these tests is affected by the dimensionality.

For our test in nonlinear regression models, we use the least square method to estimate the unknown parameter β_0 in the regression function. In the nonparametric regression case, we adopt the standard normal density function as the kernel function. To assess the effect of bandwidth h for our test $\hat{U}_n$ in nonparametric cases, we conduct a simple simulation for a large bunch of bandwidths. This strategy was also adopted by Zhu, Fujikoshi, and Naito (2001), Wang and Zhou (2007), Khmaladze and Koul (2009), among many others. Let $h = cn^{-1/(p+4)}$ where c varies from 0.6 to 1.4. Figures 1-2 give the empirical sizes and powers of our test in nonparametric settings for model $H_{21} : Y = \beta_0^T X + |1 + (\theta_0^T X)^2 + a\sin(\theta_0^T X)|^{1/2}\varepsilon$. Here $\beta_0 = \theta_0 = (1, 1, ...1)^T/\sqrt{p}$ and $X \sim N(0, I_p)$ independent of the standard normal error term ε. Simulation results for other models in the following are very similar. Thus we omit them for brevity. From these figures, we can see that when the bandwidth is relatively small, our test $\hat{U}_n$ is conservative and have very low empirical powers. While if the bandwidth is too large, our test can not control the significant level either.

Thus, we adopt $h = 1.2n^{-1/(p+4)}$ in nonparametric settings in the following simulation examples.

Figures $1-2$ about here

Example 1. The data are generated from the following regression models:

$$
\begin{aligned}
H_{11}: Y &= \beta_0^T X + (1 + |\theta_0^T X + a\exp(\theta_0^T X)|)^{1/2}\varepsilon \\
H_{12}: Y &= \beta_0^T X + (1 + |\theta_1^T X| + a(\theta_1^T X))^{1/2}\varepsilon,
\end{aligned}
$$

where $X \sim N(0, I_p)$ independent of the standard normal error term ε, $\beta_0 = (1, 1, \cdots, 1)^T/\sqrt{p}$, $\theta_0 = (1, 1, \cdots, 1)^T/\sqrt{p}$, and $\theta_1 = (\underbrace{1, \cdots, 1}_{p/2}, 0, \cdots, 0)^T/\sqrt{p/2}$.

The simulation results are reported in Tables $1-4$. We can see that all these tests can maintain the significance level very well in both the nonlinear and nonparametric regression models when the dimension $p = 2, 4$. While the empirical sizes of these tests become unstable for large dimension ($p = 8$). For the empirical power, it is easy to see that our test works better than the other two competitors in most cases. However, all these tests have bad power performance when the dimension becomes large, especially in nonparametric settings. This may be due to the curse of dimensionality resulting from the nonparametric estimate of the regression function.

Tables $1-4$ about here

In the next simulation, we consider regression models with a slightly more complicated variance functions under the null hypotheses.

Example 2. The data are generated from the following regression models:

$$
\begin{aligned}
H_{21}: Y &= \beta_0^T X + |1 + (\theta_0^T X)^2 + a\sin(\theta_0^T X)|^{1/2}\varepsilon, \\
H_{22}: Y &= \beta_0^T X + (1 + |\sin(\theta_0^T X)| + a\exp(\theta_0^T X))\varepsilon,
\end{aligned}
$$

where the predictor vector X, the error term ε and the parameters β_0 and θ_0 are the same as in Example 1.

The simulation results are presented in Tables 5-8. We can observe that for model H_{21}, all the tests can control the empirical size very well in both the nonlinear and nonparametric cases. The empirical powers of our test for model H_{21} grow very quickly in most cases. While T_n^{CvM} has the worst power performance in this model. For model H_{22}, all the tests can maintain the significant level very well for $p = 2$. When p is relatively large, the empirical sizes of T_n^{CvM} and T_n^{WZ} are slightly far away from the significant level. For the empirical power in model H_{22}, T_n^{CvM} beats the other two competitors and the Wang and Zhou (2007) test T_n^{WZ} performs the worst, especially in nonparametric cases.

Tables $5-8$ about here

5.2 A real data example

In this subsection we apply the proposed test to the Esterase count data set that is obtained from Example 2.2 in Carroll and Ruppert (1988). This data set contains 108 observations obtained from a calibration experiment for measuring the concentration of an enzyme esterase with the radioimmunoassay (RIA) counts as the response variable Y and the concentration of esterase as the predictor variable X. Carroll and Ruppert (1988) first suggested to fit this data set by a linear regression model $Y = \beta_0 + \beta_1 X + \varepsilon$. To see whether there exist heteroscedasticity in this model, we give a scatter plot of $Y - \hat{\beta}_0 - \hat{\beta}_1 X$ against the fitted values $\hat{Y}$ in Figures 3(a), where $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$. This plot clearly indicates the existence of heteroscedasticity. Wang and Zhou (2007) further analysed the Esterase count data set and suggested to use a heteroscedasticity regression model for data fitting:

$$Y_i = \beta_0 + \beta_1 X_i + \sigma(\beta_0 + \beta_1 X_i)^{\theta} \varepsilon_i, \quad i = 1, \cdots, 108. \tag{5.1}$$

When applying our tests to model (5.1), we find that the p-value is about 0.968. This indicates that the model (5.1) may be plausible to fit the Esterase count data set. To further visualize this fit, Figure 3(b) gives the scatter plot of $\hat{\eta}$ against the predictor variable X where $\hat{\eta} = \frac{Y - \hat{\beta}_0 - \hat{\beta}_1 X}{\hat{\sigma}(\hat{\beta}_0 + \hat{\beta}_1 X)^{\hat{\theta}}}$. This plot also shows that there exists no trend between the residual $\hat{\eta}$ and the predictor X. Thus the heteroscedasticity model (5.1) is reasonable for fitting the Esterase count data.

Figure 3 about here

6 Discussion

In this paper we proposed a new test based on distance covariance to check the adequacy of the parametric form of the variance function in nonlinear and nonparametric regression models. Inherit from the nice properties of distance covariance, the proposed test is very easy to implement in practice and is less sensitive to the dimension of covariates. The asymptotic properties of the test statistic is investigated under the null and the alternative hypotheses. It has been shown that the proposed test is consistent against any alternative and can detect the local alternatives distinct from the null at a parametric rate $1/\sqrt{n}$, even in nonparametric regression settings. As the proposed test is not asymptotically distribution-free, we suggested a residual bootstrap to approximate the limiting null distribution of the test statistic. Simulation results show that our test can control the nominal level very well and has good power performance.

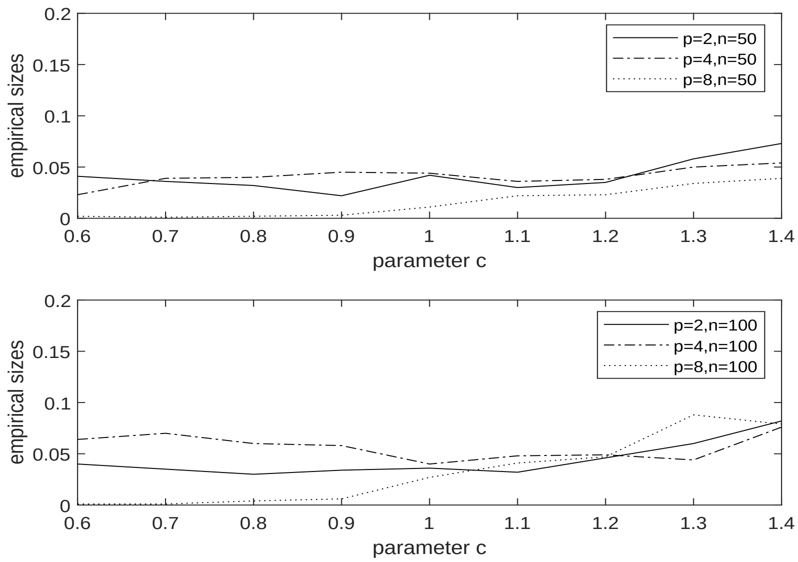

Figure 1: The empirical sizes of $\hat{U}_n$ in nonparametric cases against different bandwidths for model H_{21}.

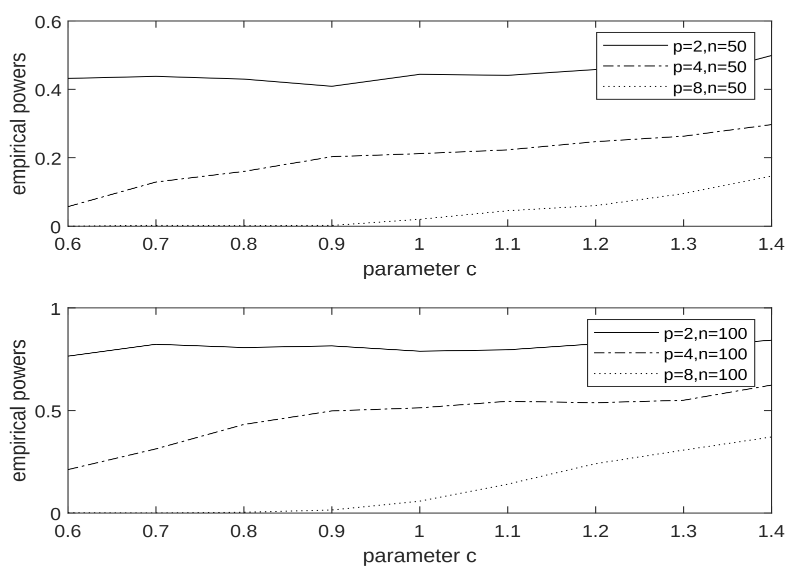

Figure 2: The empirical powers of $\hat{U}_n$ in nonparametric cases against different bandwidths when $a = 2$ for model H_{21}.

Table 1: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{11} in nonlinear cases in Example 1.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{WZ}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p=2$	0.0	0.037	0.040	0.041	0.051	0.049	0.050
	0.5	0.102	0.202	0.095	0.109	0.112	0.216
	1.0	0.210	0.466	0.109	0.164	0.222	0.449
	1.5	0.322	0.629	0.119	0.219	0.290	0.625
	2.0	0.379	0.727	0.168	0.239	0.374	0.698
	2.5	0.452	0.798	0.170	0.255	0.378	0.738
$p=4$	0.0	0.054	0.044	0.058	0.061	0.058	0.052
	0.5	0.091	0.139	0.087	0.120	0.093	0.135
	1.0	0.154	0.314	0.113	0.161	0.115	0.227
	1.5	0.215	0.464	0.120	0.192	0.155	0.347
	2.0	0.255	0.556	0.124	0.191	0.197	0.359
	2.5	0.285	0.652	0.168	0.205	0.205	0.398
$p=8$	0.0	0.046	0.055	0.087	0.095	0.051	0.043
	0.5	0.111	0.126	0.117	0.136	0.073	0.103
	1.0	0.115	0.241	0.133	0.196	0.076	0.093
	1.5	0.155	0.346	0.145	0.206	0.089	0.120
	2.0	0.186	0.402	0.147	0.184	0.082	0.141
	2.5	0.218	0.481	0.134	0.180	0.103	0.160

Table 2: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{11} in nonparametric cases in Example 1.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{WZ}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p=2$	0.0	0.036	0.053	0.039	0.048	0.032	0.044
	0.5	0.085	0.185	0.097	0.119	0.079	0.111
	1.0	0.159	0.435	0.149	0.182	0.119	0.285
	1.5	0.268	0.636	0.183	0.260	0.208	0.378
	2.0	0.327	0.755	0.193	0.318	0.234	0.481
	2.5	0.415	0.843	0.225	0.378	0.287	0.569
$p=4$	0.0	0.041	0.052	0.048	0.045	0.033	0.053
	0.5	0.044	0.093	0.088	0.114	0.053	0.067
	1.0	0.081	0.174	0.134	0.163	0.061	0.111
	1.5	0.084	0.291	0.152	0.222	0.089	0.157
	2.0	0.127	0.375	0.165	0.258	0.099	0.192
	2.5	0.153	0.470	0.153	0.266	0.123	0.227
$p=8$	0.0	0.032	0.066	0.021	0.052	0.026	0.043
	0.5	0.039	0.092	0.051	0.087	0.031	0.039
	1.0	0.055	0.149	0.066	0.122	0.033	0.055
	1.5	0.060	0.185	0.067	0.158	0.038	0.068
	2.0	0.063	0.257	0.084	0.190	0.047	0.066
	2.5	0.080	0.256	0.081	0.195	0.045	0.080

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Table 3: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{12} in nonlinear cases in Example 1.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{WZ}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p=2$	0.0	0.049	0.048	0.049	0.043	0.029	0.049
	0.2	0.050	0.047	0.049	0.059	0.042	0.069
	0.4	0.062	0.091	0.052	0.058	0.045	0.083
	0.4	0.079	0.162	0.048	0.080	0.093	0.128
	0.6	0.135	0.287	0.086	0.088	0.111	0.248
	1.0	0.219	0.474	0.075	0.143	0.153	0.383
$p=4$	0.0	0.043	0.045	0.052	0.051	0.047	0.045
	0.2	0.047	0.052	0.067	0.067	0.059	0.050
	0.4	0.058	0.071	0.055	0.084	0.058	0.069
	0.6	0.089	0.133	0.081	0.081	0.057	0.115
	0.8	0.102	0.242	0.103	0.114	0.085	0.152
	1.0	0.185	0.396	0.120	0.145	0.113	0.203
$p=8$	0.0	0.029	0.050	0.083	0.075	0.055	0.057
	0.2	0.052	0.061	0.100	0.099	0.045	0.053
	0.4	0.073	0.076	0.086	0.096	0.060	0.065
	0.6	0.074	0.120	0.114	0.135	0.077	0.072
	0.8	0.102	0.176	0.117	0.176	0.095	0.083
	1.0	0.140	0.320	0.151	0.211	0.087	0.090

Table 4: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{12} in nonparametric cases in Example 1.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{WZ}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p=2$	0.0	0.053	0.072	0.047	0.051	0.044	0.042
	0.2	0.070	0.098	0.051	0.057	0.040	0.066
	0.4	0.074	0.108	0.053	0.079	0.046	0.079
	0.6	0.101	0.161	0.067	0.097	0.062	0.134
	0.8	0.129	0.230	0.082	0.140	0.093	0.217
	1.0	0.163	0.350	0.086	0.196	0.112	0.329
$p=4$	0.0	0.049	0.063	0.046	0.050	0.040	0.048
	0.2	0.053	0.067	0.058	0.072	0.042	0.039
	0.4	0.053	0.075	0.046	0.070	0.045	0.057
	0.6	0.070	0.106	0.058	0.098	0.046	0.075
	0.8	0.103	0.128	0.077	0.125	0.062	0.093
	1.0	0.105	0.200	0.126	0.161	0.062	0.134
$p=8$	0.0	0.047	0.077	0.015	0.048	0.018	0.039
	0.2	0.044	0.059	0.018	0.033	0.023	0.029
	0.4	0.038	0.080	0.013	0.039	0.034	0.035
	0.6	0.047	0.075	0.025	0.051	0.025	0.038
	0.8	0.043	0.072	0.033	0.078	0.034	0.040
	1.0	0.054	0.085	0.045	0.116	0.033	0.061

Table 5: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{21} in nonlinear cases in Example 2.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{ZH}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p=2$	0.0	0.052	0.047	0.042	0.053	0.037	0.051
	0.5	0.053	0.090	0.053	0.047	0.064	0.067
	1.0	0.079	0.163	0.065	0.065	0.087	0.164
	1.5	0.176	0.409	0.066	0.084	0.164	0.339
	2.0	0.422	0.777	0.098	0.132	0.289	0.617
	2.5	0.666	0.960	0.140	0.205	0.400	0.749
$p=4$	0.0	0.045	0.040	0.065	0.048	0.048	0.043
	0.5	0.052	0.065	0.049	0.047	0.060	0.053
	1.0	0.061	0.124	0.057	0.073	0.074	0.083
	1.5	0.129	0.320	0.083	0.086	0.114	0.158
	2.0	0.301	0.622	0.101	0.109	0.130	0.269
	2.5	0.480	0.903	0.115	0.119	0.190	0.369
$p=8$	0.0	0.050	0.048	0.077	0.065	0.057	0.044
	0.5	0.061	0.058	0.076	0.078	0.074	0.050
	1.0	0.077	0.084	0.097	0.107	0.068	0.065
	1.5	0.115	0.201	0.123	0.137	0.096	0.103
	2.0	0.191	0.434	0.145	0.146	0.091	0.127
	2.5	0.386	0.729	0.162	0.157	0.137	0.167

Table 6: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{21} in nonparametric cases in Example 2.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{ZH}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p=2$	0.0	0.032	0.048	0.043	0.056	0.035	0.050
	0.5	0.046	0.079	0.054	0.063	0.046	0.076
	1.0	0.103	0.194	0.046	0.077	0.065	0.180
	1.5	0.209	0.453	0.079	0.132	0.117	0.413
	2.0	0.440	0.805	0.171	0.320	0.222	0.652
	2.5	0.711	0.971	0.302	0.632	0.312	0.789
$p=4$	0.0	0.034	0.042	0.042	0.044	0.037	0.045
	0.5	0.042	0.053	0.031	0.041	0.040	0.055
	1.0	0.060	0.104	0.051	0.057	0.043	0.083
	1.5	0.109	0.243	0.075	0.101	0.056	0.124
	2.0	0.252	0.524	0.115	0.167	0.100	0.231
	2.5	0.443	0.808	0.199	0.310	0.132	0.326
$p=8$	0.0	0.023	0.059	0.025	0.023	0.029	0.038
	0.5	0.041	0.065	0.028	0.033	0.034	0.025
	1.0	0.046	0.082	0.035	0.037	0.029	0.031
	1.5	0.049	0.129	0.038	0.087	0.043	0.051
	2.0	0.073	0.226	0.088	0.110	0.051	0.077
	2.5	0.129	0.420	0.151	0.218	0.072	0.085

Table 7: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{22} in nonlinear cases in Example 2.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{ZH}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p = 2$	0.0	0.047	0.044	0.047	0.046	0.056	0.053
	0.5	0.319	0.632	0.778	0.970	0.334	0.733
	1.0	0.601	0.901	0.993	1.000	0.610	0.843
	1.5	0.754	0.988	0.999	1.000	0.692	0.839
	2.0	0.860	0.993	1.000	1.000	0.679	0.810
	2.5	0.890	0.998	1.000	1.000	0.685	0.799
$p = 4$	0.0	0.062	0.043	0.114	0.165	0.094	0.133
	0.5	0.237	0.471	0.669	0.950	0.066	0.316
	1.0	0.426	0.789	0.986	1.000	0.232	0.572
	1.5	0.586	0.928	1.000	1.000	0.315	0.607
	2.0	0.676	0.974	1.000	1.000	0.342	0.587
	2.5	0.728	0.981	1.000	1.000	0.348	0.549
$p = 8$	0.0	0.059	0.043	0.104	0.166	0.077	0.112
	0.5	0.108	0.278	0.541	0.940	0.019	0.081
	1.0	0.222	0.596	0.972	1.000	0.051	0.223
	1.5	0.287	0.732	0.999	1.000	0.114	0.344
	2.0	0.337	0.835	1.000	1.000	0.171	0.388
	2.5	0.401	0.888	1.000	1.000	0.207	0.406

Table 8: Empirical sizes and powers of $\hat{U}_n$, T_n^{CvM}, and T_n^{WZ} for H_{22} in nonparametric cases in Example 2.

	a	$\hat{U}_n$		T_n^{CvM}		T_n^{ZH}	
		n=50	n=100	n=50	n=100	n=50	n=100
$p = 2$	0.0	0.044	0.055	0.047	0.065	0.040	0.045
	0.5	0.169	0.405	0.551	0.927	0.227	0.601
	1.0	0.362	0.751	0.964	1.000	0.500	0.762
	1.5	0.527	0.905	0.998	1.000	0.623	0.797
	2.0	0.632	0.958	1.000	1.000	0.676	0.800
	2.5	0.698	0.979	1.000	1.000	0.666	0.790
$p = 4$	0.0	0.052	0.056	0.084	0.097	0.070	0.098
	0.5	0.085	0.217	0.312	0.815	0.016	0.170
	1.0	0.194	0.465	0.893	1.000	0.089	0.425
	1.5	0.257	0.685	0.997	1.000	0.176	0.518
	2.0	0.317	0.775	1.000	1.000	0.261	0.565
	2.5	0.404	0.832	1.000	1.000	0.336	0.623
$p = 8$	0.0	0.020	0.038	0.032	0.073	0.053	0.093
	0.5	0.053	0.114	0.023	0.202	0.004	0.009
	1.0	0.062	0.196	0.401	0.970	0.009	0.047
	1.5	0.077	0.273	0.827	0.998	0.029	0.095
	2.0	0.077	0.355	0.967	1.000	0.062	0.177
	2.5	0.125	0.384	0.996	1.000	0.121	0.259

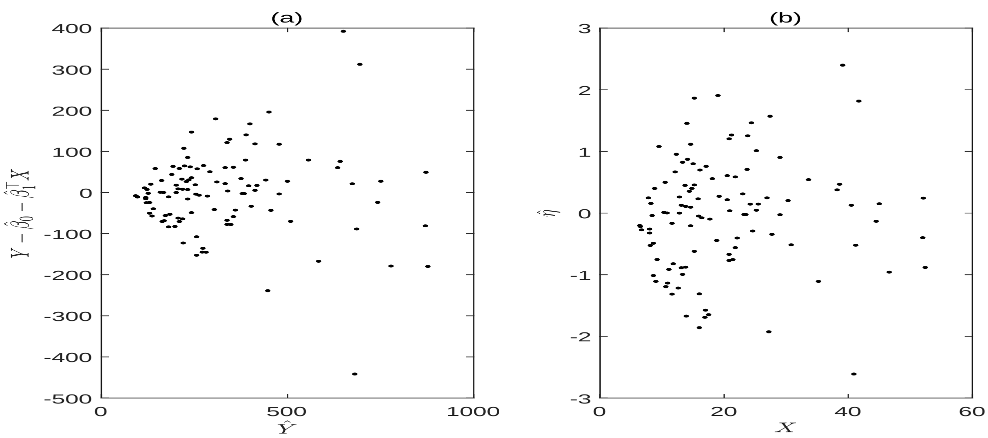

Figure 3: (a) is the scatter plot of $Y - \hat{\beta}_0 - \hat{\beta}_1 X$ against the fitted values $\hat{Y}$ and (b) is the scatter plot of $\hat{\eta}$ against the predictor variable X for the Esterase count data set.

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Appendix

The Appendix contains the proofs of the theoretical results stated in the main context.

Proof of Theorem 3.1. Recall that $\eta_i = \frac{Y_i - m(X_i, \beta_0)}{\sigma(X_i, \tilde{\theta}_0)}$ and $\hat{\eta}_i = \frac{Y_i - m(X_i, \hat{\beta}_n)}{\sigma(X_i, \hat{\theta}_n)}$ in nonlinear cases. It follows that

$$\hat{\eta}_i - \eta_i = -\frac{m(X_i, \hat{\beta}_n) - m(X_i, \beta_0)}{\sigma(X_i, \tilde{\theta}_0)} - \frac{\varepsilon_i(\sigma(X_i, \hat{\theta}_n) - \sigma(X_i, \tilde{\theta}_0))}{\sigma(X_i, \tilde{\theta}_0)} + R_i, \tag{6.1}$$

where

$$
\begin{aligned}
R_i =& \frac{\varepsilon_i[\sigma(X_i,\tilde{\theta}_0)-\sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\tilde{\theta}_0)\sigma(X_i,\hat{\theta}_n)} + \frac{m(X_i,\beta_0)-m(X_i,\hat{\beta}_n)}{\sigma^2(X_i,\tilde{\theta}_0)}[\sigma(X_i,\tilde{\theta}_0)-\sigma(X_i,\hat{\theta}_n)] \\
&+ \frac{m(X_i,\beta_0)-m(X_i,\hat{\beta}_n)}{\sigma^2(X_i,\tilde{\theta}_0)}\frac{[\sigma(X_i,\tilde{\theta}_0)-\sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\hat{\theta}_n)}.
\end{aligned}
$$

According to the identity in the proof of Theorem 1 in Knight (1998), if $x \neq 0$, then we have

$$
|x-y|-|x| = -y\{\mathbb{I}(x>0)-\mathbb{I}(x<0)\} + 2\int_0^y \{\mathbb{I}(x \le s)-\mathbb{I}(x \le 0)\}ds. \tag{6.2}
$$

Here $\mathbb{I}(A)$ is the indicator function of the set A. Note that $\eta_i = \varepsilon_i$ under the null H_0. It follows from (6.1) and (6.2) that

$$
\begin{aligned}
&|\hat{\eta}_i - \hat{\eta}_j| \\
&= |\varepsilon_i - \varepsilon_j| - \{\frac{m(X_i,\hat{\beta}_n)-m(X_i,\beta_0)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{m(X_j,\hat{\beta}_n)-m(X_j,\beta_0)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_i[\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\tilde{\theta}_0)]}{\sigma(X_i,\tilde{\theta}_0)} \\
&- \frac{\varepsilon_j(\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\tilde{\theta}_0))}{\sigma(X_j,\tilde{\theta}_0)} + (R_i-R_j)\}[\mathbb{I}(\varepsilon_i > \varepsilon_j)-\mathbb{I}(\varepsilon_i < \varepsilon_j)] \qquad (6.3) \\
&+ 2\int_0^{\frac{m(X_i,\hat{\beta}_n)-m(X_i,\beta_0)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{m(X_i,\hat{\beta}_n)-m(X_j,\beta_0)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_i[\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\tilde{\theta}_0)]}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\varepsilon_j[\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\tilde{\theta}_0)]}{\sigma(X_j,\tilde{\theta}_0)} + R_i - R_j} \\
&[\mathbb{I}(\varepsilon_s - \varepsilon_t \le z) - \mathbb{I}(\varepsilon_s \le \varepsilon_t)]dz.
\end{aligned}
$$

Recall that $\hat{U}_n = \frac{1}{n(n-3)}\sum_{1\le i\ne j\le n} A_{ij}\hat{B}_{ij}$ with

$$
A_{ij} = \|X_i - X_j\| - \frac{1}{n-2}\sum_{k=1}^n \|X_i - X_k\| - \frac{1}{n-2}\sum_{l=1}^n \|X_j - X_l\| + \frac{1}{(n-1)(n-2)}\sum_{k,l=1}^n \|X_k - X_l\|
$$

$$
\hat{B}_{ij} = |\hat{\eta}_i - \hat{\eta}_j| - \frac{1}{n-2}\sum_{k=1}^n |\hat{\eta}_i - \hat{\eta}_k| - \frac{1}{n-2}\sum_{l=1}^n |\hat{\eta}_j - \hat{\eta}_l| + \frac{1}{(n-1)(n-2)}\sum_{k,l=1}^n |\hat{\eta}_k - \hat{\eta}_l|.
$$

By Lemma 1 of Yao et al. (2018), we can rewrite $\hat{U}_n$ as

$$
\hat{U}_n = \frac{1}{C_n^4}\sum_{i<j<k<l} \tilde{h}_0(\hat{Z}_i,\hat{Z}_j,\hat{Z}_k,\hat{Z}_l),
$$

where

$$
\tilde{h}_0(\hat{Z}_i,\hat{Z}_j,\hat{Z}_k,\hat{Z}_l) = \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)} |\hat{\eta}_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)} |\hat{\eta}_{st}|\|X_{su}\|, \tag{6.4}
$$

$\hat{Z}_i = (\hat{\eta}_i, X_i)$, $X_{st} = X_s - X_t$, and $\hat{\eta}_{st} = \hat{\eta}_s - \hat{\eta}_t$. Here the summation in (6.4) is over all permutations of the 4-tuples of indices (i,j,k,l). Combining (6.3) and (6.4), we have

$$
\hat{U}_n \quad = \quad \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)} |\varepsilon_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)} |\varepsilon_{st}|\|X_{su}\|\right)
$$

$$
\begin{aligned}
& +\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{1st}(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{1st}\|X_{su}\|\right)\\
& +\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{2st}(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{2st}\|X_{su}\|\right)\\
=: \;& \hat{U}_{n0}+\hat{U}_{n1}+\hat{U}_{n2}, \qquad (6.5)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{1st} &= -[\frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)}+\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)}\\
&\quad -\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}+(R_s-R_t)]\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}\\
\delta_{2st} &= 2\int_0^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}+\frac{\varepsilon_s\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)}+R_s-R_t}\\
&\quad \{\mathbb{I}(\varepsilon_s-\varepsilon_t\le z)-\mathbb{I}(\varepsilon_s\le\varepsilon_t)\}dz.
\end{aligned}
$$

First we deal with $\hat{U}_{n1}$. By Taylor's expansion, we have

$$
\begin{aligned}
& \frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}\\
= \;& \frac{(\hat{\beta}_n-\beta_0)^T\dot{m}(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{(\hat{\beta}_n-\beta_0)^T\dot{m}(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}\\
& +2^{-1}(\hat{\beta}_n-\beta_0)^T[\frac{\ddot{m}(X_s,(\hat{\beta}_n-\beta_0)\zeta+\beta_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\ddot{m}(X_t,(\hat{\beta}_n-\beta_0)\zeta+\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}](\hat{\beta}_n-\beta_0),
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}\\
= \;& \frac{(\hat{\theta}_n-\tilde{\theta}_0)^T\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{(\hat{\theta}_n-\tilde{\theta}_0)^T\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}\\
& +2^{-1}(\hat{\theta}_n-\tilde{\theta}_0)^T[\frac{\ddot{\sigma}(X_s,(\hat{\theta}_n-\tilde{\theta}_0)\zeta+\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\ddot{\sigma}(X_t,(\hat{\theta}_n-\tilde{\theta}_0)\zeta+\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}](\hat{\theta}_n-\tilde{\theta}_0),
\end{aligned}
$$

where $\zeta\in(0,1)$. Then we have

$$
\begin{aligned}
\hat{U}_{n1} &= (\hat{\beta}_n-\beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{11}(Z_i,Z_j,Z_k,Z_l)\\
&\quad +2^{-1}(\hat{\beta}_n-\beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{12}(Z_i,Z_j,Z_k,Z_l)(\hat{\beta}_n-\beta_0)\\
&\quad +2^{-1}(\hat{\beta}_n-\beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{13}(Z_i,Z_j,Z_k,Z_l)(\hat{\beta}_n-\beta_0)
\end{aligned}
$$

$$
\begin{aligned}
& +(\hat{\theta}_n - \tilde{\theta}_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{14}(Z_i, Z_j, Z_k, Z_l) \\
& +2^{-1}(\hat{\theta}_n - \tilde{\theta}_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{15}(Z_i, Z_j, Z_k, Z_l)(\hat{\theta}_n - \tilde{\theta}_0) \\
& +2^{-1}(\hat{\theta}_n - \tilde{\theta}_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{16}(Z_i, Z_j, Z_k, Z_l)(\hat{\theta}_n - \tilde{\theta}_0) \\
& +\frac{1}{C_n^4} \sum_{i<j<k<l} h_{17}(Z_i, Z_j, Z_k, Z_l) \\
=: \; & I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16} + I_{17},
\end{aligned} \tag{6.6}
$$

where $Z_i = (\varepsilon_i, X_i)$,

$$
\begin{aligned}
& h_{1m}(Z_i, Z_j, Z_k, Z_l) \\
= \; & -6^{-1} \sum_{s<t,u<v}^{(i,j,k,l)} \delta_{1mst}\{\mathbb{I}(\varepsilon_s > \varepsilon_t) - \mathbb{I}(\varepsilon_s < \varepsilon_t)\}(\|X_{st}\| + \|X_{uv}\|) \\
& +12^{-1} \sum_{(s,t,u)}^{(i,j,k,l)} \delta_{1mst}\{\mathbb{I}(\varepsilon_s > \varepsilon_t) - \mathbb{I}(\varepsilon_s < \varepsilon_t)\}\|X_{su}\|, \quad \text{for } m = 1, \cdots, 7
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{11st} &= \frac{\dot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\dot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)} \\
\delta_{12st} &= \frac{\ddot{m}(X_s, (\hat{\beta}_n - \beta_0)\zeta + \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\ddot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\beta}_0)} - [\frac{\ddot{m}(X_t, (\hat{\beta}_n - \beta_0)\zeta + \beta_0)}{\sigma(X_t, \tilde{\theta}_0)} - \frac{\ddot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)}] \\
\delta_{13st} &= \frac{\ddot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\ddot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)} \\
\delta_{14st} &= \frac{\varepsilon_s \dot{\sigma}(X_s, \tilde{\theta}_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_t \dot{\sigma}(X_t, \tilde{\theta}_0)}{\sigma(X_t, \tilde{\theta}_0)} \\
\delta_{15st} &= \frac{\varepsilon_s \ddot{\sigma}(X_s, (\hat{\theta}_n - \tilde{\theta}_0)\zeta + \theta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_s \ddot{\sigma}(X_s, \tilde{\theta}_0)}{\sigma(X_s, \tilde{\theta}_0)} - [\frac{\varepsilon_t \ddot{\sigma}(X_t, (\hat{\theta}_n - \tilde{\theta}_0)\zeta + \tilde{\theta}_0)}{\sigma(X_t, \tilde{\theta}_0)} - \frac{\varepsilon_t \ddot{\sigma}(X_t, \tilde{\theta}_0)}{\sigma(X_t, \tilde{\theta}_0)}] \\
\delta_{16st} &= \frac{\varepsilon_s \ddot{\sigma}(X_s, \tilde{\theta}_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_t \ddot{\sigma}(X_t, \tilde{\theta}_0)}{\sigma(X_t, \tilde{\theta}_0)} \\
\delta_{17st} &= R_s - R_t.
\end{aligned}
$$

For the term I_{12}, recall that

$$
I_{12} = 2^{-1}(\hat{\beta}_n - \beta_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{12}(Z_i, Z_j, Z_k, Z_l)(\hat{\beta}_n - \beta_0).
$$

Next we will show that $\frac{1}{C_n^4} \sum_{i<j<k<l} h_{12}(Z_i, Z_j, Z_k, Z_l) = o_p(1)$. For this, set $\mathcal{B} = \{\beta : \sqrt{n}\|\beta - \beta_0\| \leq$

$C\}$, $U_{12}(\beta) = \frac{1}{C_n^4} \sum\limits_{i<j<k<l} h_{12}(\beta, Z_i, Z_j, Z_k, Z_l)$, and

$$\begin{aligned} h_{12}(\beta, Z_i, Z_j, Z_k, Z_l) &= -6^{-1} \sum_{s<t,u<v}^{(i,j,k,l)} [\frac{\ddot{m}(X_s, \beta) - \ddot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\ddot{m}(X_t, \beta) - \ddot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)}] \\ & \times [\mathbb{I}(\varepsilon_s > \varepsilon_t) - \mathbb{I}(\varepsilon_s < \varepsilon_t)](\|X_{st}\| + \|X_{uv}\|) \\ & +12^{-1} \sum_{(s,t,u)}^{(i,j,k,l)} [\frac{\ddot{m}(X_s, \beta) - \ddot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\ddot{m}(X_t, \beta) - \ddot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)}] \\ & \times [\mathbb{I}(\varepsilon_s > \varepsilon_t) - \mathbb{I}(\varepsilon_s < \varepsilon_t)] \|X_{su}\|. \end{aligned}$$

It is easy to see that

$$\frac{1}{C_n^4} \sum_{i<j<k<l} h_{12}(Z_i, Z_j, Z_k, Z_l) = U_{12}((\hat{\beta}_n - \beta_0)\zeta + \beta_0).$$

To show that $\frac{1}{C_n^4} \sum\limits_{i<j<k<l} h_{12}(Z_i, Z_j, Z_k, Z_l) = o_p(1)$, since $\sqrt{n}(\hat{\beta}_n - \beta_0) = O_p(1)$, it remains to show that $U_{12}(\beta) = o_p(1)$ uniformly in $\beta \in \mathcal{B}$. By Assumption 2, we have

$$\|\frac{\ddot{m}(X_s, \beta) - \ddot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\ddot{m}(X_t, \beta) - \ddot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)}\| \leq C[\kappa_1(X_s) + \kappa_1(X_t)],$$

whence

$$\begin{aligned} & \|h_{12}(\beta, Z_i, Z_j, Z_k, Z_l)\| \leq C(\kappa_1(X_i) + \kappa_1(X_j) + \kappa_1(X_k) + \kappa_1(X_l))(\|X_i\| + \|X_j\| + \|X_k\| + \|X_l\|). \\ & E[\|h_{12}(\beta, Z_i, Z_j, Z_k, Z_l)\|^2] \leq CE\{[\kappa_1^2(X) + \kappa_1^2(X)]E\|X\|^2\} < \infty \end{aligned}$$

Note that $\{U_{12}(\beta) : \beta \in \mathcal{B}\}$ is a non-degenerate U-process of order 4. Together with lemma 2.13 of Pakes and Pollard (1989), we obtain that $\{U_{12}(\beta) : \beta \in \mathcal{B}\}$ is Euclidean for a squared-integrable envelope. Then it follows from the main corollary and corollary 4 of Sherman (1994) that $U_{12}(\beta) = o_p(1)$ uniformly in $\beta \in \mathcal{B}$. By Slutsky's theorem, we obtain that

$$nI_{12} = \frac{n}{2}(\hat{\beta}_n - \beta_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{12}(Z_i, Z_j, Z_k, Z_l)(\hat{\beta}_n - \beta_0) = o_p(1).$$

For the term I_{13}, it is readily seen that $\frac{1}{C_n^4} \sum\limits_{i<j<k<l} h_{13}(Z_i, Z_j, Z_k, Z_l)$ is non-degenerate U-statistic of order 4. By the independence between ε and X, we have

$$E\{h_{13}(Z_i, Z_j, Z_k, Z_l)\} = E[(\frac{\ddot{m}(X_1, \beta_0)}{\sigma(X_1, \tilde{\theta}_0)} - \frac{\ddot{m}(X_2, \beta_0)}{\sigma(X_2, \tilde{\theta}_0)})(\mathbb{I}(\varepsilon_1 > \varepsilon_2) - \mathbb{I}(\varepsilon_1 < \varepsilon_2))C_x(X_1, X_2)] = 0,$$

where $C_x(X_1, X_2) = \|X_1 - X_2\| - E(\|X_1 - X_2\||X_1) - E(\|X_1 - X_2\||X_2) + E(\|X_1 - X_2\|)$. By the law of large numbers for U-statistics, we obtain that

$$\frac{1}{C_n^4} \sum_{i<j<k<l} h_{13}(Z_i, Z_j, Z_k, Z_l) = o_p(1).$$

Combining this with $\sqrt{n}(\hat{\beta}_n - \beta_0) = O_p(1)$, we have

$$nI_{13} = n(\hat{\beta}_n - \beta_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{13}(Z_i, Z_j, Z_k, Z_l)(\hat{\beta}_n - \beta_0) = o_p(1).$$

Similarly to the arguments for I_{12} and I_{13}, we can show that

$$nI_{15} = o_p(1), \quad nI_{16} = o_p(1), \quad \text{and} \quad nI_{17} = o_p(1).$$

Consequently, we obtain that

$$n\hat{U}_{n1} = n(\hat{\beta}_n - \beta_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{11}(Z_i, Z_j, Z_k, Z_l) + n(\hat{\theta}_n - \tilde{\theta}_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h_{14}(Z_i, Z_j, Z_k, Z_l) + o_p(1). \tag{6.7}$$

Next we deal with the term $\hat{U}_{n2}$. Decompose it as

$$\begin{aligned} \hat{U}_{n2} &= \frac{1}{C_n^4} \sum_{i<j<k<l} h_{21}(Z_i, Z_j, Z_k, Z_l) + \frac{1}{C_n^4} \sum_{i<j<k<l} h_{22}(Z_i, Z_j, Z_k, Z_l) \\ &=: \hat{U}_{n21} + \hat{U}_{n22}, \end{aligned}$$

where

$$\begin{aligned} h_{21}(Z_i, Z_j, Z_k, Z_l) &= \frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} E(\delta_{2st}|X_s, X_t)(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} E(\delta_{2st}|X_s, X_t)\|X_{su}\|, \\ h_{22}(Z_i, Z_j, Z_k, Z_l) &= \frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} [\delta_{2st} - E(\delta_{2st}|X_s, X_t)](\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} [\delta_{2st} - E(\delta_{2st}|X_s, X_t)]\|X_{su}\|. \end{aligned}$$

For the term $\hat{U}_{n21}$, similar to the arguments in Theorem 1 of Xu and Cao (2021), we can show that uniformly over $1 \le s, t \le n$,

$$\begin{aligned} & E(\delta_{2st}|X_s, X_t) \\ = \;& 2E\{\int_0^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(x_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)} + R_s - R_t} \\ & [\mathbb{I}(\varepsilon_s - \varepsilon_t \le z) - \mathbb{I}(\varepsilon_s \le \varepsilon_t)]dz|X_s, X_t\} \\ = \;& 2E[\int_0^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}} \mathbb{I}(\varepsilon_s - \varepsilon_t \le z) - \mathbb{I}(\varepsilon_s - \varepsilon_t \le 0)dz|X_s, X_t] \\ & +2E[\int_{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)}}^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)}} \\ & [\mathbb{I}(\varepsilon_s - \varepsilon_t \le z) - \mathbb{I}(\varepsilon_s - \varepsilon_t \le 0)]dz|X_s, X_t] \\ & +2E[\int_{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)}}^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)} + R_s - R_t} \\ & [\mathbb{I}(\varepsilon_s - \varepsilon_t \le z) - \mathbb{I}(\varepsilon_s - \varepsilon_t \le 0)]dz|X_s, X_t] \end{aligned}$$

$$
\begin{aligned}
&= Q_\varepsilon\left([\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\tilde\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\tilde\theta_0)}]^T(\hat\beta_n-\beta_0)\right)^2+2A_\varepsilon(\hat\theta_n-\tilde\theta_0)^T[\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}]\\
&\quad +2A_\varepsilon\left([\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}]^T(\hat\theta_n-\tilde\theta_0)\right)^2+o_p(\frac{1}{n}),
\end{aligned}
$$

where $Q_\varepsilon=E[f_\varepsilon(\varepsilon)]$ and $A_\varepsilon=E[\varepsilon F_\varepsilon(\varepsilon)]$. Consequently, we obtain that

$$
\begin{aligned}
\hat U_{n21} &= Q_\varepsilon(\hat\beta_n-\beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{211}(Z_i,Z_j,Z_k,Z_l)(\hat\beta_n-\beta_0)\\
&\quad +2A_\varepsilon(\hat\theta_n-\tilde\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{212}(Z_i,Z_j,Z_k,Z_l)\\
&\quad +2A_\varepsilon(\hat\theta_n-\tilde\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{213}(Z_i,Z_j,Z_k,Z_l)(\hat\theta_n-\tilde\theta_0)+o_p(\frac{1}{n}),\\
&=: Q_\varepsilon\hat U_{n211}+2A_\varepsilon\hat U_{n212}+2A_\varepsilon\hat U_{n213}+o_p(\frac{1}{n}),
\end{aligned}
$$

where

$$
\begin{aligned}
& h_{211}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\tilde\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\tilde\theta_0)}\}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\tilde\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\tilde\theta_0)}\}^T(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\tilde\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\tilde\theta_0)}\}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\tilde\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\tilde\theta_0)}\}^T\|X_{su}\|,\\
& h_{212}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}\}(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}\}\|X_{su}\|,\\
& h_{213}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}\}\{\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}\}^T(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}\}\{\frac{\dot\sigma(X_s,\tilde\theta_0)}{\sigma(X_s,\tilde\theta_0)}+\frac{\dot\sigma(X_t,\tilde\theta_0)}{\sigma(X_t,\tilde\theta_0)}\}^T\|X_{su}\|,
\end{aligned}
$$

For the term $\hat U_{n211}$, it is easy to see that $\frac{1}{C_n^4}\sum_{i<j<k<l}h_{211}(Z_i,Z_j,Z_k,Z_l)$ is a non-degenerate U-statistic. By the law of large numbers, we have

$$
\frac{1}{C_n^4}\sum_{i<j<k<l}h_{211}(Z_i,Z_j,Z_k,Z_l)\longrightarrow M_1,\quad \text{in probability},
$$

where

$$M_1 = E[\{\frac{\dot m(X_1,\beta_0)}{\sigma(X_1,\tilde\theta_0)} - \frac{\dot m(X_2,\beta_0)}{\sigma(X_2,\tilde\theta_0)}\}\{\frac{\dot m(X_1,\beta_0)}{\sigma(X_1,\tilde\theta_0)} - \frac{\dot m(X_2,\beta_0)}{\sigma(X_2,\tilde\theta_0)}\}^T C_x(X_1,X_2)],$$

where $C_x(X_1,X_2) = \|X_1-X_2\| - E(\|X_1-X_2\||X_1) - E(\|X_1-X_2\||X_2) + E(\|X_1-X_2\|)$. Consequently, we obtain that

$$n\hat U_{n211} = \sqrt{n}(\hat\beta_n-\beta_0)^T M_1 \sqrt{n}(\hat\beta_n-\beta_0) + o_p(1). \tag{6.8}$$

For the term $\hat U_{n212}$, by the standard theory of U-statistics (see Subsection 5.3.1 in Serfling (2009),for instance), we have

$$\begin{aligned}
n\hat U_{n212} &= \sqrt{n}(\hat\theta_n-\tilde\theta_0)\sqrt{n}\frac{1}{C_n^4}\sum_{i<j<k<l} h_{212}(Z_i,Z_j,Z_k,Z_l)\\
&= \sqrt{n}(\hat\theta_n-\tilde\theta_0)\{\frac{4}{\sqrt n}\sum_{i=1}^n E[h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i] + o_p(1)\}\\
&= \sqrt{n}(\hat\theta_n-\tilde\theta_0)\frac{1}{\sqrt n}\sum_{i=1}^n 4E[\{\frac{\dot\sigma(X_i,\beta_0)}{\sigma(X_i,\tilde\theta_0)} + \frac{\dot\sigma(X,\beta_0)}{\sigma(X,\tilde\theta_0)}\}C_x(X_i,X)|X_i] + o_p(1). \qquad (6.9)
\end{aligned}$$

For the term $\hat U_{n213}$, similar to the arguments for $\hat U_{n211}$, we have

$$\frac{1}{C_n^4}\sum_{i<j<k<l} h_{213}(Z_i,Z_j,Z_k,Z_l) \longrightarrow M_2, \quad \text{in probability},$$

where

$$M_2 = E[\{\frac{\dot\sigma(X_1,\tilde\theta_0)}{\sigma(X_1,\tilde\theta_0)} + \frac{\dot\sigma(X_2,\tilde\theta_0)}{\sigma(X_2,\tilde\theta_0)}\}\{\frac{\dot\sigma(X_1,\tilde\theta_0)}{\sigma(X_1,\tilde\theta_0)} + \frac{\dot\sigma(X_2,\tilde\theta_0)}{\sigma(X_2,\tilde\theta_0)}\}^T C_x(X_1,X_2)].$$

Consequently,

$$n\hat U_{n213} = \sqrt{n}(\hat\theta_n-\tilde\theta_0)^T M_2 \sqrt{n}(\hat\theta_n-\tilde\theta_0) + o_p(1). \tag{6.10}$$

Thus we obtain that

$$\begin{aligned}
n\hat U_{n21} &= Q_\varepsilon\sqrt{n}(\hat\beta_n-\beta_0)^T M_1\sqrt{n}(\hat\beta_n-\beta_0) + 2A_\varepsilon\sqrt{n}(\hat\theta_n-\tilde\theta_0)^T M_2\sqrt{n}(\hat\theta_n-\tilde\theta_0)\\
&\quad + 2A_\varepsilon\sqrt{n}(\hat\theta_n-\tilde\theta_0)\frac{1}{\sqrt n}\sum_{i=1}^n 4E[\{\frac{\dot\sigma(X_i,\beta_0)}{\sigma(X_i,\tilde\theta_0)} + \frac{\dot\sigma(X,\beta_0)}{\sigma(X,\tilde\theta_0)}\}C_x(X_i,X)|X_i] + o_p(1).
\end{aligned}$$

For the term $\hat U_{n22}$, following the same line as Theorem 1 inXu and Cao (2021), we can show that $n\hat U_{n22} = o_p(\frac{1}{n})$. Altogether we obtain that

$$\begin{aligned}
n\hat U_n &= n\hat U_{n0} + n(\hat\beta_n-\beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{11}(Z_i,Z_j,Z_k,Z_l)\\
&\quad + n(\hat\theta_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{14}(Z_i,Z_j,Z_k,Z_l)\\
&\quad + 2A_\varepsilon\sqrt{n}(\hat\theta_n-\tilde\theta_0)\frac{1}{\sqrt n}\sum_{i=1}^n 4E[h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i]\\
&\quad + Q_\varepsilon\sqrt{n}(\hat\beta_n-\beta_0)^T M_1\sqrt{n}(\hat\beta_n-\beta_0) + 2A_\varepsilon\sqrt{n}(\hat\theta_n-\tilde\theta_0)^T M_2\sqrt{n}(\hat\theta_n-\tilde\theta_0) + o_p(1),
\end{aligned} \tag{6.11}$$

where $E[h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i] = E\{[\frac{\dot\sigma(X_i,\beta_0)}{\sigma(X_i,\tilde\theta_0)} + \frac{\dot\sigma(X,\beta_0)}{\sigma(X,\tilde\theta_0)}]C_x(X_i,X)|X_i\}$. For the term $\hat U_{n0}$, recall that

$$\begin{aligned}\hat U_{n0} &= \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\varepsilon_{st}|(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\varepsilon_{st}|\|X_{su}\|\right)\\ &=: \frac{1}{C_n^4}\sum_{i<j<k<l} h_0(Z_i,Z_j,Z_k,Z_l).\end{aligned}$$

It is easy to verify that U_0 is degenerate. By Hoeffding decomposition in the technical appendix of Yao et al. (2018), we can show that

$$E[h_0(Z_i,Z_j,Z_k,Z_l)|Z_i,Z_j] = \frac{1}{6}C_\varepsilon(\varepsilon_i,\varepsilon_j)C_x(X_i,X_j),$$

where

$$\begin{aligned}C_\varepsilon(\varepsilon_i,\varepsilon_j) &= |\varepsilon_i-\varepsilon_j| - E(|\varepsilon_i-\varepsilon_j\|\varepsilon_i) - E(|\varepsilon_i-\varepsilon_j\|\varepsilon_j) + E(|\varepsilon_i-\varepsilon_j|)\\ C_x(X_i,X_j) &= \|X_i-X_j\| - E(\|X_i-X_j\||X_i) - E(\|X_i-X_j\||X_j) + E(\|X_i-X_j\|).\end{aligned}$$

It follows from the arguments in Section 5.3 of Serfling (2009) that

$$\begin{aligned}n\hat U_{n0} &= \frac{6}{n-1}\sum_{i=1}^n\sum_{j\neq i}^n E\{h_0(Z_i,Z_j,Z_k,Z_l)|Z_i,Z_j\} + o_p(1)\\ &= \frac{1}{n-1}\sum_{i=1}^n\sum_{j\neq i}^n C_\varepsilon(\varepsilon_i,\varepsilon_j)C_x(X_i,X_j) + o_p(1).\end{aligned}$$

For the second and third terms in (6.11), set

$$\begin{aligned}\hat U_{n11} &= \frac{1}{C_n^4}\sum_{i<j<k<l} h_{11}(Z_i,Z_j,Z_k,Z_l)\\ \hat U_{n14} &= \frac{1}{C_n^4}\sum_{i<j<k<l} h_{14}(Z_i,Z_j,Z_k,Z_l).\end{aligned}$$

By some elementary calculations, we can show that $\hat U_{n11}$ and $\hat U_{n14}$ are non-degenerate and

$$\begin{aligned}E[h_{11}(Z_1,Z_2,Z_3,Z_4)|Z_1] &= -(F_\varepsilon(\varepsilon_1)-1/2)E[\{\frac{\dot m(X_1,\beta_0)}{\sigma(X_1,\tilde\theta_0)} - \frac{\dot m(X_2,\beta_0)}{\sigma(X_2,\tilde\theta_0)}\}C_x(X_1,X_2)|X_1]\\ E[h_{14}(Z_1,Z_2,Z_3,Z_4)|Z_1] &= -\frac{1}{2}E\{[\frac{\varepsilon_1\dot\sigma(x_1,\tilde\theta_0)}{\sigma(X_1,\tilde\theta_0)} - \frac{\varepsilon_2\dot\sigma(X_2,\tilde\theta_0)}{\sigma(X_2,\tilde\theta_0)}][\mathbb{I}(\varepsilon_1>\varepsilon_2) - \mathbb{I}(\varepsilon_1<\varepsilon_2)]C_x(X_1,X_2)|Z_1\},\end{aligned}$$

where $Z_i = (X_i,\varepsilon_i)$. By the standard theory of U-statistics (see Section 5.3 in Serfling (2009), for instance), we have

$$\begin{aligned}\sqrt{n}U_{n11} &= \frac{4}{\sqrt{n}}\sum_{i=1}^n E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i\} + o_p(1)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n 4[1/2-F_\varepsilon(\varepsilon_i)]E[\{\frac{\dot m(X_i,\beta_0)}{\sigma(X_i,\tilde\theta_0)} - \frac{\dot m(X,\beta_0)}{\sigma(X,\tilde\theta_0)}\}C_x(X_i,X)|X_i] + o_p(1)\end{aligned}$$

$$
\begin{aligned}
\sqrt{n}U_{14} &= \frac{4}{\sqrt{n}}\sum_{i=1}^{n} E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\} + o_p(1)\\
&= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n} 2E[\{\frac{\varepsilon_i\dot{\sigma}(X_i,\tilde{\theta}_0)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\varepsilon\dot{\sigma}(X,\tilde{\theta}_0)}{\sigma(X,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon)-\mathbb{I}(\varepsilon_i<\varepsilon)\}C_x(X_i,X)|Z_i] + o_p(1).
\end{aligned}
$$

To obtain the limiting distribution of $n\hat{U}_n$, it remains to derive the asymptotic expansion of $\hat{\beta}_n-\beta_0$ and $\hat{\theta}_n-\tilde{\theta}_0$. By Assumption 1 and Proposition 3 of Tan et al. (2022), we have

$$
\begin{aligned}
\sqrt{n}(\hat{\beta}_n-\beta_0) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} l(Y_i,X_i,\beta_0) + o_p(1)\\
\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i)\varepsilon_i^2-\sigma^2(X_i,\tilde{\theta}_0)]\Sigma^{-1}\dot{\sigma}^2(X_i,\tilde{\theta}_0) + o_p(1).
\end{aligned}
$$

Altogether we obtain that

$$
\begin{aligned}
& n\hat{U}_n\\
= \; & \frac{6}{n-1}\sum_{i=1}^{n}\sum_{j\neq i}^{n} E\{h_0(Z_i,Z_j,Z_k,Z_l)|Z_i,Z_j\}\\
& +4\frac{1}{\sqrt{n}}\sum_{i=1}^{n} l(Y_i,X_i,\beta_0)^T\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E[h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i]\\
& +4\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde{\theta}_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\tilde{\theta}_0)^T\Sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E[h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i]\\
& +8A_{\varepsilon}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde{\theta}_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\tilde{\theta}_0)^T\Sigma^{-1}E[h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i]\\
& +2A_{\varepsilon}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde{\theta}_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\tilde{\theta}_0)^T\Sigma^{-1}M_2\Sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde{\theta}_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\tilde{\theta}_0)\\
& +Q_{\varepsilon}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} l(Y_i,X_i,\beta_0)^T M_1\frac{1}{\sqrt{n}}\sum_{i=1}^{n} l(Y_i,X_i,\beta_0) + o_p(1).
\end{aligned}
$$

Since $E[h_0(Z_i,Z_j,Z_k,Z_l)]^2 \le CE\|X\|^2E(\varepsilon^2) < \infty$, it follows that

$$
n\hat{U}_n \longrightarrow \sum_{k=1}^{\infty}\lambda_k(\mathcal{Z}_k^2-1)+4\mathcal{N}^\top\mathcal{P}_1+4\mathcal{W}^\top\Sigma^{-1}\mathcal{P}_2+8A_{\varepsilon}\mathcal{W}^\top\Sigma^{-1}\mathcal{P}_3+2A_{\varepsilon}\mathcal{W}^\top\Sigma^{-1}M_2\Sigma^{-1}\mathcal{W}+Q_{\varepsilon}\mathcal{N}^\top M_1\mathcal{N},
$$

where $\mathcal{Z}_1,\mathcal{Z}_2,\cdots$ are independent standard normal random variables, the eigenvalues $\{\lambda_q\}_{q=1}^{\infty}$ are the solutions of the integral equation

$$
\int C_{\varepsilon}(\varepsilon_i,\varepsilon_j)C_x(X_i,X_j)\phi_q(Z_j)dF(Z_j) = \lambda_q\phi_q(Z_i),
$$

$\{\phi_i(\cdot)\}_{i=1}^{\infty}$ are orthonormal eigenfunctions and $F(\cdot)$ is the cumulative distribution function of of Z, and $(\mathcal{Z}_i,\mathcal{N},\mathcal{W},\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)\in\mathbb{R}^{5p+1}$ are jointly Gaussian random variables with zero-mean and the covariance matrix satisfying

$$
var(Z_i) \;=\; 1,\; var(\mathcal{N}) = var(l(Y_i,X_i,\beta_0))
$$

$$
\begin{aligned}
var(\mathcal{P}_1) &= var(E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
var(\mathcal{P}_2) &= var(E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
var(\mathcal{P}_3) &= var(E\{h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
var(\mathcal{W}) &= var([\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\theta_0))\\
cov(Z_i,\mathcal{P}_1) &= cov(\phi_i(Z_i),E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(Z_i,\mathcal{P}_2) &= cov(\phi_i(Z_i),E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(Z_i,\mathcal{P}_3) &= cov(\phi_i(Z_i),E\{h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(Z_i,\mathcal{N}) &= cov(\phi_i(Z_i),l(Y_i,X_i,\beta_0))\\
cov(Z_i,\mathcal{W}) &= cov(\phi_i(Z_i),[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\theta_0))\\
cov(\mathcal{P}_1,\mathcal{N}) &= cov(l(Y_i,X_i,\beta_0),E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i\}\\
cov(\mathcal{P}_2,\mathcal{N}) &= cov(l(Y_i,X_i,\beta_0),E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\}\\
cov(\mathcal{P}_3,\mathcal{N}) &= cov(l(Y_i,X_i,\beta_0),E\{h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i\}\\
cov(\mathcal{W},\mathcal{N}) &= cov([\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\theta_0),l(Y_i,X_i,\beta_0))\\
cov(\mathcal{P}_1,\mathcal{P}_2) &= cov(E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i,E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{P}_1,\mathcal{P}_3) &= cov(E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i,E\{h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{P}_3,\mathcal{P}_2) &= cov(E\{h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i\},E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{W},\mathcal{P}_1) &= cov([\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\theta_0),E\{h_{11}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{W},\mathcal{P}_2) &= cov([\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\theta_0),E\{h_{14}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{W},\mathcal{P}_3) &= cov([\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\theta_0),E\{h_{212}(Z_i,Z_j,Z_k,Z_l)|Z_i\}).
\end{aligned}
$$

Hence we complete the proof of Theorem 3.1. □

Proof of Theorem 3.2. (1) Under the local alternatives H_{1n}, recall that $\sigma^2(X)=\sigma^2(X,\theta_0)+\frac{1}{\sqrt{n}}s(X)$ and $\hat{\eta}_i=\frac{Y_i-m(X_i,\hat{\beta}_n)}{\sigma(X_i,\hat{\theta}_n)}$ in nonlinear cases. It follows from (6.2) in the proof of Theorem 3.1 that

$$
\begin{aligned}
& |\hat{\eta}_i-\hat{\eta}_j|\\
= & |\varepsilon_i-\varepsilon_j|\\
& -\{P_{ij}+\frac{\varepsilon_i s(X_i)[\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\theta_0)]}{2\sqrt{n}\sigma^3(X_i,\theta_0)}-\frac{\varepsilon_j s(X_j)[\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\theta_0)]}{2\sqrt{n}\sigma^3(X_j,\theta_0)}\}[\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)]\\
& +2\int_0^{P_{ij}+\frac{\varepsilon_i s(X_i)[\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\theta_0)]}{2\sqrt{n}\sigma^3(X_i,\theta_0)}-\frac{\varepsilon_j s(X_j)[\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\theta_0)]}{2\sqrt{n}\sigma^3(X_j,\theta_0)}}\{\mathbb{I}(\varepsilon_i-\varepsilon_j\le z)-\mathbb{I}(\varepsilon_i\le\varepsilon_j)\}dz, \qquad (6.12)
\end{aligned}
$$

where

$$
\begin{aligned}
P_{ij} &= \frac{m(X_i,\hat{\beta}_n)-m(X_i,\beta_0)}{\sigma(X_i,\theta_0)}-\frac{m(X_j,\hat{\beta}_n)-m(X_j,\beta_0)}{\sigma(X_j,\theta_0)}+\frac{\varepsilon_i(\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\theta_0))}{\sigma(X_i,\theta_0)}\\
&\quad -\frac{\varepsilon_j(\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\theta_0))}{\sigma(X_j,\theta_0)}+(R_i-R_j),\\
R_i &= \frac{\varepsilon_i[\sigma(X_i,\theta_0)-\sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\tilde{\theta}_0)\sigma(X_i,\hat{\theta}_n)}+\frac{m(X_i,\beta_0)-m(X_i,\hat{\beta}_n)}{\sigma^2(X_i,\theta_0)}[\sigma(X_i,\theta_0)-\sigma(X_i,\hat{\theta}_n)]\\
&\quad +\frac{m(X_i,\beta_0)-m(X_i,\hat{\beta}_n)}{\sigma^2(X_i,\theta_0)}\frac{[\sigma(X_i,\theta_0)-\sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\hat{\theta}_n)}.
\end{aligned}
$$

Similar to the arguments for (6.5) in the proof of Theorem 3.1, $\hat{U}_n$ can be decomposed as

$$
\begin{aligned}
\hat{U}_n &= \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\varepsilon_{st}|(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\varepsilon_{st}|\|X_{su}\|\right)\\
&\quad+\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{1st}(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{1st}\|X_{su}\|\right)\\
&\quad+\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{2st}(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{2st}\|X_{su}\|\right)\\
&\quad+\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{3st}(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{3st}\|X_{su}\|\right)\\
&=: \hat{U}_{n0}+\hat{U}_{n1}+\hat{U}_{n2}+\hat{U}_{n3}, \qquad (6.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{1st} &= -\Big[\frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{\sigma(X_s,\theta_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{\sigma(X_t,\theta_0)}+\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}\\
&\quad-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}+(R_s-R_t)\Big]\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}\\
\delta_{2st} &= -\Big\{\frac{\varepsilon_s s(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{2\sqrt{n}\sigma^3(X_s,\theta_0)}-\frac{\varepsilon_t s(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{2\sqrt{n}\sigma^3(X_t,\theta_0)}\Big\}[\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)].\\
\delta_{3st} &= 2\int_0^{P_{st}+\frac{\varepsilon_s s(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{2\sqrt{n}\sigma^3(X_s,\theta_0)}-\frac{\varepsilon_t s(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{2\sqrt{n}\sigma^3(X_t,\theta_0)}}[\mathbb{I}(\varepsilon_s-\varepsilon_t\le z)-\mathbb{I}(\varepsilon_s\le\varepsilon_t)]dz.
\end{aligned}
$$

First we deal with the term $\hat{U}_{n2}$. By Taylor expansion, $\hat{U}_{n2}$ can be decomposed as

$$
\begin{aligned}
\hat{U}_{n2} &= \frac{1}{\sqrt{n}}(\hat{\theta}_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{21}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+\frac{1}{2\sqrt{n}}(\hat{\theta}_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{22}(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\theta_0)\\
&\quad+\frac{1}{2\sqrt{n}}(\hat{\theta}_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{23}(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\theta_0)\\
&=: I_{21}+I_{22}+I_{23},
\end{aligned}
$$

where $Z_i=(\varepsilon_i,X_i)$,

$$
\begin{aligned}
h_{2m}(Z_i,Z_j,Z_k,Z_l) &= -6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{2mst}\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}(\|X_{st}\|+\|X_{uv}\|)\\
&\quad+12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{2mst}\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}\|X_{su}\|, \quad \text{for } m=1,2,3
\end{aligned}
$$

and

$$\begin{aligned}
\delta_{21st} &= \frac{\varepsilon_s s(X_s)\dot{\sigma}(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)} - \frac{\varepsilon_t s(X_t)\dot{\sigma}(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)} \\
\delta_{22st} &= \frac{\varepsilon_s s(X_s)\ddot{\sigma}(X_s,(\hat{\theta}_n-\theta_0)\zeta+\theta_0)}{2\sigma^3(X_s,\theta_0)} - \frac{\varepsilon_s s(X_s)\ddot{\sigma}(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)} \\
&\quad - \frac{\varepsilon_t s(X_t)\ddot{\sigma}(X_t,(\hat{\theta}_n-\theta_0)\zeta+\theta_0)}{2\sigma^3(X_t,\theta_0)} - \frac{\varepsilon_t s(X_t)\ddot{\sigma}(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)} \\
\delta_{23st} &= \frac{\varepsilon_s s(X_s)\ddot{\sigma}(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)} - \frac{\varepsilon_t s(X_t)\ddot{\sigma}(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)},
\end{aligned}$$

for some $\zeta \in (0,1)$. For the term I_{21}, we have

$$nI_{21} = \sqrt{n}(\hat{\theta}_n-\theta_0)^T \frac{1}{C_n^4}\sum_{i<j<k<l} h_{21}(Z_i,Z_j,Z_k,Z_l),$$

Note that

$$\begin{aligned}
& E[h_{21}(Z_i,Z_j,Z_k,Z_l)] \\
= & -E[\{\frac{\varepsilon_1 s(X_1)\dot{\sigma}(X_1,\theta_0)}{2\sigma^3(X_1,\theta_0)} - \frac{\varepsilon_2 s(X_2)\dot{\sigma}(X_2,\theta_0)}{2\sigma^3(X_2,\theta_0)}\}\{\mathbb{I}(\varepsilon_1>\varepsilon_2)-\mathbb{I}(\varepsilon_1<\varepsilon_2)\}C_x(X_1,X_2)] \\
= & -2E[\{\frac{\varepsilon_1 s(X_1)\dot{\sigma}(X_1,\theta_0)}{2\sigma^3(X_1,\theta_0)}\}\{\mathbb{I}(\varepsilon_1>\varepsilon_2)-\mathbb{I}(\varepsilon_1<\varepsilon_2)\}C_x(X_1,X_2)] \\
= & -2E[\varepsilon_1\{\mathbb{I}(\varepsilon_1>\varepsilon_2)-\mathbb{I}(\varepsilon_1<\varepsilon_2)\}]E[\frac{s(X_1)\dot{\sigma}(X_1,\theta_0)}{2\sigma^3(X_1,\theta_0)}E\{C_x(X_1,X_2)|X_1\}] = 0.
\end{aligned}$$

where $C_x(X_1,X_2) = \|X_1-X_2\| - E(\|X_1-X_2\||X_1) - E(\|X_1-X_2\||X_2) + E(\|X_1-X_2\|)$. Since $\sqrt{n}(\hat{\theta}_n-\theta_0) = O_p(1)$, it follows that $nI_{21} = o_p(1)$. For the terms I_{22} and I_{23}, similar to the arguments for I_{12} and I_{13} in the proof of Theorem 3.1, we can show $nI_{22} = o_p(1)$ and $nI_{23} = o_p(1)$. Consequently, we obtain that

$$n\hat{U}_{n2} = nI_{21} + nI_{22} + nI_{23} = o_p(1).$$

Next we deal with the term $\hat{U}_{n3}$, decomposed it as

$$\begin{aligned}
\hat{U}_{n3} &= \frac{1}{C_n^4}\sum_{i<j<k<l} h_{31}(Z_i,Z_j,Z_k,Z_l) + \frac{1}{C_n^4}\sum_{i<j<k<l} h_{32}(Z_i,Z_j,Z_k,Z_l) \\
&=: \hat{U}_{n31} + \hat{U}_{n32},
\end{aligned}$$

where

$$\begin{aligned}
h_{31}(Z_i,Z_j,Z_k,Z_l) &= \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)} E(\delta_{3st}|X_s,X_t)(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)} E(\delta_{3st}|X_s,X_t)\|X_{su}\|, \\
h_{32}(Z_i,Z_j,Z_k,Z_l) &= \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)} [\delta_{3st} - E(\delta_{3st}|X_s,X_t)](\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)} [\delta_{3st} - E(\delta_{3st}|X_s,X_t)]\|X_{su}\|.
\end{aligned}$$

For the term $\hat{U}_{n32}$, similar to arguments for the term $\hat{U}_{n22}$ in the proof of Theorem 3.1, we have $n\hat{U}_{n32} = o_p(1)$. For the term $\hat{U}_{n31}$, following the same line as Theorem 1 of Xu and Cao (2021), we can show that uniformly over $1 \le s, t \le n$,

$$
\begin{aligned}
& E(\delta_{3st}|X_s, X_t) \\
= \;& 2E[\int_0^{(P_{st}+\frac{\varepsilon_s s(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{2\sqrt{n}\sigma^3(X_s,\theta_0)}-\frac{\varepsilon_t s(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{2\sqrt{n}\sigma^3(X_t,\theta_0)})+R_s-R_t} [\mathbb{I}(\varepsilon_s-\varepsilon_t \le z) - \mathbb{I}(\varepsilon_s \le \varepsilon_t)]dz|X_s, X_t] \\
= \;& 2E[\int_0^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}} \mathbb{I}(\varepsilon_s-\varepsilon_t \le z) - \mathbb{I}(\varepsilon_s-\varepsilon_t \le 0)dz|X_s, X_t] \\
& +2E[\int_{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}}^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}+\frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{\sigma(X_s,\theta_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{\sigma(X_t,\theta_0)}} \\
& [\mathbb{I}(\varepsilon_s-\varepsilon_t \le z) - \mathbb{I}(\varepsilon_s-\varepsilon_t \le 0)]dz|X_s, X_t] \\
& +2E[\int_{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}+\frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{\sigma(X_s,\theta_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{\sigma(X_t,\theta_0)}}^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}+\frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{\sigma(X_s,\theta_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{\sigma(X_t,\theta_0)}+R_s-R_t} \\
& [\mathbb{I}(\varepsilon_s-\varepsilon_t \le z) - \mathbb{I}(\varepsilon_s-\varepsilon_t \le 0)]dz|X_s, X_t] \\
& +2E[\int_{P_{st}}^{(P_{st}+\frac{\varepsilon_s s(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{2\sqrt{n}\sigma^3(X_s,\theta_0)}-\frac{\varepsilon_t s(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{2\sqrt{n}\sigma^3(X_t,\theta_0)})} \{\mathbb{I}(\varepsilon_s-\varepsilon_t \le z) - \mathbb{I}(\varepsilon_s \le \varepsilon_t)\}dz|X_s, X_t] \\
= \;& Q_\varepsilon\left([\frac{\dot{m}(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{\dot{m}(X_t,\beta_0)}{\sigma(X_t,\theta_0)}]^T(\hat{\beta}_n-\beta_0)\right)^2 + 2A_\varepsilon(\hat{\theta}_n-\theta_0)^T[\frac{\dot{\sigma}(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot{\sigma}(X_t,\theta_0)}{\sigma(X_t,\theta_0)}] \\
& +2A_\varepsilon\left([\frac{\dot{\sigma}(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot{\sigma}(X_t,\theta_0)}{\sigma(X_t,\theta_0)}]^T(\hat{\theta}_n-\theta_0)\right)^2 + 2A_\varepsilon(\hat{\theta}_n-\theta_0)^T[\frac{s(X_s)\dot{\sigma}(X_s,\theta_0)}{2\sqrt{n}\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot{\sigma}(X_t,\theta_0)}{2\sqrt{n}\sigma^3(X_t,\theta_0)}] \\
& 2A_\varepsilon\left([\frac{s(X_s)\dot{\sigma}(X_s,\theta_0)}{2\sqrt{n}\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot{\sigma}(X_t,\theta_0)}{2\sqrt{n}\sigma^3(X_t,\theta_0)}]^T(\hat{\theta}_n-\theta_0)\right)^2 + o_p(\frac{1}{n}),
\end{aligned}
$$

where $Q_\varepsilon = E[f_\varepsilon(\varepsilon)]$ and $A_\varepsilon = E[\varepsilon F_\varepsilon(\varepsilon)]$. Consequently, we obtain that

$$
\begin{aligned}
\hat{U}_{n31} \;=\;& Q_\varepsilon(\hat{\beta}_n-\beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{311}(Z_i,Z_j,Z_k,Z_l)(\hat{\beta}_n-\beta_0) \\
& +2A_\varepsilon(\hat{\theta}_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{312}(Z_i,Z_j,Z_k,Z_l) \\
& +2A_\varepsilon(\hat{\theta}_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{313}(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\theta_0) \\
& +2A_\varepsilon(\hat{\theta}_n-\theta_0)^T\frac{1}{\sqrt{n}}\frac{1}{C_n^4}\sum_{i<j<k<l} h_{314}(Z_i,Z_j,Z_k,Z_l) \\
& +2A_\varepsilon(\hat{\theta}_n-\theta_0)^T\frac{1}{n}\frac{1}{C_n^4}\sum_{i<j<k<l} h_{315}(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\theta_0) + o_p(\frac{1}{n}) \\
=:\;& Q_\varepsilon\hat{U}_{n311} + 2A_\varepsilon\hat{U}_{n312} + 2A_\varepsilon\hat{U}_{n313} + 2A_\varepsilon\hat{U}_{n314} + 2A_\varepsilon\hat{U}_{n315} + o_p(\frac{1}{n}), \qquad (6.14)
\end{aligned}
$$

where

$$h_{311}(Z_i,Z_j,Z_k,Z_l)$$

$$
\begin{aligned}
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}\}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}\}^T(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}\}\{\frac{\dot m(X_s,\beta_0)}{\sigma(X_s,\theta_0)}-\frac{\dot m(X_t,\beta_0)}{\sigma(X_t,\theta_0)}\}^T\|X_{su}\|,\\
&\quad h_{312}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}(\|X_{st}\|+\|X_{uv}\|)-12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}\|X_{su}\|,\\
&\quad h_{213}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}^T(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}^T\|X_{su}\|,\\
&\quad h_{314}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\}(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\}\|X_{su}\|,\\
&\quad h_{315}(Z_i,Z_j,Z_k,Z_l)\\
&= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\{\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\}\{\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\}^T(\|X_{st}\|+\|X_{uv}\|)\\
&\quad -12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\{\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\}\{\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\}^T\|X_{su}\|.
\end{aligned}
$$

For the terms $\hat U_{n311}$, $\hat U_{n312}$ and $\hat U_{n313}$, similar to the arguments for (6.8), (6.9) and (6.10) in the proof of Theorem 3.1, we have

$$
\begin{aligned}
n\hat U_{n311} &= \sqrt{n}(\hat\beta_n-\beta_0)^T M_1\sqrt{n}(\hat\beta_n-\beta_0)+o_p(1).\\
n\hat U_{n312} &= \sqrt{n}(\hat\theta_n-\tilde\theta_0)\frac{1}{\sqrt{n}}\sum_{i=1}^n 4E[h_{312}(Z_i,Z_j,Z_k,Z_l)|Z_i]+o_p(1).\\
n\hat U_{n313} &= \sqrt{n}(\hat\theta_n-\tilde\theta_0)^T M_2\sqrt{n}(\hat\theta_n-\theta_0)+o_p(1),
\end{aligned}
$$

where $E[h_{312}(Z_i,Z_j,Z_k,Z_l)|Z_i]=E[\{\frac{\dot\sigma(X_i,\beta_0)}{\sigma(X_i,\theta_0)}+\frac{\dot\sigma(X,\beta_0)}{\sigma(X,\theta_0)}\}C_x(X_i,X)|X_i]$, M_1 and M_2 are given in Theorem 3.1.

For the term $\hat U_{n314}$, recall that

$$
n\hat U_{n314} = \sqrt{n}(\hat\theta_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{314}(Z_i,Z_j,Z_k,Z_l).
$$

By some elementary calculations, we have

$$E[h_{314}(Z_i, Z_j, Z_k, Z_l)] = E[\{\frac{s(X_1)\dot{\sigma}(X_1, \theta_0)}{\sigma^3(X_1, \theta_0)} + \frac{s(X_2)\dot{\sigma}(X_2, \theta_0)}{\sigma^3(X_2, \theta_0)}\}^T C_x(X_1, X_2)] = 0$$

with $C_x(X_1, X_2) = \|X_1 - X_2\| - E(\|X_1 - X_2\||X_1) - E(\|X_1 - X_2\||X_2) + E(\|X_1 - X_2\|)$. Consequently, we obtain that $n\hat{U}_{n314} = o_p(1)$. Similarly, we have $n\hat{U}_{n315} = o_p(1)$. Hence we obtain

$$\begin{aligned} n\hat{U}_{n3} &= Q_\varepsilon \sqrt{n}(\hat{\beta}_n - \beta_0)^T M_1 \sqrt{n}(\hat{\beta}_n - \beta_0) + 2A_\varepsilon \sqrt{n}(\hat{\theta}_n - \theta_0)^T M_2 \sqrt{n}(\hat{\theta}_n - \theta_0) \\ &\quad + 2A_\varepsilon \sqrt{n}(\hat{\theta}_n - \theta_0) \frac{1}{\sqrt{n}} \sum_{i=1}^n 4E[h_{312}(Z_i, Z_j, Z_k, Z_l)|Z_i] + o_p(1). \end{aligned}$$

According to the arguments in Theorem 3.1, we have

$$\begin{aligned} n\hat{U}_{n0} &= \frac{6}{n-1} \sum_{i=1}^n \sum_{j \neq i}^n E\{h_0(Z_i, Z_j, Z_k, Z_l)|Z_i, Z_j\} + o_p(1) \\ n\hat{U}_{n1} &= \sqrt{n}(\hat{\beta}_n - \beta_0)^T \frac{4}{\sqrt{n}} \sum_{i=1}^n E\{h_{11}(Z_i, Z_j, Z_k, Z_l)|Z_i\} + o_p(1) \\ &\quad + \sqrt{n}(\hat{\theta}_n - \theta_0)^T \frac{4}{\sqrt{n}} \sum_{i=1}^n E\{h_{14}(Z_i, Z_j, Z_k, Z_l)|Z_i\} + o_p(1), \end{aligned}$$

with $E\{h_0(Z_1, Z_2, Z_3, Z_4)|Z_1, Z_2\} = 6^{-1} C_\varepsilon(\varepsilon_1, \varepsilon_2) C_x(X_1, X_2)$ and

$$\begin{aligned} E[h_{11}(Z_1, Z_2, Z_3, Z_4)|Z_1] &= -(F_\varepsilon(\varepsilon_1) - 1/2) E[\{\frac{\dot{m}(X_1, \beta_0)}{\sigma(X_1, \theta_0)} - \frac{\dot{m}(X_2, \beta_0)}{\sigma(X_2, \theta_0)}\} C_x(X_1, X_2)|X_1] \\ E[h_{14}(Z_1, Z_2, Z_3, Z_4)|Z_1] &= -\frac{1}{2} E\{[\frac{\varepsilon_1 \dot{\sigma}(x_1, \theta_0)}{\sigma(X_1, \theta_0)} - \frac{\varepsilon_2 \dot{\sigma}(X_2, \theta_0)}{\sigma(X_2, \theta_0)}][\mathbb{I}(\varepsilon_1 > \varepsilon_2) - \mathbb{I}(\varepsilon_1 < \varepsilon_2)] C_x(X_1, X_2)|Z_1\}. \end{aligned}$$

Altogether we obtain that

$$\begin{aligned} n\hat{U}_n &= \frac{6}{n-1} \sum_{i=1}^n \sum_{j \neq i}^n E\{h_0(Z_i, Z_j, Z_k, Z_l)|Z_i, Z_j\} \\ &\quad \sqrt{n}(\hat{\beta}_n - \beta_0)^T \frac{4}{\sqrt{n}} \sum_{i=1}^n E\{h_{11}(Z_i, Z_j, Z_k, Z_l)|Z_i\} + o_p(1) \\ &\quad + \sqrt{n}(\hat{\theta}_n - \theta_0)^T \frac{4}{\sqrt{n}} \sum_{i=1}^n E\{h_{14}(Z_i, Z_j, Z_k, Z_l)|Z_i\} \\ &\quad Q_\varepsilon \sqrt{n}(\hat{\beta}_n - \beta_0)^T M_1 \sqrt{n}(\hat{\beta}_n - \beta_0) + 2A_\varepsilon \sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0)^T M_2 \sqrt{n}(\hat{\theta}_n - \theta_0) \\ &\quad + 2A_\varepsilon \sqrt{n}(\hat{\theta}_n - \theta_0) \frac{1}{\sqrt{n}} \sum_{i=1}^n 4E[h_{312}(Z_i, Z_j, Z_k, Z_l)|Z_i] + o_p(1). \end{aligned}$$

By Assumption 1 and Proposition 4 of Tan et al. (2022), we have

$$\sqrt{n}(\hat{\beta}_n - \beta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l(Y_i, X_i, \beta_0) + o_p(1)$$

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i, \theta_0)(\varepsilon_i^2 - 1)] \Sigma_\sigma^{-1} \dot{\sigma}^2(X_i, \theta_0) + \Sigma_\sigma^{-1} E[s(X) \dot{\sigma}^2(X_i, \theta_0)] + o_p(1).$$

Consequently,

$$\begin{aligned}
& n\hat{U}_n \\
= & \frac{6}{n-1} \sum_{i=1}^{n} \sum_{j \neq i}^{n} E\{h_0(Z_i, Z_j, Z_k, Z_l)|Z_i, Z_j\} \\
& +4 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(Y_i, X_i, \beta_0)^T \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[h_{11}(Z_i, Z_j, Z_k, Z_l)|Z_i] \\
& +4 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i, \theta_0)(\varepsilon_i^2 - 1)] \dot{\sigma}^2(X_i, \theta_0)^T \Sigma_\sigma^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[h_{14}(Z_i, Z_j, Z_k, Z_l)|Z_i] \\
& +8 A_\varepsilon \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i, \theta_0)(\varepsilon_i^2 - 1)] \dot{\sigma}^2(X_i, \theta_0)^T \Sigma_\sigma^{-1} E[h_{312}(Z_i, Z_j, Z_k, Z_l)|Z_i] \\
& +8 A_\varepsilon E[s(X) \dot{\sigma}^2(X_i, \theta_0)]^T \Sigma_\sigma^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[h_{312}(Z_i, Z_j, Z_k, Z_l|Z_i)] \\
& +4 E[s(X) \dot{\sigma}^2(X_i, \theta_0)]^T \Sigma_\sigma^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[h_{14}(Z_i, Z_j, Z_k, Z_l|Z_i)] \\
& +2 A_\varepsilon \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i, \theta_0)(\varepsilon_i^2 - 1)] \dot{\sigma}^2(X_i, \theta_0)^T \Sigma_\sigma^{-1} M_2 \Sigma_\sigma^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i, \theta_0)(\varepsilon_i^2 - 1)] \dot{\sigma}^2(X_i, \theta_0) \\
& +4 A_\varepsilon \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i, \theta_0)(\varepsilon_i^2 - 1)] \dot{\sigma}^2(X_i, \theta_0)^T \Sigma_\sigma^{-1} M_2 \Sigma_\sigma^{-1} E[s(X) \dot{\sigma}^2(X_i, \theta_0)] \\
& +2 A_\varepsilon E[s(X) \dot{\sigma}^2(X_i, \theta_0)]^T \Sigma_\sigma^{-1} M_2 \Sigma_\sigma^{-1} E[s(X) \dot{\sigma}^2(X_i, \theta_0)] \\
& +Q_\varepsilon \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(Y_i, X_i, \beta_0)^T M_1 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(Y_i, X_i, \beta_0) + o_p(1).
\end{aligned}$$

Since $E[h_0(Z_i, Z_j, Z_k, Z_l)]^2 \leq CE\|X\|^2 E(\varepsilon^2) < \infty$, it follows that

$$\begin{aligned}
n\hat{U}_n \longrightarrow & \sum_{k=1}^{\infty} \lambda_k (\mathcal{Z}_k^2 - 1) + 4\mathcal{N}^\top \mathcal{P}_1 + 4\mathcal{W}^\top \Sigma_\sigma^{-1} \mathcal{P}_2 + 8A_\varepsilon \mathcal{W}^\top \Sigma_\sigma^{-1} \mathcal{P}_3 + 2A_\varepsilon \mathcal{W}^\top \Sigma_\sigma^{-1} M_2 \Sigma_\sigma^{-1} \mathcal{W} \\
& + Q_\varepsilon \mathcal{N}^\top M_1 \mathcal{N} + 4E[s(X) \dot{\sigma}^2(X_i, \theta_0)]^\top \Sigma_\sigma^{-1} \mathcal{P}_2 + 8A_\varepsilon E[s(X) \dot{\sigma}^2(X_i, \theta_0)]^\top \Sigma_\sigma^{-1} \mathcal{P}_3 \\
& + 2A_\varepsilon E[s(X) \dot{\sigma}^2(X_i, \theta_0)]^T \Sigma_\sigma^{-1} M_2 \Sigma_\sigma^{-1} E[s(X) \dot{\sigma}^2(X_i, \theta_0)] \\
& + 4A_\varepsilon \mathcal{W}^T M_2 \Sigma_\sigma^{-1} E[s(X) \dot{\sigma}^2(X_i, \theta_0)]
\end{aligned}$$

where $\Sigma_\sigma = E[\dot{\sigma}^2(X_i, \theta_0) \dot{\sigma}^2(X_i, \theta_0)^T]$, A_ε, Q_ε, $\lambda_i, \mathcal{Z}_i, \mathcal{N}, \mathcal{W}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, M_1 and M_2 are defined in Theorem 3.1. Hence we complete the proof of the first part of Theorem 3.2.

(2) Recall that $\eta_i = \frac{Y_i - m(X_i, \beta_0)}{\sigma(X_i, \tilde{\theta}_0)}$ and $\hat{\eta}_i = \frac{Y_i - m(X_i, \hat{\beta}_n)}{\sigma(X_i, \hat{\theta}_n)}$ in nonlinear cases. Under the global alternative H_1, we have $\eta_i = \frac{\varepsilon_i \sigma(X_i)}{\sigma(X_i, \tilde{\theta}_0)}$. It follows from (6.1) and (6.2) in the proof of Theorem 3.1 that

$$|\hat{\eta}_i - \hat{\eta}_j|$$

$$
\begin{aligned}
&= |\eta_i - \eta_j| - [\frac{m(X_i,\hat{\beta}_n) - m(X_i,\beta_0)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{m(X_j,\hat{\beta}_n) - m(X_j,\beta_0)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_i\sigma(X_i)(\sigma(X_i,\hat{\theta}_n) - \sigma(X_i,\tilde{\theta}_0))}{\sigma^2(X_i,\tilde{\theta}_0)} \\
&- \frac{\varepsilon_j\sigma(X_j)(\sigma(X_j,\hat{\theta}_n) - \sigma(X_j,\tilde{\theta}_0))}{\sigma^2(X_j,\tilde{\theta}_0)} + (R_i - R_j)][\mathbb{I}(\eta_i > \eta_j) - \mathbb{I}(\eta_i < \eta_j)] \\
&+2\int_0^{\frac{m(X_i,\hat{\beta}_n)-m(X_i,\beta_0)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{m(X_i,\hat{\beta}_n)-m(X_j,\beta_0)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_i\sigma(X_i)(\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\tilde{\theta}_0))}{\sigma^2(X_i,\tilde{\theta}_0)} - \frac{\varepsilon_j\sigma(X_j)(\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\theta_0))}{\sigma^2(X_j,\tilde{\theta}_0)} + R_i - R_j} \\
&[\mathbb{I}(\eta_i - \eta_j \le z) - \mathbb{I}(\eta_i \le \eta_j)]dz. \qquad (6.15)
\end{aligned}
$$

where

$$
\begin{aligned}
R_i =& \frac{\varepsilon_i[\sigma(X_i,\tilde{\theta}_0) - \sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\tilde{\theta}_0)\sigma(X_i,\hat{\theta}_n)} + \frac{m(X_i,\beta_0) - m(X_i,\hat{\beta}_n)}{\sigma^2(X_i,\tilde{\theta}_0)}[\sigma(X_i,\tilde{\theta}_0) - \sigma(X_i,\hat{\theta}_n)] \\
&+ \frac{m(X_i,\beta_0) - m(X_i,\hat{\beta}_n)}{\sigma^2(X_i,\tilde{\theta}_0)} \frac{[\sigma(X_i,\tilde{\theta}_0) - \sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\hat{\theta}_n)}.
\end{aligned}
$$

By the analog to (6.5) in the proof of Theorem 3.1, $\hat{U}_n$ can be decomposed into three parts:

$$
\begin{aligned}
\hat{U}_n &= \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\eta_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\eta_{st}|\|X_{su}\|\right) \\
&+\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{5st}(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{5st}\|X_{su}\|\right) \\
&+\frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{6st}(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{6st}\|X_{su}\|\right) \\
&=: \hat{U}_{n4} + \hat{U}_{n5} + \hat{U}_{n6}, \qquad (6.16)
\end{aligned}
$$

where $\eta_{st} = \eta_s - \eta_t$,

$$
\begin{aligned}
\delta_{5st} &= -[\frac{\varepsilon_s\sigma(X_s)(\sigma(X_s,\hat{\theta}_n) - \sigma(X_s,\tilde{\theta}_0))}{\sigma^2(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t\sigma(X_t)(\sigma(X_t,\hat{\theta}_n) - \sigma(x_t,\tilde{\theta}_0))}{\sigma^2(X_t,\tilde{\theta}_0)} \\
&+\frac{m(X_s,\hat{\beta}_n) - m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n) - m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)} + (R_s - R_t)][\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)], \\
\delta_{6st} &= 2\int_0^{\frac{m(X_s,\hat{\beta}_n)-m(X_s,\beta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{m(X_t,\hat{\beta}_n)-m(X_t,\beta_0)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s\sigma(x_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma^2(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t\sigma(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma^2(X_t,\tilde{\theta}_0)} + R_s - R_t} \\
&[\mathbb{I}(\eta_s - \eta_t \le z) - \mathbb{I}(\eta_s \le \eta_t)]dz.
\end{aligned}
$$

For the term $\hat{U}_{n5}$, similar to the arguments for $\hat{U}_{n1}$ in the proof of Theorem 3.1, we have

$$
\begin{aligned}
\hat{U}_{n5} &= (\hat{\beta}_n - \beta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{51}(Z_i,Z_j,Z_k,Z_l) \\
&+(\hat{\theta}_n - \tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{52}(Z_i,Z_j,Z_k,Z_l) + o_p(\frac{1}{n})
\end{aligned}
$$

where

$$
\begin{aligned}
& h_{51}(Z_i, Z_j, Z_k, Z_l) \\
= \; & -6^{-1} \sum_{s<t,u<v}^{(i,j,k,l)} [\frac{\dot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\dot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)}][\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)](\|X_{st}\| + \|X_{uv}\|) \\
& +12^{-1} \sum_{(s,t,u)}^{(i,j,k,l)} [\frac{\dot{m}(X_s, \beta_0)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\dot{m}(X_t, \beta_0)}{\sigma(X_t, \tilde{\theta}_0)}][\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)]\|X_{su}\| \\
& h_{52}(Z_i, Z_j, Z_k, Z_l) \\
= \; & -6^{-1} \sum_{s<t,u<v}^{(i,j,k,l)} [\frac{\varepsilon_s \sigma(X_s)\dot{\sigma}(X_s, \tilde{\theta}_0)}{\sigma^2(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_t \sigma(X_t)\dot{\sigma}(X_t, \tilde{\theta}_0)}{\sigma^2(X_t, \tilde{\theta}_0)}][\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)](\|X_{st}\| + \|X_{uv}\|) \\
& +12^{-1} \sum_{(s,t,u)}^{(i,j,k,l)} [\frac{\varepsilon_s \sigma(X_s)\dot{\sigma}(X_s, \tilde{\theta}_0)}{\sigma^2(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_t \sigma(X_t)\dot{\sigma}(X_t, \tilde{\theta}_0)}{\sigma^2(X_t, \tilde{\theta}_0)}][\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)]\|X_{su}\|.
\end{aligned}
$$

It is easy to see that $\frac{1}{C_n^4} \sum_{i<j<k<l} h_{51}(Z_i, Z_j, Z_k, Z_l)$ and $\frac{1}{C_n^4} \sum_{i<j<k<l} h_{52}(Z_i, Z_j, Z_k, Z_l)$ are non-degenerate U-statistic of order 4. By some elementary calculations, we have

$$
\begin{aligned}
E[h_{51}(Z_i, Z_j, Z_k, Z_l)] &= -2E[(\frac{\dot{m}(X_1, \beta_0)}{\sigma(X_1, \tilde{\theta}_0)} - \frac{\dot{m}(x_2, \beta_0)}{\sigma(X_2, \tilde{\theta}_0)})\mathbb{I}(\eta_1 > \eta_2)C_x(X_1, X_2)] \stackrel{def}{=} 2K_1 \\
E[h_{52}(Z_i, Z_j, Z_k, Z_l)] &= -2E[(\frac{\varepsilon_1 \sigma(X_1)\dot{\sigma}(X_1, \tilde{\theta}_0)}{\sigma^2(X_1, \tilde{\theta}_0)} - \frac{\varepsilon_2 \sigma(X_2)\dot{\sigma}(X_2, \tilde{\theta}_0)}{\sigma^2(X_2, \tilde{\theta}_0)})\mathbb{I}(\eta_1 > \eta_2)C_x(X_1, X_2)] \stackrel{def}{=} 2K_2,
\end{aligned}
$$

where $C_x(X_1, X_2) = \|X_1 - X_2\| - E(\|X_1 - X_2\||X_1) - E(\|X_1 - X_2\||X_2) + E(\|X_1 - X_2\|)$. Thus we obtain that

$$
\hat{U}_{n5} = (\hat{\beta}_n - \beta_0)^T K_1 + (\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + O_p(\frac{1}{n}).
$$

Following the same line for the term $\hat{U}_{n2}$ in the proof of Theorem 3.1, we can show that $\sqrt{n}\hat{U}_{n6} = o_p(1)$. Altogether we obtain that

$$
\begin{aligned}
\hat{U}_n &= \hat{U}_{n4} + 2(\hat{\beta}_n - \beta_0)^T K_1 + 2(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(\frac{1}{\sqrt{n}}) \\
&=: \frac{1}{C_n^4} \sum_{i<j<k<l} h_4(Z_i, Z_j, Z_k, Z_l) + 2(\hat{\beta}_n - \beta_0)^T K_1 + 2(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(\frac{1}{\sqrt{n}}),
\end{aligned}
$$

where $Z_i = (\eta_i, X_i)$ and $h_4(Z_i, Z_j, Z_k, Z_l) = \frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} |\eta_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} |\eta_{st}|\|X_{su}\|$. Consequently,

$$
\sqrt{n}[\hat{U}_n - dCov^2(\eta, X)] = \sqrt{n}[\hat{U}_{n4} - dCov^2(\eta, X)] + 2\sqrt{n}(\hat{\beta}_n - \beta_0)^T K_1 + 2\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(1).
$$

By Assumption 1 and Proposition 3 of Tan et al. (2022), we have

$$
\sqrt{n}(\hat{\beta}_n - \beta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l(Y_i, X_i, \beta_0) + o_p(1)
$$

$$\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i)\varepsilon_i^2 - \sigma^2(X_i,\tilde{\theta}_0)]\Sigma^{-1}\dot{\sigma}^2(X_i,\tilde{\theta}_0) + o_p(1).$$

It follows that

$$\begin{aligned}
& \sqrt{n}[\hat{U}_n - dCov^2(\eta, X)] \\
= & \sqrt{n}\frac{1}{C_n^4}\sum_{i<j<k<l}[h_4(Z_i,Z_j,Z_k,Z_l) - dCov(\eta,X)] \\
& + \frac{2}{\sqrt{n}}\sum_{i=1}^{n}K_1^T l(Y_i,X_i,\beta_0) + \frac{2}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i)\varepsilon_i^2 - \sigma^2(X_i,\tilde{\theta}_0)]\Sigma^{-1}K_2^T\dot{\sigma}^2(X_i,\tilde{\theta}_0) + o_p(1).
\end{aligned}$$

It is easy to verify that $E[h_4(Z_i,Z_j,Z_k,Z_l)] = dCov^2(\eta,X)$ and $\hat{U}_{n4}$ is non-degenerate. According to technical appendix 1.1 of Yao et al. (2018), we can obtain

$$E[h_4(Z_1,Z_2,Z_3,Z_4|Z_1)] = \frac{1}{2}\{E[C_\eta(\eta_1,\eta_2)C_x(X_1,X_2)|Z_1] + dCov^2(\eta,X)\}$$

where $dCov^2(\eta,X) = E[C_\eta(\eta_i,\eta_j)C_x(X_i,X_j)]$,

$$\begin{aligned}
C_\eta(\eta_i,\eta_j) &= |\eta_i - \eta_j| - E(|\eta_i - \eta_j\|\eta_i) - E(|\eta_i - \eta_j\|\eta_j) + E(|\eta_i - \eta_j|) \\
C_x(X_i,X_j) &= \|X_i - X_j\| - E(\|X_i - X_j\||X_i) - E(\|X_i - X_j\||X_j) + E(\|X_i - X_j\|).
\end{aligned}$$

According to the formula (2) in Section 5.3.4 of Serfling (1984), we have

$$\begin{aligned}
& \sqrt{n}[\hat{U}_n - dCov^2(\eta,X)] \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}2\{\mathcal{G}(\eta_i,X_i) + K_1^T l(Y_i,X_i,\beta_0) + [\sigma^2(X_i)\varepsilon_i^2 - \sigma^2(X_i,\tilde{\theta}_0)]K_2^T\Sigma^{-1}\dot{\sigma}^2(X_i,\tilde{\theta}_0)\} + o_p(1),
\end{aligned}$$

it follows that

$$\sqrt{n}[\hat{U}_n - dCov^2(\eta,X)] \longrightarrow N(0,\sigma_1^2),$$

where $\sigma_1^2 = 4var\{\mathcal{G}(\eta,X) + K_1^T l(Y,X,\beta_0) + [\sigma^2(X)\varepsilon^2 - \sigma^2(X,\tilde{\theta}_0)]K_2^T\Sigma^{-1}\dot{\sigma}^2(X,\tilde{\theta}_0)\}$ with

$$\begin{aligned}
K_1 &= -E[(\frac{\dot{m}(X_1,\beta_0)}{\sigma(X_1,\tilde{\theta}_0)} - \frac{\dot{m}(X_2,\beta_0)}{\sigma(X_2,\tilde{\theta}_0)})I(\eta_1 > \eta_2)C_x(X_1,X_2)], \\
K_2 &= -E[(\frac{\eta_1\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma(X_1,\tilde{\theta}_0)} - \frac{\eta_2\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma(X_2,\tilde{\theta}_0)})I(\eta_1 > \eta_2)C_x(X_1,X_2)], \\
\mathcal{G}(\eta_1,X_1) &= E[C_\eta(\eta_1,\eta_2)C_x(X_1,X_2)|Z_1] - dCov^2(\eta_1,X_1) \\
C_\eta(\eta_i,\eta_j) &= |\eta_i - \eta_j| - E(|\eta_i - \eta_j||\eta_i) - E(|\eta_i - \eta_j||\eta_j) + E(|\eta_i - \eta_j|),
\end{aligned}$$

Hence we complete the proof of Theorem 3.2. □

Proof of Theorem 3.3. (1) First we discuss the asymptotic properties of $n\hat{U}_n$ under the null hypothesis in the nonparametric models. Recall that $\eta_i = \frac{Y_i - m(X_i)}{\sigma(X_i,\tilde{\theta}_0)}$ and $\hat{\eta}_i = \frac{Y_i - \hat{m}(X_i)}{\sigma(X_i,\hat{\theta}_n)}$ in nonparametric cases. It follows from (6.1) and (6.2) in the proof of Theorem 3.1 that

$$
\begin{aligned}
&|\hat{\eta}_i - \hat{\eta}_j| \\
=&|\varepsilon_i - \varepsilon_j| - [\frac{\hat{m}(X_i) - m(X_i)}{\sigma(X_i, \tilde{\theta}_0)} - \frac{\hat{m}(X_j) - m(X_j)}{\sigma(X_j, \tilde{\theta}_0)} + \frac{\varepsilon_i(\sigma(X_i, \hat{\theta}_n) - \sigma(X_i, \tilde{\theta}_0))}{\sigma(X_i, \tilde{\theta}_0)} \\
&- \frac{\varepsilon_j(\sigma(X_j, \hat{\theta}_n) - \sigma(X_j, \tilde{\theta}_0))}{\sigma(X_j, \tilde{\theta}_0)} + (R_i - R_j)][\mathbb{I}(\varepsilon_i < \varepsilon_j) - \mathbb{I}(\varepsilon_i > \varepsilon_j)] \\
&+ 2\int_0^{\frac{\hat{m}(X_i)-m(X_i)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\hat{m}(X_j)-m(X_j)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_i(\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\tilde{\theta}_0))}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\varepsilon_j(\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\tilde{\theta}_0))}{\sigma(X_j,\tilde{\theta}_0)} + R_i - R_j} \\
&[\mathbb{I}(\varepsilon_i - \varepsilon_j \le z) - \mathbb{I}(\varepsilon_i \le \varepsilon_j)]dz.
\end{aligned}
\tag{6.17}
$$

where

$$
\begin{aligned}
R_i =& \frac{\varepsilon_i[\sigma(X_i, \tilde{\theta}_0) - \sigma(X_i, \hat{\theta}_n)]^2}{\sigma(X_i, \tilde{\theta}_0)\sigma(X_i, \hat{\theta}_n)} + \frac{\hat{m}(X_i) - m(X_i)}{\sigma^2(X_i, \tilde{\theta}_0)}[\sigma(X_i, \tilde{\theta}_0) - \sigma(X_i, \hat{\theta}_n)] \\
&+ \frac{\hat{m}(X_i) - m(X_i)}{\sigma^2(X_i, \tilde{\theta}_0)} \frac{[\sigma(X_i, \tilde{\theta}_0) - \sigma(X_i, \hat{\theta}_n)]^2}{\sigma(X_i, \hat{\theta}_n)}.
\end{aligned}
$$

Similar to the arguments in Theorem 3.1, we can rewrite $\hat{U}_n$ as

$$
\hat{U}_n = \frac{1}{C_n^4} \sum_{i<j<k<l} \tilde{h}_0(\hat{Z}_i, \hat{Z}_j, \hat{Z}_k, \hat{Z}_l),
$$

where

$$
\tilde{h}_0(\hat{Z}_i, \hat{Z}_j, \hat{Z}_k, \hat{Z}_l) = \frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} |\hat{\eta}_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} |\hat{\eta}_{st}|\|X_{su}\|, \tag{6.18}
$$

$\hat{Z}_i = (\hat{\eta}_i, X_i)$, $X_{st} = X_s - X_t$, and $\hat{\eta}_{st} = \hat{\eta}_s - \hat{\eta}_t$. Here the summation in (6.18) is over all permutations of the 4-tuples of indices (i, j, k, l). By the analog to (6.5) in the proof of Theorem 3.1, we have

$$
\begin{aligned}
\hat{U}_n =& \frac{1}{C_n^4} \sum_{i<j<k<l} \left(\frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} |\varepsilon_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} |\varepsilon_{st}|\|X_{su}\| \right) \\
&+ \frac{1}{C_n^4} \sum_{i<j<k<l} \left(\frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} \delta_{1st}^*(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} \delta_{1st}^*\|X_{su}\| \right) \\
&+ \frac{1}{C_n^4} \sum_{i<j<k<l} \left(\frac{1}{6} \sum_{s<t,u<v}^{(i,j,k,l)} \delta_{2st}^*(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12} \sum_{(s,t,u)}^{(i,j,k,l)} \delta_{2st}^*\|X_{su}\| \right) \\
=:& \hat{U}_{n0}^* + \hat{U}_{n1}^* + \hat{U}_{n2}^*,
\end{aligned}
\tag{6.19}
$$

where

$$
\delta_{1st}^* = -[\frac{\varepsilon_s(\sigma(X_s, \hat{\theta}_n) - \sigma(X_s, \tilde{\theta}_0))}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t, \hat{\theta}_n) - \sigma(X_t, \tilde{\theta}_0))}{\sigma(X_t, \tilde{\theta}_0)} + \frac{\hat{m}(X_s) - m(X_s)}{\sigma(X_s, \tilde{\theta}_0)}
$$

$$
\begin{aligned}
& -\frac{\hat{m}(X_t)-m(X_t)}{\sigma(X_t,\tilde{\theta}_0)}+(R_s-R_t)]\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\} \\
\delta^*_{2st} \quad = \quad & 2\int_0^{\frac{\hat{m}(X_s)-m(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\hat{m}(X_t)-m(X_t)}{\sigma(X_t,\tilde{\theta}_0)}+\frac{\varepsilon_s\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)}+R_s-R_t} \\
& \{\mathbb{I}(\varepsilon_s-\varepsilon_t\le z)-\mathbb{I}(\varepsilon_s\le\varepsilon_t)\}dz.
\end{aligned}
$$

First we deal with the term $\hat{U}^*_{n1}$. Recall that $\hat{g}(X_t)=\frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)Y_j$, $\hat{f}_X(X_t)=\frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)$, and $\hat{m}(X_t)=\hat{g}(X_t)/\hat{f}_X(X_t)$, it follows that

$$
\begin{aligned}
\hat{m}(X_t)-m(X_t) \quad = \quad & \frac{\hat{g}(X_t)-g(X_t)}{f_X(X_t)}-m(X_t)\frac{\hat{f}_X(X_t)-f_X(X_t)}{f_X(X_t)} \\
& -\frac{[\hat{g}(X_t)-g(X_t)][\hat{f}_X(X_t)-f_X(X_t)]}{f_X(X)\hat{f}_X(X_t)}+\frac{m(X_t)[\hat{f}_X(X_t)-f_X(X_t)]^2}{f_X(X_t)\hat{f}_X(X_t)} \qquad (6.20)
\end{aligned}
$$

where h is the bandwidth, $K_h(\cdot)=K(\cdot/h)/h^p$, and $K(\cdot)$ is a kernel function.

For $\hat{g}(X_t)-g(X_t)$, we have

$$
\begin{aligned}
& \hat{g}(X_t)-g(X_t) \\
= \quad & \frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)Y_j-g(X_t) \\
= \quad & \frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)m(X_j)-g(X_t)+\frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)\sigma(X_j)\varepsilon_j \\
= \quad & \frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)m(X_j)-E[K_h(X_t-X_j)m(X_j)|X_t] \\
& +E[K_h(X_t-X_j)m(X_j)|X_t]-g(X_t)+\frac{1}{n-1}\sum_{j=1,j\neq t}^n K_h(X_t-X_j)\sigma(X_j)\varepsilon_j. \qquad (6.21)
\end{aligned}
$$

Using Taylor expansion, we obtain that

$$
\begin{aligned}
& E[K_h(X_t-X_j)m(X_j)|X_t]-g(X_t) \\
= \quad & \int K_h(X_t-X)m(X)f_X(X)dx-g(X_t) \\
= \quad & \int \frac{K(u)g(X_t+uh)h^p}{h^p}du-g(X_t) \\
= \quad & C\frac{h^k}{k!}\int \frac{\partial g^{(k)}(X_t)}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_p^{l_p}}u_1^{l_1}u_2^{l_2}\cdots u_p^{l_p}K(u)du+o_p(h^k) \\
= \quad & h^kD_1(X_t)+o_p(h^k).
\end{aligned}
$$

Similarly, we have

$$
\hat{f}_X(X_t)-f_X(X_t)
$$

$$
\begin{aligned}
&= \frac{1}{n-1}\sum_{j=1,j\neq t}^{n}\{K_h(X_t-X_j)-E[K_h(X_t-X_j)|X_t]\}+E[K_h(X_t-X_j)|X_t]-f_X(X_t)\\
&= C\frac{h^k}{k!}\int\frac{\partial f_X^{(k)}(X_t)}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_p^{l_p}}u_1^{l_1}u_2^{l_2}\cdots u_p^{l_p}K(u)du+\frac{1}{n-1}\sum_{j=1,j\neq t}^{n}\{K_h(X_t-X_j)\\
&\quad -E[K_h(X_t-X_j)|X_t]\}+o_p(h^k)\\
&= \frac{1}{n-1}\sum_{j=1,j\neq t}^{n}\{K_h(X_t-X_j)-E[K_h(X_t-X_j)|X_t]\}+h^kD_2(X_t)+o_p(h^k).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\hat{m}(X_t)-m(X_t)\\
= &\frac{1}{n-1}\sum_{j=1,j\neq t}^{n}\{\omega_{t,j}m(X_j)-E[\omega_{t,j}m(X_j)|X_t]\}+\frac{1}{n-1}\sum_{j=1,j\neq t}^{n}(\omega_{t,j}-E[\omega_{t,j}|X_t])m(X_t)\\
&+\frac{1}{n-1}\sum_{j=1,j\neq t}^{n}\omega_{t,j}\sigma(X_j)\varepsilon_j+h^kD(X_t)+o_p(h^k),
\end{aligned}
$$

where $D(X_t)=\frac{D_1(X_t)+D_2(X_t)m(X_t)}{f_X(X_t)}$ and $\omega_{i,j}=\frac{K_h(X_i-X_j)}{f_X(X_i)}$. By Taylor expansion and the decomposition of $\hat{m}(X_t)-m(X_t)$, we can decompose $\hat{U}_{n1}^*$ as

$$
\begin{aligned}
\hat{U}_{n1}^* &= \frac{1}{C_n^4}\sum_{i<j<k<l}h^kh_{11}^*(Z_i,Z_j,Z_k,Z_l)+\frac{1}{C_n^4}\sum_{i<j<k<l}h_{12}^*(Z_i,Z_j,Z_k,Z_l)\\
&\quad+\frac{1}{C_n^4}\sum_{i<j<k<l}h_{13}^*(Z_i,Z_j,Z_k,Z_l)+\frac{1}{C_n^4}\sum_{i<j<k<l}h_{14}^*(Z_i,Z_j,Z_k,Z_l)\\
&\quad+(\hat{\theta}_n-\tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{15}^*(Z_i,Z_j,Z_k,Z_l)\\
&\quad+2^{-1}(\hat{\theta}_n-\tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{16}^*(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\tilde{\theta}_0)\\
&\quad+2^{-1}(\hat{\theta}_n-\tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{17}^*(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\tilde{\theta}_0)\\
&\quad+\frac{1}{C_n^4}\sum_{i<j<k<l}h_{18}^*(Z_i,Z_j,Z_k,Z_l)\\
&=: I_{11}^*+I_{12}^*+I_{13}^*+I_{14}^*+I_{15}^*+I_{16}^*+I_{17}^*+I_{18}^*,
\end{aligned}
$$

where $Z_i=(\varepsilon_i,X_i)$ and

$$
\begin{aligned}
&h_{1m}^*(Z_i,Z_j,Z_k,Z_l)\\
= &-6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{1mst}^*\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}(\|X_{st}\|+\|X_{uv}\|)\\
&+12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{1mst}^*\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}\|X_{su}\|, \quad \text{for } m=1,2,\cdots,8
\end{aligned}
$$

with

$$\begin{aligned}
\delta^*_{11st} &= \frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)} \\
\delta^*_{12st} &= \frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n \omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)}, \\
\delta^*_{13st} &= \frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n (\omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s])}{\sigma(X_s,\tilde{\theta}_0)} - \frac{(\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t])}{\sigma(X_t,\tilde{\theta}_0)}, \\
\delta^*_{14st} &= \frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n (\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n (\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde{\theta}_0)}, \\
\delta^*_{15st} &= \frac{\varepsilon_s\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)} \\
\delta^*_{16st} &= \frac{\varepsilon_s\ddot{\sigma}(X_s,(\hat{\theta}_n-\tilde{\theta}_0)\zeta+\theta_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_s\ddot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)} - [\frac{\varepsilon_t\ddot{\sigma}(X_t,(\hat{\theta}_n-\tilde{\theta}_0)\zeta+\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)} - \frac{\varepsilon_t\ddot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}] \\
\delta^*_{17st} &= \frac{\varepsilon_s\ddot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t\ddot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)} \\
\delta^*_{18st} &= R_s - R_t.
\end{aligned}$$

For the term I^*_{11}, it is easy to see that $E[h^*_{11}(Z_i,Z_j,Z_k,Z_l)] = 0$ and $\frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{11}(Z_i,Z_j,Z_k,Z_l)$ is non-degenerate. Combining this with the assumption 6(d), we have

$$nI^*_{11} = \sqrt{n}h^k\frac{\sqrt{n}}{C_n^4}\sum_{i<j<k<l} h^*_{11}(Z_i,Z_j,Z_k,Z_l) = o_p(1).$$

For the term I^*_{12}, decomposed it as

$$\begin{aligned}
& I^*_{12} \\
=\ & \frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^n\sum_{j=1.j\neq i}^n\sum_{k=1,k\neq i,j}^n\sum_{l=1,l\neq i,j,k}^n\sum_{p=1,p\neq i}^n\sum_{q=1,q\neq j}^n\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\
& -\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
=\ & \frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^n\sum_{j\neq i}^n\sum_{k\neq i,j}^n\sum_{l\neq i,j,k}^n\sum_{p=q\neq i,j}^n\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\
& \times\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
& +\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^n\sum_{j\neq i}^n\sum_{k\neq i,j}^n\sum_{l\neq i,j,k}^n\sum_{p=j}\sum_{q=i}\{\frac{\omega_{j,i}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{i,j}\sigma(X_i,\tilde{\theta}_0)\varepsilon_i}{\sigma(X_j,\tilde{\theta}_0)}\} \\
& \times\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
& +\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^n\sum_{j\neq i}^n\sum_{k\neq i,j}^n\sum_{l\neq i,j,k}^n\sum_{p=j}^n\sum_{q\neq i,j}^n\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{p,j}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_j,\tilde{\theta}_0)}\} \\
& \times\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p\neq i,j}^{n}\sum_{q=i}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{p,j}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_j,\tilde{\theta}_0)}\}\\
&\times\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
&+\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p\neq i\neq j\neq k\neq l}^{n}\sum_{q\neq i\neq j\neq k\neq l\neq p}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}\\
&-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
=:&\quad I^*_{121}+I^*_{122}+I^*_{123}+I^*_{124}+I^*_{125}
\end{aligned}
$$

For the term I^*_{121}, we decompose it as

$$
\begin{aligned}
I^*_{121} \quad=\quad &\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p=q=k}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}\\
&-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
&+\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p=q=l}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}\\
&-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
&+\frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p=q\neq i,j,l,k}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}\\
&-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
=:\quad &I^*_{1211}+I^*_{1212}+I^*_{1213}.
\end{aligned}
$$

For the term I^*_{1211},

$$
\begin{aligned}
nI^*_{1211} \quad=\quad &\frac{1}{(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\{\frac{\omega_{k,i}\sigma(X_k,\tilde{\theta}_0)\varepsilon_k}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{k,j}\sigma(X_k,\tilde{\theta}_0)\varepsilon_k}{\sigma(X_j,\tilde{\theta}_0)}\}\\
&\times\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
=\quad &\frac{n}{(n-1)^2}\frac{1}{C_n^4}\sum\sum\sum\sum_{1\le i<j<k<l\le n}\delta^*_{1211}(Z_i,Z_j,Z_k,Z_l),
\end{aligned}
$$

where

$$
\begin{aligned}
\delta^*_{1211}(Z_i,Z_j,Z_k,Z_l) \quad=\quad &\frac{1}{24}\sum_{(s,t,u,v)}^{(i,j,k,l)}\{\frac{\omega_{u,s}\sigma(X_u,\tilde{\theta}_0)\varepsilon_u}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\omega_{u,t}\sigma(X_u,\tilde{\theta}_0)\varepsilon_u}{\sigma(X_t,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}\\
&(\|X_{st}\|+\|X_{uv}\|-2\|X_{su}\|).
\end{aligned}
$$

Since $E|\delta^*_{1211}(Z_i,Z_j,Z_k,Z_l)|<\infty$, it follows from the law of large numbers for U-statistics that $nI^*_{1211}=o_p(1)$. Similarly, we can show that $nI^*_{1212}=o_p(1)$ and $nI^*_{1213}=o_p(1)$. Consequently,

$$nI^*_{121}=nI^*_{1211}+nI^*_{1212}+nI^*_{1213}=o_p(1).$$

For the term I^*_{122}, recall that

$$\begin{aligned} nI^*_{122} &= \frac{1}{(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \{\frac{\omega_{j,i}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{i,j}\sigma(X_i,\tilde{\theta}_0)\varepsilon_i}{\sigma(X_j,\tilde{\theta}_0)}\} \\ &\quad \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \end{aligned}$$

By the law of large numbers for U-statistics, it is readily seen that $nI^*_{122} = o_p(1)$.

For the term I^*_{123}, recall that

$$\begin{aligned} I^*_{123} &= \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j} \sum_{q\neq i,j}^n \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\ &\quad \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|). \end{aligned}$$

Decompose I^*_{123} as follows,

$$\begin{aligned} I^*_{123} &= \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j} \sum_{q=k} \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\ &\quad \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\ &\quad + \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j} \sum_{q=l} \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\ &\quad \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\ &\quad + \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j} \sum_{q\neq i,j,k,l} \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\ &\quad \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\ &=: I^*_{1231} + I^*_{1232} + I^*_{1233} \end{aligned}$$

Similar to the arguments for I^*_{122}, we have $nI^*_{1231} = o_p(1)$ and $nI^*_{1232} = o_p(1)$. For the term I^*_{1233},

$$\begin{aligned} nI^*_{1233} &= \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j} \sum_{q\neq i,j,k,l} \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\ &\quad - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\ &= \frac{n(n-4)}{(n-1)^2} \frac{1}{C_n^5} \sum\sum\sum\sum\sum_{1\leq i<j<k<l<r\leq n} \delta^*_{1233}(Z_i, Z_j, Z_k, Z_l, Z_r), \end{aligned}$$

where

$$\begin{aligned} \delta^*_{1233}(Z_i, Z_j, Z_k, Z_l, Z_r) &= \frac{1}{120} \sum_{(s,t,u,v,r)}^{(i,j,k,l,q)} \{\frac{\omega_{t,s}\sigma(X_t,\tilde{\theta}_0)\varepsilon_t}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\omega_{r,t}\sigma(X_r,\tilde{\theta}_0)\varepsilon_r}{\sigma(X_t,\tilde{\theta}_0)}\} \\ &\quad \times\{\mathbb{I}(\varepsilon_s > \varepsilon_t) - \mathbb{I}(\varepsilon_s < \varepsilon_t)\}(\|X_{st}\| + \|X_{uv}\| - 2\|X_{su}\|). \end{aligned}$$

By the law of large numbers for U-statistics, it follows that $nI^*_{1233} \to 2A_\varepsilon E\|X_1 - X_2\|$, where $A_\varepsilon = E[\varepsilon F_\varepsilon(\varepsilon)]$. Consequently, we obtain that $nI^*_{123} \to 2A_\varepsilon E\|X_1 - X_2\|$. Similarly, we can show that

$$nI^*_{124} \longrightarrow 2A_\varepsilon E\|X_1 - X_2\|.$$

For the term I^*_{125}, recall that

$$\begin{aligned}
I^*_{125} &= \frac{1}{n(n-1)^3(n-2)(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p\neq i\neq j\neq k\neq l}^{n}\sum_{q\neq i\neq j\neq k\neq l\neq p}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\
&\quad -\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
&= \frac{(n-4)(n-5)}{(n-1)^2}\frac{1}{C_n^6}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p\neq i\neq j\neq k\neq l}^{n}\sum_{q\neq i\neq j\neq k\neq l\neq p}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\
&\quad -\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)\}(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
&= \frac{(n-4)(n-5)}{(n-1)^2}\frac{1}{C_n^6}\sum\sum\sum\sum\sum\sum_{1\leq i<j<k<l<r<m\leq n}\delta^*_{125}(Z_i,Z_j,Z_k,Z_l,Z_r,Z_m),
\end{aligned}$$

where

$$\begin{aligned}
\delta^*_{125}(Z_i,Z_j,Z_k,Z_l,Z_r,Z_m) &= \frac{1}{6!}\sum_{(s,t,u,v,r,m)}^{(i,j,k,l,p,q)}\{\frac{\omega_{r,s}\sigma(X_r,\tilde{\theta}_0)\varepsilon_r}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\omega_{m,t}\sigma(X_m,\tilde{\theta}_0)\varepsilon_m}{\sigma(X_t,\tilde{\theta}_0)}\} \\
&\quad \times\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\}(\|X_{st}\|+\|X_{uv}\|-2\|X_{su}\|).
\end{aligned}$$

Some elementary calculations show that I^*_{125} is degenerate of order 1. By the arguments in Section 5.3.4 of Serfling (2009), we can obtain

$$nI^*_{125} = \frac{2}{(n-1)}\sum\sum_{1\leq i<j\leq n}h^*_1(Z_i,Z_j)+o_p(1),$$

where $h^*_1(Z_i,Z_j)=\frac{1}{2}(H_{1ij}+H_{2ij})$,

$$\begin{aligned}
H_{1ij} &= \frac{1}{4!}E[\sum_{(k_1,l_1,p_1,q_1)}^{(k,l,p,q)}(\frac{\omega_{p_1,q_1}\sigma(X_{p_1},\tilde{\theta}_0)\varepsilon_{p_1}}{\sigma(X_{q_1},\tilde{\theta}_0)}-\frac{\omega_{i,j}\sigma(X_i,\tilde{\theta}_0)\varepsilon_i}{\sigma(X_j,\tilde{\theta}_0)})\{\mathbb{I}(\varepsilon_{q_1}>\varepsilon_j)-\mathbb{I}(\varepsilon_{q_1}<\varepsilon_j)\} \\
&\quad \times(\|X_{q_1j}\|+\|X_{kl}\|-2\|X_{q_1k}\|)|Z_i,Z_j], \\
H_{2ij} &= \frac{1}{4!}E[\sum_{(k_1,l_1,p_1,q_1)}^{(k,l,p,q)}(\frac{\omega_{j,i}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{q_1,p_1}\sigma(X_{q_1},\tilde{\theta}_0)\varepsilon_{q_1}}{\sigma(X_{p_1},\tilde{\theta}_0)})\{\mathbb{I}(\varepsilon_i>\varepsilon_{p_1})-\mathbb{I}(\varepsilon_i<\varepsilon_{p_1})\} \\
&\quad \times(\|X_{ip_1}\|+\|X_{k_1l_1}\|-2\|X_{ik_1}\|)|Z_i,Z_j].
\end{aligned}$$

Hence we obtain that

$$nI^*_{12} = \frac{2}{(n-1)}\sum\sum_{1\leq i<j\leq n}h^*_1(Z_i,Z_j)+4A_\varepsilon E\|X_1-X_2\|+o_p(1).$$

For the term I^*_{13} and I^*_{14}, similar to the arguments for I^*_{12}, we can show that

$$nI^*_{13} = \frac{2}{n-1}\sum\sum_{1\leq i<j\leq n}h^*_2(Z_i,Z_j)+o_p(1),$$

$$nI_{14}^* = \frac{2}{n-1}\sum\sum_{1\le i<j\le n} h_3^*(Z_i, Z_j) + o_p(1),$$

where $h_2^*(Z_i, Z_j) = \frac{1}{2}(H_{3ij} + H_{4ij})$, $h_3^*(Z_i, Z_j) = \frac{1}{2}(H_{5ij} + H_{6ij})$,

$$\begin{aligned}
H_{3ij} &= \frac{1}{4!}\sum_{(k_1,l_1,p_1,q_1)}^{(k,l,p,q)} E[(\frac{(\omega_{j,i}m(X_j) - E[\omega_{j,i}m(X_j)|X_i])}{\sigma(X_i,\tilde{\theta}_0)} - \frac{(\omega_{q_1,p_1}m(X_{q_1}) - E[\omega_{q_1,p_1}m(X_{q_1})|X_{p_1}])}{\sigma(X_j,\tilde{\theta}_0)}) \\
&\quad \times\{\mathbb{I}(\varepsilon_i > \varepsilon_{p_1}) - \mathbb{I}(\varepsilon_i < \varepsilon_{p_1})\}(\|X_{(i)}^{(p_1)}\| + \|X_{(k_1)}^{(l_1)}\| - 2\|X_{(i)}^{(k_1)}\|)|Z_i, Z_j], \\
H_{4ij} &= \frac{1}{4!}\sum_{(k_1,l_1,p_1,q_1)}^{(k,l,p,q)} E[(\frac{(\omega_{p_1,q_1}m(X_{p_1}) - E[\omega_{p_1,q_1}m(X_{p_1})|X_{q_1}])}{\sigma(X_{q_1},\tilde{\theta}_0)} - \frac{(\omega_{j,i}m(X_j) - E[\omega_{j,i}m(X_j)|X_i])}{\sigma(X_i,\tilde{\theta}_0)}) \\
&\quad \times\{\mathbb{I}(\varepsilon_{q_1} > \varepsilon_j) - \mathbb{I}(\varepsilon_{q_1} < \varepsilon_j)\}(\|X_{(q_1)}^{(j)}\| + \|X_{(k_1)}^{(l_1)}\| - 2\|X_{(q_1)}^{(k_1)}\|)|Z_i, Z_j], \\
H_{5ij} &= \frac{1}{4!}E[\sum_{(k_1,l_1,p_1,q_1)}^{(k,l,p,q)}\{\frac{(\omega_{j,i} - E[\omega_{j,i}|X_i])m(X_i)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{(\omega_{q_1,p_1} - E[\omega_{q_1,p_1}|X_{p_1}])m(X_j)}{\sigma(X_j,\tilde{\theta}_0)}\} \\
&\quad \{\mathbb{I}(\varepsilon_i > \varepsilon_{p_1}) - \mathbb{I}(\varepsilon_i < \varepsilon_{p_1})\}(\|X_{ip_1}\| + \|X_{k_1l_1}\| - 2\|X_{ik_1}\|)|Z_i, Z_j], \\
H_{6ij} &= \frac{1}{4!}E[\sum_{(k_1,l_1,p_1,q_1)}^{(k,l,p,q)}\{\frac{(\omega_{p_1,q_1} - E[\omega_{p_1,q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)} - \frac{(\omega_{j,i} - E[\omega_{j,i}|X_i])m(X_i)}{\sigma(X_i,\tilde{\theta}_0)}\} \\
&\quad \{\mathbb{I}(\varepsilon_{q_1} > \varepsilon_j) - \mathbb{I}(\varepsilon_{q_1} < \varepsilon_j)\}(\|X_{q_1j}\| + \|X_{k_1l_1}\| - 2\|X_{q_1k_1}\|)|Z_i, Z_j].
\end{aligned}$$

For the term I_{15}^*, I_{16}^*, I_{17}^*, I_{18}^*, similar to the arguments for I_{14}, I_{15}, I_{16}, I_{17} in the proof of Theorem 3.1, we can show that $nI_{16}^* = o_p(1), nI_{17}^* = o_p(1), nI_{18}^* = o_p(1)$, and

$$I_{15}^* = (\hat{\theta}_n - \tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{15}^*(Z_i, Z_j, Z_k, Z_l) + o_p(1).$$

Hence we obtain that

$$\begin{aligned}
n\hat{U}_{n1}^* &= \frac{2}{n-1}\sum\sum_{1\le i<j\le n}[h_1^*(Z_i, Z_j) + h_2^*(Z_i, Z_j) + h_3^*(Z_i, Z_j)] \\
&\quad + n(\hat{\theta}_n - \tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l} h_{15}^*(Z_i, Z_j, Z_k, Z_l) + 4A_\varepsilon E\|X_1 - X_2\| + o_p(1)
\end{aligned}$$

Now we consider the term $\hat{U}_{n2}^*$. Recall that

$$\hat{U}_{n2}^* = \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{2st}^*(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{2st}^*\|X_{su}\|\right),$$

where

$$\begin{aligned}
\delta_{2st}^* &= 2\int_0^{\frac{\hat{m}(X_s)-m(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\hat{m}(X_t)-m(X_t)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)} + R_s - R_t} \\
&\quad \{\mathbb{I}(\varepsilon_s - \varepsilon_t \le z) - \mathbb{I}(\varepsilon_s \le \varepsilon_t)\}dz.
\end{aligned}$$

Following the same line as that for the term $\hat{U}_{n2}$ in Theorem 3.1, it can be decomposed as

$$\begin{aligned}
\hat{U}^*_{n2} &= \frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{21}(Z_i,Z_j,Z_k,Z_l) + \frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{22}(Z_i,Z_j,Z_k,Z_l) \\
&=: \hat{U}^*_{n21} + \hat{U}^*_{n22},
\end{aligned}$$

where

$$\begin{aligned}
h^*_{21}(Z_i,Z_j,Z_k,Z_l) &= \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)} E(\delta^*_{2st}|X_s,X_t)(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)} E(\delta^*_{2st}|X_s,X_t)\|X_{su}\|, \\
h^*_{22}(Z_i,Z_j,Z_k,Z_l) &= \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)} [\delta^*_{2st} - E(\delta^*_{2st}|X_s,X_t)](\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)} [\delta^*_{2st} - E(\delta^*_{2st}|X_s,X_t)]\|X_{su}\|.
\end{aligned}$$

For $\hat{U}^*_{n21}$, similar to the arguments in U_{21} in Theorem 3.1, we have uniformly over $1\le s,t\le n$,

$$\begin{aligned}
& E[\delta^*_{2st}|X_s,X_t] \\
= \; & 2E[\int_0^{\frac{\hat{m}(X_s)-m(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\hat{m}(X_t)-m(X_t)}{\sigma(X_t,\tilde{\theta}_0)} + \frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma(X_t,\tilde{\theta}_0)} + R_s - R_t} \{\mathbb{I}(\varepsilon_s-\varepsilon_t\le z) \\
& -\mathbb{I}(\varepsilon_s\le\varepsilon_t)\}dz|X_s,X_t] \\
= \; & Q_\varepsilon h^{2k}(\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)})^T(\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}) \\
& +Q_\varepsilon h^k(\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)})^T(\frac{\frac{1}{n-1}\sum_{p=1,p\ne s,t}^n \omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\ne s,t}^n \omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)}) \\
& +Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\ne s,t}^n \omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\ne s,t}^n \omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)})^T \\
& (\frac{\frac{1}{n-1}\sum_{p=1,p\ne s,t}^n \omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\ne s,t}^n \omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)}) \\
& +Q_\varepsilon h^k(\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)})^T(\frac{\frac{1}{n-1}\sum_{p=1,p\ne s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde{\theta}_0)} \\
& -\frac{\frac{1}{n-1}\sum_{q=1,q\ne t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde{\theta}_0)}) \\
& +Q_\varepsilon h^k(\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)})^T(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\ne s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\ne t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde{\theta}_0)}) \\
& +Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\ne s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\ne t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde{\theta}_0)})^T \\
& \times(\frac{\frac{1}{n-1}\sum_{p=1,p\ne s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\ne t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde{\theta}_0)}) \\
& +Q_\varepsilon(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\ne s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\ne t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde{\theta}_0)})^T
\end{aligned}$$

$$\times(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^{n}(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^{n}(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde{\theta}_0)})$$
$$+Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^{n}\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^{n}\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde{\theta}_0)})^T$$
$$\times(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^{n}(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^{n}(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde{\theta}_0)})$$
$$+Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^{n}\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^{n}\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde{\theta}_0)})^T$$
$$\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^{n}\omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^{n}\omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)})$$
$$+Q_\varepsilon(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^{n}(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^{n}(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde{\theta}_0)})^T$$
$$\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^{n}\omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^{n}\omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)})$$
$$+2A_\varepsilon\left([\frac{\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}]^T(\hat{\theta}_n-\tilde{\theta}_0)\right)^2+2A_\varepsilon(\hat{\theta}_n-\tilde{\theta}_0)^T[\frac{\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}]+o_p(\frac{1}{n}),$$

where $Q_\varepsilon=E[f_\varepsilon(\varepsilon)]$ and $A_\varepsilon=E[\varepsilon F_\varepsilon(\varepsilon)]$. Consequently, we obtain that

$$\begin{aligned}
\hat{U}^*_{n21} &= h^{2k}Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{211}(Z_i,Z_j,Z_k,Z_l)+h^kQ_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{212}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+h^kQ_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{213}(Z_i,Z_j,Z_k,Z_l)+h^kQ_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{214}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{215}(Z_i,Z_j,Z_k,Z_l)+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{216}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{217}(Z_i,Z_j,Z_k,Z_l)+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{218}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{219}(Z_i,Z_j,Z_k,Z_l)+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{2110}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+2A_\varepsilon(\hat{\theta}_n-\tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{2111}(Z_i,Z_j,Z_k,Z_l)\\
&\quad+2A_\varepsilon(\hat{\theta}_n-\tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{2112}(Z_i,Z_j,Z_k,Z_l)(\hat{\theta}_n-\tilde{\theta}_0)+o_p(\frac{1}{n}),\\
&=: h^{2k}Q_\varepsilon\hat{U}^*_{n211}+h^kQ_\varepsilon\hat{U}^*_{n212}+h^kQ_\varepsilon\hat{U}^*_{n213}+h^kQ_\varepsilon\hat{U}^*_{n214}+Q_\varepsilon\hat{U}^*_{n215}+Q_\varepsilon\hat{U}^*_{n216}+Q_\varepsilon\hat{U}^*_{n217}\\
&\quad+Q_\varepsilon\hat{U}^*_{n218}+Q_\varepsilon\hat{U}^*_{n219}+Q_\varepsilon\hat{U}^*_{n2110}+2A_\varepsilon\hat{U}^*_{n2111}+2A_\varepsilon\hat{U}^*_{n2112}+o_p(\frac{1}{n}), \qquad (6.22)
\end{aligned}$$

where

$$h^*_{21m}(Z_i,Z_j,Z_k,Z_l)$$

$$= 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\delta^*_{21mst}(\|X_{st}\|+\|X_{uv}\|) - 12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\delta^*_{21mst}\|X_{su}\|, \quad \text{for } m=1,\cdots,12$$

and

$$\delta^*_{211st} = (\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)})(\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)})^T$$

$$\delta^*_{212st} = \{\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n \omega_{s,p}\sigma(X_p,\tilde\theta_0)\varepsilon_p}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n \omega_{t,q}\sigma(X_q,\tilde\theta_0)\varepsilon_q}{\sigma(X_t,\tilde\theta_0)}\}\{\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)}\}^T$$

$$\delta^*_{213st} = (\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)})^T(\frac{\frac{m(X_p)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s])}{\sigma(X_s,\tilde\theta_0)}$$
$$-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t])}{\sigma(X_t,\tilde\theta_0)})$$

$$\delta^*_{214st} = (\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)} - \frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)})^T(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde\theta_0)})$$

$$\delta^*_{215st} = \{\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n \omega_{s,p}\sigma(X_p,\tilde\theta_0)\varepsilon_p}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n \omega_{t,q}\sigma(X_q,\tilde\theta_0)\varepsilon_q}{\sigma(X_t,\tilde\theta_0)}\}$$
$$\times\{\frac{\frac{1}{n-1}\sum_{p'=1,p'\neq s,t}^n \omega_{s,p'}\sigma(X_{p'},\tilde\theta_0)\varepsilon_{p'}}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q'=1,q'\neq s,t}^n \omega_{t,q'}\sigma(X_{q'},\tilde\theta_0)\varepsilon_{q'}}{\sigma(X_t,\tilde\theta_0)}\}^T$$

$$\delta^*_{216st} = (\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde\theta_0)})^T$$
$$\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde\theta_0)})$$

$$\delta^*_{217st} = (\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde\theta_0)})^T$$
$$\times(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde\theta_0)})$$

$$\delta^*_{218st} = (\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde\theta_0)})^T$$
$$\times(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde\theta_0)})$$

$$\delta^*_{219st} = (\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n \omega_{s,p}m(X_p) - E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}m(X_q) - E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\tilde\theta_0)})^T$$
$$\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n \omega_{s,p}\sigma(X_p,\tilde\theta_0)\varepsilon_p}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n \omega_{t,q}\sigma(X_q,\tilde\theta_0)\varepsilon_q}{\sigma(X_t,\tilde\theta_0)})$$

$$\delta^*_{2110st} = (\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s,\tilde\theta_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t,\tilde\theta_0)})^T$$

$$\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^{n}\omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^{n}\omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)})$$

$$\delta^*_{2111st} = \frac{\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}$$

$$\delta^*_{2112st} = \{\frac{\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}\}\{\frac{\dot{\sigma}(X_s,\tilde{\theta}_0)}{\sigma(X_s,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_t,\tilde{\theta}_0)}{\sigma(X_t,\tilde{\theta}_0)}\}^T.$$

For the term $\hat{U}^*_{n211}$, by the law of large numbers, we have

$$\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{211}(Z_i,Z_j,Z_k,Z_l)\longrightarrow E[h^*_{211}(Z_i,Z_j,Z_k,Z_l)],\quad \text{in probability},$$

where

$$E[h^*_{211}(Z_i,Z_j,Z_k,Z_l)] = E[(\frac{D(X_1)}{\sigma(X_1,\tilde{\theta}_0)}-\frac{D(X_2)}{\sigma(X_2,\tilde{\theta}_0)})(\frac{D(X_1)}{\sigma(X_1,\tilde{\theta}_0)}-\frac{D(X_2)}{\sigma(X_2,\tilde{\theta}_0)})^T C_x(X_1,X_2)],$$

and $C_x(X_1,X_2)=\|X_1-X_2\|-E(\|X_1-X_2\||X_1)-E(\|X_1-X_2\||X_2)+E(\|X_1-X_2\|)$. Together with the assumption 6(d), we have

$$nh^{2k}Q_\varepsilon\hat{U}^*_{n211} = nh^{2k}Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{211}(Z_i,Z_j,Z_k,Z_l)=o_p(1).$$

For the term $\hat{U}^*_{n212}$, it can be decomposed as

$$\begin{aligned}
\hat{U}^*_{n212} &= \frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j=1.j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\sum_{l=1,l\neq i,j,k}^{n}\sum_{p=1,p\neq i,j}^{n}\sum_{q=1,q\neq i,j}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}\\
&\quad -\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-\|X_{ik}\|)\\
&= \frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p=q\neq i,j}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\\
&\quad \times\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
&\quad +\frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p=k}\sum_{q\neq i,j,k}^{n}\{\frac{\omega_{k,i}\sigma(X_k,\tilde{\theta}_0)\varepsilon_k}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\\
&\quad \times\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
&\quad +\frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p=l}\sum_{q\neq i,j,l}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{p,j}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_j,\tilde{\theta}_0)}\}\\
&\quad \times\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{q=k}\sum_{p\neq i,j,k}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\
&\times\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
&+\frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{q=l}\sum_{p\neq i,j,l}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)}-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\} \\
&\times\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
&+\frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p\neq i\neq j\neq k\neq l}^{n}\sum_{q\neq i\neq j\neq k\neq l\neq p}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\
&-\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
=:\ & O_p(\frac{1}{n})+\hat{U}^*_{n2121},
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{U}^*_{n2121} &= \frac{1}{n(n-1)^2(n-2)^2(n-3)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n}\sum_{l\neq i,j,k}^{n}\sum_{p\neq i\neq j\neq k\neq l}^{n}\sum_{q\neq i\neq j\neq k\neq l\neq p}^{n}\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\
&\quad -\frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|) \\
&= \frac{(n-4)(n-5)}{(n-1)(n-2)}\frac{1}{C_n^6}\sum\sum\sum\sum\sum\sum_{1\leq i<j<k<l<p<q\leq n}\delta^*_{2121}(Z_i,Z_j,Z_k,Z_l,Z_p,Z_q),
\end{aligned}
$$

with

$$
\begin{aligned}
&\delta^*_{2121}(Z_i,Z_j,Z_k,Z_l,Z_p,Z_q) \\
=\ & \frac{1}{6!}\sum_{(s,t,u,v,r,m)}^{(i,j,k,l,p,q)}\{\frac{\omega_{r,s}\sigma(X_r,\tilde{\theta}_0)\varepsilon_r}{\sigma(X_s,\tilde{\theta}_0)}-\frac{\omega_{m,t}\sigma(X_m,\tilde{\theta}_0)\varepsilon_m}{\sigma(X_t,\tilde{\theta}_0)}\}\{\frac{D(X_s)}{\sigma(X_s,\tilde{\theta}_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde{\theta}_0)}\}^T \\
& (\|X_{st}\|+\|X_{uv}\|-2\|X_{su}\|).
\end{aligned}
$$

Note that $E[\delta^*_{2121}(Z_i,Z_j,Z_k,Z_l,Z_p,Z_q)]=0$, it follows that $nh^k\hat{U}^*_{n2121}=o_p(1)$. Consequently, we obtain that

$$nh^k\hat{U}^*_{n212}=o_p(1).$$

Similarly, we can show that

$$nh^k\hat{U}^*_{n213}=o_p(1) \quad \text{and} \quad nh^k\hat{U}^*_{n214}=o_p(1).$$

For the term $\hat{U}^*_{n215}$, recall that

$$\hat{U}^*_{n215} = \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta^*_{215}(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta^*_{215}\|X_{su}\|\right)$$

with

$$\delta^*_{215} = \{\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^{n}\omega_{s,p}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^{n}\omega_{t,q}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_t,\tilde{\theta}_0)}\}$$
$$\times\{\frac{\frac{1}{n-1}\sum_{p'=1,p'\neq s,t}^{n}\omega_{s,p'}\sigma(X_{p'},\tilde{\theta}_0)\varepsilon_{p'}}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q'=1,q'\neq s,t}^{n}\omega_{t,q'}\sigma(X_{q'},\tilde{\theta}_0)\varepsilon_{q'}}{\sigma(X_t,\tilde{\theta}_0)}\}^T$$

By some tedious calculations, we have

$$\begin{aligned}
&\hat{U}^*_{n215}\\
=\;& \frac{1}{n(n-1)^3(n-2)^3(n-3)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\sum_{l=1,l\neq i,j,k}^{n}\sum_{p=1,p\neq i,j}^{n}\sum_{q=1,q\neq i,j}^{n}\sum_{p'=1,p'\neq i,j}^{n}\sum_{q'=1,q'\neq i,j}^{n}\\
&\{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\frac{\omega_{p',i}\sigma(X_{p'},\tilde{\theta}_0)\varepsilon_{p'}}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{q',j}\sigma(X_{q'},\tilde{\theta}_0)\varepsilon_{q'}}{\sigma(X_j,\tilde{\theta}_0)}\}^T(\|X_{ij}\|+\|X_{kl}\|-2\|X_{ik}\|)\\
=\;& \frac{(n-4)(n-5)(n-6)(n-7)}{(n-1)^2(n-2)^2}\frac{1}{C_n^8}\sum\sum\sum\sum\sum\sum\sum\sum_{1\le i<j<k<l<p<p'<q<q'\le n}\delta^*_{215}(Z_i,Z_j,Z_k,Z_l,Z_p,Z_{p'},Z_q,Z_{q'}) + o_p(\frac{1}{n})\\
=:\;& \hat{U}^*_{n2151} + o_p(\frac{1}{n}),
\end{aligned}$$

where

$$\begin{aligned}
&\delta^*_{215}(Z_i,Z_j,Z_k,Z_l,Z_p,Z_{p'},Z_q,Z_{q'})\\
=\;& \frac{1}{8!}\sum_{(s,t,u,v,r,m,n,e)}^{(i,j,k,l,p,p',q,q')}\{\frac{\omega_{r,s}\sigma(X_r,\tilde{\theta}_0)\varepsilon_r}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\omega_{m,t}\sigma(X_m,\tilde{\theta}_0)\varepsilon_m}{\sigma(X_t,\tilde{\theta}_0)}\}\{\frac{\omega_{n,s}\sigma(X_n,\tilde{\theta}_0)\varepsilon_n}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\omega_{e,t}\sigma(X_e,\tilde{\theta}_0)\varepsilon_e}{\sigma(X_t,\tilde{\theta}_0)}\}\\
&\times(\|X_{st}\|+\|X_{uv}\|-2\|X_{su}\|).
\end{aligned}$$

Some elementary calculations show that $E[\delta^*_{215}(Z_i,Z_j,Z_k,Z_l,Z_p,Z_{p'},Z_q,Z_{q'})] = 0$ and $\hat{U}^*_{n2151}$ is degenerate of order 1. By the arguments in Section 5.3.4 of Serfling (2009), we can obtain

$$n\hat{U}^*_{n2151} = \frac{2}{(n-1)}\sum\sum_{1\le i<j\le n}\tilde{h}^*_5(Z_i,Z_j) + o_p(1),$$

where $\tilde{h}^*_5(Z_i,Z_j) = \frac{1}{2}(H_{7ij}+H_{8ij})$ and

$$\begin{aligned}
H_{7ij} =\;& \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}\sigma(X_i,\tilde{\theta}_0)\varepsilon_i}{\sigma(X_{p_1},\tilde{\theta}_0)} - \frac{\omega_{q_1,p_1'}\sigma(X_{q_1},\tilde{\theta}_0)\varepsilon_{q_1}}{\sigma(X_{p_1'},\tilde{\theta}_0)}\}\{\frac{\omega_{j,p_1}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_{p_1},\tilde{\theta}_0)} - \frac{\omega_{q_1',p_1'}\sigma(X_{q_1'},\tilde{\theta}_0)\varepsilon_{q_1'}}{\sigma(X_{p_1}',\tilde{\theta}_0)}\}\\
&(\|X_{(p_1)}^{(p_1')}\|+\|X_{(k_1)}^{(l_1)}\|-2\|X_{(p_1)}^{(k_1)}\|)|Z_i,Z_j],\\
H_{8ij} =\;& \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}\sigma(X_i,\tilde{\theta}_0)\varepsilon_i}{\sigma(X_{p_1},\tilde{\theta}_0)} - \frac{\omega_{q_1,q_1'}\sigma(X_{q_1},\tilde{\theta}_0)\varepsilon_{q_1}}{\sigma(X_{q_1'},\tilde{\theta}_0)}\}\{\frac{\omega_{p_1',p_1}\sigma(X_{p_1'},\tilde{\theta}_0)\varepsilon_{p_1'}}{\sigma(X_{p_1},\tilde{\theta}_0)} - \frac{\omega_{j,q_1'}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_{q_1'},\tilde{\theta}_0)}\}\\
&(\|X_{(p_1)}^{(q_1')}\|+\|X_{(k_1)}^{(l_1)}\|-2\|X_{(p_1)}^{(k_1)}\|)|Z_i,Z_j].
\end{aligned}$$

For the term $\hat{U}^*_{n21m}, m = 6,\cdots,10$, similar to the arguments for $\hat{U}^*_{n215}$, we have

$$n\hat{U}^*_{n216} = \frac{2}{n-1}\sum\sum_{1\le i<j\le n}\tilde{h}^*_6(Z_i,Z_j) + o_p(1),$$

$$n\hat{U}^*_{n217} = \frac{2}{n-1}\sum\sum_{1\le i<j\le n}\tilde{h}^*_7(Z_i,Z_j)+o_p(1),$$
$$n\hat{U}^*_{n218} = \frac{2}{n-1}\sum\sum_{1\le i<j\le n}\tilde{h}^*_8(Z_i,Z_j)+o_p(1),$$
$$n\hat{U}^*_{n219} = \frac{2}{n-1}\sum\sum_{1\le i<j\le n}\tilde{h}^*_9(Z_i,Z_j)+o_p(1),$$
$$n\hat{U}^*_{n2110} = \frac{2}{n-1}\sum\sum_{1\le i<j\le n}\tilde{h}^*_{10}(Z_i,Z_j)+o_p(1),$$

where $\tilde{h}^*_6(Z_i,Z_j)=\frac{1}{2}(H_{9ij}+H_{10ij})$, $\tilde{h}^*_7(Z_i,Z_j)=\frac{1}{2}(H_{11ij}+H_{12ij})$, $\tilde{h}^*_8(Z_i,Z_j)=\frac{1}{2}(H_{13ij}+H_{14ij})$, $\tilde{h}^*_9(Z_i,Z_j)=\frac{1}{2}(H_{15ij}+H_{16ij})$, $\tilde{h}^*_{10}(Z_i,Z_j)=\frac{1}{2}(H_{17ij}+H_{18ij})$,

$$\begin{aligned}
H_{9ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}m(X_i)-E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{p_1',q_1}m(X_{p_1'})-E[\omega_{p_1',q_1}m(X_{p_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\}\\
&\quad\times\{\frac{\omega_{j,p_1}m(X_j)-E[\omega_{j,p_1}m(X_j)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{q_1',q_1}m(X_{q_1'})-E[\omega_{q_1',q_1}m(X_{q_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|\\
&\quad-2\|X_{p_1k_1}\|)|Z_i,Z_j],\\
H_{10ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}m(X_i)-E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{p_1',q_1}m(X_{p_1'})-E[\omega_{p_1',q_1}m(X_{p_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\}\\
&\quad\times\{\frac{\omega_{q_1',p_1}m(X_{q_1'})-E[\omega_{q_1',p_1}m(X_{q_1'})|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{j,q_1}m(X_j)-E[\omega_{j,q_1}m(X_j)|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|\\
&\quad-2\|X_{p_1k_1}\|)|Z_i,Z_j],\\
H_{11ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{(\omega_{i,p_1}-E[\omega_{i,p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{p_1',q_1}-E[\omega_{p_1',q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\}\\
&\quad\times\{\frac{(\omega_{j,p_1}-E[\omega_{j,p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{q_1',q_1}-E[\omega_{q_1',q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1)}\|\\
&\quad+\|X_{k_1l_1}\|-2\|X_{p_1k_1}\|)|Z_i,Z_j],\\
H_{12ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{(\omega_{i,p_1}-E[\omega_{i,p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{p_1',q_1}-E[\omega_{p_1',q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\}\\
&\quad\times\{\frac{(\omega_{q_1',p_1}-E[\omega_{q_1',p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{j,q_1}-E[\omega_{j,q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|\\
&\quad+\|X_{k_1l_1}\|-2\|X_{p_1k_1}\|)|Z_i,Z_j],\\
H_{13ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}m(X_i)-E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{p_1',q_1}m(X_{p_1'})-E[\omega_{p_1',q_1}m(X_{p_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\}\\
&\quad\times\{\frac{(\omega_{j,p_1}-E[\omega_{j,p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{q_1',q_1}-E[\omega_{q_1',q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|\\
&\quad-2\|X_{p_1k_1}\|)|Z_i,Z_j],
\end{aligned}$$

$$\begin{aligned}
H_{14ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}m(X_i)-E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{p_1',q_1}m(X_{p_1'})-E[\omega_{p_1',q_1}m(X_{p_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\} \\
&\times\{\frac{(\omega_{q_1',p_1}-E[\omega_{q_1',p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{j,q_1}-E[\omega_{j,q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\| \\
&-2\|X_{p_1k_1}\|)|Z_i,Z_j], \\
H_{15ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}m(X_i)-E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{p_1',q_1}m(X_{p_1'})-E[\omega_{p_1',q_1}m(X_{p_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\} \\
&\times\{\frac{\omega_{j,p_1}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{q_1',q_1}\sigma(X_{q_1'},\tilde{\theta}_0)\varepsilon_{q_1'}}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|-2\|X_{p_1k_1}\|)|Z_i,Z_j], \\
H_{16ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{\omega_{i,p_1}m(X_i)-E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{p_1',q_1}m(X_{p_1'})-E[\omega_{p_1',q_1}m(X_{p_1'})|X_{q_1}]}{\sigma(X_{q_1},\tilde{\theta}_0)}\} \\
&\times\{\frac{\omega_{q_1',p_1}\sigma(X_{q_1'},\tilde{\theta}_0)\varepsilon_{q_1'}}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{j,q_1}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|-2\|X_{p_1k_1}\|)|Z_i,Z_j], \\
H_{17ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{(\omega_{i,p_1}-E[\omega_{i,p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{p_1',q_1}-E[\omega_{p_1',q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\} \\
&\times\{\frac{\omega_{j,p_1}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{q_1',q_1}\sigma(X_{q_1'},\tilde{\theta}_0)\varepsilon_{q_1'}}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|-2\|X_{p_1k_1}\|)|Z_i,Z_j], \\
H_{18ij} &= \frac{1}{6!}E[\sum_{(k_1,l_1,p_1,q_1,p_1',q_1')}^{(k,l,p,q,p',q')}\{\frac{(\omega_{i,p_1}-E[\omega_{i,p_1}|X_{p_1}])m(X_{p_1})}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{(\omega_{p_1',q_1}-E[\omega_{p_1',q_1}|X_{q_1}])m(X_{q_1})}{\sigma(X_{q_1},\tilde{\theta}_0)}\} \\
&\times\{\frac{\omega_{q_1',p_1}\sigma(X_{q_1'},\tilde{\theta}_0)\varepsilon_{q_1'}}{\sigma(X_{p_1},\tilde{\theta}_0)}-\frac{\omega_{j,q_1}\sigma(X_j,\tilde{\theta}_0)\varepsilon_j}{\sigma(X_{q_1},\tilde{\theta}_0)}\}^T(\|X_{p_1q_1}\|+\|X_{k_1l_1}\|-2\|X_{p_1k_1}\|)|Z_i,Z_j].
\end{aligned}$$

For the last two terms $\hat{U}^*_{n2111}$ and $\hat{U}^*_{n2112}$, similar to the arguments for $\hat{U}_{n212}$ and $\hat{U}_{n213}$ in the proof of Theorem 3.1, we can obtain that

$$\begin{aligned}
n\hat{U}^*_{n2111} &= \sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}4E[h^*_{2111}(Z_i,Z_j,Z_k,Z_l|Z_i)]+o_p(1). \\
n\hat{U}^*_{n2112} &= \sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)^TM_2\sqrt{n}(\hat{\theta}_n-\theta_0)+o_p(1).
\end{aligned}$$

where

$$E[h^*_{2111}(Z_i,Z_j,Z_k,Z_l)|Z_i]=E[\{\frac{\dot{\sigma}(X_i,\beta_0)}{\sigma(X_i,\theta_0)}+\frac{\dot{\sigma}(X,\beta_0)}{\sigma(X,\theta_0)}\}C_x(X_i,X)|X_i]$$

$$M_2=E[\{\frac{\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma(X_1,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma(X_2,\tilde{\theta}_0)}\}\{\frac{\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma(X_1,\tilde{\theta}_0)}+\frac{\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma(X_2,\tilde{\theta}_0)}\}^TC_x(X_1,X_2)].$$

with $C_x(X_1,X_2)=\|X_1-X_2\|-E(\|X_1-X_2\||X_1)-E(\|X_1-X_2\||X_2)+E(\|X_1-X_2\|)$. Hence we obtain that

$$nU^*_{n21}$$

$$
\begin{aligned}
&= \frac{2Q_\varepsilon}{n-1}\sum\sum_{1\le i<j\le n}[\tilde{h}_5^*(Z_i,Z_j)+\tilde{h}_6^*(Z_i,Z_j)+\tilde{h}_7^*(Z_i,Z_j)+\tilde{h}_8^*(Z_i,Z_j)+\tilde{h}_9^*(Z_i,Z_j)+\tilde{h}_{10}^*(Z_i,Z_j)] \\
&\quad +\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)\frac{1}{\sqrt{n}}\sum_{i=1}^n 4E[h_{2111}^*(Z_i,Z_j,Z_k,Z_l)|Z_i]+\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)^T M_2\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)+o_p(1).
\end{aligned}
$$

For the term U_{n22}^*, similar to the arguments in Theorem 1 of Xu and Cao (2021), we can also show that $nU_{n22}^*=o_p(1)$. Hence we obtain that

$$
\begin{aligned}
n\hat{U}_{n2}^* &= \frac{2Q_\varepsilon}{n-1}\sum\sum_{1\le i<j\le n}\sum_{k=5}^{10}\tilde{h}_k^*(Z_i,Z_j)+\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)\frac{1}{\sqrt{n}}\sum_{i=1}^n 4E[h_{2111}^*(Z_i,Z_j,Z_k,Z_l)|Z_i] \\
&\quad +\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)^T M_2\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)+o_p(1).
\end{aligned}
$$

For the term $\hat{U}_{n0}^*$, note that it is the same as the term $\hat{U}_{n0}$ in the proof of Theorem 3.1. Consequently,

$$
\begin{aligned}
n\hat{U}_{n0}^* &= \frac{6}{n-1}\sum_{i=1}^n\sum_{j\neq i}^n E\{h_0(Z_i,Z_j,Z_k,Z_l)|Z_i,Z_j\}+o_p(1) \\
&= \frac{1}{n-1}\sum_{i=1}^n\sum_{j\neq i}^n C_\varepsilon(\varepsilon_i,\varepsilon_j)C_x(X_i,X_j)+o_p(1).
\end{aligned}
$$

Here $h_0(Z_i,Z_j,Z_k,Z_l)=\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\varepsilon_{st}|(\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\varepsilon_{st}|\|X_{su}\|$. Hence we obtain that

$$
\begin{aligned}
n\hat{U}_n^* &= \frac{1}{n-1}\sum_{i=1}^n\sum_{j\neq i}^n C_\varepsilon(\varepsilon_i,\varepsilon_j)C_x(X_i,X_j)+\frac{2}{n-1}\sum\sum_{1\le i<j\le n}\sum_{k=1}^{3}h_k^*(Z_i,Z_j) \\
&\quad +n(\hat{\theta}_n-\tilde{\theta}_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h_{15}^*(Z_i,Z_j,Z_k,Z_l)+\frac{2Q_\varepsilon}{n-1}\sum\sum_{1\le i<j\le n}\sum_{k=5}^{10}\tilde{h}_k^*(Z_i,Z_j) \\
&\quad +\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)\frac{1}{\sqrt{n}}\sum_{i=1}^n 4E[h_{2111}^*(Z_i,Z_j,Z_k,Z_l)|Z_i] \\
&\quad +\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)^T M_2\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0)+4A_\varepsilon E\|X_1-X_2\|+o_p(1).
\end{aligned}
$$

To obtain the limiting distribution of $n\hat{U}_n$, it remains to derive the asymptotic expansion of $\hat{\theta}_n-\tilde{\theta}_0$. By proof of theorem 1 in Appendix A of Dette et al. (2007), we have

$$
\sqrt{n}(\hat{\theta}_n-\tilde{\theta}_0) = \frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i)\varepsilon_i^2-\sigma^2(X_i,\tilde{\theta}_0)]\Sigma^{-1}\dot{\sigma}^2(X_i,\tilde{\theta}_0)+o_p(1).
$$

Altogether we obtain that

$$
\begin{aligned}
&n\hat{U}_n \\
&= \frac{1}{n-1}\sum_{i=1}^n\sum_{j\neq i}^n H(Z_i,Z_j)+4\frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i,\tilde{\theta}_0)(\varepsilon_i^2-1)]\dot{\sigma}^2(X_i,\tilde{\theta}_0)^T\Sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n E[h_{15}^*(Z_i,Z_j,Z_k,Z_l)|Z_i]
\end{aligned}
$$

$$
\begin{aligned}
&+8A_\varepsilon\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\tilde\theta_0)^T\Sigma^{-1}E[h^*_{2111}(Z_i,Z_j,Z_k,Z_l)|Z_i]\\
&+2A_\varepsilon\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\tilde\theta_0)^T\Sigma^{-1}M_2\Sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\sigma^2(X_i,\tilde\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\tilde\theta_0)\\
&+4A_\varepsilon E\|X_1-X_2\|+o_p(1),
\end{aligned}
$$

where $H(Z_i,Z_j)=C_\varepsilon(\varepsilon_i,\varepsilon_j)C_x(X_i,X_j)+\sum_{k=1}^{3}h^*_k(Z_i,Z_j)+Q_\varepsilon\sum_{k=5}^{10}\tilde h^*_k(Z_i,Z_j)$. Finally, we obtain that

$$
\begin{aligned}
n\hat U_n \quad \longrightarrow \quad & \sum_{k=1}^{\infty}\lambda_k(\mathcal{Z}_k^2-1)+4\mathcal{W}^T\Sigma^{-1}\mathcal{P}_2+8A_\varepsilon\mathcal{W}^T\Sigma^{-1}\mathcal{P}_3\\
&+2A_\varepsilon\mathcal{W}^\top\Sigma^{-1}M_2\Sigma^{-1}\mathcal{W}+4A_\varepsilon E\|X_1-X_2\|,
\end{aligned}
$$

where $\mathcal{Z}_1,\mathcal{Z}_2,..$ are independent standard normal random variables, the eigenvalues $\{\lambda_q\}_{q=1}^{\infty}$ are the solutions of the integral equation

$$
\int H(Z_i,Z_j)\psi_q(Z_j)dF(Z_j)=\lambda_q\psi_q(Z_i),
$$

with $\{\psi_i(\cdot)\}_{i=1}^{\infty}$ being the orthonormal eigenfunctions and $F_Z(\cdot)$ being the cumulative distribution function of Z, and $(\mathcal{Z}_i,\mathcal{W},\mathcal{P}_2,\mathcal{P}_3)\in\mathbb{R}^{3p+1}$ is a Gaussian random vector with zero-mean and the covariance matrix satisfying

$$
\begin{aligned}
var(\mathcal{Z}_i) &= 1\\
var(\mathcal{P}_2) &= var(E\{h^*_{15}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
var(\mathcal{P}_3) &= var(E\{h^*_{2111}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
var(\mathcal{W}) &= var([\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0))\\
cov(\mathcal{Z}_i,\mathcal{P}_2) &= cov(\psi_i(Z_i),E\{h^*_{15}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{Z}_i,\mathcal{P}_3) &= cov(\psi_i(Z_i),E\{h^*_{2111}(Z_i,Z_j,Z_k,Z_l)|Z_i\})\\
cov(\mathcal{Z}_i,\mathcal{W}) &= cov(\psi_i(Z_i),[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0))\\
cov(\mathcal{P}_2,\mathcal{W}) &= cov(E\{h^*_{15}(Z_i,Z_j,Z_k,Z_l)|Z_i\},[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0))\\
cov(\mathcal{P}_3,\mathcal{W}) &= cov(E\{h^*_{2111}(Z_i,Z_j,Z_k,Z_l)|Z_i\},[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0))\\
cov(\mathcal{P}_2,\mathcal{P}_3) &= cov(E\{h^*_{15}(Z_i,Z_j,Z_k,Z_l)|Z_i\},E\{h^*_{2111}(Z_i,Z_j,Z_k,Z_l)|Z_i\}).
\end{aligned}
$$

Hence we complete the proof of the first part of Theorem 3.3.

(2) In this part we discuss the asymptotic properties of $n\hat U_n$ under the local alternatives H_{1n}. Recall that $\eta_i=\frac{Y_i-m(X_i)}{\sigma(X_i,\theta_0)}$ and $\hat\eta_i=\frac{Y_i-\hat m(X_i)}{\sigma(X_i,\hat\theta_n)}$ in nonparametric cases. It follows from (6.1) and (6.2) that

$$
\begin{aligned}
&|\hat\eta_i-\hat\eta_j|\\
&=|\varepsilon_i-\varepsilon_j|\\
&-\{P^*_{ij}+\frac{\varepsilon_i s(X_i)[\sigma(X_i,\hat\theta_n)-\sigma(X_i,\theta_0)]}{2\sqrt{n}\sigma^3(X_i,\theta_0)}-\frac{\varepsilon_j s(X_j)[\sigma(X_j,\hat\theta_n)-\sigma(X_j,\theta_0)]}{2\sqrt{n}\sigma^3(X_j,\theta_0)}\}[\mathbb{I}(\varepsilon_i>\varepsilon_j)-\mathbb{I}(\varepsilon_i<\varepsilon_j)]\\
&+2\int_0^{P^*_{ij}+\frac{\varepsilon_i s(X_i)[\sigma(X_i,\hat\theta_n)-\sigma(X_i,\theta_0)]}{2\sqrt{n}\sigma^3(X_i,\theta_0)}-\frac{\varepsilon_j s(X_j)[\sigma(X_j,\hat\theta_n)-\sigma(X_j,\theta_0)]}{2\sqrt{n}\sigma^3(X_j,\theta_0)}}\{\mathbb{I}(\varepsilon_i-\varepsilon_j\le z)-\mathbb{I}(\varepsilon_i\le\varepsilon_j)\}dz,
\end{aligned}
\tag{6.23}
$$

where

$$
\begin{aligned}
P_{ij}^* &= \frac{\hat{m}(X_i)-m(X_i)}{\sigma(X_i,\theta_0)} - \frac{\hat{m}(X_j)-m(X_j)}{\sigma(X_j,\theta_0)} + \frac{\varepsilon_i(\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\theta_0))}{\sigma(X_i,\theta_0)} - \frac{\varepsilon_j(\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\theta_0))}{\sigma(X_j,\theta_0)} \\
&\quad +(R_i-R_j). \\
R_i &= \frac{\varepsilon_i[\sigma(X_i,\theta_0)-\sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\theta_0)\sigma(X_i,\hat{\theta}_n)} + \frac{\hat{m}(X_i)-m(X_i)}{\sigma^2(X_i,\theta_0)}[\sigma(X_i,\theta_0)-\sigma(X_i,\hat{\theta}_n)] \\
&\quad + \frac{\hat{m}(X_i)-m(X_i)}{\sigma^2(X_i,\theta_0)} \frac{[\sigma(X_i,\theta_0)-\sigma(X_i,\hat{\theta}_n)]^2}{\sigma(X_i,\hat{\theta}_n)}.
\end{aligned}
$$

Similar to the arguments for (6.5) in the proof of Theorem 3.1, $n\hat{U}_n$ can be decomposed as

$$
\begin{aligned}
\hat{U}_n &= \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\varepsilon_{st}|(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\varepsilon_{st}|\|X_{su}\|\right) \\
&\quad + \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{1st}^*(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{1st}^*\|X_{su}\|\right) \\
&\quad + \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{2st}^*(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{2st}^*\|X_{su}\|\right) \\
&\quad + \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{3st}^*(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{3st}^*\|X_{su}\|\right) \\
&=: \hat{U}_{n0}^* + \hat{U}_{n1}^* + \hat{U}_{n2}^* + \hat{U}_{n3}^*, \qquad (6.24)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{1st}^* &= -[\frac{\varepsilon_s(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{\sigma(X_s,\theta_0)} - \frac{\varepsilon_t(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{\sigma(X_t,\theta_0)} + \frac{\hat{m}(X_s)-m(X_s)}{\sigma(X_s,\theta_0)} - \frac{\hat{m}(X_t)-m(X_t)}{\sigma(X_t,\theta_0)} \\
&\quad +(R_s-R_t)]\{\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)\} \\
\delta_{2st}^* &= -\{\frac{\varepsilon_s s(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{2\sqrt{n}\sigma^3(X_s,\theta_0)} - \frac{\varepsilon_t s(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{2\sqrt{n}\sigma^3(X_t,\theta_0)}\}[\mathbb{I}(\varepsilon_s>\varepsilon_t)-\mathbb{I}(\varepsilon_s<\varepsilon_t)]. \\
\delta_{3st}^* &= 2\int_0^{P_{st}^*+\frac{\varepsilon_s s(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\theta_0))}{2\sqrt{n}\sigma^3(X_s,\theta_0)}-\frac{\varepsilon_t s(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\theta_0))}{2\sqrt{n}\sigma^3(X_t,\theta_0)}}[\mathbb{I}(\varepsilon_s-\varepsilon_t\le z)-\mathbb{I}(\varepsilon_s\le\varepsilon_t)]dz.
\end{aligned}
$$

For the term $\hat{U}_{n3}^*$, decomposed it as

$$
\begin{aligned}
\hat{U}_{n3}^* &= \frac{1}{C_n^4}\sum_{i<j<k<l}h_{31}^*(Z_i,Z_j,Z_k,Z_l) + \frac{1}{C_n^4}\sum_{i<j<k<l}h_{32}^*(Z_i,Z_j,Z_k,Z_l) \\
&=: \hat{U}_{n31}^* + \hat{U}_{n32}^*,
\end{aligned}
$$

where

$$
h_{31}^*(Z_i,Z_j,Z_k,Z_l) = \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}E(\delta_{3st}^*|X_s,X_t)(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}E(\delta_{3st}^*|X_s,X_t)\|X_{su}\|,
$$

$$h^*_{32}(Z_i,Z_j,Z_k,Z_l) \;=\; \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}[\delta^*_{3st}-E(\delta^*_{3st}|X_s,X_t)](\|X_{st}\|+\|X_{uv}\|)-\frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}[\delta^*_{3st}-E(\delta^*_{3st}|X_s,X_t)]\|X_{su}\|.$$

For the term $\hat{U}^*_{n31}$, similar to the arguments for $\hat{U}^*_{n21}$ in the proof of Theorem 3.2(1), we have uniformly over $1\le s,t\le n$,

$$\begin{aligned}
& E[\delta^*_{3st}|X_s,X_t] \\
= \;& 2E[\int_0^{\frac{\hat m(X_s)-m(X_s)}{\sigma(X_s,\theta_0)}-\frac{\hat m(X_t)-m(X_t)}{\sigma(X_t,\theta_0)}+\frac{\varepsilon_s(\sigma(X_s,\hat\theta_n)-\sigma(X_s,\theta_0))}{\sigma(X_s,\theta_0)}-\frac{\varepsilon_t(\sigma(X_t,\hat\theta_n)-\sigma(X_t,\theta_0))}{\sigma(X_t,\theta_0)}+R_s-R_t}\{\mathbb{I}(\varepsilon_s-\varepsilon_t\le z) \\
& -\mathbb{I}(\varepsilon_s\le\varepsilon_t)\}dz|X_s,X_t] \\
& +2E[\int_0^{(P_{st}+\frac{\varepsilon_i s(X_i)(\sigma(X_i,\hat\theta_n)-\sigma(X_i,\theta_0))}{2\sqrt{n}\sigma^3(X_i,\theta_0)}-\frac{\varepsilon_j s(X_j)(\sigma(X_j,\hat\theta_n)-\sigma(X_j,\tilde\theta_0))}{2\sqrt{n}\sigma^3(X_j,\theta_0)})}\{\mathbb{I}(\varepsilon_s-\varepsilon_t\le z)-\mathbb{I}(\varepsilon_s\le\varepsilon_t)\}dz|X_s,X_t] \\
= \;& Q_\varepsilon h^{2k}(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)})^T(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon h^k(\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)})^T(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)})^T \\
& (\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon h^k(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)})^T(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)} \\
& -\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon h^k(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)})^T(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)})^T \\
& \times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)})^T \\
& \times(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)})^T \\
& \times(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}) \\
& +Q_\varepsilon(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)})^T
\end{aligned}$$

$$
\begin{aligned}
&\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^{n}\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^{n}\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)})\\
&+Q_\varepsilon(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^{n}(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^{n}(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)})^T\\
&\times(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^{n}\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^{n}\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)})\\
&+2A_\varepsilon\left([\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}]^T(\hat\theta_n-\theta_0)\right)^2+2A_\varepsilon(\hat\theta_n-\theta_0)^T[\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)}+\frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}]\\
&+2A_\varepsilon(\hat\theta_n-\theta_0)^T[\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sqrt{n}\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sqrt{n}\sigma^3(X_t,\theta_0)}]\\
&+2A_\varepsilon\left([\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sqrt{n}\sigma^3(X_s,\theta_0)}+\frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sqrt{n}\sigma^3(X_t,\theta_0)}]^T(\hat\theta_n-\theta_0)\right)^2+o_p(\frac{1}{n})
\end{aligned}
$$

where $Q_\varepsilon=E[f_\varepsilon(\varepsilon)]$ and $A_\varepsilon=E[\varepsilon F_\varepsilon(\varepsilon)]$. Consequently,

$$
\begin{aligned}
\hat U^*_{n31} \quad = \quad & h^{2k}Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{311}(Z_i,Z_j,Z_k,Z_l)+h^kQ_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{312}(Z_i,Z_j,Z_k,Z_l)\\
&+h^kQ_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{313}(Z_i,Z_j,Z_k,Z_l)+h^kQ_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{314}(Z_i,Z_j,Z_k,Z_l)\\
&+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{315}(Z_i,Z_j,Z_k,Z_l)+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{316}(Z_i,Z_j,Z_k,Z_l)\\
&+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{317}(Z_i,Z_j,Z_k,Z_l)+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{318}(Z_i,Z_j,Z_k,Z_l)\\
&+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{319}(Z_i,Z_j,Z_k,Z_l)+Q_\varepsilon\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{3110}(Z_i,Z_j,Z_k,Z_l)\\
&+2A_\varepsilon(\hat\theta_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{3111}(Z_i,Z_j,Z_k,Z_l)\\
&+2A_\varepsilon(\hat\theta_n-\theta_0)^T\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{3112}(Z_i,Z_j,Z_k,Z_l)(\hat\theta_n-\theta_0)\\
&+2A_\varepsilon(\hat\theta_n-\theta_0)^T\frac{1}{\sqrt{n}}\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{3113}(Z_i,Z_j,Z_k,Z_l)\\
&+2A_\varepsilon(\hat\theta_n-\theta_0)^T\frac{1}{n}\frac{1}{C_n^4}\sum_{i<j<k<l}h^*_{3114}(Z_i,Z_j,Z_k,Z_l)(\hat\theta_n-\theta_0)+o_p(\frac{1}{n}),\\
=: \quad & h^{2k}Q_\varepsilon\hat U^*_{n311}+h^kQ_\varepsilon\hat U^*_{n312}+h^kQ_\varepsilon\hat U^*_{n313}+h^kQ_\varepsilon\hat U^*_{n314}+Q_\varepsilon\sum_{k=5}^{10}\hat U^*_{n31k}\\
&+2A_\varepsilon\sum_{k=11}^{14}\hat U^*_{n31k}+o_p(\frac{1}{n}),
\end{aligned}
$$

where

$$h^*_{31m}(Z_i,Z_j,Z_k,Z_l)$$

$$= \quad 6^{-1}\sum_{s<t,u<v}^{(i,j,k,l)}\delta_{31mst}(\|X_{st}\|+\|X_{uv}\|)-12^{-1}\sum_{(s,t,u)}^{(i,j,k,l)}\delta_{31mst}\|X_{su}\|, \quad \text{for } m=1,\cdots,14$$

and

$$\begin{aligned}
\delta^*_{311st} &= \Big(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)}\Big)\Big(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)}\Big)^T\\
\delta^*_{312st} &= \Big\{\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)}\Big\}\Big\{\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)}\Big\}^T\\
\delta^*_{313st} &= \Big(\frac{D(X_s)}{\sigma(X_s,\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\theta_0)}\Big)^T\Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}\\
&\quad-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}\Big)\\
\delta^*_{314st} &= \Big(\frac{D(X_s)}{\sigma(X_s,\tilde\theta_0)}-\frac{D(X_t)}{\sigma(X_t,\tilde\theta_0)}\Big)^T\Big(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}\Big)\\
\delta^*_{315st} &= \Big\{\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)}\Big\}\\
&\quad\times\Big\{\frac{\frac{1}{n-1}\sum_{p'=1,p'\neq s,t}^n\omega_{s,p'}\sigma(X_{p'},\theta_0)\varepsilon_{p'}}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q'=1,q'\neq s,t}^n\omega_{t,q'}\sigma(X_{q'},\theta_0)\varepsilon_{q'}}{\sigma(X_t,\theta_0)}\Big\}^T\\
\delta^*_{316st} &= \Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}\Big)^T\\
&\quad\times\Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}\Big)\\
\delta^*_{317st} &= \Big(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}\Big)^T\\
&\quad\times\Big(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}\Big)\\
\delta^*_{318st} &= \Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}\Big)^T\\
&\quad\times\Big(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}\Big)\\
\delta^*_{319st} &= \Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n\omega_{s,p}m(X_p)-E[\omega_{s,p}m(X_p)|X_s]}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n\omega_{t,q}m(X_q)-E[\omega_{t,q}m(X_q)|X_t]}{\sigma(X_t,\theta_0)}\Big)^T\\
&\quad\times\Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)}\Big)\\
\delta^*_{3110st} &= \Big(\frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n(\omega_{s,p}-E[\omega_{s,p}|X_s])}{\sigma(X_s,\theta_0)}-\frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n(\omega_{t,q}-E[\omega_{t,q}|X_t])}{\sigma(X_t,\theta_0)}\Big)^T\\
&\quad\times\Big(\frac{\frac{1}{n-1}\sum_{p=1,p\neq s,t}^n\omega_{s,p}\sigma(X_p,\theta_0)\varepsilon_p}{\sigma(X_s,\theta_0)}-\frac{\frac{1}{n-1}\sum_{q=1,q\neq s,t}^n\omega_{t,q}\sigma(X_q,\theta_0)\varepsilon_q}{\sigma(X_t,\theta_0)}\Big)
\end{aligned}$$

$$
\begin{aligned}
\delta^*_{3111st} &= \{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)} + \frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\} \\
\delta^*_{3112st} &= \{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)} + \frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}\{\frac{\dot\sigma(X_s,\theta_0)}{\sigma(X_s,\theta_0)} + \frac{\dot\sigma(X_t,\theta_0)}{\sigma(X_t,\theta_0)}\}^T \\
\delta^*_{3113st} &= \frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)} + \frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)} \\
\delta^*_{3114st} &= \frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)} + \frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}\frac{s(X_s)\dot\sigma(X_s,\theta_0)}{2\sigma^3(X_s,\theta_0)} + \frac{s(X_t)\dot\sigma(X_t,\theta_0)}{2\sigma^3(X_t,\theta_0)}^T.
\end{aligned}
$$

For the term $\hat U^*_{n311}$, by the law of large numbers and some elementary calculations, we have

$$
\frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{311}(Z_i,Z_j,Z_k,Z_l) \longrightarrow E[(\frac{D(X_1)}{\sigma(X_1,\theta_0)} - \frac{D(X_2)}{\sigma(X_2,\theta_0)})(\frac{D(X_1)}{\sigma(X_1,\theta_0)} - \frac{D(X_2)}{\sigma(X_2,\theta_0)})^T C_x(X_1,X_2)],
$$

in probability, where

$$
E[h^*_{311}(Z_i,Z_j,Z_k,Z_l)] = E[(\frac{D(X_1)}{\sigma(X_1,\theta_0)} - \frac{D(X_2)}{\sigma(X_2,\theta_0)})(\frac{D(X_1)}{\sigma(X_1,\theta_0)} - \frac{D(X_2)}{\sigma(X_2,\theta_0)})^T C_x(X_1,X_2)]
$$

and $C_x(X_1,X_2) = \|X_1-X_2\| - E(\|X_1-X_2\||X_1) - E(\|X_1-X_2\||X_2) + E(\|X_1-X_2\|)$. Together with the assumption 6(d), we have

$$
nh^{2k}\hat U^*_{n311} = nh^{2k}\frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{311}(Z_i,Z_j,Z_k,Z_l) = o_p(1).
$$

For the term $\{\hat U^*_{n31m}\}_{m=2}^{10}$, similar to the arguments for $\{\hat U^*_{n21m}\}_{m=2}^{10}$ in the proof of Theorem 3.3(1), we have $nh^k\hat U^*_{n31m} = o_p(1)$ for $m = 2,3,4$ and

$$
n\hat U^*_{n31m} = \frac{2}{n-1}\sum\sum_{1\le i<j\le n} \tilde h^*_m(Z_i,Z_j) + o_p(1), \quad m = 5,\cdots,10,
$$

where $\tilde h^*_m(Z_i,Z_j)$ is given in the first part of this proof.

For the terms $\hat U^*_{n3111}$ and $\hat U^*_{n3112}$, similar to the arguments for $\hat U^*_{n2111}$ and $\hat U^*_{n2112}$ in the proof of Theorem 3.3(1), we have

$$
\begin{aligned}
n\hat U^*_{n3111} &= \sqrt{n}(\hat\theta_n - \tilde\theta_0)\frac{1}{\sqrt n}\sum_{i=1}^n 4E[h^*_{3111}(Z_i,Z_j,Z_k,Z_l|Z_i)] + o_p(1). \\
n\hat U^*_{n3112} &= \sqrt{n}(\hat\theta_n - \tilde\theta_0)^T M_2\sqrt{n}(\hat\theta_n - \theta_0) + o_p(1).
\end{aligned}
$$

where

$$
\begin{aligned}
E[h^*_{3111}(Z_i,Z_j,Z_k,Z_l)|Z_i] &= E[\{\frac{\dot\sigma(X_i,\beta_0)}{\sigma(X_i,\theta_0)} + \frac{\dot\sigma(X,\beta_0)}{\sigma(X,\theta_0)}\}C_x(X_i,X)|X_i] \\
M_2 &= E[\{\frac{\dot\sigma(X_1,\theta_0)}{\sigma(X_1,\theta_0)} + \frac{\dot\sigma(X_2,\theta_0)}{\sigma(X_2,\theta_0)}\}\{\frac{\dot\sigma(X_1,\theta_0)}{\sigma(X_1,\theta_0)} + \frac{\dot\sigma(X_2,\theta_0)}{\sigma(X_2,\theta_0)}\}^T C_x(X_1,X_2)].
\end{aligned}
$$

with $C_x(X_1,X_2) = \|X_1-X_2\| - E(\|X_1-X_2\||X_1) - E(\|X_1-X_2\||X_2) + E(\|X_1-X_2\|)$.

For the term $\hat{U}^*_{n3113}$ and $\hat{U}^*_{n3114}$ similar to the arguments for $\hat{U}^*_{n314}$ and $\hat{U}^*_{n315}$ in the proof of Theorem 3.2(1), we have

$$n\hat{U}^*_{n3113} = o_p(1), \quad \text{and} \quad n\hat{U}^*_{n3114} = o_p(1).$$

Thus we obtain that

$$\begin{aligned}
n\hat{U}^*_{n31} &= Q_\varepsilon \frac{2}{n-1} \sum\sum_{1\le i<j\le n} \sum_{m=5}^{10} \tilde{h}^*_m(Z_i, Z_j) \\
&\quad + \sqrt{n}(\hat{\theta}_n - \theta_0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} 4E[h^*_{3111}(Z_i, Z_j, Z_k, Z_l)|Z_i] \\
&\quad + \sqrt{n}(\hat{\theta}_n - \theta_0)^T M_2 \sqrt{n}(\hat{\theta}_n - \theta_0) + o_p(1).
\end{aligned}$$

Following the same line as $\hat{U}^*_{n22}$ in the first part of this proof, we can show that $\hat{U}^*_{n32} = o_p(1)$. It follows that

$$\begin{aligned}
n\hat{U}^*_{n3} &= Q_\varepsilon \frac{2}{n-1} \sum\sum_{1\le i<j\le n} \sum_{m=5}^{10} \tilde{h}^*_m(Z_i, Z_j) \\
&\quad + 8A_\varepsilon \sqrt{n}(\hat{\theta}_n - \theta_0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[h^*_{3111}(Z_i, Z_j, Z_k, Z_l)|Z_i] \\
&\quad + \sqrt{n}(\hat{\theta}_n - \theta_0)^T M_2 \sqrt{n}(\hat{\theta}_n - \theta_0) + o_p(1).
\end{aligned}$$

Note that $\hat{U}^*_{n0}$ and $\hat{U}^*_{n1}$ are the same as the terms $\hat{U}^*_{n0}$ and $\hat{U}^*_{n1}$ in the first part of this proof. Consequently,

$$\begin{aligned}
n\hat{U}^*_{n0} &= \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j\ne i}^{n} C_\varepsilon(\varepsilon_i, \varepsilon_j) C_x(X_i, X_j) + o_p(1), \\
n\hat{U}^*_{n1} &= \frac{2}{n-1} \sum\sum_{1\le i<j\le n} [h^*_1(Z_i, Z_j) + h^*_2(Z_i, Z_j) + h^*_3(Z_i, Z_j)] \\
&\quad + n(\hat{\theta}_n - \theta_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h^*_{15}(Z_i, Z_j, Z_k, Z_l) + 4A_\varepsilon E\|X_1 - X_2\| + o_p(1).
\end{aligned}$$

For the term $\hat{U}^*_{n2}$, similar to the arguments in Theorem 1 of Xu and Cao (2021), we have

$$n\hat{U}^*_{n2} = o_p(1).$$

Altogether we obtain that

$$\begin{aligned}
n\hat{U}^*_{n} &= \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j\ne i}^{n} C_\varepsilon(\varepsilon_i, \varepsilon_j) C_x(X_i, X_j) \\
&\quad \frac{2}{n-1} \sum\sum_{1\le i<j\le n} [h^*_1(Z_i, Z_j) + h^*_2(Z_i, Z_j) + h^*_3(Z_i, Z_j)] \\
&\quad + n(\hat{\theta}_n - \theta_0)^T \frac{1}{C_n^4} \sum_{i<j<k<l} h^*_{15}(Z_i, Z_j, Z_k, Z_l) + 4A_\varepsilon E\|X_1 - X_2\|
\end{aligned}$$

$$
\begin{aligned}
&+Q_\varepsilon \frac{2}{n-1} \sum\sum_{1\le i<j\le n} \sum_{m=5}^{10} \tilde{h}^*_m(Z_i,Z_j) \\
&+8A_\varepsilon \sqrt{n}(\hat\theta_n-\theta_0)\frac{1}{\sqrt{n}}\sum_{i=1}^n E[h^*_{3111}(Z_i,Z_j,Z_k,Z_l|Z_i)] \\
&+2A_\varepsilon \sqrt{n}(\hat\theta_n-\theta_0)^T M_2 \sqrt{n}(\hat\theta_n-\theta_0)+o_p(1).
\end{aligned}
$$

According to the proof of theorem 1 in Appendix A of Dette et al. (2007), we have

$$
\sqrt{n}(\hat\theta_n-\theta_0) = \frac{1}{\sqrt{n}}\sum_{i=1}^n [\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\Sigma_\sigma^{-1}\dot\sigma^2(X_i,\theta_0)+\Sigma_\sigma^{-1}E[s(X)\dot\sigma^2(X_i,\theta_0)]+o_p(1),
$$

where $\Sigma_\sigma = E[\dot\sigma^2(X_i,\theta_0)\dot\sigma^2(X_i,\theta_0)^T]$. Hence we obtain that

$$
\begin{aligned}
& n\hat{U}_n \\
= \; & \frac{1}{n-1}\sum_{i=1}^n\sum_{j\ne i}^n H(Z_i,Z_j)+4\frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0)^T\Sigma_\sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n E[h^*_{15}(Z_i,Z_j,Z_k,Z_l)|Z_i] \\
& +8A_\varepsilon \frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0)^T\Sigma_\sigma^{-1}E[h^*_{3111}(Z_i,Z_j,Z_k,Z_l)|Z_i] \\
& +2A_\varepsilon \frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0)^T\Sigma_\sigma^{-1}M_2\Sigma_\sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0) \\
& +4A_\varepsilon \frac{1}{\sqrt{n}}\sum_{i=1}^n[\sigma^2(X_i,\theta_0)(\varepsilon_i^2-1)]\dot\sigma^2(X_i,\theta_0)^T\Sigma_\sigma^{-1}M_2\Sigma_\sigma^{-1}E[s(X)\dot\sigma^2(X_i,\theta_0)] \\
& +2A_\varepsilon E[s(X)\dot\sigma^2(X_i,\theta_0)]^T\Sigma_\sigma^{-1}M_2\Sigma_\sigma^{-1}E[s(X)\dot\sigma^2(X_i,\theta_0)] \\
& +8A_\varepsilon E[s(X)\dot\sigma^2(X_i,\theta_0)]^T\Sigma_\sigma^{-1}\{\frac{1}{\sqrt{n}}\sum_{i=1}^n E\{h^*_{3111}(Z_i,Z_j,Z_k,Z_l|Z_i)\}+4A_\varepsilon E\|X_1-X_2\| \\
& +4E[s(X)\dot\sigma^2(X_i,\theta_0)]^T\Sigma_\sigma^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n E\{h^*_{15}(Z_i,Z_j,Z_k,Z_l|Z_i)\}+o_p(1),
\end{aligned}
$$

whence

$$
\begin{aligned}
n\hat{U}_n \longrightarrow \; & \sum_{k=1}^\infty \lambda_k(\mathcal{Z}_k^2-1)+4\mathcal{W}^T\Sigma^{-1}\mathcal{P}_2+8A_\varepsilon\mathcal{W}^T\Sigma^{-1}\mathcal{P}_3+2A_\varepsilon\mathcal{W}^\top\Sigma^{-1}M_2\Sigma^{-1}\mathcal{W}+4A_\varepsilon E\|X_1-X_2\| \\
& +4E[s(X)\dot\sigma^2(X_i,\theta_0)]^\top\Sigma^{-1}\mathcal{P}_2+8A_\varepsilon E[s(X)\dot\sigma^2(X_i,\theta_0)]^\top\Sigma^{-1}\mathcal{P}_3+4A_\varepsilon\mathcal{W}^TM_2\Sigma_\sigma^{-1}E[s(X)\dot\sigma^2(X_i,\theta_0)] \\
& +2A_\varepsilon E[s(X)\dot\sigma^2(X_i,\theta_0)]^T\Sigma_\sigma^{-1}M_2\Sigma_\sigma^{-1}E[s(X)\dot\sigma^2(X_i,\theta_0)],
\end{aligned}
$$

where $H(Z_i,Z_j)$, A_ε, Q_ε, $\lambda_i, \mathcal{Z}_i, \mathcal{N}, \mathcal{W}, \mathcal{P}_2, \mathcal{P}_3$ and M_2 are defined in Theorem 3.3(1). Hence we complete the proof of the second part of Theorem 3.2.

(3) Now we discuss the asymptotic properties of $n\hat{U}_n$ under the global alternative H_1 in nonparametric models. Recall that $\eta_i = \frac{Y_i-m(X_i)}{\sigma(X_i,\tilde\theta_0)} = \frac{\varepsilon_i\sigma(X_i)}{\sigma(X_i,\theta_0)}$ and $\hat\eta_i = \frac{Y_i-\hat m(X_i)}{\sigma(X_i,\hat\theta_n)}$ in nonparametric cases.

Under the global alternative H_1, applying (6.1) and (6.2) in the proof of Theorem 3.1 again, we have

$$\begin{aligned}
&|\hat{\eta}_i - \hat{\eta}_j| \\
=&|\eta_i - \eta_j| - [\frac{\hat{m}(X_i) - m(X_i)}{\sigma(X_i, \tilde{\theta}_0)} - \frac{\hat{m}(X_j) - m(X_j)}{\sigma(X_j, \tilde{\theta}_0)} + \frac{\varepsilon_i \sigma(X_i)(\sigma(X_i, \hat{\theta}_n) - \sigma(X_i, \tilde{\theta}_0))}{\sigma^2(X_i, \tilde{\theta}_0)} \\
&- \frac{\varepsilon_j \sigma(X_j)(\sigma(X_j, \hat{\theta}_n) - \sigma(X_j, \tilde{\theta}_0))}{\sigma^2(X_j, \tilde{\theta}_0)} + (R_i - R_j)]\{\mathbb{I}(\eta_i > \eta_j) - \mathbb{I}(\eta_i < \eta_j)\} \qquad (6.25) \\
&+ 2\int_0^{\frac{\hat{m}(X_i)-m(X_i)}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\hat{m}(X_j)-m(X_j)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_i\sigma(X_i)(\sigma(X_i,\hat{\theta}_n)-\sigma(X_i,\tilde{\theta}_0))}{\sigma^2(X_i,\tilde{\theta}_0)} - \frac{\varepsilon_j\sigma(X_j)(\sigma(X_j,\hat{\theta}_n)-\sigma(X_j,\tilde{\theta}_0))}{\sigma^2(X_j,\tilde{\theta}_0)} + R_i - R_j} \\
&\{\mathbb{I}(\eta_i - \eta_j \leq z) - \mathbb{I}(\eta_i \leq \eta_j)\}dz,
\end{aligned}$$

By the analog to (6.13), $\hat{U}_n$ can be decomposed into three parts:

$$\begin{aligned}
\hat{U}_n &= \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\eta_{st}|(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\eta_{st}|\|X_{su}\|\right) \\
&+ \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta^*_{5st}(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta^*_{5st}\|X_{su}\|\right) \\
&+ \frac{1}{C_n^4}\sum_{i<j<k<l}\left(\frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}\delta^*_{6st}(\|X_{st}\| + \|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}\delta^*_{6st}\|X_{su}\|\right) \\
&=: \hat{U}^*_{n4} + \hat{U}^*_{n5} + \hat{U}^*_{n6}, \qquad (6.26)
\end{aligned}$$

where

$$\begin{aligned}
\delta^*_{5st} &= -[\frac{\varepsilon_s\sigma(X_s)(\sigma(X_s,\hat{\theta}_n) - \sigma(X_s,\tilde{\theta}_0))}{\sigma^2(X_s,\theta_0)} - \frac{\varepsilon_t\sigma(X_t)(\sigma(X_t,\hat{\theta}_n) - \sigma(X_t,\tilde{\theta}_0))}{\sigma^2(X_t,\theta_0)} + \frac{\hat{m}(X_i) - m(X_i)}{\sigma(X_i,\tilde{\theta}_0)} \\
&\quad - \frac{\hat{m}(X_j) - m(X_j)}{\sigma(X_j,\tilde{\theta}_0)} + (R_s - R_t)]\{\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)\}. \\
\delta^*_{6st} &= 2\int_0^{\frac{\hat{m}(X_i)-m(X_i)}{\sigma(X_i,\theta_0)} - \frac{\hat{m}(X_j)-m(X_j)}{\sigma(X_j,\tilde{\theta}_0)} + \frac{\varepsilon_s\sigma(X_s)(\sigma(X_s,\hat{\theta}_n)-\sigma(X_s,\tilde{\theta}_0))}{\sigma^2(X_s,\tilde{\theta}_0)} - \frac{\varepsilon_t\sigma(X_t)(\sigma(X_t,\hat{\theta}_n)-\sigma(X_t,\tilde{\theta}_0))}{\sigma^2(X_t,\tilde{\theta}_0)} + R_s - R_t} \\
&\quad \{\mathbb{I}(\eta_s - \eta_t \leq z) - \mathbb{I}(\eta_s \leq \eta_t)\}dz.
\end{aligned}$$

For the term $\hat{U}^*_{n5}$, similar to the arguments for $\hat{U}^*_{n1}$ in the proof of Theorem 3.3(1), we have

$$\begin{aligned}
\hat{U}^*_{n5} &= \frac{1}{C_n^4}\sum_{i<j<k<l} h^k h^*_{51}(Z_i, Z_j, Z_k, Z_l) + \frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{52}(Z_i, Z_j, Z_k, Z_l) \\
&+ \frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{53}(Z_i, Z_j, Z_k, Z_l) + \frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{54}(Z_i, Z_j, Z_k, Z_l) \\
&+ (\hat{\theta}_n - \tilde{\theta}_0)^T \frac{1}{C_n^4}\sum_{i<j<k<l} h^*_{55}(Z_i, Z_j, Z_k, Z_l) + o_p(\frac{1}{\sqrt{n}}) \\
&= : h^k \hat{U}^*_{n51} + \hat{U}^*_{n52} + \hat{U}^*_{n53} + \hat{U}^*_{n54} + \hat{U}^*_{n55} + o_p(\frac{1}{\sqrt{n}})
\end{aligned}$$

where

$$\begin{aligned}
& h^*_{5m}(Z_i, Z_j, Z_k, Z_l) \\
= \quad & -6^{-1} \sum_{s<t,u<v}^{(i,j,k,l)} \delta^*_{5mst}\{\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)\}(\|X_{st}\| + \|X_{uv}\|) \\
& +12^{-1} \sum_{(s,t,u)}^{(i,j,k,l)} \delta^*_{5mst}\{\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)\}\|X_{su}\|, \quad \text{for } m = 1, 2, 3, 4
\end{aligned}$$

with

$$\begin{aligned}
\delta^*_{51st} &= \frac{D(X_s)}{\sigma(X_s, \tilde{\theta}_0)} - \frac{D(X_t)}{\sigma(X_t, \tilde{\theta}_0)} \\
\delta^*_{52st} &= \frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n \omega_{s,p}\sigma(X_p, \tilde{\theta}_0)\varepsilon_p}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q}\sigma(X_q, \tilde{\theta}_0)\varepsilon_q}{\sigma(X_t, \tilde{\theta}_0)}, \\
\delta^*_{53st} &= \frac{\frac{1}{n-1}\sum_{p=1,p\neq s}^n (\omega_{s,p} m(X_p) - E[\omega_{s,p} m(X_p)|X_s])}{\sigma(X_s, \tilde{\theta}_0)} - \frac{(\frac{1}{n-1}\sum_{q=1,q\neq t}^n \omega_{t,q} m(X_q) - E[\omega_{t,q} m(X_q)|X_t])}{\sigma(X_t, \tilde{\theta}_0)}, \\
\delta^*_{54st} &= \frac{\frac{m(X_s)}{n-1}\sum_{p=1,p\neq s}^n (\omega_{s,p} - E[\omega_{s,p}|X_s])}{\sigma(X_s, \tilde{\theta}_0)} - \frac{\frac{m(X_t)}{n-1}\sum_{q=1,q\neq t}^n (\omega_{t,q} - E[\omega_{t,q}|X_t])}{\sigma(X_t, \tilde{\theta}_0)}, \\
\delta^*_{55st} &= \{\frac{\varepsilon_s \sigma(X_s)\dot{\sigma}(X_s, \tilde{\theta}_0)}{\sigma^2(X_s, \tilde{\theta}_0)} - \frac{\varepsilon_t \sigma(X_t)\dot{\sigma}(X_t, \tilde{\theta}_0)}{\sigma^2(X_t, \tilde{\theta}_0)}\}.
\end{aligned}$$

For the term $\hat{U}^*_{n51}$, by the law of large numbers for U-statistics and the assumption 6(d),we have

$$\sqrt{n}h^k \hat{U}^*_{n51} = \sqrt{n}h^k E[h^*_{51}(Z_i, Z_j, Z_k, Z_l)] = o_p(1).$$

For the term $\hat{U}^*_{n52}$, similar to the arguments for I^*_{12} in the proof of Theorem 3.3(1), $\hat{U}^*_{n52}$ can be decomposed as

$$\begin{aligned}
& \hat{U}^*_{n52} \\
= \quad & \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j=1.j\neq i}^n \sum_{k=1,k\neq i,j}^n \sum_{l=1,l\neq i,j,k}^n \sum_{p=1,p\neq i}^n \sum_{q=1,q\neq j}^n \{\frac{\omega_{p,i}\sigma(X_p, \tilde{\theta}_0)\varepsilon_p}{\sigma(X_i, \tilde{\theta}_0)} \\
& -\frac{\omega_{q,j}\sigma(X_q, \tilde{\theta}_0)\varepsilon_q}{\sigma(X_j, \tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\
= \quad & \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=q\neq i,j}^n \{\frac{\omega_{p,i}\sigma(X_p, \tilde{\theta}_0)\varepsilon_p}{\sigma(X_i, \tilde{\theta}_0)} - \frac{\omega_{q,j}\sigma(X_q, \tilde{\theta}_0)\varepsilon_q}{\sigma(X_j, \tilde{\theta}_0)}\} \\
& \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\
& +\frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j} \sum_{q=i} \{\frac{\omega_{j,i}\sigma(X_j, \tilde{\theta}_0)\varepsilon_j}{\sigma(X_i, \tilde{\theta}_0)} - \frac{\omega_{i,j}\sigma(X_i, \tilde{\theta}_0)\varepsilon_i}{\sigma(X_j, \tilde{\theta}_0)}\} \\
& \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\
& +\frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^n \sum_{j\neq i}^n \sum_{k\neq i,j}^n \sum_{l\neq i,j,k}^n \sum_{p=j}^n \sum_{q\neq i,j}^n \{\frac{\omega_{p,i}\sigma(X_p, \tilde{\theta}_0)\varepsilon_p}{\sigma(X_i, \tilde{\theta}_0)} - \frac{\omega_{p,j}\sigma(X_p, \tilde{\theta}_0)\varepsilon_p}{\sigma(X_j, \tilde{\theta}_0)}\}
\end{aligned}$$

$$
\begin{aligned}
& \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\
& + \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \sum_{k\neq i,j}^{n} \sum_{l\neq i,j,k}^{n} \sum_{p\neq i,j}^{n} \sum_{q=i}^{n} \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} - \frac{\omega_{p,j}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_j,\tilde{\theta}_0)}\} \\
& \times\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\
& + \frac{1}{n(n-1)^3(n-2)(n-3)} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \sum_{k\neq i,j}^{n} \sum_{l\neq i,j,k}^{n} \sum_{p\neq i\neq j\neq k\neq l}^{n} \sum_{q\neq i\neq j\neq k\neq l\neq p}^{n} \{\frac{\omega_{p,i}\sigma(X_p,\tilde{\theta}_0)\varepsilon_p}{\sigma(X_i,\tilde{\theta}_0)} \\
& - \frac{\omega_{q,j}\sigma(X_q,\tilde{\theta}_0)\varepsilon_q}{\sigma(X_j,\tilde{\theta}_0)}\}\{\mathbb{I}(\varepsilon_i > \varepsilon_j) - \mathbb{I}(\varepsilon_i < \varepsilon_j)\}(\|X_{ij}\| + \|X_{kl}\| - 2\|X_{ik}\|) \\
=: \ & \hat{U}^*_{n521} + \hat{U}^*_{n522} + \hat{U}^*_{n523} + \hat{U}^*_{n524} + \hat{U}^*_{n525}
\end{aligned}
$$

For the term $\{\hat{U}^*_{n52m}\}_{m=2}^4$, similar to the arguments for $\{I^*_{12m}\}_{m=2}^4$ in the proof of Theorem 3.3(1), we obtain that $\sqrt{n}\hat{U}^*_{n52m} = o_p(1), m = 2, 3, 4$.

For the term $\hat{U}^*_{n525}$, we have

$$
\hat{U}^*_{n525} = \frac{(n-4)(n-5)}{(n-1)^2} \frac{1}{C_n^6} \sum\sum\sum\sum\sum\sum_{1\leq i<j<k<l<r<m\leq n} \delta^*_{525}(Z_i, Z_j, Z_k, Z_l, Z_r, Z_m)
$$

where

$$
\begin{aligned}
& \delta^*_{525}(Z_i, Z_j, Z_k, Z_l, Z_r, Z_m) \\
= \ & \frac{1}{6!} \sum_{(s,t,u,v,r,m)}^{(i,j,k,l,p,q)} \{\frac{\omega_{r,s}\sigma(X_r,\tilde{\theta}_0)\varepsilon_r}{\sigma(X_s,\tilde{\theta}_0)} - \frac{\omega_{m,t}\sigma(X_m,\tilde{\theta}_0)\varepsilon_m}{\sigma(X_t,\tilde{\theta}_0)}\}\{\mathbb{I}(\eta_s > \eta_t) - \mathbb{I}(\eta_s < \eta_t)\}(\|X_{st}\| + \|X_{uv}\| - 2\|X_{su}\|).
\end{aligned}
$$

Some elementary calculations show that $\hat{U}^*_{n525}$ is non-degenerate. By the standard theory of U-statistics (see Section 5.3 in Serfling (2009), for instance), we have

$$
\begin{aligned}
\sqrt{n}\hat{U}^*_{n525} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[\delta^*_{525}(Z_i, Z_j, Z_k, Z_l, Z_r, Z_m)|Z_i] + o_p(1) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (H_{11i} + H_{12i}) + o_p(1)
\end{aligned}
$$

where

$$
\begin{aligned}
H_{11i} &= \frac{1}{5!} \sum_{(j_1,k_1,l_1,p_1,q_1)}^{(j,k,l,p,q)} E[\{\frac{\omega_{i,p_1}\sigma(X_i,\theta_0)\varepsilon_i}{\sigma(X_{p_1},\theta_0)} - \frac{\omega_{q_1,j_1}\sigma(X_{q_1},\theta_0)\varepsilon_{q_1}}{\sigma(X_{j_1},\theta_0)}\}\{\mathbb{I}(\eta_{p_1} > \eta_{j_1}) - \mathbb{I}(\eta_{p_1} < \eta_{j_1})\}(\|X_{p_1 j_1}\| \\
&\quad + \|X_{k_1 l_1}\| - 2\|X_{p_1 k_1}\|)|Z_i] \\
H_{12i} &= \frac{1}{5!} \sum_{(j_1,k_1,l_1,p_1,q_1)}^{(j,k,l,p,q)} E[\{\frac{\omega_{p_1,q_1}\sigma(X_{p_1},\theta_0)\varepsilon_{p_1}}{\sigma(X_{q_1},\theta_0)} - \frac{\omega_{i,j_1}\sigma(X_i,\theta_0)\varepsilon_i}{\sigma(X_{j_1},\theta_0)}\}\{\mathbb{I}(\eta_{q_1} > \eta_{j_1}) - \mathbb{I}(\eta_{q_1} < \eta_{j_1})\}(\|X_{q_1 j_1}\| \\
&\quad + \|X_{k_1 l_1}\| - 2\|X_{q_1 k_1}\|)|Z_i].
\end{aligned}
$$

Similarly, we have

$$
\sqrt{n}\hat{U}^*_{n53} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[\delta^*_{535}(Z_i, Z_j, Z_k, Z_l, Z_r, Z_m)|Z_i] + o_p(1) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (H_{13i} + H_{14i}) + o_p(1)
$$

$$\sqrt{n}\hat{U}^*_{n54} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} E[\delta^*_{545}(Z_i,Z_j,Z_k,Z_l,Z_r,Z_m)|Z_i] + o_p(1) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(H_{15i}+H_{16i}) + o_p(1)$$

where

$$\begin{aligned}
H_{13i} &= \frac{1}{5!}\sum_{(j_1,k_1,l_1,p_1,q_1)}^{(j,k,l,p,q)} E[\{\frac{\omega_{i,p_1}m(X_i) - E[\omega_{i,p_1}m(X_i)|X_{p_1}]}{\sigma(X_{p_1},\theta_0)} - \frac{\omega_{q_1,j_1}m(X_{q_1}) - E[\omega_{q_1,j_1}m(X_{q_1})|X_{j_1}]}{\sigma(X_{j_1},\theta_0)}\},\\
&\quad \times\{\mathbb{I}(\eta_{p_1} > \eta_{j_1}) - \mathbb{I}(\eta_{p_1} < \eta_{j_1})\}(\|X_{p_1j_1}\| + \|X_{k_1l_1}\| - 2\|X_{p_1k_1}\|)|Z_i],\\
H_{14i} &= \frac{1}{5!}\sum_{(j_1,k_1,l_1,p_1,q_1)}^{(j,k,l,p,q)} E[\{\frac{\omega_{p_1,q_1}m(X_{p_1}) - E[\omega_{p_1,q_1}m(X_{p_1})|X_{q_1}]}{\sigma(X_{q_1},\theta_0)} - \frac{\omega_{i,j_1}m(X_i) - E[\omega_{i,j_1}m(X_i)|X_{j_1}]}{\sigma(X_{j_1},\theta_0)}\}\\
&\quad \times\{\mathbb{I}(\eta_{q_1} > \eta_{j_1}) - \mathbb{I}(\eta_{q_1} < \eta_{j_1})\}(\|X_{q_1j_1}\| + \|X_{k_1l_1}\| - 2\|X_{q_1k_1}\|)|Z_i],\\
H_{15i} &= \frac{1}{5!}\sum_{(j_1,k_1,l_1,p_1,q_1)}^{(j,k,l,p,q)} E[\{\frac{\omega_{i,p_1} - E[\omega_{i,p_1}|X_{p_1}]}{\sigma(X_{p_1},\theta_0)} - \frac{\omega_{q_1,j_1}m(X_{q_1}) - E[\omega_{q_1,j_1}|X_{j_1}]}{\sigma(X_{j_1},\theta_0)}\}\{\mathbb{I}(\eta_{p_1} > \eta_{j_1})\\
&\quad -\mathbb{I}(\eta_{p_1} < \eta_{j_1})\}(\|X_{p_1j_1}\| + \|X_{k_1l_1}\| - 2\|X_{p_1k_1}\|)|Z_i],\\
H_{16i} &= \frac{1}{5!}\sum_{(j_1,k_1,l_1,p_1,q_1)}^{(j,k,l,p,q)} E[\{\frac{\omega_{p_1,q_1}m(X_{p_1}) - E[\omega_{p_1,q_1}|X_{q_1}]}{\sigma(X_{q_1},\theta_0)} - \frac{\omega_{i,j_1} - E[\omega_{i,j_1}|X_{j_1}]}{\sigma(X_{j_1},\theta_0)}\}\\
&\quad \times\{\mathbb{I}(\eta_{q_1} > \eta_{j_1}) - \mathbb{I}(\eta_{q_1} < \eta_{j_1})\}(\|X_{q_1j_1}\| + \|X_{k_1l_1}\| - 2\|X_{q_1k_1}\|)|Z_i].
\end{aligned}$$

For the term $\hat{U}^*_{n55}$, it is easy to see that $\frac{1}{C_n^4}\sum_{i<j<k<l} h_{55}(Z_i,Z_j,Z_k,Z_l)$ is non-degenerate U-statistic of order 4. By some elementary calculations, we have

$$E[h_{55}(Z_i,Z_j,Z_k,Z_l)] = -2E[(\frac{\varepsilon_1\sigma(X_1)\dot{\sigma}(X_1,\tilde{\theta}_0)}{\sigma^2(X_1,\tilde{\theta}_0)} - \frac{\varepsilon_2\sigma(X_2)\dot{\sigma}(X_2,\tilde{\theta}_0)}{\sigma^2(X_2,\tilde{\theta}_0)})\mathbb{I}(\eta_1 > \eta_2)C_x(X_1,X_2)] \overset{def}{=} 2K_2,$$

where $C_x(X_1,X_2) = \|X_1 - X_2\| - E(\|X_1-X_2\||X_1) - E(\|X_1-X_2\||X_2) + E(\|X_1-X_2\|)$. Thus we obtain that

$$\sqrt{n}\hat{U}^*_{n5} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{k=1}^{6} H_{1ki} + 2\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(1).$$

Following the same line for the term $\hat{U}^*_{n2}$ in the proof of Theorem 3.3(1), we can show that $\sqrt{n}\hat{U}^*_{n6} = o_p(1)$. Altogether we obtain that

$$\begin{aligned}
\sqrt{n}\hat{U}_n &= \sqrt{n}\hat{U}^*_{n4} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{k=1}^{6} H_{1ki} + 2\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(1)\\
&=: \frac{\sqrt{n}}{C_n^4}\sum_{i<j<k<l} h^*_4(Z_i,Z_j,Z_k,Z_l) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{k=1}^{6} H_{1ki} + 2\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(1),
\end{aligned}$$

where $Z_i = (\eta_i, X_i)$ and $h^*_4(Z_i,Z_j,Z_k,Z_l) = \frac{1}{6}\sum_{s<t,u<v}^{(i,j,k,l)}|\eta_{st}|(\|X_{st}\|+\|X_{uv}\|) - \frac{1}{12}\sum_{(s,t,u)}^{(i,j,k,l)}|\eta_{st}|\|X_{su}\|$. Consequently,

$$\sqrt{n}[\hat{U}_n - dCov^2(\eta,X)] = \sqrt{n}[\hat{U}_{n4} - dCov^2(\eta,X)] + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{k=1}^{6} H_{1ki} + 2\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0)^T K_2 + o_p(1).$$

To obtain the limiting distribution of $n\hat{U}_n$, it remains to derive the asymptotic expansion of $\hat{\theta}_n - \theta_0$. According to the arguments of Theorem 1 in Appendix A of Dette et al. (2007), we have

$$\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0) \quad = \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\sigma^2(X_i)\varepsilon_i^2 - \sigma^2(X_i, \tilde{\theta}_0)]\Sigma^{-1}\dot{\sigma}^2(X_i, \tilde{\theta}_0) + o_p(1).$$

Altogether we obtain that

$$\begin{aligned} & \sqrt{n}[\hat{U}_n - dCov^2(\eta_i, X_i)] \\ = \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} 2\{\mathcal{G}(\eta_i, X_i) + \mathcal{I}_i(\eta_i, X_i) + [\sigma^2(X_i)\varepsilon_i^2 - \sigma^2(X_i, \tilde{\theta}_0)]K_2^T\Sigma^{-1}\dot{\sigma}^2(X_i, \tilde{\theta}_0)\} + o_p(1), \end{aligned}$$

it follows that

$$\sqrt{n}[\hat{U}_n - dCov^2(\eta, X)] \longrightarrow N(0, \sigma_2^2),$$

where $\sigma_2^2 = 4var\{\mathcal{G}(\eta, X) + \mathcal{I}_1(\eta, X) + [\sigma^2(X)\varepsilon^2 - \sigma^2(X, \tilde{\theta}_0)]K_2^T\Sigma^{-1}\dot{\sigma}^2(X, \tilde{\theta}_0)\}$ with

$$\begin{aligned} \mathcal{G}(\eta_1, X_1) &= E[C_\eta(\eta_1, \eta_2)C_x(X_1, X_2)|Z_1] - dCov^2(\eta_1, X_1) \\ C_\eta(\eta_i, \eta_j) &= |\eta_i - \eta_j| - E(|\eta_i - \eta_j||\eta_i) - E(|\eta_i - \eta_j||\eta_j) + E(|\eta_i - \eta_j|) \\ C_x(X_i, X_j) &= \|X_i - X_j\| - E(\|X_i - X_j\||X_i) - E(\|X_i - X_j\||X_j) + E(\|X_i - X_j\|) \\ dCov^2(\eta, X) &= E[C_\eta(\eta_i, \eta_j)C_x(X_i, X_j)], \end{aligned}$$

$\mathcal{I}_1(\eta_i, X_i) = \frac{1}{2}\sum_{k=1}^{6} H_{1ki}$ and K_2 are defined in Theorem 3.2(2). Hence we complete the proof of Theorem 3.3. □

Proof of Theorem 4.1. This proof follows the same line as Theorem 4 in Xu and Cao (2021). Thus we omit the details here. □