General Cut-Generating Procedures for the Stable Set Polytope¹

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Abstract

We propose general separation procedures for generating cuts for the stable set polytope, inspired by a procedure by Rossi and Smriglio and applying a lifting method by Xavier and Campêlo. In contrast to existing cut-generating procedures, ours generate both rank and non-rank valid inequalities, hence they are of a more general nature than existing methods. This is accomplished by iteratively solving a lifting problem, which consists of a maximum weighted stable set problem on a smaller graph. Computational experience on DIMACS benchmark instances shows that the proposed approach may be a useful tool for generating cuts for the stable set polytope.

1. Introduction

Let G = (V, E) be an undirected graph with node set V and edge set E. A stable set in G is a subset of pairwise non-adjacent vertices of G. Given a graph G, the maximum stable set (MSS) problem asks for a stable set S in G of maximum cardinality. The stability number of G is this maximum cardinality and is denoted by $\alpha(G)$. The MSS problem is computationally hard to solve in practice, being in NP-Hard unless the graph G has some special structure. For an arbitrary input graph G, a number of exact methods have been developed to solve it through several combinatorial or mathematical programming-based techniques. For a survey of these theoretical and practical aspects of the MSS problem, see [3] and references therein.

Enumerative combinatorial algorithms have shown to be efficient to solve the MSS problem exactly for moderately sized graphs (for an overview, see [22]). Typically, such algorithms perform a search in a tree with the employment of simple and fast, but still effective, bounding procedures for pruning purposes. In this vein, the most successful approach involves the use of approximate

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colorings of selected subgraphs of \overline{G} (the complement of G). This bound is based on the following remark: if \overline{G} admits an ℓ -coloring, then $\alpha(G) \leq \ell$. This is a relatively weak bound and, consequently, the procedure to compute it is generally applied at numerous nodes of the search tree. However, it can be computed quickly by means of a greedy coloring heuristic implemented with bit parallelism operations [7, 18, 20].

An alternative to combinatorial algorithms is the use of sophisticated mathematical programming techniques to handle the combinatorial properties of the polytope associated with the formulation $\alpha(G) = \max\{\sum_{v \in V} x_v \mid x_u + x_v \leq 1, uv \in E, x_v \in \{0, 1\}, v \in V\}$. Although combinatorial methods for the MSS problem from the literature outperform mathematical programming-based algorithms devised so far, it is of great interest to continue the search for efficient polyhedral methods for this problem. Despite its natural theoretical relevance, there are other motivations of algorithmic nature, namely: (a) the algorithms can be easily extended to the weighted version of the MSS problem, (b) MSS constraints frequently appear as a sub-structure in many combinatorial optimization problems, (c) in many situations, probing strategies gives MSS valid inequalities on conflicting variables for general mixed integer programs (see, e.g., [1, 4]), and (d) real applications may need specific versions of the MSS problem with additional constraints. In this context, procedures for valid inequalities generation often turns out to be effective.

There are two main directions of research when polyhedral techniques, in particular procedures for valid inequalities generation, are concerned. The first direction is usually referred to as the *lift-and-project* method [6], which consists in three steps: first, a lifting operator is applied to the initial formulation to obtain a lifted formulation in a higher dimensional space; second, the lifted formulation is strengthened by means of additional valid inequalities; and third, a strengthened relaxation of the initial formulation is finally obtained as a result of an appropriate projection of the strengthened lifted formulation onto the original space. Several upper bounds for $\alpha(G)$ have been stated in connection of this method, such as the ones based on semidefinite programming (SDP, for short) relaxations described in [8, 13], which can be rather time-consuming to compute in practice. More recently, a new relaxation was introduced in [10] which preserves some theoretical properties of SDP relaxations in generating effective cuts but is computationally more tractable for a range of synthetic instances.

The second direction of research is integer programming, which in turn have followed two main approaches. The first approach consists in developing strong cuts coming from facet-inducing inequalities associated with special structures in the input graph (for instance, cliques, odd holes, webs, among others) and searching for specialized separation techniques for these families of inequalities (an up to date list of references for this approach can be found in [16]). The second approach relies on general cut-generating procedures which, starting from a fractional solution, search for a violated inequality with no prespecified structure. Such procedures were either shown to be effective in practice [16, 17] or to generate provably strong inequalities [23]. The main contribution of this work are general procedures that are both effective and generate inequalities that can be proved to be facet-inducing under quite general conditions.

We now discuss existing works following the second approach, i.e., procedures generating cuts with no prespecified structure. Mannino and Sassano [14] introduced in 1996 the idea of edge projections as a specialization of Lovász and Plummer's clique projection operation [12]. Many properties of edge projections are discussed in [14] and, based on these properties, a procedure computing an upper bound for the MSS problem is developed. This bound is then incorporated in a branch and bound scheme. Rossi and Smriglio take these ideas into an integer programming environment in [17], where a separation procedure based on edge projection is proposed. Finally, Pardalos et al. [16] extend the theory of edge projection by explaining the facetness properties of the inequalities obtained by this procedure. The authors give a branch and cut algorithm that uses edge projections as a separation tool, as well as several specific families of valid inequalities such as the odd hole inequalities (with a polynomial-time exact separation algorithm), the clique inequalities (with heuristic separation procedures), and mod-{2,3,5,7} cuts.

Rossi and Smriglio propose in [17] to employ a sequence of edge projection operations to reduce the original graph G and make it denser at the same time, allowing for a faster identification of clique inequalities on the reduced graph G'. This procedure iteratively removes and projects edges with certain properties, and heuristically finds violated rank inequalities (i.e., inequalities of the form $\sum_{v \in A} x_v \leq \alpha(G[A])$, where $A \subseteq V$ and G[A] is the subgraph of G induced by A). A key step for achieving this is to be able to establish how $\alpha(G)$ is affected by these edge projections, or, in other words, how exactly $\alpha(G)$ relates to $\alpha(G')$. We aim at generalizing Rossi and Smriglio's procedure by projecting cliques instead of edges, so we also need to show how $\alpha(G)$ changes as a result of this operation. Our method allows thus to establish a more general relation between Gand the graph resulting from the clique projection.

In this article we propose the use of clique projections as a general method for cutting plane generation for the MSS, along with new clique lifting operations that lead to stronger inequalities than those obtained with the edge projection method. The proposed method is able to generate both rank and weighted rank valid inequalities (to be defined below), by resorting to the general lifting operation introduced in [23]. This approach allows to produce cuts of a quite general nature, including cuts from the known families of valid inequalities for the MSS polytope. Based upon the projection and lifting operations, we give a separation procedure that departs from the usual template-based paradigm for generating cuts, and seeks to unify and generalize the separation procedures for the known cuts. In this sense, our main goal is to provide a more complete understanding of the maximum stable set polytope, which may help also in the solution of other combinatorial optimization problems. Experimental results are provided to validate the general procedure we propose.

This work is organized as follows. In Section 2 we define the MSS polytope STAB(G), we define the operation of clique projection and we explore some basic facts on this operation. Section 3 introduces the crucial concept of *clique lifting*, based on the results in [23]. In Sections 4 and 5 we introduce our cut-generating method, by applying the lifting method presented in [23]. Finally, in Section 6 we present some computational experience on the DIMACS and randomly generated instances, which show that the method is competitive. The paper is closed with some concluding remarks in Section 7.

2. The Stable Set Polytope and the Clique Projection Operation

Let n := |V|, $N_G(v)$ be the neighborhood of v in graph G, and $\mathcal{S}(G) \subseteq \{0,1\}^n$ be the set of all characteristic vectors of stable sets of G. We write simply N(v) and \mathcal{S} , respectively, when G is clear from context. For $W \subseteq V$, $\mathcal{S}(G[W])$ stands for the characteristic vectors of stable sets of Ginvolving vertices in W only. The polytope of stable sets of G is denoted by

$$STAB(G) = \operatorname{conv}\{x \mid x \in \mathcal{S}(G)\}.$$

Note that the stability number of G is $\alpha(G) = \max\{\sum_{v \in V} x_v \mid x \in STAB(G)\}$. If $c \in \mathbb{R}^n$, then the weighted stability number of G, according to c is $\alpha(G, c) = \max\{c^{\top}x \mid x \in STAB(G)\}$. The general form of a facet-inducing inequality of STAB(G) is

$$c^{\top}x \le \alpha(G[H], c), \tag{1}$$

where $c \in \mathbb{R}^n$, $c \ge 0$, $H = \{v \in V \mid c_v > 0\}$, and (G[H], c) is a so-called *facet-subgraph* of G [11]. Note that if $c \in \{0, 1\}^n$ then we have the rank inequality mentioned in the Introduction.

Our interest is to build inequalities of type (1) by means of the following operation.

Definition 1 (Clique Projection [12]). Let $W \subseteq V$, $|W| \ge 2$, be a clique in G. The clique projection of W gives the graph $G \mid W = (V, E \mid W)$ in which $E \mid W = E \cup \{uv \notin E \mid W \subseteq N(u) \cup N(v)\}$.

Figure 1 shows an example of this operation. The edges in $(E \mid W) \setminus E$ (i.e., the added edges after the projection) are called *false edges*. These are the edges simulated by W in the sense stated in the lemma below. For $W \subseteq V$, we define $x_W = \sum_{v \in W} x_v$.



Figure 1: Projected graph $G \mid W$ (with 3 false edges) is obtained after projecting $W = \{1, 2, 3\}$.

Lemma 1. $F_W = \{x \in STAB(G) \mid x_W = 1\} \subseteq STAB(G \mid W) \subseteq STAB(G).$

Proof. For the first inclusion, we show that $x \in (STAB(G) \setminus STAB(G \mid W)) \cap S(G)$ implies $x \notin F_W$. For such an x, there is a false edge $wz \in (E \mid W) \setminus E$ such that $x_{\{w\}} = x_{\{z\}} = 1$. By definition of clique projection, $W \subseteq N(w) \cup N(z)$. Hence, for every $v \in W$, either $v \in N(w)$ or $v \in N(z)$ holds, leading to $x_W = 0$. The second inclusion stems directly from the definition of clique projection. \Box

A clique projection of an edge uv is also referred to as *edge projection*. This term is employed in [16, 17] with a slightly different meaning since, in those papers, the vertices in $(N(u) \cap N(v)) \cup$ $\{u, v\}$ are removed when performing the projection. A fundamental property of edge projection is the following.

Definition 2 ([17]). An edge $uv \in E$ is projectable in G if there exists a maximum stable set S in G such that $S \cap \{u, v\} \neq \emptyset$.

Lemma 2 ([14]). If $uv \in E$ is a projectable edge in G, then $\alpha(G) = \alpha(G \mid \{u, v\})$.

Proof. Since uv is projectable, we get $\alpha(G) \leq \alpha(G \mid \{u, v\})$. On the other hand, $\alpha(G) \geq \alpha(G \mid \{u, v\})$ due to Lemma 1.

Results presented in [14, 17] yield that if $N(u) - \{v\}$ is a clique, then uv is projectable in every induced subgraph of G containing u and v. Indeed, in such a situation the projection of uv is equivalent to the projection of the clique $\{u, v\} \cup (N(u) \cap N(v))$. Thus, define the subgraph \tilde{G} as the graph obtained from G by removing the edges connecting u to all the vertices in $N(u) \setminus W$, where $W \subseteq N(u) - \{v\}$ induces a clique in G. Lemma 2 can then be used to write

$$\alpha(\tilde{G} \mid \{u, v\}[H]) = \alpha(\tilde{G}[H]) \ge \alpha(G[H]),$$

for every $H \subseteq V$ such that $\{u, v\} \subseteq H$. A direct consequence is that the rank inequality $x_H \leq \alpha(\tilde{G} \mid \{u, v\}[H])$, valid for $\tilde{G} \mid \{u, v\}$, is also valid for G.

3. The Clique Lifting Operation

In this section we lay a lifting operation that can be applied to valid inequalities of a projected graph to obtain valid inequalities for STAB(G). Given an inequality

$$c^{\top}x = \sum_{v \in H} c_v x_v \le \beta \tag{2}$$

with $H \subseteq V$, $\beta \in \mathbb{R}$, and $c \in \mathbb{R}^n$ such that $c_v \neq 0$ if and only if $v \in H$, we say that H is the support of (2). We are now in position of stating the lifting lemma on which our cut-generating procedure is based. For ease of presentation, we will restrict ourselves to a simplified version of this result in terms of the stable set polytope, and we refer the reader to [23] for the general result. In order to keep this work self-contained, we also provide a proof of this simplified version.

Lemma 3 (Simplified version of the Lifting Lemma [23]). Let $W \subseteq V$ be a clique of G. If $c^{\top}x \leq d$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, is a valid inequality for $F_W = \{x \in STAB(G) \mid x_W = 1\}$ with support $H \subseteq V$, then

$$f(x) = (c^{\top} x - d) - \lambda (x_W - 1) \le 0,$$
(3)

with the lifting factor λ being such that

$$\lambda \le d - \alpha(G[H \setminus W], c), \tag{4}$$

is a valid inequality for STAB(G). In addition, if W is a maximal clique, $c^{\top}x \leq d$ is facet-defining for F_W , and λ satisfies (4) at equality, then (3) is facet-defining for STAB(G).

Proof. To prove validity, it is sufficient to show that (3) holds for any $x \in S$. If $x \in F_W$, then $f(x) \leq 0$ holds because $c^{\top}x \leq d$ is valid for F_W . Otherwise, $x_W = 0$ and, by definition,

$$f(x) = (c^{\top}x - d) + \lambda \le c^{\top}x - \alpha(G[H \setminus W], c) \le 0.$$

Now, assume that W is a maximal clique (so F_W is a facet of STAB(G)) and $c^{\top}x \leq d$ is facetdefining for F_W , and let x^1, \ldots, x^{n-1} be n-1 affinely independent vectors in $\{x \in F_W \mid c^{\top}x = d\}$. Clearly, $f(x^i) = 0$, for all $i \in \{1, \ldots, n-1\}$. Additionally, let x^n be the characteristic vector of a maximum weight independent set of $G[H \setminus W]$ according to c. It stems from $\lambda = d - \alpha(G[H \setminus W], c)$ that

$$f(x^n) = (c^{\top}x^n - d) - \lambda (x_W^n - 1) = (\alpha (G[H \setminus W], c) - d) + d - \alpha (G[H \setminus W], c) = 0.$$

Finally, since $x^n \notin F_W$, x^1, \ldots, x^n are affinely independent vectors in $\{x \in STAB(G) \mid f(x) = 0\}$.

The lifting operation in [17] corresponds to a special case of Lemma 3 in which inequality $c^{\top}x \leq d$ is a rank inequality of a projected graph's clique with empty intersection with W. More precisely, it includes the projected edge uv, a clique \tilde{W} in the projected graph \tilde{G} , and $N(u) \cap N(v)$ such that $\tilde{W} \cap N(u) \cap N(v) = \emptyset$, $\tilde{W} \cap \{u, v\} = \emptyset$, and $N(u) \cap N(v)$ is a clique of \tilde{G} to produce the valid inequality

$$x_{\tilde{W}} + x_{\{u,v\}} + x_{N(u)\cap N(v)} \le 2 \tag{5}$$

for STAB(G). It is straightforward to check that Lemma 3, with $W = \{u, v\} \cup (N(u) \cap N(v))$, $c^{\top}x \leq d$ being the clique inequality of \tilde{W} , and $\lambda = -1$, establishes that (5) is a valid inequality for STAB(G).

The clique projection operation and the corresponding clique-lifting operations according to Lemma 3 lead to stronger inequalities than those that can be obtained with the edge projection method proposed in [17]. As an illustration, consider the structure in Figure 2(a) and $W = \{d, e, f\}$. The projection of de in this graph adds the false edge ac, and if we then lift the clique $\{a, b, c\}$ of $G \mid de$ we get the rank inequality $x_a + x_b + x_c + x_d + x_e + x_f \leq 2$. The same inequality is obtained with Lemma 3 if we take as $c^{\top}x \leq d$ the clique inequality of G + de for $\{a, b, c\}$. Nevertheless, even in this simple example, there is an inequality that cannot be derived with the method in [17]. If we take $\{a, b, c, f\}$ as the clique inducing set of vertices associated with $c^{\top}x \leq d$ in Lemma 3, then we get $x_a + x_b + x_c + x_d + x_e + 2x_f \leq 2$ as a valid (indeed, facet-defining [5]) inequality for STAB(G).



Figure 2: Structures leading to stronger inequalities than edge projection.

The structure in Figure 2(b) (assuming that it induces a rank inequality of G [2, 5]) also illustrates the fact that $\sum_{v \in W} x_v \leq 1$ being facet-defining for STAB(G) is not necessary to derive another facet of STAB(G). To show this, we choose $W = \{d, e\}$ and again take the clique inequality of G + de associated with $\{a, b, c, f\}$. With such a configuration, Lemma 3 gives the rank inequality $x_a + x_b + x_c + x_d + x_e + x_f \leq 2$ as well. Observe that this inequality is not derived by the method in [17] if edge *ae* is deleted before projecting *de* $(x_b + x_c + x_d + x_e + x_f \leq 2$ would be generated instead).

4. Two Procedures for Generating Valid Inequalities

We are now in position of introducing the general cut-generating procedures based on the previous definitions and lemmas. We shall introduce two procedures. The first one is a simple procedure directly based on Lemma 3, whereas the second one is a strenghtening based on the results in [23]. In both procedures, the generation of a valid inequality consists of the following two steps.



Figure 3: Projected graph G_3 (with false edges) is obtained after projecting $W_1 = \{1, 2, 3\}, W_2 = \{1, 3, 4\}$, and $W_3 = \{1, 4, 5\}$, in this order.

Step 1: Sequence of clique projections. Define $G_0 := G$ and determine distinct subsets W_1, \ldots, W_r of V such that, for every $t \in \{1, \ldots, r\}$, W_t is a clique of G_{t-1} and G_t is the projected graph $G_{t-1} \mid W_t$. An illustration of such a sequence is depicted in Figure 3.

Step 2: Sequence of clique lifting operations. Let $W_{r+1} \subseteq V$ be a clique of G_r . The lifting procedure starts with the clique inequality $f_r(x) = x_{W_{r+1}} \leq 1$, which is valid for $STAB(G_r)$, and iteratively for $t = r - 1, \ldots, 0$ applies a specific version of Lemma 3 in order to generate $f_t(x) = f_{t+1}(x) + \lambda_{t+1}(x_{W_{t+1}} - 1) \leq 1$ in such a way that $f_0(x) \leq 1$ is valid for $STAB(G_0 = G)$.

4.1. Basic Procedure

The basic specific version of Lemma 3, stated below, leads the general method to generate valid inequalities for all projected graphs G_0, \ldots, G_r . For $t \in \{1, \ldots, r\}$, let

$$F_{W_t} = \{ x \in STAB(G_{t-1}) \mid x_{W_t} = 1 \}.$$

Lemma 4. The inequality

$$x_{W_{r+1}} + \sum_{t=1}^{r} \lambda_t^B(x_{W_t} - 1) \le 1$$
(6)

is valid for STAB(G), where $P_t := \{x \in STAB(G_{t-1}) \mid x_{W_t} = 0\}$ and

$$\lambda_t^B := \max\left\{ x_{W_{r+1}} + \sum_{i=t+1}^r \lambda_i^B(x_{W_i} - 1) \mid x \in P_t \right\} - 1.$$

Proof. This result is obtained by iteratively applying Lemma 3 on P_t , for t = r, ..., 1 (i.e., in reverse order). At step t, the first inclusion of Lemma 1 assures that $x_{W_{t+1}} \leq 1$ is valid for $F_{W_{t+1}}$. Hence, being the lifting factor λ_t^B calculated according to the definition in (4), the obtained inequality is valid.

Consider the projected graph G_3 in Figure 3(b), and let $W_4 = \{2, 5, 6, 7, 8\}$. The inequality $x_{W_4} \leq 1$ is trivially valid for $STAB(G_3)$, and is also valid for $STAB(G_2)$ since $\lambda_3^B = 0$. This comes from $\lambda_3^B = \max \{x_{W_4} \mid x_{W_3} = 0\} - 1$, which has $\{8\}$ as an optimal solution. The remaining lifting operations are illustrated in Figure 4, finally giving rise to the inequality $x_{\{4,5,6,7,8\}} + 2x_{\{1,2,3\}} \leq 3$, which is valid for STAB(G).





(b) The optimal solution {4,5,6} of $\max \{x_{W_4} + x_{W_2} \mid x_{W_1} = 0\}$ yields $\lambda_1^B = 1$ and $x_{W_4} + x_{W_2} + x_{W_1} \le 3$ valid for $STAB(G_0)$.

Figure 4: Basic lifting procedure starting with the clique inequality of $W_4 = \{2, 5, 6, 7, 8\}$ to generate $x_{\{4,5,6,7,8\}} + 2x_{\{1,2,3\}} \leq 3$.

4.2. Strengthened Procedure

Let $F_0 := STAB(G)$ and, for $t \in \{1, \dots, r\}$, $F_t = \{x \in F_{t-1} \mid x_{W_t} = 1\} = \{x \in STAB(G) \mid x_{W_j} = 1, j = 1, \dots, t\}.$

Clearly, the integral elements of F_t are stable sets of G. A property of the clique projection operation is that they are also stable sets of G_t . Define E_t as the edge set of G_t , for $t = 1, \ldots, r$.

Lemma 5. $F_t \subseteq STAB(G_t)$, for all $t \in \{0, \ldots, r\}$.

Proof. We use induction on t to show that $F_t \cap \mathcal{S}(G) \subseteq STAB(G_t) \cap \mathcal{S}(G)$. The case t = 0 is trivial. For $t \ge 1$, $F_{t-1} \subseteq STAB(G_{t-1})$ by the induction hypothesis and, consequently, $F_t = F_{t-1} \cap F_{W_t}$. The results follow from the first inclusion of Lemma 1.

By definition, $x_{W_t} \leq 1$ is valid for $STAB(G_{t-1})$. The previous lemma implies that it is also valid for the stable sets of G that intersect W_1, \ldots, W_{t-1} .

Corollary 1. $x_{W_t} \leq 1$ is valid for F_{t-1} .

Our strengthened lifting procedure is as follows. We assume that $\max \emptyset = 0$.

Lemma 6. Let $c^{\top}x \leq d$, $c, x \in \mathbb{R}^n$ and $d \in \mathbb{R}$, be a valid inequality for $STAB(G_r)$. Then, $f_t(x) \leq d$ is valid for F_t , where, for $t \in \{0, \ldots, r\}$,

$$f_t(x) = c^{\top} x + \sum_{\ell=t+1}^r \lambda_{\ell}^S(x_{W_{\ell}} - 1),$$

 $\mathcal{S}_t = \mathcal{S}(G) \cap F_t$, and

$$\lambda_{\ell}^{S} = \max\left\{f_{\ell}(x) - d \mid x \in \mathcal{S}_{\ell-1}, x_{W_{\ell}} = 0\right\}.$$

Proof. We show that $f_t(x) \leq d$ is valid for F_t by induction on t. For t = r, the result follows since $f_r(x) = c^{\top}x \leq d$ is valid for $STAB(G_r)$ and $F_r \subseteq STAB(G_r)$ by Lemma 5. For t < r, by induction hypothesis, $f_{t+1}(x) \leq d$ is valid for $F_{t+1} = \{x \in F_t \mid x_{W_{t+1}} = 1\}$. Applying the inequality construction, we get $\lambda_{t+1}^S = 0$ if $\{x \in S_t \mid x_{W_{t+1}} = 0\} = \emptyset$, and

$$\lambda_{t+1}^{S} = \max\left\{ c^{\top} x + \sum_{i=t+2}^{r} \lambda_{i}^{S}(x_{W_{i}} - 1) \mid \begin{array}{c} x \in \mathcal{S}_{t}, \\ x_{W_{t+1}} = 0 \end{array} \right\} - d$$

otherwise. We now apply Lemma 3, considering that $x_{W_{t+1}} \leq 1$ is valid for F_t by Corollary 1 and $f_{t+1}(x) \leq d$ is valid for F_{t+1} , and then obtain that $f_t(x) \leq d$ is valid for F_t .

Let us take Figure 5 as an example of a sequence of r = 3 clique liftings of the projected graph depicted in Figure 3(b). For t = 3, lifting $f_3(x) = x_{W_4} \leq 1$ with $\lambda_3^S = -1$ generates $f_2(x) = x_{\{2,6,7,8\}} - x_{\{1,4\}} + 1 \leq 1$, which is valid for $F_2 = \{x \in STAB(G) \mid x_{W_1} = x_{W_2} = 1\}$. The iterative procedure of Lemma 6 may generate stronger valid inequalities than the basic procedure in Lemma 4. For instance, for the example in Figure 4, $x_{\{4,5,6,7,8\}} + 2x_{\{1,2,3\}} \leq 3$ is a linear combination between $x_{\{1,2,4,6,7,8\}} + 2x_{\{3\}} \leq 2$ and the clique inequality $x_{\{1,2,5\}} \leq 1$..



Figure 5: Strengthened lifting procedure starting with the clique inequality of $W_4 = \{2, 5, 6, 7, 8\}$ to generate $x_{\{1,2,4,6,7,8\}} + 2x_{\{3\}} \leq 2$.

4.3. On the Strength of Lemma 6

The following results state that the strengthened procedure is, in some sense, related to facetsubgraphs of G. The first result indicates that, in general, the inequality produced by the strengthened procedure is stronger than the one produced by the basic procedure. **Lemma 7.** Let $f_0(x) \leq 1$ be the inequality produced by the strengthened procedure. Then,

$$1 + \sum_{\ell=1}^r \lambda_\ell^S = \alpha(G[H^S], c^S) \le \alpha(G[H^B], c^B) \le 1 + \sum_{\ell=1}^r \lambda_\ell^B,$$

where $H^S \subseteq \bigcup_{t=1}^{r+1} W_t$ and $H^B \supseteq H^S$ are the support, and c^S and c^B are the coefficient vectors, of $f_0(x) \leq 1$ and (6), respectively.

Proof. Let us examine the first equality. The inequality $1 + \sum_{\ell=1}^{r} \lambda_{\ell}^{S} \geq \alpha(G[H^{S}], c^{S})$ holds since $f_{0}(x) \leq 1$ is valid by Lemma 6. For the converse inequality, a stable set of H^{S} of weight $1 + \sum_{\ell=1}^{r} \lambda_{\ell}^{S}$ can be constructed by including a subset of W_{r+1} of weight λ_{r}^{S} and a vertex of each W_{t} , for all $t \in \{0, \ldots, r-1\}$, by the definition of F_{r-1} and λ_{r}^{S} .

The comparison between the basic and strengthened procedures is given by the next two inequalities. The former holds because $H^S \subseteq H^B$ and $\lambda_t^B \ge \lambda_t^S$, for all $t \in \{1, \ldots, r\}$, whereas the validity of (6) (by Lemma 4) implies the latter.

The second result establishes sufficient conditions for the generated inequalities to be facet defining. These conditions are slightly weaker than those in [23] due to two differences. First, the subsets W_3, \ldots, W_r are not required to be cliques of G. Second, we use clique projection and we assume condition 3 to impose appropriate false edges in $G_{t'}$ instead of the auxiliary contracted graph defined in [23] to determine W_{r+1} . Since the proof is very similar to the one in that paper, it is left to the appendix.

Theorem 1. If $f_r(x) = x_{W_{r+1}}$ and

- 1. $|W_t| = k$, the subgraph of G_{t-1} induced by $\bigcup_{i=1}^t W_i$ is k-partite with vertex classes V_t^1, \ldots, V_t^k , and W_{r+1} is a maximal clique of G_r such that $W_{r+1} \cap V_r^k = \emptyset$,
- 2. $T_t := (V_t, \mathcal{W}_t)$ is a strong hypertree defined by $V_t := \bigcup_{i=1}^k V_t^i$ and $\mathcal{W}_t := \{W_1, \ldots, W_t\}$. More precisely, either $\mathcal{W}_t = \{V_t\}$ or there is a $v \in V_t$ incident to a hyperedge $W_i \in \mathcal{W}_t$ sharing exactly k 1 vertices with some other hyperedge of T_t such that $(V_t \setminus \{v\}, \mathcal{W}_t \setminus \{W_i\})$ is also a strong hypertree,
- 3. for every $i \in \{1, \ldots, k-1\}$ and $w \in V_r^0$ such that $N_{G_r}(w) \cap V_r^i \neq \emptyset$, one of the following holds: $v \in W_t \cap V_t^i$ is a neighbor of w in G or there exists $t' \in \{1, \ldots, r\}$ such that W_t is a clique of $G_{t'-1}$, W_t and $W_{t'}$ are adjacent in T_r , $v \notin W_{t'}$, and $v' \in W_{t'} \cap V_r^i$ is a neighbor of w in $G_{t'-1}$,
- 4. no $v \in V_t^k$ has neighbors in V_r^0 , i.e. $N_{G_r}(v) \cap V_r^0 = \emptyset$,

hold for some k > 0 and for all $t \in \{1, \ldots, r\}$, then $f_t(x) \leq 1$ is facet defining for F_t , for all $t \in \{1, \ldots, r\}$.

It can be noticed that the graph and the cliques W_1, W_2, W_3, W_4 in Figure 3 satisfy the sufficient conditions with r = 3 and k = 3. The above theorem implies that the inequality generated by the strengthened procedure (as shown in Figure 5) is indeed facet defining for the graph of the example.

5. The Separation Procedure

We present in this section a separation procedure based on the valid inequality generation procedures presented in the last section. Algorithm 1 summarizes the proposed separation procedure. Besides the graph G, the input of this algorithm is a fractional solution \bar{x} to be separated and a set W of maximal cliques of G. The variable F, initially empty, stores the set of violated inequalities that are generated by the separation procedure and returned at the end of its execution. For each clique W_1 in W, we proceed by generating a sequence $S = \langle W_1, \ldots, W_{r+1} \rangle$ of distinct maximal cliques, with the corresponding sequence $\langle G_0, \ldots, G_r \rangle$ of projected graphs, and a set T of indices t such that the clique inequality associated with W_t is violated for $STAB(G_{t-1})$. The generation of the sequence S continues until a certain number of projections is performed and a violated clique inequality of the current projected graph is found. At this point, all subsequences $\langle W_1, \ldots, W_t \rangle$ of S defined by a violated clique are lifted in reverse order as follows: for each $t \in T$, we apply Lemma 4 or Lemma 6 iteratively to W_t in order to generate a valid inequality for the original graph. The computation of the lifting factors is accomplished with the algorithm in [15]. The set of violated valid inequalities so generated (stored in F) is then returned.

Algorithm 1 Separation procedure: SEPFORSTAB (G, \bar{x}, W)

```
Input: Graph G, fractional solution \bar{x} and set \mathcal{W} of maximal cliques of G
Output: A set of clique, rank, or weighted rank cuts
 1:\ k \gets 0
 2:\ F \leftarrow \emptyset
 3: G_0 \leftarrow G
 4: repeat
         k \leftarrow k + 1
 5:
 6:
         Select, and remove, a starting clique W_1 in W
 7:
         t \leftarrow 0
 8:
         T \gets \emptyset
         while (\bar{x}_{W_{t+1}} \leq 1 + MINVIOLATION \text{ or } t \leq MINDEPTH) and t \leq MAXDEPTH do
9:
10:
             Project the clique W_{t+1}, getting the graph G_{t+1}
11:
              while |W_{t+1}| > 2 and E_{t+1} = E_t do
12:
                  Remove a vertex from W_{t+1}
13:
                  Project the clique W_{t+1}, getting the graph G_{t+1}
14:
             if E_{t+1} \neq E_t then
                   \begin{array}{ll} \mbox{if } \bar{x}_{W_{t+1}} > 1 + MINVIOLATION \mbox{ then } \\ T \leftarrow T \cup \{t\} \end{array} 
15:
16:
17:
                  t \leftarrow t + 1
             Find a maximal clique W_{t+1} of G_t
18:
19:
         for all t \in T do
20:
              f_t(x) \leftarrow x_{W_{t+1}}
              while t > 0 do
21:
22:
                  t \leftarrow t - 1
23:
                  Compute \lambda_{t+1} on G_t
24:
                  f_t(x) \leftarrow f_{t+1}(x) + \lambda_{t+1}(x_{W_{t+1}} - 1)
             if f_0(\bar{x}) > 1 + MINVIOLATION then
25:
26:
                  F \leftarrow F \cup \{f_0\}
27: until \mathcal{W} = \emptyset or |F| \ge MAXNCUTS or k = MAXITER
28: return F
```

The execution of the separation procedure is governed by five parameters. Two parameters control the number of iterations k of the main loop. According to the condition checked at line 27,

the loop is iterated at most MAXITER times, and this as long as there are cliques left in W and the number of violated inequalities encountered is at most MAXNCUTS. At each iteration, the size r + 1 of the sequence of projections performed is at least MINDEPTH. The greater is the sequence size r + 1, the larger is either the number of variables or the coefficients involved in the valid inequality generated after the lifting process. An inequality is considered violated only if its violation is greater than the threshold MINVIOLATION. An iteration fails if no violated clique inequality is found after MINDEPTH clique projections. The number of projections are bounded from above by parameter MAXDEPTH because of possible failed iterations, which seldom occurs in practice.

The aim of the set \mathcal{W} of maximal cliques given as input to the separation procedure is to yield different sequences of clique projections. The cliques in \mathcal{W} are generated with two versions of a greedy algorithm. In both versions, the generation of a clique consists in selecting an initial vertex v and then determining a maximal clique in the subgraph induced by N(v). This clique in N(v)is greedily built by considering the vertices sorted in a certain order. In one version, vertices are sorted in a nonincreasing order of weight, where the weight of a vertex v is the value $\bar{x}_{\{v\}}$. A clique built with this version tends to have a relatively large intersection with previous cliques. In the other version, vertices not covered by previous cliques in \mathcal{W} have priority with the purpose of generating cliques with small intersections. In order to avoid repetitions of cliques, the initial vertex of both versions is one not covered by previous cliques. In order to maintain a good probability of generating valid inequalities violated by \bar{x} , only cliques W with $\bar{x}_W \geq 0.65$ are kept in \mathcal{W} .

Some remarks with respect to the generation of maximal cliques at line 18 are the following. The heuristic used to generate the maximal clique W_{t+1} is similar to the one used to generate the cliques in W. There are two differences, though: W_{t+1} is guaranteed to include both a vertex that does not appear in W_0, \ldots, W_t and a false edge in $E_t \setminus E_{t-1}$, when t > 0. For every K clique projections, we employ the algorithm in [21] to search for a violated clique. We do not generate all cliques, but stop when a prespecified number of cliques is enumerated instead. It might be the case that G_{t+1} contains no false edges, relative to G_t , which means that no false edges are generated by the clique projection of W_{t+1} . In such a situation, vertices are iteratively removed from W_{t+1} until either a false edge is generated in the projection of W_{t+1} is discarded.

6. Computational Experiments and Analysis of Results

In this section we provide some results of computational experiments conducted in order to explore whether the proposed method is useful as a cut-generating tool for the MSS problem. Our main goal is not to provide a competitive algorithm for the MSS problem, since combinatorial algorithms are much more effective than cutting-plane algorithms for this problem [7, 19]. As already pointed out in [17], the facts that other combinatorial problems may be formulated including stable set constraints, either explicitly or devised to address their vertex packing relaxation, are motivations to the investigation of efficient polyhedral methods for the stable set problem. In this context, we intend to assess whether the proposed procedure is effective at generating rank or weighted rank cuts for the STAB polytope, and the nature of the obtained cuts. To this end, we performed three cutting-plane method implementations attached to the COIN-CLP linear programming solver to compute a strengthened upper bound for the MSS problem [9]. In these implementations, a clique cover of all edges in E is first determined and the corresponding inequalities constitute the initial

model. The method consists in iteratively solving the current model. Whenever a fractional solution is found, we first select the set \mathcal{W} of maximal cliques of G. The violated clique inequalities encountered in this process are added to the model. Then, the separation procedure of Algorithm 1 is executed. In addition to the separation procedure, we also implemented the rounding heuristic proposed in [16] and employed it to compute lower bounds.

The first implementation, called SFS_C , only employs the clique cuts found in the generation of \mathcal{W} and aims to serve as a reference for evaluation of the two other implementations. These are implementations that include a call to SEPFORSTAB after the generation of \mathcal{W} . The difference between these two implementations is restricted to how the lifting operation at line 23 is performed. Version SFS_B uses the procedure established in Lemma 4 while Lemma 6 is employed in version SFS_S . Various configurations of the parameters of SEPFORSTAB were tested and we chose to report the results corresponding to the following setup: MINVIOLATION = 0.03, MINDEPTH = 10, K = 10, MAXITER = 50, and MAXNCUTS = 20. These were the values that produced the best average results. All the implementations were written in C++, compiled with g++ -std=c++11 -m64 -0 -fPIC, and run on a Intel(R) Core(TM) i7-4790K CPU clocked at 4.00GHz.

Table 1 summarizes the results of experiments with some instances from the DIMACS benchmark and for random graphs with 100–300 vertices. The notation G(n, d) specifies random graphs with n vertices and a density of $d \in [0, 1]$, and for these instances we report the average results over five randomly-generated instances. The experiments were performed on a 64-bit personal computer, with a time limit of two minutes. The first four columns contain the instance name, the number of vertices, the graph density, and its stability number. The following columns contain data for the cutting-plane method: the column "LB" contains the lower bound found by the rounding heuristic, the columns "Upper bound" contain the upper bound obtained with the three implementations (in addition, the upper bound Z_0 corresponding to the linear relaxation of the initial model is also indicated), and the columns "Time" report the total time spent, in seconds.

We can observe the following facts from the data in Table 1. The graphs on which the reduction in the upper bound obtained with SEPFORSTAB is significant when compared to $SFS_{-}C$ are brock200_2, brock200_4, C125.9, C250.9 and DSJC500.5. For graphs c-fat200-5 and MANN_a45, clique cuts are not capable of improving the bound obtained with respect to Z_0 . However, the rank and weighted rank cuts added with SEPFORSTAB made versions SFS_B and SFS_S capable of improving the upper bound, attaining the optimal value in the first case. The only case where the versions SFS_B e SFS_S do not get better upper bounds than SFS_C is the graph san400_0.5_1. The reason for this phenomenon is that cliques become large in projected graphs at depth 6 and beyond, making the calculation of lifting factors very time consuming. Thus, the time limit is reached before violated inequalities are found. In Table 2, the results are presented with a depth limit of 3 projections, where we can observe the improvement of the bounds with respect to SFS_{-C} . In general, there is a tendency of the version SFS_S to produce upper bounds slightly better than the version SFS_B . In particular, the graph DSJC500.5 is the case where the difference is more pronounced. Unlike the graphs with particular structures, random graphs present a homogenous behavior, with both versions SFS_B and SFS_S having better performance than SFS_C , to the advantage of version SFS_S. With regard to the comparison of the processing time between the versions with projection of cliques, we observed a trend to an increase in version SFS_{-S} with respect to version SFS_B. There are, however, 3 exceptions: p_hat300-2, p_hat300-3 and san200_0.7_2. In such cases, there is a significant reduction in processing time, with slight improvement in the upper bound obtained. This confirms the special case of gen400_p0.9_55 and gen400_p0.9_55 where the integer programming approach has performance far superior to combinatorial algorithms. The

Instance	Upper bound							Time (sec.)			
Graph	n/Dens.	α	LB	Z_0	SFS_C	SFS_B	SFS_S	SFS_B	SFS_S		
brock200 1	200/25	21	20	45.48	38.57	35.18	34.85	41.76	44.86		
brock200 2	200/50	12	12	28.69	22.05	17.24	16.29	120.29	120.23		
brock200 3	200/40	15	14	35.66	28.21	24.22	23.26	120.22	120.47		
brock200 4	200/34	17	16	37.82	31.17	27.77	26.81	109.81	117.72		
brock400.2	$\frac{200}{25}$	12	24	78.95	65.68	63.85	63 27	120.67	120.95		
brock400 4	400/25	17	23	79.53	65.99	63.61	63 17	120.01	120.80		
c-fat200-1	200/92	12	12	12	12	12	12	0.04	0.04		
c-fat200-2	200/84	24	24	24	24	24	24	0.02	0.02		
c-fat200-5	200/57	58	58	66 66	66 66	58	58	22.67	27.10		
c-fat500-1	500/96	14	14	14	14	14	14	1.68	1.65		
c-fat500-10	500/81	126	126	126	126	126	126	0.97	0.97		
c-fat500-2	500/93	26	26	26	26	26	26	0.69	0.69		
c-fat500-5	500/96	64	64	64	64	64	64	0.81	0.81		
C125 9	125/10	34	34	44 37	43 21	38 79	38.84	1.05	1.20		
C250.9	250/10	44	43	77.25	71.78	65.71	65.35	11.88	18.04		
DS IC125 1	125/90	3/	3/	45.28	43.22	39.23	39.41	1.00	0.94		
DSJ0125.1	125/50	10	10	20.80	15.98	12.34	11 97	32.44	20.25		
DS IC500 5	500/50	13	13	58.94	46.16	39.44	35 54	123.20	123.29		
gen400 p0 9 55	400/90	55	55	93.07	55	55	55	1.09	1.22		
gen400 p0 9 65	400/90	65	65	103 59	65	65	65	1.00	1.22		
gen400_p0.9_05	400/90	75	75	106.54	75	75	75	2.59	2.84		
hamming6-4	64/65	4	4	7 57	5 33	4 28	4	0.24	0.23		
hamming8-4	256/36	16	16	16	16	16	16	0.18	0.16		
keller4	171/35	11	11	25.48	14.82	14 24	14 20	12 71	16 78		
MANN 227	378/1	126	125	135	135	135	135	0.11	0.11		
MANN a45	1035/0.4	345	341	363	363	360	360	1.39	1 43		
MANN a9	45/7	16	16	18	18	18	18	0.00	0.00		
n hat300-1	300/75	8	8	24 12	16.06	12.94	12 43	121.91	120.62		
p_hat300-2	300/51	25	25	45.34	34.06	33.64	33 50	37.18	30.29		
p_hat300-3	300/26	36	35	66.67	55.67	51.95	51.82	67.61	57.77		
san200.0.7.2	200/30	18	18	27.24	19.07	18 73	18 54	20.37	6.83		
san200_0_9_1	200/10	70	70	70.29	70	70	70	0.03	0.04		
san200_0.9_2	200/10	60	60	65.81	60	60	60	0.26	0.24		
san200.0.9.3	$\frac{200}{10}$	44	44	59.75	44	44	44	0.17	0.23		
san400.0.5.1	400/50	13	9	16.41	13.60	15.20	15.18	142.37	124.81		
san400 0.9 1	400/10	100	100	119.24	100.11	100	100	2.57	2.87		
sanr200.0.7	200/30	18	18	41.30	33.83	30.92	30.21	73.48	77.88		
sanr200 0.9	$\frac{200}{10}$	42	42	64.19	59.98	54.30	54.18	8.98	10.77		
C(100, 10)	100/10	21.0		97.50	20.05	88.07	88.00	0.25	0.20		
G(100, 10)	100/10	31.2	31	37.00	30.20	33.07	33.00	0.35	0.38		
G(100, 20) G(100, 20)	100/20	20.2	20.2	29.28	20.53	23.87	23.80	1.00	1.07		
G(100, 50)	100/50	10	10	24.91	21.04	10.40	10.23	5.09	4.08		
G(100, 50) G(150, 10)	100/50	9.2	9.2	17.19	13.45	10.83	10.60	0.17	0.52		
G(150, 10) G(150, 20)	150/10	37.4	37	31.31	48.00	44.07	43.99	2.17	2.00		
G(150, 20) G(150, 20)	150/20	22.4	22.4	39.84	35.37	32.20	32.00	8.10	10.35		
G(150, 30)	150/30	10.0	10.0	02.88 02.14	27.80	24.07 14.15	24.19 19 € 9	23.49	20.22		
G(100, 50) G(200, 10)	150/50	10.2	10.2	23.14	18.06	14.15	13.33	59.00	50.63 8.07		
G(200, 10) G(200, 20)	200/10	41.0	41 2F 4	63.99 50.72	00.86	20.11	00.04 40.65	(.12	8.07		
G(200, 20)	200/20	20	20.4	50.73	44.00	40.87	40.00	23.76	30.20		
G(200, 30)	200/30	18	11	40.70	34.33	31.21	30.49	00.03	04.12		
G(200, 50)	200/50	11	11	29.13	22.32	17.62	16.65	115.00	112.76		
G(300, 10)	300/10		44.8	90.70	82.81	75.81	(0.91	33.74	35.91		
G(300, 20)	300/20	28.4	21.2	69.84 F6.70	59.97	20.25	30.38	120.60	84.00		
G(300, 30)	300/30	20.2	19.6	20.79	47.20	43.93	43.31	120.60	120.66		
G(300, 50)	300/50	12	12	40.07	30.68	24.47	22.90	121.00	121.29		

Table 1: Comparison of upper bounds and processing times among SFS_C , SFS_B , and SFS_S .

only combinatorial algorithm that solves these graphs is in [7].

Upper bound				Time	Time			SFS_B N. of cuts			SFS_S N. of cuts		
LB	SFS_B	SFS_S	[16]	SFS_B	SFS_S		Rank	W-Rank		Rank	W-Rank		
9	13.61	13.59	13.24	113.60	120.46		1437	1072		1331	1121		

Table 2: Behavior of SFS_B and SFS_S with graph san400_0.5_1 when the depth limit is 3.

As shown in Table 3, the procedure is able to generate a large number of cuts, and provides upper bounds that are competitive with those generated in [16], [17], and [10] for a representative sample of benchmark graphs. It is worth remarking that the upper bound obtained with our approaches is tighter, in comparison the ones from the literature, when the density is between 40% and 75%. The columns "Upper bound" contain the upper bound attained in [17], [16], and [10] in the root node of their branch-and-cut algorithms, respectively. Finally, the last three columns contain the number of generated clique cuts, violated rank inequalities, and violated weighted rank inequalities, respectively. Similarly to existing procedures, our cut-generating algorithm finds a large number of violated clique inequalities, and is also able to find many violated rank inequalities. The number of weighted rank inequalities generated by the procedure is smaller, but nevertheless provides an interesting set of additional and non-trivial valid inequalities. In Table 4, similar results can be observed for random generated graphs.

Some characteristics of the results presented in Table 3 have been also observed in [16] in the context of a comparison between the approach adopted in that paper of combinining several separation heuristics and the one of edge projection of [17]. It was observed that in some peculiar cases (notably, the sparse graphs C125.9 and C250.9), the edge projection alone performed better than the combination of cuts with respect to the upper bound obtained. The intriguing question that deserves further clarification is how a strategy which involves a number of approximations (for instance, in the removal of edges and in determining the righthand side of the cut resulting of the lifting operation) results in stronger cuts. An analysis of the description given in [17] indicates an inaccuracy in the proposed procedure that may generate non-valid inequalities, although the authors asserted in a personal communication that the cuts generated in the reported experiments are verified to be valid. This fact leaves the possibility of generating cuts that, though valid, are not generated by procedures that ensure the viability of *all* generated cuts. However, the cuts used in [17] can be considered as a reference of strong cuts for some sparse graphs.

Instance			Upper bound				SFS_B:	Number	of cuts	SFS_S:	SFS_S: Number of cuts		
Graph	n/Dens.	α	SFS_B	SFS_S	[16]	[17]	[10]	Clique	Rank	W-Rank	Clique	Rank	W-Rank
brock200_1	200/25	21	35.18	34.85	-	-	33.59	849	728	2116	847	752	2083
brock200_2	200/50	12	17.24	16.29	20.99	22.01	18.27	2957	591	6261	2893	628	6512
brock200_3	200/40	15	24.22	23.26	-	-	23.55	1873	468	6065	2086	563	6115
brock200_4	200/34	17	27.77	26.81	29.93	30.87	26.77	1491	531	5531	1568	622	5354
brock400_2	400/25	12	63.85	63.27	63.84	67.66	-	2696	843	1394	2657	862	1686
brock400_4	400/25	17	63.61	63.17	63.89	67.98	-	2685	778	1634	2706	937	1998
c-fat200-1	200/92	12	12	12	12.71	12.86	-	-	-	-	-	-	-
c-fat200-2	200/84	24	24	24	24	24	-	-	-	-	_	-	-
c-fat200-5	200/57	58	58	58	58.89	65.25	58	97	843	84	149	866	79
c-fat500-1	500/96	14	14	14	14	14.98	_	107	13	9	72	15	11
c-fat500-10	500/81	126	126	126	126	223.29	_	_	-	_	_	-	_
c-fat500-2	500/93	26	26	26	26.97	57.78	_	_	-	_	_	_	_
c-fat500-5	500/96	64	64	64	64.70	67.08	_	-	-	-	_	-	_
C125.9	125/10	34	38.79	38.84	41.26	37.40	37.81	61	397	148	67	430	132
C250.9	250/10	44	65.71	65.35	69.76	58.30	63.95	389	1111	399	380	1093	402
DSJC125.1	125/90	34	39.23	39.41	_	-	38.22	55	387	149	51	365	147
DSJC125.5	125/50	10	12.34	11.97	-	-	13.21	918	255	4182	968	261	3270
DSJC500.5	500/50	13	39.44	35.54	-	52.95	-	5556	257	1066	5538	299	1291
gen400_p0.9_55	400/90	55	55	55	-	56.20	-	513	254	33	523	257	32
gen400_p0.9_65	400/90	65	65	65	-	65.25	-	598	246	23	620	269	17
gen400_p0.9_75	400/90	75	75	75	_	75	—	866	408	55	901	475	55
hamming6-4	64/65	4	4.28	4	—	_	4.64	291	120	383	310	164	417
hamming8-4	256/36	16	16	16	16	16	_	173	-	_	173	-	-
keller4	171/35	11	14.24	14.20	14.83	14.95	14.29	560	464	1076	542	535	1172
MANN_a27	378/1	126	135	135	-	134.86	132.44	_	-	_	_	-	_
MANN_a45	1035/0.4	345	360	360	-	360	355.86	17	22	-	17	22	_
MANN_a9	45/7	16	18	18	-	_	17.11	_	-	_	_	-	_
p_hat300-1	300/75	8	12.94	12.43	_	-	13.45	2851	141	4046	2703	174	4342
p_hat300-2	300/51	25	33.64	33.50	33.81	34.19	30.73	923	89	1197	933	82	1153
p_hat300-3	300/26	36	51.95	51.82	54.12	53.19	49.79	995	764	1327	1086	759	1341
san200_0.7_2	200/30	18	18.73	18.54	18.50	19.18	18	747	265	815	676	259	531
san200_0.9_1	200/10	70	70	70	70	70	—	32	22	8	32	22	8
san200_0.9_2	200/10	60	60	60	60	60	_	163	113	10	148	101	5
san200_0.9_3	200/10	44	44	44	44	44.80	—	159	165	20	169	179	25
san400_0.5_1	400/50	13	15.20	15.18	13.24	17.14	_	217	35	57	214	30	57
san400_0.9_1	400/10	100	100	100	100	100.40	_	661	241	42	671	264	50
sanr200_0.7	200/30	18	30.92	30.21	-	-	29.45	1196	593	3446	1272	642	3830
sanr200_0.9	200/10	42	54.30	54.18	-	-	54.52	265	880	287	250	973	332

Table 3: Upper bounds obtained with SFS_B and SFS_S, and their comparison with the ones from [16], [17], and [10].

Instance		Upper bo	ound	SFS_B	: Number	of cuts	SFS_S	SFS_S : Number of cuts			
Graph	α	SFS_B	SFS_S	Clique	Rank	W-Rank	Clique	Rank	W-Rank		
G(100, 10)	31.2	33.07	33.06	35.2	300.4	83.4	34.2	297.4	93.2		
G(100, 20)	20.2	23.87	23.86	158.2	393.8	406.4	158	389.2	455		
G(100, 30)	15	18.45	18.23	295	289.4	1083.2	297	319.6	1073.2		
G(100, 50)	9.2	10.83	10.60	571	152.4	2066	556.4	156.6	1889.6		
G(150, 10)	37.4	44.07	43.99	117	559.4	217.8	117.6	573.2	224.2		
G(150, 20)	22.4	32.20	32.06	338.6	611.6	821.6	348.4	636.2	830.2		
G(150, 30)	16.6	24.57	24.19	684.2	465.6	2268.4	706.4	489.6	2294.2		
G(150, 50)	10.2	14.15	13.53	1322.8	348	5012	1305.8	350.8	4670.4		
G(200, 10)	41.6	55.11	55.04	242.8	858.4	320	235.2	878.6	329.2		
G(200, 20)	26	40.87	40.65	583.4	847.2	1089.2	592.2	886	1181.2		
G(200, 30)	18	31.21	30.49	1159.6	575	3255.4	1224	667.4	3824.6		
G(200, 50)	11	17.62	16.65	2861.8	558	6133.2	2856.8	560.2	6219.2		
G(300, 10)	-	75.81	75.91	520.8	1429.2	490.2	523.4	1458	480.2		
G(300, 20)	28.4	56.55	56.38	1228.4	1092.4	1223.4	1269	1120	1287.4		
G(300, 30)	20.2	43.93	43.31	2166.6	669	3903.2	2378.4	740.6	4041		
G(300, 50)	12	24.47	22.90	5550.8	610	3550.2	5541.4	666.2	3707.8		

Table 4: Upper bounds obtained and number of cuts applied with SFS_B and SFS_S.

7. Conclusions

In this work we have presented general cut-generating procedures for the standard formulation of the maximum stable set polytope, which are able to generate both violated rank and generalized rank inequalities. The main objective of these procedures is to generalize existing ones based on edge projection, and employ a lifting procedure in order to construct general valid inequalities from an initial clique inequality by undoing the operation of clique projection in the original graph. The computational experiments show that the proposed procedures are effective at generating general cuts, and may be competitive in a general setting.

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Appendix A. Sufficient Conditions for Faceteness

Consider the subsets W_1, \ldots, W_{r+1} . We show in this section that if there exists k > 0 such that the following conditions hold, for all $t \in \{1, \ldots, r\}$:

- (I) $|W_t| = k$ and the subgraph of G_{t-1} induced by $\bigcup_{i=1}^t W_i$ is k-partite with vertex classes V_t^1, \ldots, V_t^k ,
- (II) $T_t := (V_t, \mathcal{W}_t)$ is a strong hypertree defined by $V_t := \bigcup_{i=1}^k V_t^i$ and $\mathcal{W}_t := \{W_1, \dots, W_t\},$
- (III) for all $w \in V_t^0 := V \setminus V_t$, there exists $i \in \{1, \dots, k\}$ such that $N_{G_{t-1}}(w) \cap V_t^i = \emptyset$,

then

(i) $x_{W_t} \leq 1$ is facet defining for F_{t-1} .

If, in addition to conditions (I)–(II), we assume that

- (IV) for every $i \in \{1, \ldots, k-1\}$ and $w \in V_r^0$ such that $N_{G_r}(w) \cap V_r^i \neq \emptyset$, one of the following holds: $v \in W_t \cap V_t^i$ is a neighbor of w in G or there exists $t' \in \{1, \ldots, r\}$ such that W_t is a clique of $G_{t'-1}$, W_t and $W_{t'}$ are adjacent in T_r , $v \notin W_{t'}$, and $v' \in W_{t'} \cap V_r^i$ is a neighbor of win $G_{t'-1}$,
- (V) no $v \in V_t^k$ has neighbors in V_r^0 , *i.e.* $N_{G_r}(v) \cap V_r^0 = \emptyset$,

then we prove that

(*ii*) $f_t(x) \leq 1$ is facet defining for F_t ,

considering that $f_r(x) = x_{W_{r+1}}$ and W_{r+1} is a maximal clique of G_r such that $W_{r+1} \cap V_r^k = \emptyset$.

Appendix A.1. Proof of (i)

The proof of (i) depends on the dimension of F_t , established next, which in turn depends on conditions (I)–(III).

Lemma 8. If (I) holds, then $|W_{\ell} \cap V_t^i| = 1$, for all $\ell \in \{1, ..., t\}$ and $i \in \{1, ..., k\}$.

Proof. Since W_{ℓ} is a clique and V_t^i is a stable set, $|W_{\ell} \cap V_t^i| \leq 1$. By condition (I), $|W_{\ell}| = k$ and there are at least k stable sets intersecting W_{ℓ} . Therefore, $|W_{\ell} \cap V_t^i| \geq 1$.

Lemma 9 (Adapted from Lemma 3.1 of [23]). If (I)-(III) hold, then $dim(F_t) = n - t$.

Proof. Because the incidence matrix of T_t has rank t due to conditions (II), it follows that $dim(F_t) \leq n-t$. To prove that $dim(F_t) \geq n-t$, we exhibit n-t+1 affinely independent vectors of F_t . For this purpose, define x^i to be the incidence vector of V_t^i , for every $i \in \{1, \ldots, k\}$. Clearly, $x^i \in STAB(G)$ and, by Lemma 8, $x_{W_\ell}^i = 1$ for all $\ell \in \{1, \ldots, t\}$, which means that $x^i \in F_t$. For every $v \in V_t^0$, define $y_v = x^i + e_v$ where $i \in \{1, \ldots, k\}$ is such that v is not adjacent to any vertex in V_t^i , by condition (III), and e_v is the incidence vector of $\{v\}$. Again, it is easy to see that $y_v \in F_t$. The $|V_t^0| + k = n - (k + t - 1) + k = n - t + 1$ points $\{x^i\}_{i=1}^k \cup \{y_v\}_{v \in V_t^0}$ are affinely independent. \Box

Theorem 2. If (I)–(III) hold, then (i) holds for all $t \in \{1, \ldots, r\}$.

Proof. By Lemma 9, $dim(F_t) = dim(F_{t-1}) - 1$. Thus, by Corollary 1, $F_t = \{x \in F_{t-1} \mid x_{W_t} = 1\}$ is a facet of F_{t-1} .

The reader might observe that conditions (I)–(III) are sufficient for (*ii*) when t < r.

Theorem 3. If (I)-(III) hold for all $t \in \{1, ..., r\}$, $f_r(x) = x_{W_r}$, and d = 1, then (ii) holds for all $t \in \{1, ..., r-1\}$.

Proof. By induction on t. For t = r, $x_{W_{r+1}} \leq 1$ is facet defining for F_r by (i). For t < r, $x_{W_{t+1}} \leq 1$ is facet defining for F_t by (i) and $f_{t+1}(x)$ is facet defining for $F_{t+1} = \{x \in F_t \mid x_{W_{t+1}} = 1\}$ by induction hypothesis. Thus, the result follows by Lemma 3.

Appendix A.2. Proof of (ii)

Conditions (I)–(II) imply the following property of G_r and the stable sets of G that cover W_1, \ldots, W_r . A strong hyperpath is a strong hypertree with exactly two vertices of degree 1.

Lemma 10 (Adapted from Lemma 3.2 of [23]). If (I)–(II) hold and $x \in F_t$, then $x_{\{u\}} = x_{\{v\}}$, for all $i \in \{1, \ldots, k\}$ and $\{u, v\} \subseteq V_t^i$.

Proof. Considering that conditions (I)–(II) hold, let W_{t_1}, \ldots, W_{t_q} be the strong hyperpath in T_t connecting u, v. We prove the result by induction on q. If q = 2, then $x_{W_{t_1}} - x_{W_{t_2}} = x_{\{u\}} - x_{\{v\}} = 0$. Otherwise, q > 2. Let $w \in W_{t_2} \setminus W_{t_1}$. Since $u \notin W_{t_2}$, we conclude that $w \in V_{t_2}^i$ by Lemma 8. Hence, W_{t_1}, W_{t_2} is a strong hyperpath with 2 hyperedges connecting u, w, which gives $x_{\{u\}} = x_{\{w\}}$. Moreover, W_{t_p}, \ldots, W_{t_q} , for $p = \max\{j \mid w \in W_{t_j}\}$, is a strong hyperpath with less than q hyperedges connecting two vertices of V_t^i . By inductive hypothesis, $x_{\{w\}} = x_{\{v\}}$. Therefore, $x_{\{u\}} = x_{\{v\}}$.

Differently from [23], we determine W_{r+1} in G_r (instead of defining an auxiliary graph). For this purpose, we use the following property of G_r due to conditions (I)–(II) and (IV).

Lemma 11. Let $v \in V_t^i \cap W_t$, for some $i \in \{1, \ldots, k-1\}$, and $w \in V_r^0$ be such that $V_r^i \cap N_{G_r}(w) \neq \emptyset$. If (I)-(II) and (IV) hold, then $vw \in E_r$.

Proof. If $vw \in E$, then the lemma is trivially valid. Otherwise, by condition (IV), let $t' \in \{1, \ldots, r\}$ be such that W_t is a clique of $G_{t'-1}$, W_t and $W_{t'}$ are adjacent in T_r (considering conditions (I)–(II)), $v \notin W_{t'}$ and $v' \in W_{t'} \cap V_r^i$ is a neighbor of w in $G_{t'-1}$. In this situation, $W_{t'} \subseteq N_{G_{t'-1}}(v) \cup N_{G_{t'-1}}(w)$ implies $vw \in E_{t'} \subseteq E_r$ by the clique projection of $W_{t'}$.

To show (*ii*), we still need the following property of the subgraph of G_r induced by V_r^0 and a certain subset of vertices. Notation \cong denotes the affine isomorphism relation [23].

Lemma 12 (Adapted from Lemma 3.3 of [23]). If (I)–(V) hold for t = r, then $F_r \cong STAB(G_r[V_r^0 \cup R])$ where $R \subseteq V$ is such that $|R \cap V_r^i| = 1$, for all $i \in \{1, \ldots, k-1\}$, and $R \cap V_r^k = \emptyset$.

Proof. In what follows, we denote by v_i the unique vertex in $R \cap V_r^i$, for all $i \in \{1, \ldots, k-1\}$.

 $STAB(G_r[V_r^0 \cup R]) \to F_r$: Take a point in $y \in STAB(G_r[V_r^0 \cup R])$. For each $u \in V$, set $x_u = y_u$ if $u \in V_r^0$; $x_u = y_{v_i}$, if $u \in V_r^i$, $i \in \{1, \ldots, k-1\}$; and $x_u = 1 - \sum_{i=1}^{k-1} y_{v_i}$, if $u \in V_r^k$. We prove that $x \in F_r$. First, to show that $x \in STAB(G)$, take $uw \in E$. If $u, w \in V_r^0 \cup R$, then $x_u + x_w \leq 1$ trivially holds. If $u, w \in V_r$, then $x_u + x_w \leq 1$ because $\{u, w\} \setminus V_r^i \neq \emptyset$, for all $i \in \{1, \ldots, k\}$. Otherwise, assume without loss of generality that $u \in V_r \setminus R$ and $w \in V_r^0 \setminus R$. It turns out that $u \notin V_r^k$ by condition (V). Then, use Lemma 11 to conclude that $v_i w \in E_r$ and, consequently, $x_u + x_w \leq 1$. To show that $x_{W_\ell} = 1$, for $\ell \in \{1, \ldots, t\}$, we use Lemma 8 to write $x_{W_\ell} = \sum_{i=1}^{k-1} y_{v_i} + (1 - \sum_{i=1}^{k-1} y_{v_i}) = 1$.

 $F_r \to STAB(G_r[V_r^0 \cup R])$: Take $x \in F_r$. For each $v \in V_r^0$, set $y_v = x_v$, and for each $i \in \{1, \ldots, k-1\}$, set $y_{v_i} = x_{v_i}$. This mapping is injective due to Lemma 10. Take $vw \in E[V_r^0 \cup R]$. It is straightforward to check that $y_v + y_w \leq 1$.

Claim (*ii*) follows directly from Lemma 12 combined with the Lifting Lemma, as follows.

Theorem 4. If (I)-(II) and (IV)-(V) hold, $f_r(x) = x_{W_{r+1}}$, and W_{r+1} is a maximal clique of G_r such that $W_{r+1} \cap V_r^k = \emptyset$, then (ii) holds for all $t \in \{1, \ldots, r\}$.

Proof. By induction on t. For t = r, $x_{W_{r+1}} \leq 1$ is facet defining for $STAB(G_r[V_r^0 \cup R])$, for every $R \subseteq V$ such that $|R \cap V_r^i| = 1$, for all $i \in \{1, \ldots, k-1\}$, $R \cap V_r^k = \emptyset$, and $W_{r+1} \subseteq R$. Such an R exists since $W_{r+1} \cap V_r^k = \emptyset$. Hence, by Lemma 12, $f_r(x) = x_{W_{r+1}} \leq 1$ is facet defining for F_r . For t < r, $x_{W_{t+1}} \leq 1$ is facet defining for F_t by (i) (notice that (V) implies (III) and therefore we can use Theorem 2) and $f_{t+1}(x)$ is facet defining for $F_{t+1} = \{x \in F_t \mid x_{W_{t+1}} = 1\}$ by induction hypothesis. Thus, the result follows by Lemma 3.