# Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes 

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#### Abstract

The Fibonacci cube of dimension $n$, denoted as $\Gamma_{n}$, is the subgraph of $n$-cube $Q_{n}$ induced by vertices with no consecutive 1's. In this short note we give an immediate proof that asymptotically all vertices of $\Gamma_{n}$ are covered by a maximum set of disjoint subgraphs isomorphic to $Q_{k}$, answering an open problem proposed in [2] and solved with a longer proof in [3].


Keywords: Fibonacci cube, Fibonacci numbers.
AMS Subj. Class. (2010):

## 1 Introduction

Let $n$ be a positive integer and denote $[n]=\{1, \ldots, n\}$, and $[n]_{0}=\{0, \ldots, n-1\}$. The $n$-cube, denoted as $Q_{n}$, is the graph with vertex set

$$
V\left(Q_{n}\right)=\left\{x_{1} x_{2} \ldots x_{n} \mid x_{i} \in[2]_{0} \text { for } i \in[n]\right\},
$$

where two vertices are adjacent in $Q_{n}$ if the corresponding strings differ in exactly one position. The Fibonacci $n$-cube, denoted by $\Gamma_{n}$, is the subgraph of $Q_{n}$ induced by vertices with no consecutive 1's. Let $\left\{F_{n}\right\}$ be the Fibonacci numbers: $F_{0}=0$, $F_{1}=1, F_{n}=F_{n-1}+F_{n-2}$ for $n \geq 2$. The number of vertices of $\Gamma_{n}$ is $\left|V\left(\Gamma_{n}\right)\right|=F_{n+2}$. Fibonacci cubes have been investigated from many points of view and we refer to the survey [1] for more information about them. Let $q_{k}(n)$ be the maximum number of disjoint subgraphs isomorphic to $Q_{k}$ in $\Gamma_{n}$. This number is studied in a recent paper [2]. The authors obtained the following recursive formula

Theorem 1.1 For every $k \geq 1$ and $n \geq 3 q_{k}(n)=q_{k-1}(n-2)+q_{k}(n-3)$.

[^0]In [3] Elif Saygı and Ömer Eğecioğlu, solved an open problem proposed by the authors of [2]. They proved that asymptotically all vertices of $\Gamma_{n}$ are covered by a maximum set of disjoint subgraphs isomorphic to $Q_{k}$ thus that
Theorem 1.2 For every $k \geq 1, \lim _{n \rightarrow \infty} \frac{q_{k}(n)}{\left|V\left(\Gamma_{n}\right)\right|}=\frac{1}{2^{k}}$.
The ingenious, but long, proof they proposed is a nine cases study of the decomposition of the generating function of $q_{k}(n)$. The purpose of this short note is to deduce from Theorem 1.1 a recursive formula for the number of non covered vertices by a maximum set of disjoint hypercubes. We obtain as a consequence an immediate proof of Theorem 1.2 .

## 2 Number of non covered vertices

Definition 2.1 Let $\left\{P_{k}(n)\right\}_{k=1}^{\infty}$ be the family of sequences of integers defined by (i) $P_{k}(n+3)=P_{k}(n)+2 P_{k-1}(n+1)$ for $k \geq 2$ and $n \geq 0$
(ii) $P_{k}(0)=1, P_{k}(1)=2, P_{k}(2)=3$, for $k \geq 2$
(iii) $P_{1}(n)=0$ if $n \equiv 1[3]$ and $P_{1}(n)=1$ if $n \equiv 0[3]$ or $n \equiv 2[3]$.

Solving the recursion consecutively for the first values of $k$ and each class of $n$ modulo 3 we obtain the first values of $P_{k}(n)$.

| $n \bmod 3$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| $P_{1}(n)$ | 1 | 0 | 1 |
| $P_{2}(n)$ | 1 | $\frac{2}{3} n+\frac{4}{3}$ | $\frac{2}{3} n+\frac{5}{3}$ |
| $P_{3}(n)$ | $\frac{2}{9} n^{2}+\frac{2}{3} n+1$ | $\frac{2}{9} n^{2}+\frac{8}{9} n+\frac{8}{9}$ | $\frac{2}{3} n+\frac{5}{3}$ |
| $P_{4}(n)$ | $\frac{4}{81} n^{3}+\frac{2}{9} n^{2}+\frac{2}{9} n+1$ | $\frac{2}{9} n^{2}+\frac{8}{9} n+\frac{8}{9}$ | $\frac{4}{81} n^{3}+\frac{4}{27} n^{2}+\frac{10}{27} n+\frac{103}{81}$ |

Table 1: $P_{k}(n)$ for $k=1, \ldots, 4$

Proposition 2.2 Let $n=3 p+r$ with $r=0,1$ or 2 . For a fixed $r, P_{k}(n)$ is a polynomial in $n$ of degree at most $k-1$.

Proof. From (i) we can write

$$
P_{k}(n)=2 \sum_{i=0}^{p-1} P_{k-1}(n-2-3 i)+P_{k}(r) .
$$

For any integer $d$ the classical Faulhaber's formula expresses the sum $\sum_{m=0}^{n} m^{d}$ as a polynomial in $n$ of degree $d+1$. Thus if $Q(n)$ is a polynomial of degree at most $d$ then $\sum_{m=0}^{n} Q(m)$ is a polynomial in $n$ of degree at most $d+1$. Let $Q^{\prime}(m)=Q(m)$ if
$m \equiv 0[3]$ and 0 otherwise. Applying this to $Q^{\prime}$ we obtain that $\sum_{m=0, m \equiv 0[3]}^{n} Q(m)$ is also a polynomial in $n$ of degree at most $d+1$. Thus if $P_{k-1}(n)$ is a polynomial in $n$ of degree at most $k-2$ then $\sum_{i=0}^{p-1} P_{k-1}(n-2-3 i)$ is a polynomial of degree at most $k-1$. Since for a fixed $r P_{1}(n)$ is a constant, by induction on $k, P_{k}(n)$ is a polynomial in $n$ of degree at most $k-1$.

Theorem 2.3 The number of non covered vertices of $\Gamma_{n}$ by $q_{k}(n)$ disjoint $Q_{k}$ 's is $P_{k}(n)$.

Proof. This is true for $k=1$ since the Fibonacci cube $\Gamma_{n}$ has a perfect matching for $n \equiv 1[3]$ and a maximum matching missing a vertex otherwise.
For $k>1$ this is true for $n=0,1,2$ since the values of $P_{k}(n)$ are respectively $1,2,3$ thus are equal to $\left|V\left(\Gamma_{n}\right)\right|$ and there is no $Q_{k}$ in $\Gamma_{n}$.
Assume the property is true for some $k \geq 1$ and any $n$. Then consider $k+1$. By induction on $n$ we can assume that the property is true for $\Gamma_{n-3}$. Let us prove it for $\Gamma_{n}$.
From Theorem 1.1] we have $q_{k+1}(n)=q_{k}(n-2)+q_{k+1}(n-3)$.
Thus the number of non covered vertices of $\Gamma_{n}$ by $q_{k+1}(n)$ disjoint $Q_{k+1}$ 's is

$$
\left|V\left(\Gamma_{n}\right)\right|-2^{k+1} q_{k+1}(n)=F_{n+2}-2^{k+1}\left[q_{k}(n-2)+q_{k+1}(n-3)\right]=F_{n+2}-2 \cdot 2^{k} q_{k}(n-2)-2^{k+1} q_{k+1}(n-3) .
$$

Using equalities $P_{k}(n-2)=F_{n}-2^{k} q_{k}(n-2)$ and $P_{k+1}(n-3)=F_{n-1}-2^{k+1} q_{k+1}(n-3)$ we obtain

$$
\left|V\left(\Gamma_{n}\right)\right|-2^{k+1} q_{k+1}(n)=F_{n+2}+2\left(P_{k}(n-2)-F_{n}\right)+P_{k+1}(n-3)-F_{n-1} .
$$

From $F_{n+2}-2 F_{n}-F_{n-1}=0$ and $2 P_{k}(n-2)+P_{k+1}(n-3)=P_{k+1}(n)$ the number of non covered vertices is $P_{k+1}(n)$. So the theorem is proved.

For any $k$, since the number of non covered vertices is polynomial in $n$ and $\left|V\left(\Gamma_{n}\right)\right|=$ $F_{n+2} \sim \frac{3+\sqrt{5}}{2 \sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}$ we obtain, like in [3], that

$$
\lim _{n \rightarrow \infty} \frac{P_{k}(n)}{\left|V\left(\Gamma_{n}\right)\right|}=0
$$

thus

$$
\lim _{n \rightarrow \infty} \frac{q_{k}(n)}{\left|V\left(\Gamma_{n}\right)\right|}=\frac{1}{2^{k}}
$$

## References

[1] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (2011) 1-18.
[2] Sylvain Gravier, Michel Mollard, Simon Špacapan, Sara Sabrina Zemljič, On disjoint hypercubes in Fibonacci cubes, Discrete Applied Mathematics, Volumes 190191(2015) 50-55, http://dx.doi.org/10.1016/j.dam.2015.03.016.
[3] Elif Saygı, Ömer Eğecioğlu, Counting Disjoint Hypercubes in Fibonacci cubes, Discrete Applied Mathematics, Volume 215 (2016) 231-237, http://dx.doi.org/10.1016/j.dam.2016.07.004


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