Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes

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Abstract

The Fibonacci cube of dimension n, denoted as Γ_n , is the subgraph of n-cube Q_n induced by vertices with no consecutive 1's. In this short note we give an immediate proof that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k , answering an open problem proposed in [2] and solved with a longer proof in [3].

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1 Introduction

Let n be a positive integer and denote $[n] = \{1, ..., n\}$, and $[n]_0 = \{0, ..., n-1\}$. The *n*-cube, denoted as Q_n , is the graph with vertex set

 $V(Q_n) = \{x_1 x_2 \dots x_n \mid x_i \in [2]_0 \text{ for } i \in [n]\},\$

where two vertices are adjacent in Q_n if the corresponding strings differ in exactly one position. The *Fibonacci n-cube*, denoted by Γ_n , is the subgraph of Q_n induced by vertices with no consecutive 1's. Let $\{F_n\}$ be the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. The number of vertices of Γ_n is $|V(\Gamma_n)| = F_{n+2}$. Fibonacci cubes have been investigated from many points of view and we refer to the survey [1] for more information about them. Let $q_k(n)$ be the maximum number of disjoint subgraphs isomorphic to Q_k in Γ_n . This number is studied in a recent paper [2]. The authors obtained the following recursive formula

Theorem 1.1 For every $k \ge 1$ and $n \ge 3$ $q_k(n) = q_{k-1}(n-2) + q_k(n-3)$.

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In [3] Elif Saygi and Ömer Eğecioğlu, solved an open problem proposed by the authors of [2]. They proved that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k thus that

Theorem 1.2 For every $k \ge 1$, $\lim_{n\to\infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$.

The ingenious, but long, proof they proposed is a nine cases study of the decomposition of the generating function of $q_k(n)$. The purpose of this short note is to deduce from Theorem 1.1 a recursive formula for the number of non covered vertices by a maximum set of disjoint hypercubes. We obtain as a consequence an immediate proof of Theorem 1.2.

2 Number of non covered vertices

Definition 2.1 Let $\{P_k(n)\}_{k=1}^{\infty}$ be the family of sequences of integers defined by $(i)P_k(n+3) = P_k(n) + 2P_{k-1}(n+1)$ for $k \ge 2$ and $n \ge 0$ $(ii)P_k(0) = 1, P_k(1) = 2, P_k(2) = 3$, for $k \ge 2$ $(iii)P_1(n) = 0$ if $n \equiv 1[3]$ and $P_1(n) = 1$ if $n \equiv 0[3]$ or $n \equiv 2[3]$.

Solving the recursion consecutively for the first values of k and each class of n modulo 3 we obtain the first values of $P_k(n)$.

$\begin{bmatrix} P_1(n) & 1 & 0 & 1 \\ P_2(n) & 1 & \frac{2}{3}n + \frac{4}{3} & \frac{2}{3}n + \frac{5}{3} \end{bmatrix}$	
$P_2(n)$ 1 $\frac{2}{2}n + \frac{4}{2}$ $\frac{2}{2}n + \frac{5}{2}$	
$P_{3}(n) = \frac{2}{9}n^{2} + \frac{2}{3}n + 1 = \frac{2}{9}n^{2} + \frac{8}{9}n + \frac{8}{9} = \frac{2}{3}n + \frac{5}{3}$	
$P_4(n) = \frac{4}{81}n^3 + \frac{2}{9}n^2 + \frac{2}{9}n + 1 = \frac{3}{9}n^2 + \frac{8}{9}n + \frac{8}{9} = \frac{4}{81}n^3 + \frac{4}{27}n^2 + \frac{10}{27}n - \frac{10}{27}n^2 + \frac{10}{27}n^2 $	$+\frac{103}{81}$

Table 1: $P_k(n)$ for k = 1, ..., 4

Proposition 2.2 Let n = 3p+r with r = 0, 1 or 2. For a fixed r, $P_k(n)$ is a polynomial in n of degree at most k - 1.

Proof. From (i) we can write

$$P_k(n) = 2\sum_{i=0}^{p-1} P_{k-1}(n-2-3i) + P_k(r).$$

For any integer d the classical Faulhaber's formula expresses the sum $\sum_{m=0}^{n} m^{d}$ as a polynomial in n of degree d + 1. Thus if Q(n) is a polynomial of degree at most d then $\sum_{m=0}^{n} Q(m)$ is a polynomial in n of degree at most d + 1. Let Q'(m) = Q(m) if

 $m \equiv 0[3]$ and 0 otherwise. Applying this to Q' we obtain that $\sum_{m=0,m\equiv0[3]}^{n} Q(m)$ is also a polynomial in n of degree at most d+1. Thus if $P_{k-1}(n)$ is a polynomial in n of degree at most k-2 then $\sum_{i=0}^{p-1} P_{k-1}(n-2-3i)$ is a polynomial of degree at most k-1. Since for a fixed $r P_1(n)$ is a constant, by induction on k, $P_k(n)$ is a polynomial in n of degree at most k-1.

Theorem 2.3 The number of non covered vertices of Γ_n by $q_k(n)$ disjoint Q_k 's is $P_k(n)$.

Proof. This is true for k = 1 since the Fibonacci cube Γ_n has a perfect matching for $n \equiv 1[3]$ and a maximum matching missing a vertex otherwise.

For k > 1 this is true for n = 0, 1, 2 since the values of $P_k(n)$ are respectively 1,2,3 thus are equal to $|V(\Gamma_n)|$ and there is no Q_k in Γ_n .

Assume the property is true for some $k \geq 1$ and any n. Then consider k + 1. By induction on n we can assume that the property is true for Γ_{n-3} . Let us prove it for Γ_n .

From Theorem 1.1 we have $q_{k+1}(n) = q_k(n-2) + q_{k+1}(n-3)$. Thus the number of non covered vertices of Γ_n by $q_{k+1}(n)$ disjoint Q_{k+1} 's is

$$|V(\Gamma_n)| - 2^{k+1}q_{k+1}(n) = F_{n+2} - 2^{k+1}[q_k(n-2) + q_{k+1}(n-3)] = F_{n+2} - 2 \cdot 2^k q_k(n-2) - 2^{k+1}q_{k+1}(n-3).$$

Using equalities $P_k(n-2) = F_n - 2^k q_k(n-2)$ and $P_{k+1}(n-3) = F_{n-1} - 2^{k+1} q_{k+1}(n-3)$ we obtain

$$|V(\Gamma_n)| - 2^{k+1}q_{k+1}(n) = F_{n+2} + 2(P_k(n-2) - F_n) + P_{k+1}(n-3) - F_{n-1}$$

From $F_{n+2} - 2F_n - F_{n-1} = 0$ and $2P_k(n-2) + P_{k+1}(n-3) = P_{k+1}(n)$ the number of non covered vertices is $P_{k+1}(n)$. So the theorem is proved.

For any k, since the number of non covered vertices is polynomial in n and $|V(\Gamma_n)| = F_{n+2} \sim \frac{3+\sqrt{5}}{2\sqrt{5}} (\frac{1+\sqrt{5}}{2})^n$ we obtain, like in [3], that

$$\lim_{n \to \infty} \frac{P_k(n)}{|V(\Gamma_n)|} = 0$$

thus

$$\lim_{n \to \infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$$

References

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