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A Class of Spectral Bounds for Max k-Cut[☆]

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Abstract

In this paper we introduce a new class of bounds for the maximum k-cut problem on undirected edge-weighted simple graphs. The bounds involve eigenvalues of the weighted adjacency matrix together with geometrical parameters. They generalize previous results on the maximum (2-)cut problem and we demonstrate that they can strictly improve over other eigenvalue bounds from the literature. We also report computational results illustrating the potential impact of the new bounds.

Keywords: Max k-cut, Adjacency matrix eigenvalues, Adjacency matrix eigenvectors

1. Introduction

The partitioning of graphs is an important theme of combinatorial optimization that emerges as a natural modeling of many practical problems from very diverse fields such as VLSI design [4], physical statistics [20], or network planning [14]. Basically, it consists in finding a partition of the node set of a graph that maximizes some objective function and possibly satisfies some side

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constraints, e.g., with respect to the cardinality of the partition or the number of nodes in its subsets. In this paper we consider the *maximum k-cut problem* denoted by Max k-Cut, where k denotes a positive integer. The objective is to find a partition of the node set into k subsets so as to maximize the sum of the weights of the edges having their endpoints in different subsets.

Our contribution is a new family of bounds on the optimal objective value of this problem which generalizes previous results for Max 2-Cut and can improve over other bounds introduced recently for Max k-Cut, for any integer $k \geq 2$. We also provide computational results that illustrate the potential impact of these new bounds.

The paper is organized as follows. We establish some notation at the end of this section. In Section 2 we proceed to a literature review. The new bounds are introduced in Section 3 and we prove in Section 4 that it is possible to define perturbations of the weighted adjacency matrix such that these bounds dominate (not strictly) a bound stemming from a classical semidefinite relaxation. We then investigate in Section 5 the efficient computation of distances involved in the expression of the new bounds and also present connections with Max 2-Cut. The computational results are reported in Section 6, and we conclude in Section 7.

We now close the section with some notation. Given a positive integer n, let $[n]$ stand for the set of integers $\{1, 2, \dots, n\}$. Let $G = (V, E)$ be an undirected simple graph having node set $V = [n]$, edge set E, and let $w \in \mathbb{R}^E$ denote a weight function on the edges. The weighted adjacency matrix, denoted by $W \in \mathbb{R}^{n \times n}$, is a symmetric matrix with entries defined by $W_{ij} = w_{ij}$ if $ij \in E$ and $W_{ij} = 0$ otherwise, for all $(i, j) \in V^2$. Let k denote a positive integer. Given any partition $(V_1, V_2, \dots, V_k)$ of V into k subsets $V_1, V_2, \dots, V_k$ (some of which may be empty), the k-cut defined by this partition is the set $\delta(V_1, V_2, \dots, V_k)$ of edges in E having their endpoints in different subsets of the partition, and the weight of the k-cut is the sum of the weights of the edges it contains. The maximum weight of a k-cut in G is denoted by $mc_k(G, W)$.

Given two disjoint node subsets A, B, let $w[A, B]$ denote the sum of the weights of the edges having one endpoint in A and the other in B: $w[A,B] = \sum_{(i,j)\in A\times B\,:\, ij\in E} w_{ij}$. Similarly, let $w[A]$ represent the sum of the weights of the edges with both endpoints in A: $w[A] = \sum_{\substack{(i,j)\in A^2:\\ ij\in E, i<j}} w_{ij}$. Given a real symmetric matrix $M \in \mathbb{R}^{n\times n}$, let $\lambda_1(M) \leq \lambda_2(M) \leq \ldots \leq \lambda_n(M)$ denote the eigenvalues of M in increasing order and let $\nu_1(M), \nu_2(M), \ldots, \nu_n(M)$ be the corresponding orthonormal eigenvectors. For the particular case when $M = W$, we shall more simply use λ_i (resp. ν_i) instead of $\lambda_i(W)$ (resp. $\nu_i(W)$) for all $i \in [n]$. For any positive integer q, let $\vec{1}_q$ stand for the q-dimensional all-ones vector. Given any vector $x \in \mathbb{R}^n$, $Diag(x)$ stands for the square diagonal matrix of order n, having x for diagonal. The *Laplacian matrix* is $L = Diag(W\vec{1}_n) - W$. The inner scalar product in $\mathbb{R}^n$ is denoted by $\langle\cdot,\cdot\rangle$, and the Euclidean norm by $\|\cdot\|$. Given $\alpha \in \mathbb{R}$ and a matrix X, the notation αX represents the matrix obtained by multiplying all the entries of X by α. Given two matrices X, Y in $\mathbb{R}^{n\times n}$, $X \bullet Y$ stands for the quantity $\sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}Y_{ij}$.

2. Related work

Max k-Cut is a notorious $\mathcal{NP}$-hard problem [23]. In particular, there exists no polynomial time approximation scheme for Max k-Cut for any $k \geq 2$ unless $\mathcal{P} = \mathcal{NP}$ [2, 23]. Also, there can be no polynomial time approximation algorithm with performance ratio $1 - \frac{1}{34k}$, unless $\mathcal{P} = \mathcal{NP}$ [19]. The challenging task of developing methods for solving this problem has generated many works stemming from the communities in discrete mathematics and operations research. The main developed approaches include heuristics [12], approximation [15] and exact algorithms [1]. In this paper, we are interested in different ways of computing bounds for Max k-Cut using information about the spectrum of W (possibly *perturbed*, as described later). Therefore the literature review that follows is focused on the works which, to the best of our knowledge, are the most relevant ones in this respect. We point out that all the approximation guarantees mentioned hereafter for randomized algorithms only apply to the restricted version of Max k-Cut with all the edge weights nonnegative.

The simple randomized algorithm, which consists of assigning each vertex uniformly at random to one of the k subsets, has an approximation guarantee of $\left(1 - \frac{1}{k}\right)$. For the particular case $k = 2$, i.e., the Max Cut problem, Goemans and Williamson [17] designed a 0.87856-approximation algorithm based on a semidefinite relaxation of the problem. Their work was subsequently extended to $k \geq 2$ by Frieze and Jerrum [15] making use also of a semidefinite relaxation which can be formulated as follows:

$$(k\text{-}SDP)\begin{cases} Z^*_{kSDP} = \max \dfrac{k-1}{k} \displaystyle\sum_{ij \in E} w_{ij} \left(1 - X_{ij}\right) & \\ \quad s.t. \quad X_{ii} = 1, & \forall i, \quad (1) \\ \qquad\quad X_{ij} \geq \dfrac{-1}{k-1}, & \forall i < j, \quad (2) \\ \qquad\quad X \succeq 0, X \in \mathbb{R}^{n \times n}, & \quad (3) \end{cases}$$

where the constraint $X \succeq 0$ means that the matrix X is symmetric and positive semidefinite. Note that, for $k = 2$, the inequalities (2) can be removed from the formulation since they are implied by the other constraints. Removing them leads to the SDP relaxation of Max Cut used by Goemans and Williamson [17], and in fact the randomized algorithm proposed by Frieze and Jerrum coincides with the one by Goemans and Williamson for $k = 2$. De Klerk et al. presented another randomized algorithm for Max k-Cut based on a semidefinite formulation of the Lovász theta function from [13]. They show their algorithm has the same approximation guarantee as Frieze and Jerrum's method [15] for $k \in \{3, \ldots, 10\}$; and a consequence of Conjecture 9.1 in [13] is that this also holds for any $k \geq 3$ (if the conjecture is true). We report in Table 1 the approximation guarantee of Frieze & Jerrum's algorithm for Max k-Cut, denoted by α_k, for some values of k (these results were proved by Goemans and Williamson for $k = 2, 3$ [17, 18], and De Klerk et al. [13] for $k \geq 3$). From the analysis carried out in [17, 18, 13], it follows that the upper bound on $mc_k(G, W)$ given by the relaxation (k-SDP) satisfies $mc_k(G, W) \geq \alpha_k Z^*_{kSDP}$.

The semidefinite relaxation (k-SDP) may be strengthened with different families of linear inequalities that are valid for the k-cut polytope, i.e., the

Table 1: Approximation guarantees of Frieze and Jerrum's algorithm for Max k-Cut in the case when all the edge weights are nonnegative [17, 18, 13]

k	2	3	4	5	6	7	8	9	10
α_k	0.87856	0.836008	0.857487	0.876610	0.891543	0.903259	0.912664	0.920367	0.926788

convex hull of all the incidence vectors in $\mathbb{R}^E$ of the k-cuts in G (see, e.g., [3, 24] for $k = 2$ and [10, 11, 26] for $k \geq 3$).

Differently from the works mentioned above which rely on a semidefinite relaxation, another line of research [28, 22] derives upper bounds for Max k-Cut from the spectrum of the Laplacian or the weighted adjacency matrix. Our work investigates further this last line of research that we now present.

For the particular case when $k = 2$, Mohar and Poljak [21] proved the inequality $mc_2(G, W) \leq \frac{n}{4}\lambda_n(L)$. More recently, van Dam and Sotirov [28] generalized this result for Max k-Cut:

Theorem 2.1. [28]

$$mc_k(G, W) \leq \frac{n(k-1)}{2k}\lambda_n(L). \tag{4}$$

They also provide several graphs for which this bound is tight together with some comparisons with other bounds stemming from semidefinite relaxations. Also recently, Nikiforov [22] introduced an upper bound for the maximum cardinality of a k-cut in G (i.e., the maximum k-cut problem with $w_e = 1$, for all $e \in E$) that is easily extended to the weighted case and can be expressed as follows.

Theorem 2.2. [22]

$$mc_k(G, W) \leq \frac{k-1}{k}\left(w[V] - \frac{\lambda_1 n}{2}\right). \tag{5}$$

As noted in [22], the bounds from Theorems 2.1 and 2.2 are equivalent for regular graphs but different in general. For $k = 2$ (Max Cut), an upper bound on $mc_2(G, W)$ that is at least as good as (5) was introduced in [5], making use of the eigenvalues and eigenvectors of W. Let d_j denote the distance between the set of vectors in $\{-1,1\}^n$ and the linear subspace $\text{lin}(\nu_1, \nu_2, \ldots, \nu_j)$ that is generated by the first j eigenvectors of W. Then, the result can be formulated as follows.

Theorem 2.3. **[5]** *The following inequality holds:*

$$mc_2(G,W) \leq \frac{1}{2}w[V] - \frac{1}{4}\left(\lambda_1 n - \sum_{l\in[n-1]} (\lambda_{l+1} - \lambda_l)\, d_l^2\right). \tag{6}$$

In the next section we shall introduce a generalization of the bound (6) for Max k-Cut that is less than or equal to (5) for any integer $k \geq 2$. This is obtained by combining ideas from the proofs in [22, 5] leading to Theorems 2.2 and 2.3, and by considering particular pertubations of the entries of the matrix W. The new bounds share with (6) the drawback that computing all the terms involved in their expression is generally $\mathcal{NP}$-hard. However, we show that *truncated variants* (obtained by removing some terms of the last sum appearing in their expression), which are still no greater than (5), can be computed in polynomial time (see Section 5.2).

3. Spectral bounds for Max *k*-Cut

For our purposes and with no loss of generality, we assume that the graph G is complete (setting zero weights on edges not present in G). In order to formulate the new bounds, we now introduce quantities to extend the expression of the bound (5) (implying a potential improvement, i.e., decrease of its value) by involving almost all of the eigenvalues and eigenvectors of the matrix $\widehat{W} = W + Q$, where $Q \in \mathbb{R}^n \times \mathbb{R}^n$ stands for a symmetric matrix satisfying some conditions to be specified later (the zero matrix is a possible choice for Q). Let $\widehat{\lambda}_1 \leq \widehat{\lambda}_2 \leq \ldots \leq \widehat{\lambda}_n$ stand for the eigenvalues of the matrix $W+Q$ in increasing order, and let $\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_n$ be the corresponding unit and pairwise orthogonal eigenvectors. Given $r \in \mathbb{R} \setminus \{1\}$, let $\widehat{d}_{j,r}$ denote the distance between the set of vectors $\{r,1\}^n$ and the linear subspace $\text{lin}(\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_j)$ that is generated by the first j eigenvectors of $\widehat{W}$:

$$\widehat{d}_{j,r} = \min\left\{\|z-y\| : z \in \{r,1\}^n, y \in \text{lin}\left(\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_j\right)\right\}. \tag{7}$$

In the definition of $\widehat{d}_{j,r}$, a vector in $\{r, 1\}^n$ may be interpreted as the incidence vector of a node subset (that may itself be interpreted as a subset of a partition), where the nodes in the subset correspond exactly to the entries with value r (as will be the case in the proof establishing the bound). We now formulate our main general result providing an upper bound on $mc_k(G, W)$.

Theorem 3.1. *For any $r \in \mathbb{R} \setminus \{1\}$ and any symmetric matrix $Q \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $Q_{ij} \leq 0$, for all $i \neq j$, the following inequality holds.*

$$mc_k(G, W) \leq \frac{1}{2(r-1)^2} \left(A_r - B_r\right), \tag{8}$$

with A_r and B_r defined as follows,

$$\left\{ \begin{array}{l} A_r = \left(r^2 + k - 1\right)\left(2w[V] - \widehat{\lambda}_1 n + 2\sum_{i=1}^{n} Q_{ii}\right) + 2(2r + k - 2)\sum_{(i,j)\in[n]^2:\, i<j} Q_{ij}, \\ B_r = k\sum_{l\in[n-1]} \left(\widehat{\lambda}_{l+1} - \widehat{\lambda}_l\right)\left(\widehat{d}_{l,r}\right)^2. \end{array} \right. \tag{9}$$

Proof. Let $(V_1, V_2, \ldots, V_k)$ denote a partition of V corresponding to an optimal solution of Max k-Cut.

For all $i \in [k]$, let the vector $y^i \in \{r, 1\}^n$ be defined as follows: $y^i_l = r$ if $l \in V_i$ and 1 otherwise. We have:

$$\begin{aligned} \langle y^i, W y^i \rangle \;=\; & 2r^2 w[V_i] + 2\sum_{j\in[k]\setminus\{i\}} w[V_j] + 2r\sum_{j\in[k]\setminus\{i\}} w[V_i, V_j] + \\ & 2\sum_{\substack{(j,l)\in([k]\setminus\{i\})^2:\\ j<l}} w[V_j, V_l]. \end{aligned} \tag{10}$$

Let us now compute the sum of each term occurring in the right-hand-side of (10) over all $i \in [k]$.

$$\begin{aligned} \sum_{i\in[k]} 2r^2 w[V_i] &= 2r^2\left(w[V] - mc_k(G, W)\right), \\ \sum_{i\in[k]} 2\sum_{j\in[k]\setminus\{i\}} w[V_j] &= 2\left(k-1\right)\left(w[V] - mc_k(G, W)\right), \\ \sum_{i\in[k]} 2r\sum_{j\in[k]\setminus\{i\}} w[V_i, V_j] &= 4r\ mc_k(G, W), \\ \sum_{i\in[k]} 2\sum_{\substack{(j,l)\in([k]\setminus\{i\})^2:\\ j<l}} w[V_j, V_l] &= 2(k-2)mc_k(G, W). \end{aligned}$$

Thus, we deduce

$$\sum_{i\in[k]} \langle y^i, W y^i \rangle = 2mc_k(G, W)(-r^2 + 2r - 1) + 2w[V](r^2 + k - 1). \tag{11}$$

Also, we have for all $i \in [k]$,

$$\langle y^i, Qy^i \rangle = r^2 \sum_{j \in V_i} Q_{jj} + 2r^2 \sum_{(j,l) \in V_i^2 :\, j<l} Q_{jl} + 2r \sum_{j \in [n] \setminus V_i} Q_{ij} + 2 \sum_{\substack{(j,l) \in ([n] \setminus V_i)^2 : \\ j<l}} Q_{jl}. \tag{12}$$

Now let us consider the summation over all $i \in [k]$ of the right-hand side in equation (12). Observe that the coefficient of any

- diagonal entry Q_{jj} is $r^2 + k - 1$,
- non-diagonal entry Q_{jl} is

$$\begin{cases} r^2 + k - 1 \text{ if there exists } q \in [k] \text{ such that } \{j, l\} \subseteq V_q, \text{ and} \\ 2r + k - 2 \text{ otherwise.} \end{cases}$$

Let J_1 stand for set of pairs $(j, l) \in [n]^2$ such that $j < l$ and there exists $q \in [k]$ satisfying $\{j, l\} \subseteq V_q$. Similarly, let J_2 stand for set of pairs $(j, l) \in [n]^2$ such that j and l belong to different subsets of the partition $(V_1, V_2, \ldots, V_k)$ and $j < l$. Using the observation given earlier and the fact that Q is symmetric, we deduce:

$$\sum_{i \in [k]} \langle y^i, Qy^i \rangle = \left(r^2 + k - 1\right) \left(\sum_{j \in [n]} Q_{jj} + 2 \sum_{(j,l) \in J_1} Q_{jl} \right) + 2\left(2r + k - 2\right) \sum_{(j,l) \in J_2} Q_{jl}.$$

Then, using the inequality $r^2 + k - 1 \geq 2r + k - 2$ together with the fact that the non-diagonal coefficients of Q are nonpositive, we obtain:

$$\sum_{i \in [k]} \langle y^i, Qy^i \rangle \leq \left(r^2 + k - 1\right) \sum_{j \in [n]} Q_{jj} + 2\left(2r + k - 2\right) \sum_{(j,l) \in J_1 \cup J_2} Q_{jl}. \tag{13}$$

We now derive a lower bound on $\langle y^i, \widehat{W} y^i \rangle$, where $\widehat{W} = W + Q$, by making use of the spectrum of $\widehat{W}$. First, we mention some preliminary properties. Since $\widehat{W}$ is a real symmetric matrix, we may assume that the eigenvectors $\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_n$ form an orthonormal basis. Considering the expression of y^i in this basis: $y^i = \sum_{l \in [n]} \alpha_l \widehat{\nu}_l$ with $\alpha \in \mathbb{R}^n$, we have $\|y^i\|^2 = \sum_{l \in [n]} \alpha_l^2 = n + |V_i|(r^2 - 1)$. Using this equation, we deduce

$$\begin{aligned}\langle y^i, \widehat{W} y^i\rangle &= \textstyle\sum_{l\in[n]} \widehat{\lambda}_l \alpha_l^2 \\ &= \widehat{\lambda}_1 \left(n + |V_i|(r^2-1) - \textstyle\sum_{l=2}^n \alpha_l^2\right) + \textstyle\sum_{l=2}^n \widehat{\lambda}_l \alpha_l^2 \\ &= \widehat{\lambda}_1 \left(n + |V_i|(r^2-1)\right) + \textstyle\sum_{l=2}^n (\widehat{\lambda}_l - \widehat{\lambda}_1)\alpha_l^2.\end{aligned}$$

Note that the quantity $\sum_{l=j}^n \alpha_l^2$ can be interpreted as the distance between the vector y^i and the subspace $\mathrm{lin}\,(\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_{j-1})$. From the definition of the distances defined above, we have $\left(\widehat{d}_{j-1,r}\right)^2 \le \sum_{l=j}^n \alpha_l^2$, for all $j \in \{2, 3, \ldots, n\}$, thus implying $\alpha_j^2 \ge \left(\widehat{d}_{j-1,r}\right)^2 - \sum_{l=j+1}^n \alpha_j^2$. By iteratively using the latter inequality for $j = 2, \ldots, n$ in the expression of $\langle y^i, \widehat{W} y^i\rangle$ above, we deduce

$$\langle y^i, \widehat{W} y^i\rangle \ \ge \widehat{\lambda}_1 \left(n + |V_i|(r^2-1)\right) + \sum_{l\in[n-1]} (\widehat{\lambda}_{l+1} - \widehat{\lambda}_l) \left(\widehat{d}_{l,r}\right)^2.$$

Summing up these inequalities for all $i \in [k]$ we obtain

$$\sum_{i\in[k]} \langle y^i, \widehat{W} y^i\rangle \ge \widehat{\lambda}_1 n \left(k + r^2 - 1\right) + k\left(\sum_{l\in[n-1]} (\widehat{\lambda}_{l+1} - \widehat{\lambda}_l) \left(\widehat{d}_{l,r}\right)^2\right). \qquad (14)$$

Combining (11), (13) and (14), the result follows. □

Remark Enforcing the value '1' among the two possible values for the components of the vectors used in the definition of the distances (7) is done only for simplicity of the presentation. We are basically interested in the distance between $\mathrm{lin}(\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_j)$ and a set of vectors whose components are restricted to take any of two nonzero values. If we denote by $\widehat{d}_{j,r_1,r_2}$ the distance between $\mathrm{lin}(\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_j)$ and the set of vectors $\{r_1, r_2\}^n$ with $(r_1, r_2) \in \mathbb{R}^2$ and $0 \ne r_1 \ne r_2$, then $d_{j,r_1,r_2} = |r_1| d_{j,\frac{r_2}{r_1}}$, for all $j \in [n]$, and the results obtained by using such vectors are equivalent to the ones presented.

Note that all the terms occurring in the expression of B_r (see (9) above) are nonnegative, so that even after removing some or all of the terms involved in the sum, the expression (8) obtained still provides an upper bound on $mc_k(G, W)$. This is relevant with respect to complexity aspects (see Section 5.2).

In view of the bound (8) on $mc_k(G, W)$, one may ask for the best choice for the parameter r and the matrix Q. Firstly with respect to the parameter

r and assuming Q is fixed: if we consider the truncated bound obtained from (8) by removing B_r, it is straightforward to check that the ratio $\frac{r^2+k-1}{(r-1)^2}$ is minimized for $r = 1 - k$. Still assuming Q is fixed, if we now consider the whole expression of the bound (8), computational experiments show that other values of r may lead to strictly better bounds, depending on the instance. However the improvements that we observed in our experiments by considering other values than $1 - k$ for r in (8) tend to be rather small (by comparison with the choice $r = 1 - k$), and $r = 1 - k$ seems to be a fairly robust choice (more on that in Section 6).

In the statement of Theorem 3.1, the matrix Q may be interpreted as a perturbation of the coefficients of the matrix W. We shall see later how such perturbations may lead to a value of the bound (8) that is less than or equal to the bound stemming from Frieze and Jerrum's semidefinite relaxation (see Section 4), and we will also observe that the matrix Q may have an important impact on the value of the bound (Section 6).

Taking $Q = 0$ in Theorem 3.1, we obtain the following upper bound with a simpler expression, where the terms $d_{l,r}$ are defined similarly as $\widehat{d}_{l,r}$ above but using the linear subspace $\operatorname{lin}(\nu_1, \nu_2, \ldots, \nu_l)$ instead of $\operatorname{lin}(\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_l)$.

Corollary 3.2. *For any $r \in \mathbb{R} \setminus \{1\}$, the following inequality holds.*

$$mc_k(G, W) \leq \frac{1}{2(r-1)^2} \left((r^2 + k - 1)(2w[V] - \lambda_1 n) - k \sum_{l \in [n-1]} (\lambda_{l+1} - \lambda_l) d_{l,r}^2 \right). \tag{15}$$

Observe that the upper bound (6) is a particular case of (15), obtained by setting $k = 2$ and $r = 1 - k = -1$.

We conclude this section by mentioning that the approach for proving Theorem 3.1 can also be used to obtain lower bounds on the weight of any k-cut and to generalize results from [6, Section 2.1] for $k = 2$. Let $lc_k(G, W)$ denote the minimum weight of a k-cut in G and let $\overline{d}_{j,r}$ denote the distance between the set of vectors $\{r, 1\}^n$ and the subspace $\operatorname{lin}(\nu_j, \nu_{j+1}, \ldots, \nu_n)$ that is generated by the last $n-j+1$ eigenvectors of W:

$$\overline{d}_{j,r} = \min\left\{\|z-y\| : z \in \{r,1\}^n, y \in \operatorname{lin}(\nu_j, \nu_{j+1}, \ldots, \nu_n)\right\}. \tag{16}$$

Proposition 3.3.

$$lc_k(G,W) \geq \frac{1}{2(r-1)^2}\left((r^2+k-1)(2w[V]-\lambda_n n) + k\sum_{l\in[n-1]}(\lambda_{l+1}-\lambda_l)\overline{d}^2_{l+1,r}\right). \tag{17}$$

Proof. Similar to that of Theorem 3.1 taking $Q=0$. Alternatively, apply Theorem 3.1 with $Q=0$ and the weight matrix $-W$ instead of W, which gives an upper bound on $-lc_k(G,W)$. □

Corollary 3.2 and Proposition 3.3 lead to the definition of the *spectral bound gap*, which is the difference between the upper and lower spectral bounds:

$$sbgc_k = \frac{1}{2(r-1)^2}\left[\left(r^2+k-1\right)n\left(\lambda_n-\lambda_1\right) - k\sum_{l\in[n-1]}\left(\lambda_{l+1}-\lambda_l\right)\left(\overline{d}^2_{l+1,r} + d^2_{l,r}\right)\right].$$

4. Improving on the bound Z^*_{kSDP}

In this section we show that by making use of a dual optimal solution of (k-SDP) it is possible to define a matrix Q in Theorem 3.1 such that the bound given by (8) is less than or equal to the bound Z^*_{kSDP}.

First note that the upper bound from (k-SDP) can be expressed as follows: $Z^*_{kSDP} = \frac{k-1}{k}\left(w[V] + Z^*_1\right)$, where Z^*_1 stands for the optimal objective value of the SDP problem

$$(SDP_P)\begin{cases} Z^*_1 = & \max \quad \left(-\frac{1}{2}W\right)\bullet X \\ & s.t. \quad X_{ii} = 1, \qquad \forall i\in[n], \\ & \qquad\;\; X_{ij} - z_{ij} = -\frac{1}{k-1}, \qquad \forall i<j, (i,j)\in[n]^2, \\ & \qquad\;\; X \succeq 0, Diag(z) \succeq 0, \\ & \qquad\;\; X\in\mathbb{R}^{n\times n}, z\in\mathbb{R}^{\binom{n}{2}}. \end{cases}$$

The dual problem of (SDP_P) can be expressed as

$$(SDP_D)\begin{cases} Z_2^* = & \min & \sum_{i\in[n]} Y_{ii} - \frac{1}{k-1}\sum_{i<j:\,(i,j)\in[n]^2} Y_{ij} \\ & s.t. & \mathcal{B}(Y) + W \succeq 0, \\ & & Y_{ij} \leq 0, \forall i<j, (i,j)\in[n]^2, \\ & & Y \in \mathbb{R}^{n\times n}, Y \text{ symmetric}, \end{cases}$$

where $\mathcal{B}(Y)$ has entries

$$\begin{cases} \mathcal{B}(Y)_{ii} = 2Y_{ii}, \forall i \in [n], \\ \mathcal{B}(Y)_{ij} = Y_{ij}, \forall i \neq j, (i,j)\in[n]^2. \end{cases}$$

One can easily check that strong duality holds and thus $Z_1^* = Z_2^*$. Let Y^* denote an optimal solution of (SDP_D). Then the optimal objective value of $(k\text{-}SDP)$ can be expressed as: $Z^*_{kSDP} = \frac{k-1}{k}\left(w[V] + \sum_{i\in[n]} Y^*_{ii} - \frac{1}{k-1}\sum_{i<j:(i,j)\in[n]^2} Y^*_{ij}\right)$. Also, an observation to be used later is that necessarily the smallest eigenvalue of the matrix $\mathcal{B}(Y^*) + W$ is zero.

We now prove that particular upper bounds given by (8) are always less than or equal to Z^*_{kSDP}. In the statement of Theorem 3.1, let us take $Q = \mathcal{B}(Y^*)$. Then, in the expression of the spectral bound $A_r - B_r$, we have (using $\lambda_1(\mathcal{B}(Y^*) + W) = 0$):

$$A_r = \frac{1}{(r-1)^2}\left(\left(r^2+k-1\right)\left(w[V] + \sum_{i\in[n]} Y^*_{ii}\right) + (2r+k-1)\sum_{i<j:\,(i,j)\in[n]^2} Y^*_{ij}\right).$$

And taking for r the value $1-k$, we obtain: $A_{1-k} = Z^*_{kSDP}$. Since $B_{1-k} \geq 0$, the next result follows.

Corollary 4.1. *In the statement of Theorem 3.1, taking $r = 1-k$ and $Q = \mathcal{B}(Y^*)$, the spectral bound* (8) *is less than or equal to Z^*_{kSDP}.*

The last corollary leads to the next result establishing a domination relation between bounds previously introduced for Max k-Cut.

Proposition 4.2. *The best spectral bound $\widehat{Z}$ which can be obtained by the general family described in Theorem 3.1 is always no worse than the bounds given by Z^*_{kSDP},* (4) *and* (5).

Proof. By Corollary 4.1, we have $\widehat{Z} \leq Z^*_{kSDP}$.

In the statement of Theorem 3.1, considering the matrix $Q = -Diag(W\vec{1}_n)$, setting $r = 1-k$ and removing from (8) the nonnegative term B_r, we obtain the bound (4), and thus $\widehat{Z}$ dominates (4). (van Dam and Sotirov also present in [28] a strenghtened version of (4) obtained by considering a particular perturbation of the diagonal entries of the Laplacian matrix, which leads however to a bound that is dominated by (i.e. greater than or equal to) Z^*_{kSDP}. One can easily define a diagonal matrix Q such that their strengthened bound coincides with A_{1-k} in (8).)

The fact that $\widehat{Z}$ dominates (5) follows by considering the zero matrix for Q, $r = 1-k$, and removing B_r in the expression of the bound given by Theorem 3.1. $\square$

In Section 6 we will see on several instances the relevance of the perturbation of W as suggested by Proposition 4.1 and the improvement that may be obtained over Z^*_{kSDP} by using the whole expression of the bound (8).

5. On the efficient computation of distances

It has been shown [5, Proposition 4.4] that computing the distances $(d_{j,r})_{j=1}^{n-1}$ is $\mathcal{NP}$-hard in general. In this section, we shall deal with distances that can be computed efficiently. We start by considering the instances of Max k-Cut such that the all-ones vector is an eigenvector of W (Section 5.1). This case leads to simple expressions of upper bounds for Max k-Cut and permits us to identify a family of instances for which the spectral bound (8) coincides with $mc_k(G,W)$. Then, we show in Section 5.2 that, for any fixed positive integer $j \leq n-1$ and under some additional conditions, the distance $d_{j,r}$ can be computed in polynomial time.

5.1. On the case when $\vec{1}_n$ is an eigenvector of W

In this subsection we specialize Corollary 3.2 for the particular case when $\vec{1}_n$ is an eigenvector of W. In particular, we obtain simple expressions of upper bounds on $mc_k(G,W)$ that are lower than or equal to the bounds of Theorems 2.1 and 2.2. We start with an auxiliary result on the minimum squared distance between any vector in $\{1,r\}^n$ and the subspace in $\mathbb{R}^n$ that is orthogonal to $\mathrm{lin}(\vec{1}_n)$ and is denoted by $\mathrm{lin}(\vec{1}_n)^\perp$.

Proposition 5.1. *The following equation holds.*

$$\min\left\{\|y-z\|^2 : y \in \{1,r\}^n, z \in \mathrm{lin}(\vec{1}_n)^\perp\right\} = \begin{cases} n & \text{if } r \geq 1, \\ nr^2 & \text{if } 0 \leq r < 1, \\ \min(\frac{(s+r-1)^2}{n}, \frac{s^2}{n}) & \text{otherwise,} \end{cases}$$

with $n \equiv s \mod (1-r)$, $0 \leq s < 1-r$, for the case when $r < 0$.

Proof. Let $p \in \{0,1,\ldots,n\}$ and $\widehat{y} \in \{r,1\}^n$ such that $\widehat{y}$ has exactly p entries with value r. Let $\hat{d}^2$ denote the squared distance between $\widehat{y}$ and $\mathrm{lin}(\vec{1}_n)^\perp$, that is, the quantity

$$\hat{d}^2 = \frac{\langle \widehat{y}, \vec{1}_n \rangle^2}{n} = \frac{(p\,(r-1)+n)^2}{n},$$

where the first equation follows from the definition of $\hat{d}$ and the normalization of the vector $\vec{1}_n$. The minimum of $\hat{d}^2$ is obtained for $p=0$ if $r \geq 1$, for $p=n$ if $0 < r < 1$, and for $p = \left\lfloor \frac{n}{1-r} \right\rfloor$ or $p = \left\lceil \frac{n}{1-r} \right\rceil$, otherwise. □

Using Proposition 5.1 together with the fact that $d_{j,r} \geq d_{j+1,r}$, for all $j \in [n-1]$, yields the next result.

Corollary 5.2. *If $\vec{1}_n$ is an eigenvector of W associated with the eigenvalue λ_q, then*

$$mc_k(G,W) \leq \frac{1}{2(r-1)^2}\left((r^2+k-1)(2w[V]-\lambda_1 n) - \frac{k}{n}\min((s+r-1)^2, s^2)\sum_{l\in[q-1]}(\lambda_{l+1}-\lambda_l)\right) \tag{18}$$

with $r < 0$ and $n \equiv s \mod (1-r)$, $0 \leq s < 1-r$.

For the case of complete graphs with unit edge weights, taking $r = 1-k$ in (18) leads to the following simpler expression.

Corollary 5.3. *If G is a complete graph and all the edge weights are equal to one, then*

$$mc_k(G, W) \leq \frac{1}{2k}\left(\left(k-1\right)n^2 - \min\left(\left(s-k\right)^2, s^2\right)\right), \tag{19}$$

with $n \equiv s \mod k$, $0 \leq s < k$.

Proof. The eigenvalues of the adjacency matrix of the complete graph K_n are -1 with multiplicity $n-1$ and $n-1$ with multiplicity 1. The vector $\vec{1}_n$ is an eigenvector associated with the eigenvalue $\lambda_n = n-1$. The result follows from (18) with $q = n$ and $r = 1-k$. □

Corollary 5.3 gives an infinite class of graphs (complete graphs such that $\min((s-k)^2, s^2) > 0$) where Theorem 3.1 strictly improves on Theorem 2.2. The bound (19) has also the feature of coinciding with the optimal objective value of Max k-Cut for some cases. For completeness, we give the proof of the next result on the number of edges of Turán graphs, i.e., complete k-partite graphs (k integer, $k \geq 2$), whose partition sets differ in cardinality by at most one. Let $T_{n,k}$ stand for the complete k-partite graph on n vertices with partition sizes equal to $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$, and let $e(T_{n,k})$ denote its number of edges.

Proposition 5.4. *The number of edges of a k-partite Turán graph $T_{n,k}$ is*

$$e\left(T_{n,k}\right) = \frac{1}{2k}\left(\left(k-1\right)n^2 + s^2 - sk\right),$$

with $n \equiv s \mod k$, $0 \leq s < k$.

Proof. Let $n = qk + s$ with q and s nonnegative integers such that $0 \leq s < k$.

The number of edges is then equal to

$$
\begin{aligned}
& \binom{2}{n} - s\binom{2}{q+1} - (k-s)\binom{2}{q} \\
= \; & \tfrac{n(n-1)}{2} - s\tfrac{q(q+1)}{2} - (k-s)\tfrac{q(q-1)}{2} \\
= \; & \tfrac{1}{2}\left(n(n-1) - q\left(s\left(q+1\right) + \left(k-s\right)\left(q-1\right)\right)\right) \\
= \; & \tfrac{1}{2}\left(n(n-1) - q\left(2s + kq - k\right)\right) \\
= \; & \tfrac{1}{2}\big(n(n-1) - \tfrac{n-s}{k}\left(2s + (n-s) - k\right)\big) \\
= \; & \tfrac{1}{2k}\left(kn(n-1) - (n-s)\left(s+n-k\right)\right) \\
= \; & \tfrac{1}{2k}\big((k-1)n^2 + s^2 - sk\big)\,.
\end{aligned}
$$

□

Let us recall Turán's theorem [27].

Theorem 5.5. **[27]** *If G is an n-vertex K_{k+1}-free graph, then it contains at most $e\left(T_{n,k}\right)$ edges.*

By Theorem 5.5 and Proposition 5.4, the maximum cardinality of a k-cut in the complete graph K_n is $\frac{1}{2k}\big(\left(k-1\right)n^2 + s^2 - sk\big)$, with $n \equiv s \mod k$, $0 \leq s < k$. Corollary 5.3 leads to next result.

Proposition 5.6. *Let G be a complete graph with all the edge weights equal to one. If*

$$
\begin{cases}
n \equiv 0 \mod k, \text{ or} \\
n \equiv \frac{k}{2} \mod k \text{ and } k \text{ is even,}
\end{cases}
$$

then $mc_k(G, W)$ coincides with the bound (19).

Proposition 5.6 generalizes Proposition 4.4 in [6] (the latter being obtained by setting $k = 2$ and $r = -1$). Note that for $k = 2$, the bound (19) coincides with $mc_k(G, W)$ for all complete graphs with unit weights, whereas this fails for the bounds of Theorems 2.1 and 2.2 for complete graphs having an odd number of vertices. More generally, for any complete graph with unit edge weights such that $n \equiv \frac{k}{2} \mod k$, k positive and even, the bound (19) coincides with the optimal objective value $mc_k(G,W) = \frac{(k-1)n^2}{2k} - \frac{k}{8}$ and strictly improves over Z^*_{kSDP}, (4) and (5), which are all equal to $\frac{(k-1)n^2}{2k}$ for this family of instances. This is obvious for (4) and (5), since in this case we have $\lambda_n(L) = n$, $\lambda_1 = -1$. To see that $Z^*_{kSDP} = \frac{(k-1)n^2}{2k}$, note that a feasible solution of (k-SDP) is given by $X = \frac{1}{n-1}\left(nI_n - J_n\right), z = 0$, and a feasible solution of (SDP_D) is given by $Y = \frac{1}{2}I_n$, where I_n stands for the identity matrix of order n, and J_n for the all-ones matrix with order n. Since these two solutions have the same objective value $\frac{(k-1)n^2}{2k}$, the result follows. In fact, this family of instances of Max k-Cut also illustrates the fact that the gap between (19) and Z^*_{kSDP} (or equivalently (4) or (5) for this family of instances) can be arbitrarily large with increasing values of k.

5.2. Polynomial-time computable distances

Computing all the distances $(\widehat{d}_{j,r})_{j=1}^{n-1}$ involved in the expression of the bounds (8) is difficult in general. Even for $k = 2$, computing the single distance $d_{n-1,-1}$ is NP-hard in general [5]. However, given a fixed positive integer $p \leq n-1$ and assuming all the eigenvalues and eigenvectors are given and rational, it has been shown that computing the restricted set of distances $(\widehat{d}_{j,-1})_{j=1}^{p}$ can be done in polynomial time [7]. We now show that this result can be extended to the computation of the distances $(\widehat{d}_{j,r})_{j=1}^{p}$ for any $r \in \mathbb{R} \setminus \{1\}$.

Lemma 5.7. *Let $r \in \mathbb{R}\setminus\{1\}$. The computation of the distance $\widehat{d}_{j,r}$ is equivalent to an unconstrained quadratic program of the form $\min_{z\in\{-1,1\}^{n+1}} z^T\overline{Q}z$, where $\overline{Q}$ has rank at most $j+2$ and no positive diagonal entries.*

Proof. The problem of determining the squared distance $\left(\widehat{d}_{j,r}\right)^2$ can be formulated as follows:

$$(P1)\begin{cases} \min & y^TVV^Ty \\ & y \in \{r,1\}^n, \end{cases}$$

where V stands for the $n \times (n-j)$ matrix having as columns the eigenvectors $\widehat{\nu}_{j+1}, \widehat{\nu}_{j+2}, \ldots, \widehat{\nu}_n$. Using the affine transformation

$$z = \frac{2}{1-r}y - \frac{1+r}{1-r}\vec{1}_n,$$

problem $(P1)$ can be reformulated as follows:

$$(P2)\begin{cases} \min & \frac{(1-r)^2}{4} z^T V V^T z + \frac{1-r^2}{2} (\vec{1}_n)^T V V^T z + \frac{(1+r)^2}{4} (\vec{1}_n^T V V^T \vec{1}_n) \\ & z \in \{-1,1\}^n. \end{cases}$$

Using the fact that for any $z \in \{-1,1\}^n$ we have :

$$z^T V V^T z = \sum_{j+1}^{n} (z^T \nu_i)^2 = n - \sum_{i=1}^{j} (z^T \nu_i)^2,$$

problem $(P2)$ is equivalent to

$$(P3)\begin{cases} \min & z^T Q z + 2b^T z + c \\ & z \in \{-1,1\}^n, \end{cases}$$

where $Q = -\frac{(1-r)^2}{4} \overline{V}_j \overline{V}_j^T$, $\overline{V}_j$ stands for the $n \times j$ matrix whose columns correspond to the j first eigenvectors $\widehat{\nu}_1, \widehat{\nu}_2, \ldots, \widehat{\nu}_j$, $b = \frac{1-r^2}{4} V V^T \vec{1}_n$, and $c = \frac{n}{4}(1-r)^2 + \frac{1}{4}(1+r)^2 \left(\vec{1}_n^T V V^T \vec{1}_n\right)$. $(P3)$ can be reformulated as

$$(P4)\begin{cases} \min & \overline{z}^T \overline{Q} \overline{z} \\ & \overline{z} \in \{-1,1\}^{n+1}, \end{cases}$$

where $\overline{Q}$ is an $(n+1) \times (n+1)$ matrix with rows and columns indexed on $\{0, 1, \ldots, n\}$, and entries defined as follows.

$$\overline{Q}_{il} = \begin{cases} 0 & \text{if } i = l = 0, \\ b_{i+l} & \text{if } \{i,l\} \cap \{0\} \neq \emptyset \text{ and } i + l \neq 0, \\ Q_{il} & \text{otherwise.} \end{cases}$$

(Given an optimal solution $\overline{z}^*$ for $(P4)$, an optimal solution for $(P3)$ is given by $z_i^* = \overline{z}_0^* \overline{z}_i^*$, for all $i \in \{1, 2, \ldots, n\}$.)

Note that the matrix $\overline{Q}$ has rank at most $j+2$. This can be seen as follows. By definition Q has rank at most j. Adding to Q a row corresponding to vector b, the resulting $(n+1) \times n$ matrix Q' has rank at most $j+1$. Adding to Q' the column vector $(0,b)^T$, we obtain $\overline{Q}$ and its rank is at most $j+2$. $\square$

The next result directly follows from Lemma 5.7 above and Theorem 2 from [7]. It extends a polynomiality result about the complexity of computing the distance $\widehat{d}_{j,-1}$ for a fixed positive integer $j \leq n-1$ (see Corollary 2.7 in [6]).

Theorem 5.8. *Let $r \in \mathbb{R} \setminus \{1\}$. For a fixed positive integer $j \leq n-1$, assuming the eigenvalues and eigenvectors of the matrix $\overline{Q}$ as mentioned in Proposition 5.7 are given and rational, the distance $\widehat{d}_{j,r}$ can be computed in polynomial time.*

6. Computational experiments

In this section, we report computational results to illustrate the quality of the new bounds in comparison to other spectral bounds and the Frieze & Jerrum bound. We do not concern ourselves with the computational effort to obtain the distances involved in the new bounds, which we compute using a straightforward enumeration procedure. Developing efficient algorithms for determining them and dealing with large instances is a challenging matter for future research work. In the present study, we consider graphs having up to 30 nodes, and with the practical setting described hereafter, the computation of a single bound involving the distances (denoted Sp in what follows) for a fixed value of the parameter r takes about 45 minutes, whereas it is negligible (less than one second) for the other bounds (denoted FJ, vDS and N hereafter).

6.1. Practical setting

All the computational experiments were performed on a laptop using a processor Intel Core i7-2640M CPU @ 2.80GHz x 4, 7.7 Gio RAM. Our implementation is in C, and the SDPs were solved using CSDP [9]. The graphs used in our experiments are as follows, where d stands for a real value in $[0, 1]$:

- $gka1b$, $gka2b$, $gka6a$, $gka7a$: these are four instances of unconstrained binary quadratic programs of the form: $\max_{x\in\{0,1\}^n} x^T Qx$, taken from [16], where Q is a symmetric matrix of order $n \leq 30$. In our experiments, the matrix Q is re-interpreted as the weighted adjacency matrix of a graph of order n, ignoring the diagonal coefficients. The off-diagonal coefficients of Q are integers in $[0, 100]$ for $gka1b$, $gka2b$ and integers in $[-100, 100]$ for $gka6a$, $gka7a$.

- C_n: the cycle with n nodes.
- K_n: the complete graph with n nodes.
- $P_i(n, d)$: planar graphs of order n, with density parameter $d \in [0, 1]$ (so that the number of edges is about $3(n-2)d$), $i = 1, 2, \ldots, 8$.
- $R_i(n, p)$: random graphs with order n and density parameter d (so that the number of edges is about $\frac{n(n-1)}{2}d$), $i = 1, 2, \ldots, 12$.

Except for the four instances from [16], all the instances were generated using *rudy* [25]. (We indicate in Appendix A the input data to generate the instances different from the ones taken from [16].)

In our experiments, we consider both the cases of unit and non-unit edge weights. For the instances from [16], setting the nonzero entries of Q to value one is indicated by the the notation (*unit*) next to the name of the instance. For the case of non-unit edge weights, except for the four instances from [16] (for which we use the original weights), these are uniformly and randomly generated in $[-100, 100]$ using *rudy* [25].

The computational results are reported using the following notation:

- FJ: upper bound from (k-SDP).
- vDS: upper bound from [28] (Theorem 2.1).
- N: upper bound from [22] (Theorem 2.2).
- Sp: upper bound from Corollary 3.2 with the value for r that is mentioned in parentheses.
- $FJ + Sp$: upper bound from Theorem 3.1 with the value for r that is mentioned in parentheses and the matrix $Q = \mathcal{B}(Y^*)$ as described in Section 4.
- For Sp and $FJ + Sp$, we also report in additional columns the best upper bound from testing all the values $\{-k + 0.5q \colon q = 0, 1, ..., 4\}$ for r and

selecting the value r_{best} that gives the lowest bound (reported in parentheses).

6.2. Computational results

The results for instances of Max 3-Cut (resp. Max 4-Cut and Max 5-Cut) are reported in Table 2 (resp. 4, 6) for the case of unit weights and in Table 3 (resp. 5, 7) for non-unit weights (see the previous section).

If we first consider the bounds not involving semidefinite programming, i.e. vDS, N and Sp, the best results were obtained with Sp (for $k = 3, 4, 5$) with the exception of two instances from [16] for the case of unit weights. In particular for the case of non-unit weights the gaps are significant between Sp on the one hand and vDS and N on the other hand. The instance K_{30} with unit weights for Max 4-Cut (see Table 4) illustrates our discussion following Proposition 5.6 on cases when Sp coincides with $mc_k(G, W)$ and strictly improves over FJ, vDS and N.

Considering now the bounds using (k-SDP) (i.e. FJ and $FJ + Sp$), they clearly (non strictly) dominate the other bounds. The improvements over FJ obtained with $FJ + Sp$ seem modest for the case of unit weights but tend to be more important for the case of non-unit weights.

Finally, concerning the question raised in Section 3 about the best value of the parameter r and its impact on the spectral bound from Theorem 3.1, our experiments suggest that the best values may be close to $1 - k$, but possibly different. This is illustrated, for example, by the values of Sp on the planar instances from Table 3. The results also show that small modifications of this parameter may lead to substantial improvements of the bound, in particular for the case of non-unit weights and without perturbations of the weighted adjacency matrix.

Table 2: Computational results on upper bounds for Max 3-Cut and unit weights

Instance	$\|V\|$	$\|E\|$	FJ	vDS	N	Sp ($r=-2$)	$FJ+Sp$ ($r=-2$)	Sp ($r=r_{best}$)	$FJ+Sp$ ($r=r_{best}$)
gka1b (unit)	20	187	133.33	133.33	138.00	137.25	133.17	137.25 (-2.0)	133.17 (-2.0)
gka2b (unit)	30	429	300.00	300.00	306.00	304.82	300.00	304.82 (-2.0)	300.00 (-2.0)
gka6a (unit)	30	174	156.05	197.81	171.08	162.25	154.81	162.25 (-2.0)	154.81 (-2.0)
gka7a (unit)	30	211	180.69	214.56	195.26	186.87	178.97	186.87 (-2.0)	178.97 (-2.0)
C_{30}	30	30	30.00	40.00	40.00	38.75	30.00	38.17 (-2.5)	30.00 (-2.0)
K_{30}	30	435	300.00	300.00	300.00	300.00	300.00	300.00 (-2.0)	300.00 (-2.0)
$P_1(30,.7)$	30	58	57.00	116.70	72.71	65.50	56.90	65.50 (-2.0)	56.90 (-2.0)
$P_2(30,.7)$	30	58	56.34	136.83	75.05	67.06	56.23	67.06 (-2.0)	56.23 (-2.0)
$P_3(30,.9)$	30	75	70.06	143.74	81.62	76.27	69.73	76.27 (-2.0)	69.73 (-2.0)
$P_4(30,.9)$	30	75	70.29	171.67	82.84	77.03	70.09	77.03 (-2.0)	70.09 (-2.0)
$R_1(30,.25)$	30	109	104.82	138.91	119.87	112.04	104.05	112.04 (-2.0)	104.05 (-2.0)
$R_2(30,.25)$	30	109	103.95	155.33	117.09	110.07	103.16	110.07 (-2.0)	103.16 (-2.0)
$R_3(30,.5)$	30	218	187.87	218.32	199.63	192.62	186.26	192.62 (-2.0)	186.26 (-2.0)
$R_4(30,.5)$	30	218	185.84	222.24	194.68	188.89	184.24	188.89 (-2.0)	184.24 (-2.0)
$R_5(30,.8)$	30	348	270.25	285.63	280.02	275.02	268.99	275.02 (-2.0)	268.99 (-2.0)
$R_6(30,.8)$	30	348	270.26	291.34	281.34	275.60	268.98	275.60 (-2.0)	268.98 (-2.0)

Table 3: Computational results on upper bounds for MAX 3-CUT and non-unit weights

Instance	$\|V\|$	$\|E\|$	FJ	vDS	N	Sp ($r=-2$)	$FJ+Sp$ ($r=-2$)	Sp ($r=r_{best}$)	$FJ+Sp$ ($r=r_{best}$)
gka1b	20	187	7547.21	8451.14	7921.31	7698.05	7502.65	7698.05 (-2.0)	7502.65 (-2.0)
gka2b	30	429	16605.75	18564.99	17148.43	16791.01	16521.28	16791.01 (-2.0)	16521.28 (-2.0)
gka6a	30	174	2454.64	4470.14	3631.95	2972.53	2365.88	2972.53 (-2.0)	2365.88 (-2.0)
gka7a	30	211	2671.19	6728.62	3960.69	3166.80	2545.51	3166.80 (-2.0)	2545.51 (-2.0)
C_{30}	30	30	1122.00	2530.93	1638.77	1496.45	1122.00	1492.34 (-2.5)	1122.00 (-2.0)
K_{30}	30	435	4289.89	7774.93	5314.43	4565.10	4091.73	4565.10 (-2.0)	4091.73 (-2.0)
$P_5(30, .7)$	30	58	1373.12	4079.19	2541.64	2002.70	1354.35	1969.65 (-1.5)	1353.04 (-1.5)
$P_6(30, .7)$	30	58	1103.17	3565.42	2187.73	1665.76	1083.47	1652.84 (-1.5)	1079.88 (-1.5)
$P_7(30, .9)$	30	75	824.25	3470.83	1747.35	1302.61	791.01	1280.31 (-1.5)	791.01 (-2.0)
$P_8(30, .9)$	30	75	1659.94	4287.78	3429.51	2620.02	1645.23	2545.28 (-1.5)	1644.77 (-1.5)
$R_7(30, .25)$	30	109	2316.90	5051.98	3421.89	2883.97	2281.90	2883.97 (-2.0)	2281.90 (-2.0)
$R_8(30, .25)$	30	109	2286.42	5540.40	3726.62	2947.49	2221.64	2947.49 (-2.0)	2221.64 (-2.0)
$R_9(30, .5)$	30	218	2186.70	4476.53	3062.93	2524.49	2047.29	2524.49 (-2.0)	2047.29 (-2.0)
$R_{10}(30, .5)$	30	218	3112.21	6667.96	4360.54	3717.39	3013.75	3717.39 (-2.0)	3013.75 (-2.0)
$R_{11}(30, .8)$	30	348	4312.47	6578.78	5196.61	4629.66	4171.72	4629.66 (-2.0)	4171.72 (-2.0)
$R_{12}(30, .8)$	30	348	4085.90	7113.67	5390.40	4519.50	3931.45	4519.50 (-2.0)	3931.45 (-2.0)

Table 4: Computational results on upper bounds for MAX 4-CUT and unit weights

Instance	$\|V\|$	$\|E\|$	FJ	vDS	N	Sp $(r=-3)$	$FJ+Sp$ $(r=-3)$	Sp $(r=r_{best})$	$FJ+Sp$ $(r=r_{best})$
gka1b (unit)	20	187	150.00	150.00	155.25	154.67	150.00	154.67 (-3.0)	150.00 (-3.0)
gka2b (unit)	30	429	337.50	337.50	344.25	343.07	337.00	343.07 (-3.0)	337.00 (-3.0)
gka6a (unit)	30	174	168.90	222.54	192.46	180.36	168.40	180.36 (-3.0)	168.40 (-3.0)
gka7a (unit)	30	211	198.96	241.37	219.67	208.57	197.71	208.57 (-3.0)	197.71 (-3.0)
C_{30}	30	30	30.00	45.00	45.00	40.95	30.00	40.95 (-3.0)	30.00 (-3.0)
K_{30}	30	435	337.50	337.50	337.50	337.00	337.00	337.00 (-3.0)	337.00 (-3.0)
$P_1(30,.7)$	30	58	58.00	131.29	81.80	73.68	58.00	73.23 (-2.5)	58.00 (-3.0)
$P_2(30,.7)$	30	58	58.00	153.94	84.44	76.93	58.00	75.94 (-2.5)	58.00 (-3.0)
$P_3(30,.9)$	30	75	75.00	161.70	91.82	85.49	75.00	85.49 (-3.0)	75.00 (-3.0)
$P_4(30,.9)$	30	75	75.00	193.13	93.19	86.36	75.00	86.13 (-2.5)	75.00 (-3.0)
$R_1(30,.25)$	30	109	109.00	156.28	134.86	124.21	109.00	124.21 (-3.0)	109.00 (-3.0)
$R_2(30,.25)$	30	109	108.98	174.75	131.73	122.35	108.98	122.35 (-3.0)	108.98 (-3.0)
$R_3(30,.5)$	30	218	205.72	245.61	224.58	215.03	204.83	215.03 (-3.0)	204.83 (-3.0)
$R_4(30,.5)$	30	218	204.92	250.02	219.02	211.42	204.26	211.42 (-3.0)	204.26 (-3.0)
$R_5(30,.8)$	30	348	300.77	321.33	315.02	307.25	299.46	307.25 (-3.0)	299.46 (-3.0)
$R_6(30,.8)$	30	348	301.18	327.75	316.51	308.68	299.98	308.68 (-3.0)	299.98 (-3.0)

Table 5: Computational results on upper bounds for Max 4-Cut and non-unit weights

Instance	$\lvert V\rvert$	$\lvert E\rvert$	FJ	vDS	N	Sp ($r=-3$)	$FJ+Sp$ ($r=-3$)	Sp ($r=r_{best}$)	$FJ+Sp$ ($r=r_{best}$)
gka1b	20	187	8332.67	9507.53	8911.47	8652.01	8309.91	8652.01 (-3.0)	8309.91 (-3.0)
gka2b	30	429	18409.31	20885.61	19291.98	18763.12	18336.08	18763.12 (-3.0)	18336.08 (-3.0)
gka6a	30	174	2514.45	5028.91	4085.94	3271.60	2419.39	3221.83 (-2.5)	2413.12 (-2.5)
gka7a	30	211	2735.57	7569.69	4455.78	3580.26	2653.06	3515.72 (-2.5)	2618.38 (-2.0)
C_{30}	30	30	1122.00	2847.29	1843.61	1603.17	1122.00	1603.17 (-3.0)	1122.00 (-3.0)
K_{30}	30	435	4435.21	8746.79	5978.73	4956.97	4256.00	4956.97 (-3.0)	4251.25 (-2.5)
$P_5(30,.7)$	30	58	1389.23	4589.09	2859.34	2323.69	1377.79	2246.66 (-2.0)	1373.35 (-2.0)
$P_6(30,.7)$	30	58	1108.82	4011.10	2461.20	1894.45	1097.06	1840.77 (-2.5)	1090.45 (-2.0)
$P_7(30,.9)$	30	75	852.91	3904.68	1965.77	1540.86	830.20	1445.59 (-2.0)	827.43 (-2.0)
$P_8(30,.9)$	30	75	1671.66	4823.75	3858.20	3013.76	1665.97	2921.77 (-2.0)	1662.89 (-2.0)
$R_7(30,.25)$	30	109	2351.97	5683.48	3849.62	3188.21	2332.51	3188.21 (-3.0)	2331.72 (-2.5)
$R_8(30,.25)$	30	109	2330.44	6232.94	4192.45	3396.54	2283.29	3255.73 (-2.5)	2270.30 (-2.0)
$R_9(30,.5)$	30	218	2247.85	5036.09	3445.79	2705.54	2168.96	2705.54 (-3.0)	2140.88 (-2.0)
$R_{10}(30,.5)$	30	218	3203.67	7501.45	4905.60	4106.77	3120.57	4070.72 (-2.5)	3118.76 (-2.5)
$R_{11}(30,.8)$	30	348	4428.04	7401.12	5846.18	5111.59	4314.15	5111.59 (-3.0)	4308.57 (-2.5)
$R_{12}(30,.8)$	30	348	4172.81	8002.88	6064.20	4954.34	4044.38	4948.71 (-2.5)	4016.76 (-2.0)

Table 6: Computational results on upper bounds for MAX 5-CUT and unit weights

Instance	$\lvert V\rvert$	$\lvert E\rvert$	FJ	vDS	N	Sp $(r=-4)$	$FJ+Sp$ $(r=-4)$	Sp $(r=r_{best})$	$FJ+Sp$ $(r=r_{best})$
gka1b (unit)	20	187	160.00	160.00	165.60	165.12	160.00	165.12 (-4.0)	160.00 (-4.0)
gka2b (unit)	30	429	360.00	360.00	367.20	366.48	360.00	366.48 (-4.0)	360.00 (-4.0)
gka6a (unit)	30	174	173.79	237.37	205.29	192.65	173.75	191.94 (-3.5)	173.75 (-4.0)
gka7a (unit)	30	211	207.56	257.47	234.31	223.14	207.00	222.62 (-3.5)	207.00 (-4.0)
C_{30}	30	30	30.00	48.00	48.00	43.28	30.00	43.15 (-3.5)	30.00 (-4.0)
K_{30}	30	435	360.00	360.00	360.00	360.00	360.00	360.00 (-4.0)	360.00 (-4.0)
$P_1(30,.7)$	30	58	58.00	140.04	87.25	79.25	58.00	78.46 (-3.0)	58.00 (-4.0)
$P_2(30,.7)$	30	58	58.00	164.20	90.07	82.57	58.00	82.09 (-3.0)	58.00 (-4.0)
$P_3(30,.9)$	30	75	75.00	172.48	97.94	91.74	75.00	91.26 (-3.5)	75.00 (-4.0)
$P_4(30,.9)$	30	75	75.00	206.00	99.41	92.55	75.00	92.04 (-3.5)	75.00 (-4.0)
$R_1(30,.25)$	30	109	109.00	166.70	143.85	133.11	109.00	132.16 (-3.5)	109.00 (-4.0)
$R_2(30,.25)$	30	109	109.00	186.40	140.51	130.64	109.00	130.09 (-3.5)	109.00 (-4.0)
$R_3(30,.5)$	30	218	213.68	261.98	239.56	228.81	213.36	228.79 (-3.5)	213.36 (-4.0)
$R_4(30,.5)$	30	218	213.69	266.69	233.62	224.64	213.13	224.64 (-4.0)	213.13 (-4.0)
$R_5(30,.8)$	30	348	318.42	342.75	336.02	327.59	317.11	327.59 (-4.0)	317.11 (-4.0)
$R_6(30,.8)$	30	348	318.53	349.60	337.61	328.98	317.53	328.98 (-4.0)	317.53 (-4.0)

7. Conclusion

In this paper we introduced a new class of bounds for the MAX k-CUT problem involving the spectrum of the (possibly perturbed) weighted adjacency matrix. We exhibited a family of instances for which the new bounds are tight. We showed that truncated variants of the bounds can be computed in polynomial time. Computational experiments show that the new bounds compare well with other spectral bounds from the literature. We proved that particular perturbations of the weighted adjacency matrix could be used so that the bound obtained using our results dominates (non strictly) the bound from the Frieze & Jerrum semidefinite relaxation. Future research will look at developing efficient methods for computing the distances involved in the expression of the

Table 7: Computational results on upper bounds for Max 5-Cut and non-unit weights

Instance	$\|V\|$	$\|E\|$	FJ	vDS	N	Sp ($r=-4$)	$FJ+Sp$ ($r=-4$)	Sp ($r=r_{best}$)	$FJ+Sp$ ($r=r_{best}$)
$gka1b$	20	187	8708.12	10141.37	9505.57	9215.10	8699.70	9215.10 (-4.0)	8699.70 (-4.0)
$gka2b$	30	429	19373.01	22277.99	20578.12	19984.58	19314.46	19984.58 (-4.0)	19314.46 (-4.0)
$gka6a$	30	174	2532.91	5364.17	4358.33	3607.97	2478.77	3408.51 (-3.0)	2452.83 (-3.0)
$gka7a$	30	211	2755.71	8074.34	4752.83	3840.87	2705.64	3732.70 (-3.0)	2687.73 (-3.0)
C_{30}	30	30	1122.00	3037.11	1966.52	1674.95	1122.00	1664.52 (-3.5)	1122.00 (-4.0)
K_{30}	30	435	4482.41	9329.91	6377.31	5297.62	4361.68	5150.83 (-3.5)	4315.81 (-3.0)
$P_5(30,.7)$	30	58	1395.68	4895.03	3049.96	2535.54	1387.09	2428.06 (-3.0)	1383.64 (-3.0)
$P_6(30,.7)$	30	58	1111.26	4278.51	2625.28	2110.94	1103.19	1957.86 (-3.0)	1099.47 (-3.0)
$P_7(30,.9)$	30	75	864.80	4164.99	2096.82	1673.72	845.99	1598.45 (-3.0)	845.32 (-3.5)
$P_8(30,.9)$	30	75	1674.19	5145.34	4115.42	3352.09	1670.80	3124.17 (-3.0)	1669.49 (-3.0)
$R_7(30,.25)$	30	109	2358.40	6062.38	4106.26	3396.25	2345.38	3330.54 (-3.5)	2342.37 (-3.0)
$R_8(30,.25)$	30	109	2341.76	6648.47	4471.95	3719.89	2317.87	3546.93 (-3.0)	2307.28 (-3.0)
$R_9(30,.5)$	30	218	2262.05	5371.83	3675.51	2961.17	2219.45	2807.62 (-3.0)	2199.66 (-3.0)
$R_{10}(30,.5)$	30	218	3231.45	8001.55	5232.64	4457.79	3176.37	4315.86 (-3.0)	3165.59 (-3.0)
$R_{11}(30,.8)$	30	348	4450.17	7894.53	6235.93	5379.06	4399.91	5363.86 (-3.5)	4376.85 (-3.0)
$R_{12}(30,.8)$	30	348	4200.29	8536.41	6468.48	5316.22	4118.67	5175.52 (-3.5)	4084.17 (-3.0)

new bounds.

References

[1] Anjos M.F., Ghaddar B., Hupp L., Liers F., Wiegele A.: Solving k-Way Graph Partitioning Problems to Optimality: The Impact of Semidefinite Relaxations and the Bundle Method. In Facets of Combinatorial Optimization, Jünger M., Reinelt G. (Eds) . Springer, Berlin (2013) 355-386

[2] Arora, S., Lund, C., Motwani, R., Sudan, M., Szegedy, M.: Proof verification and hardness of approximation problems. Proceedings of the 33rd

IEEE Symposium on Foundations of Computer Science, IEEE Computer Society Press (1992) 14-23

[3] Barahona, F., Mahjoub, A.R.: On the cut polytope. Mathematical Programming 36 (1986) 157-173.

[4] Barahona, F., Grötschel, M., Jünger, M., Reinelt, G.: An application of combinatorial optimization to statistical physics and circuit layout design. Operations Research 36:3 (1988) 493-513

[5] Ben-Ameur, W., Neto, J.: Spectral bounds for the maximum cut problem. Networks 52 (2008) 8-13

[6] Ben-Ameur, W., Neto, J.: Spectral bounds for unconstrained $(-1, 1)$-quadratic optimization problems. European Journal of Operational Research 207 (2010) 15-24

[7] Ben-Ameur, W., Neto, J.: A polynomial-time recursive algorithm for some unconstrained quadratic optimization problems. Discrete Applied Mathematics 159 (2011) 1689-1698

[8] Ben-Ameur, W., Mahjoub, A.R., Neto, J.: The Maximum Cut Problem. In Paradigms of Combinatorial Optimization, V.T. Paschos (Ed.). John Wiley & Sons (2014) 131-172

[9] Borchers, B.: CSDP, a C Library for Semidefinite Programming. Optimization Methods and Software 11:1 (1999) 613-623

[10] Chopra, S., Rao, M.R.: The partition problem. Mathematical Programming 59:1-3 (1993) 87-115

[11] Chopra, S., Rao, M.R.: Facets of the k-partition polytope. Discrete Applied Mathematics 61:1 (1995) 27-48

[12] Coja-Oghlan, A., Moore, C., Sanwalani, V.: Max k-cut and approximating the chromatic number of random graphs. Random Structures and Algorithms 28:3 (2006) 289-322

[13] de Klerk, E., Pasechnik, D.V., Warners, J.P.: On approximate graph colouring and max-k-cut algorithms based on the θ-function. Journal of Combinatorial Optimization 8:3 (2004) 267-294

[14] Eisenblätter, A.: The semidefinite relaxation of the k-partition polytope is strong. In Cook W.J., Schulz A.S. (Eds) Integer Programming and Combinatorial Optimization. IPCO 2002. Lecture Notes in Computer Science, vol 2337. Springer, Berlin (2002) 273-290

[15] Frieze, A., Jerrum, M.: Improved approximation algorithms for max k-cut and max bisection. Algorithmica 18:1 (1997) 67-81

[16] Glover, F., Kochenberger, G., Alidaee, B.: Adaptative memory tabu search for binary quadratic programs. Management Science 44:3 (1998) 336-345

[17] Goemans, M.X., Williamson, D.P.: Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. J. ACM 42:6 (1995) 1115–1145

[18] Goemans, M.X., Williamson, D.P.: Approximation algorithms for max 3-cut and other problems via semidefinite programming. Journal of Computer and System Sciences 68:2 (2004) 442-470

[19] Kann, V., Khanna, S., Lagergren, J., Panconesi, A.: On the hardness of approximating max k-cut and its dual. Chicago Journal of Theoretical Computer Science 2 (1997) 1-18

[20] Liers, F., Jünger, M., Reinelt, G., Rinaldi, G.: Computing exact ground states of hard Ising spin glass problems by branch-and-cut. In New optimization algorithms in physics. Wiley (2004) 47-68

[21] Mohar, B., Poljak, S.: Eigenvalues and the max-cut problem. Czechoslovak Mathematical Journal 40:115 (1990) 343-352

[22] Nikiforov, V.: Max k-cut and the smallest eigenvalue. Linear Algebra and its Applications 504 (2016) 462-467

[23] Papadimitriou, C.H., Yannakakis, M.: Optimization, approximation, and complexity classes. Journal of Computer and System Sciences 43:3 (1991) 425-440

[24] Rendl, F., Rinaldi, G., Wiegele, A.: Solving max-cut to optimality by intersecting semidefinite and polyhedral relaxations. Mathematical Programming 121 (2008) 307-335

[25] Rinaldi, G. Rudy.
https://www-user.tu-chemnitz.de/~helmberg/rudy.tar.gz (1996)

[26] Rodrigues De Sousa, V.J., Anjos, M.F., Le Digabel, S.: Computational study of valid inequalities for the maximum k-cut problem. Annals of Operations Research 265:1 (2018) 5-27

[27] Turán, P.: On an extremal problem in graph theory. Matematikai és Fizikai Lapok 48 (1941) 436-452

[28] van Dam, E.R., Sotirov, R.: New bounds for the max-k-cut and chromatic number of a graph. Linear Algebra and its Applications 488 (2016) 216-234

Appendix A. Input data to generate the test graphs with "*rudy*"

Instance	Command line
C_{30}	`rudy -circuit 30`
C_{30} (W)	`rudy -circuit 30 -random -100 100 4001`
K_{30}	`rudy -clique 30 -random 1 1 1000`
K_{30} (W)	`rudy -clique 30 -random -100 100 5001`
$P_1(30, .7)$	`rudy -planar 30 70 1001`
$P_2(30, .7)$	`rudy -planar 30 70 2001`
$P_3(30, .9)$	`rudy -planar 30 90 3001`
$P_4(30, .9)$	`rudy -planar 30 90 4001`
$P_5(30, .7)$	`rudy -planar 30 70 1001 -random -100 100 1001`
$P_6(30, .7)$	`rudy -planar 30 70 2001 -random -100 100 2001`
$P_7(30, .9)$	`rudy -planar 30 90 3001 -random -100 100 3001`
$P_8(30, .9)$	`rudy -planar 30 90 4001 -random -100 100 4001`
$R_1(30, .25)$	`rudy -rnd_graph 30 25 1001`
$R_2(30, .25)$	`rudy -rnd_graph 30 25 2001`
$R_3(30, .5)$	`rudy -rnd_graph 30 50 3001`
$R_4(30, .5)$	`rudy -rnd_graph 30 50 4001`
$R_5(30, .8)$	`rudy -rnd_graph 30 80 5001`
$R_6(30, .8)$	`rudy -rnd_graph 30 80 6001`
$R_7(30, .25)$	`rudy -rnd_graph 30 25 1001 -random -100 100 1001`
$R_8(30, .25)$	`rudy -rnd_graph 30 25 2001 -random -100 100 2001`
$R_9(30, .5)$	`rudy -rnd_graph 30 50 3001 -random -100 100 3001`
$R_{10}(30, .5)$	`rudy -rnd_graph 30 50 4001 -random -100 100 4001`
$R_{11}(30, .8)$	`rudy -rnd_graph 30 80 5001 -random -100 100 5001`
$R_{12}(30, .8)$	`rudy -rnd_graph 30 80 6001 -random -100 100 6001`