On the conjecture of bijection between perfect matching and sub-hypercube in folded hypercubes *

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Abstract

Dong and Wang in [Theor. Comput. Sci. 771 (2019) 93–98] conjectured that the resulting graph of the n-dimensional folded hypercube FQ_n by deleting any perfect matching is isomorphic to the hypercube Q_n. In this paper, we show that the conjecture holds when $n = 2, 3$, and it is not true for $n \geq 4$.

Key words: Hypercube; Folded hypercube; Perfect matching; Sub-hypercube; Isomorphic

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1. Introduction

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the vertex-set of G and $E(G)$ is the edge-set of G. A *matching* of G is a set of pairwise nonadjacent edges.

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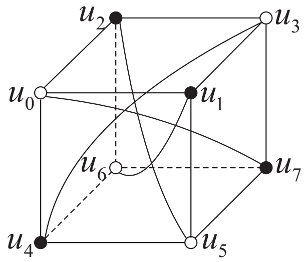

Fig. 1. The 3-dimensional folded hypercube FQ_3.

A *perfect matching* of G is a matching with size $|V(G)|/2$. A *k-factor* of G is a k-regular spanning subgraph of G. Clearly, a perfect matching, together with end vertices of its edges, forms a 1-factor of G. G is *k-factorable* if it admits a decomposition into k-factors. The distance between two vertices u and v is the number of edges in a shortest path joining u and v in G, denoted by $d(u, v)$. For any two edges uv and xy, the distance of uv and xy, denoted by $d(uv, xy)$, is $\min\{d(u, x), d(u, y), d(v, x), d(v, y)\}$. For other standard graph notations not defined here please refer to [1].

The well-known n-dimensional hypercube is a graph Q_n with 2^n vertices and $n2^{n-1}$ edges. Each vertex is labelled by an n-bit binary string. Two vertices are adjacent if their binary string differ in exactly one bit position. The folded hypercube, denoted by FQ_n, is first introduced by El-Amawy and Latifi [3] as a variant of the hypercube. FQ_n is obtained from the hypercube Q_n by adding 2^{n-1} independent edges, called *complementary edges*, each of which is between $x_1x_2\cdots x_n$ and $\overline{x}_1\overline{x}_2\cdots\overline{x}_n$, where $\overline{x}_i = 1 - x_i$, $i = 1, \cdots, n$. For convenience, the set of complementary edges of FQ_n are denoted by E_c and the set of i-dimensional edges in Q_n are denoted by E^i for each $1 \leq i \leq n$, where an edge uv is i-dimensional in Q_n if u and v differ only in the i-th position. We illustrate FQ_3 in Fig. 1.

Some attractive properties of the folded hypercube are widely studied in the literature, such as, pancyclicity [8], conditional connectivity [9], stochastic edge-fault-tolerant routing algorithm [7], conditional diagnosability [6] and conditional cycle embedding [5]. Recently, Dong and Wang [2] conjectured the following:

Conjecture 1. An subset E^m of 2^{n-1} edges of FQ_n is a perfect matching if and only if $FQ_n - E^m$ is isomorphic to Q_n.

We solve this conjecture in Section 2. Conclusions are given in Section 3.

2. Main results

The affirmative answer to Dong's conjecture for $n = 2, 3$ are shown as follows.

Theorem 2. For any perfect matching M of FQ_n, $n = 2, 3$, $FQ_n - M$ is isomorphic to Q_n.

Proof. Clearly, FQ_2 is the complete graph K_4 and Q_2 is a 4-cycle, so the statement holds when $n = 2$. Let M be a perfect matching of FQ_3. If $M = E_c$, the lemma is obviously true. Therefore, we assume that $M \neq E_c$. For convenience, we label each vertex of FQ_3 by u_i, $i \in \{0, \cdots, 7\}$, respectively (see Fig. 1). We distinguish the following cases.

Case 1. M contains no complementary edges. By symmetry of FQ_3, we may assume that $u_0u_2 \in M$. Then the subgraph induced by the hypercube edges of $FQ_3 - \{u_0, u_2\}$, say H, is a 2×1-grid with six vertices. Clearly, H has three perfect matchings $M_1 = \{u_1u_5, u_3u_7, u_4u_6\}$, $M_2 = \{u_1u_3, u_4u_5, u_6u_7\}$ and $M_3 = \{u_1u_3, u_4u_6, u_5u_7\}$. Thus, $M = \{u_0u_2\} \cup M_j$, $j = 1, 2, 3$. By direct checking, $FQ_3 - M$ is isomorphic to Q_3.

Case 2. M contains exactly one complementary edge. Suppose w.l.o.g. that $u_0u_7 \in M$. Then the subgraph induced by the hypercube edges of $FQ_3 - \{u_0, u_7\}$, say C, is a 6-cycle. Clearly, C has two perfect matchings $M_1 = \{u_1u_5, u_2u_3, u_4u_6\}$ and $M_2 = \{u_1u_3, u_2u_6, u_4u_5\}$. Thus, $M = \{u_0u_7\} \cup M_1$ or $\{u_0u_7\} \cup M_2$. By direct checking, $FQ_3 - M$ is isomorphic to Q_3.

Case 3. M contains exactly two complementary edges. By symmetry of FQ_3, we may assume that $A \subset M$ or $B \subset M$, where $A = \{u_0u_7, u_2u_5\}$ and $B = \{u_0u_7, u_3u_4\}$. Then A (resp. B) can be uniquely extended to a perfect matching of FQ_3 by adding two hypercube edges. Thus, $M = \{u_0u_7, u_2u_5, u_1u_3, u_4u_6\}$ or $\{u_0u_7, u_3u_4, u_1u_5, u_2u_6\}$. By direct checking, $FQ_3 - M$ is isomorphic to Q_3.

Case 4. M contains exactly three complementary edges. In this condition, exactly six vertices of FQ_3 are saturated by the complementary edges of M and the remaining two vertices are diagonal, which can not be saturated by any hypercube edge. So there exist no perfect matchings containing exactly three complementary edges. This completes the proof. $\square$

The following lemma is useful.

Lemma 3 [10]**.** Any two vertices in $V(FQ_n)$ exactly have two common neighbors for $n \geq 4$ if they have.

For $n \geq 4$, in fact, we prove the following theorem which characterizes the relationship between a perfect matching and the sub-hypercube of FQ_n.

Theorem 4. Let $n \geq 4$ be an integer and let M be a perfect matching of FQ_n. Then $FQ_n - M$ is isomorphic to Q_n if and only if $M = E_c$ or E^i for any $i \in \{1, 2, \cdots, n\}$.

Proof. *Sufficiency.* By the definition of FQ_n, if $M = E_c$, the statement is obviously true. Therefore, let $M = E^i$ for some $i \in \{1, 2, \cdots, n\}$. We shall show that $FQ_n - E^i$ is isomorphic to Q_n. One may consider the graph $FQ_n - E^i \cup E_c$ since $FQ_n - E^i \cup E_c$ is two disjoint copies of Q_{n-1}. For convenience, let $G = FQ_n - E^i$. The vertices of G are still labelled by n-tuple binary strings. We define a bijection $\varphi : V(G) \to V(Q_n)$ as follows: (1) $\varphi(u) = u$ if the i-th bit of u is 0; (2) $\varphi(u) = \overline{u}_1 \cdots \overline{u}_{i-1} u_i \overline{u}_{i+1} \cdots \overline{u}_n$ if the i-th bit of u is 1, where $u = u_1 \cdots u_{i-1} u_i u_{i+1} \cdots u_n$. Let $uv \in E(G)$ be an arbitrary edge. We shall verify that φ is an isomorphism.

Case 1. uv is a j-dimensional edge of G, $j \in \{1, \cdots, n\} \setminus \{i\}$. Then u and v differ only in the j-th position. We may assume that $u = u_1 \cdots u_i \cdots u_j \cdots u_n$ and $v = u_1 \cdots u_i \cdots \overline{u}_j \cdots u_n$. If $u_i = 0$, then $\varphi(u) = u$ and $\varphi(v) = v$, yielding that $\varphi(u)\varphi(v) \in E(Q_n)$. If $u_i = 1$, then $\varphi(u) = \overline{u}_1 \cdots \overline{u}_{i-1} u_i \overline{u}_{i+1} \cdots \overline{u}_j \cdots \overline{u}_n$ and $\varphi(v) = \overline{u}_1 \cdots \overline{u}_{i-1} u_i \overline{u}_{i+1} \cdots u_j \cdots \overline{u}_n$. Again, $\varphi(u)\varphi(v) \in E(Q_n)$.

Case 2. $uv \in E_c$. For convenience, let $u = u_1 \cdots u_i \cdots u_n$ and $v = \overline{u}_1 \cdots \overline{u}_i \cdots \overline{u}_n$. We may assume that $u_i = 0$. Thus, $\varphi(u) = u$ and $\varphi(v) = u_1 \cdots \overline{u}_i \cdots u_n$. Therefore, $\varphi(u)\varphi(v) \in E(Q_n)$. By above, it follows that $FQ_n - M$ is isomorphic to Q_n.

Necessity. Suppose on the contrary that $M \neq E_c$ and $M \neq E^i$ for each $i \in \{1, 2, \cdots, n\}$. We consider the following two cases.

Case 1. $M \cap E_c \neq \emptyset$. We claim that there exists a vertex u such that the complementary edge $uv \in E_c$ and one of its neighbors, say v_1, is saturated by a hypercube edge $v_1 u_1$ in M. Suppose not. If all the neighbors of any vertex u in FQ_n are saturated by complementary edges, then $M = E_c$. So the claim holds. Thus, there exists a 4-cycle $uv_1u_1v_2u$ in FQ_n, where v_1 and v_2 are two neighbors of u. Obviously, $uv_1, uv_2 \notin M$. Note that $u_1v_1 \in M$, then $u_1v_2 \notin M$. This implies that u and u_1 have exactly one common neighbor in $FQ_n - M$, contradicting the well-known fact that every two vertices in Q_n have zero or exactly two common neighbors.

Case 2. $M \cap E_c = \emptyset$. Our objective is to show that there exists a 4-cycle C of FQ_n containing exactly one edge of M. Accordingly, two diagonal vertices of C have exactly one common neighbor, which contradicts the fact that every two vertices in Q_n have zero or exactly two common neighbors. Note that $M \cap E_c = \emptyset$ and $M \neq E^i$

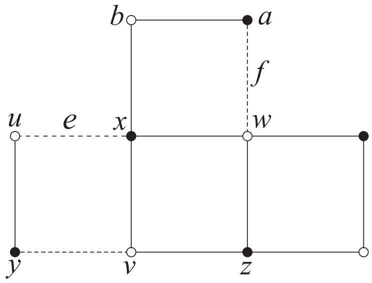

Fig. 2. Illustration for Theorem 4.

for each $i \in \{1, \cdots, n\}$, then there exists two edges $e, f \in M$ with $e \in E^i$ and $f \in E^j$ such that $d_{FQ_n}(e, f) = 1$, where $1 \leq i < j \leq n$. For clarity, let $C = uxvyu$ and $e = ux$. Suppose w.l.o.g. that all edges of C are hypercube edges. So there exists an edge xw connecting e and f, where w is an end vertex of f.

By Lemma 3, u and v have exactly two common neighbors x and y in FQ_n, and vice versa. Note that $e \notin E(FQ_n - M)$, u and v have at most one common neighbor in $FQ_n - M$. If u and v have exactly one common neighbor, say y, then we are done. So we assume that u and v have no common neighbors in $FQ_n - M$, namely $vy \in M$.

If $xv, uy \in E^j$, then there exists a 4-cycle $C' = xwzvx$ such that $f = wz$. Clearly, $xw, zv, vx \notin M$ and $wz \in M$, then we have a 4-cycle that contains exactly one edge in M, yielding that x and z have exactly one common neighbor. So we assume that $xv, uy \notin E^j$ and $f = wa \in M$. Accordingly, $xw \in E^k$, where $k \neq i, j$. Thus, there exists a cycle $C'' = xwabx$ in FQ_n. Recall that $e = ux$ and $e \in M$, thus, $bx \notin M$. Similarly, $ab, xw \notin M$. So a and x have exactly one common neighbor in G, a contradiction (see Fig. 2). Hence, the theorem holds. $\square$

By the above theorem, we have the following corollary, which disproves Dong's conjecture for $n \geq 4$.

Corollary 5. There exists a perfect matching M of FQ_n with $n \geq 4$ such that $FQ_n - M$ is not isomorphic to Q_n.

Proof. Obviously, there exists a perfect matching M of Q_n such that $M \neq E^i$ for each $1 \leq i \leq n$. Note that Q_n is a spanning subgraph of FQ_n, then $M \neq E_c$. By Theorem 4, the statement follows immediately. $\square$

3. Conclusions

In this paper, we characterize the relationship between the resulting graph of FQ_n by deleting a perfect matching and the sub-hypercube. It is interesting to study the similar property in hypercube variants which include the hypercube as their spanning subgraphs.

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