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A procedural egalitarian solution for NTU-games[☆]

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ABSTRACT

This paper introduces and analyzes a procedural egalitarian solution for nontransferable utility games. This concept is based on an egalitarian procedure in which egalitarian opportunities of coalitions are explicitly taken into account. We formulate conditions under which the new solution prescribes a core element and derive a direct expression on the class of bargaining games and the class of bankruptcy games.

1. Introduction

Egalitarianism plays a central role in fundamental principles of justice and is widely applied within several disciplines. The interpretation of egalitarianism, and which notions should exactly be equated, depends on the underlying model and its characteristics. In a general payoff space where individual utility is represented in incompatible measures, egalitarianism cannot be applied straightforwardly. To do so, it is necessary to impose assumptions which allow to compare utility not only intrapersonally, but to some extent also interpersonally.

This paper focuses on egalitarianism in the context of nontransferable utility games. Shapley and Shubik [20] introduced this model to extend the standard definition of cooperative games by dropping two substantial restrictions on the nature of utility: linearity and transferability. The economic possibilities of coalitions are now expressed in a set of attainable utility allocations. The players are assumed to cooperate in the grand coalition and a main question is to determine the utility allocation that the players agree upon or that an arbitrator recommends, while taking the opportunities of subcoalitions into account. To allow for an adequate egalitarian comparison of subcoalitions, it is required to consistently apply a fixed interpretation of egalitarianism across coalitions.

Inspired by the Kalai [11] solution for bargaining problems (cf. Nash [14]), Kalai and Samet [12] introduced egalitarian solutions for nontransferable utility games which recursively allocate payoffs in an equal way or according to exogenous weights. Inspired by the Kalai and Smorodinsky [13] solution for bargaining problems, we take a *relative* egalitarian approach and define a procedure which iteratively allocates payoffs according to endogenous weights determined by the

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utopia values of the players in the game, i.e. the maximal payoffs within the attainable individually rational allocations of the grand coalition. Contrary to Kalai and Samet [12], this approach ensures that all notions are covariant under individual rescaling of utility.

Assuming that utility is normalized in such a way that all maximal individually attainable utility levels equal zero, we consistently interpret the utopia vector as a relative egalitarian direction from the zero vector within any subcoalition. The egalitarian procedure starts assigning to any coalition the weakly Pareto efficient allocation in this direction. Players can claim their allocated payoff in a coalition if no member is allocated a higher payoff in any other coalition. In a next iteration, these players claim these payoffs in each coalition and the other members are assigned egalitarian payoffs in the utopia direction. This procedure continues and eventually all players acquire an egalitarian claim attainable in at least one coalition.

Subsequently, we introduce a procedural egalitarian solution for nontransferable utility games which takes the result of this egalitarian procedure into account to prescribe a unique egalitarian allocation for the grand coalition. The egalitarian claims can be interpreted as aspiration levels for such allocation. Players which are member of all inclusion-wise maximal egalitarian admissible coalitions are called strong egalitarian claimants. The procedural egalitarian solution prioritizes strong claimants over the other players and aims to allocate the egalitarian claims according to this priority. The possible infeasibility is modeled as a bankruptcy problem (cf. Orshan et al. [15]) in which the egalitarian claims are adopted. The constrained relative equal awards rule (cf. Dietzenbacher et al. [5]), which allocates payoffs relatively equal subject to claims boundedness, is used to solve this bankruptcy problem.

In the context of transferable utility games, the utopia values of all players coincide and the procedural egalitarian solution boils down to the Dietzenbacher et al. [4] solution, which in turn coincides with the Dutta and Ray [6] solution on the class of convex transferable utility games. On the class of bargaining games, the procedural egalitarian solution coincides with the Kalai and Smorodinsky [13] solution. On the class of bankruptcy games (cf. Dietzenbacher [3]), the procedural egalitarian solution coincides with the constrained relative equal awards rule.

This paper is organized in the following way. Section 2 formally describes nontransferable utility games. Section 3 introduces a procedural egalitarian solution and the underlying egalitarian procedure. Sections 4 and 5 analyze the procedural egalitarian solution on the class of bargaining games and on the class of bankruptcy games, respectively.

2. Preliminaries

Let N be a nonempty and finite set of *players*. The collection of all *coalitions* is denoted by $2^N = \{S \mid S \subseteq N\}$. For any $x, y \in \mathbb{R}^N$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. For any $x \in \mathbb{R}^N$ and any $S \in 2^N$, $x_S \in \mathbb{R}^S$ denotes $(x_i)_{i \in S}$. For any $S \in 2^N$, $0_S \in \mathbb{R}^S$ denotes the zero vector. For any $A, B \subseteq \mathbb{R}^N$, $A \subset B$ denotes $A \subsetneq B$. For any $A \subseteq \mathbb{R}^N$,

- the *comprehensive hull* is $\mathrm{comp}(A) = \{x \in \mathbb{R}^N \mid \exists y \in A : y \geq x\}$;
- the *weak upper contour set* is $\mathrm{WUC}(A) = \{x \in \mathbb{R}^N \mid \neg\exists y \in A : y > x\}$;
- the *weak Pareto set* is $\mathrm{WP}(A) = \{x \in A \mid \neg\exists y \in A : y > x\}$;
- the *strong Pareto set* is $\mathrm{SP}(A) = \{x \in A \mid \neg\exists y \in A, y \neq x : y \geq x\}$.

Note that $\mathrm{SP}(A) \subseteq \mathrm{WP}(A) \subseteq \mathrm{WUC}(A)$. A set $A \subseteq \mathbb{R}^N$ is *comprehensive* if $A = \mathrm{comp}(A)$, and *nonleveled* if $\mathrm{SP}(A) = \mathrm{WP}(A)$.

A *nontransferable utility game* is a pair (N, V) in which V assigns to each nonempty coalition $S \in 2^N \setminus \{\emptyset\}$ a nonempty, closed, and comprehensive set of payoff allocations $V(S) \subseteq \mathbb{R}^S$ such that

- $V(\{i\})$ is bounded from above for all $i \in N$;
- $\{x \in V(S) \mid \forall i \in N : x_i \geq \max V(\{i\})\}$ is bounded for all $S \in 2^N \setminus \{\emptyset\}$;
- $(\max V(\{i\}))_{i \in N} \notin \mathrm{WUC}(V(N))$.

Let NTU^N denote the class of all NTU-games for which $\max V(\{i\}) = 0$ for all $i \in N$. For convenience, such an NTU-game is denoted by $V \in \mathrm{NTU}^N$.

3. The procedural egalitarian solution

In this section, we introduce the procedural egalitarian solution as an egalitarian solution concept for nontransferable utility games. The procedural egalitarian solution is based on an egalitarian procedure in which coalitional opportunities are explicitly taken into account. This egalitarian procedure interprets the utopia values, the maximal payoffs within the attainable individually rational allocations of the grand coalition, as an egalitarian direction. The *egalitarian distribution* of the procedure starts assigning to any coalition the weakly Pareto efficient egalitarian allocation. Coalitions are called *egalitarian admissible* if all members are there allocated their highest payoff concerning the egalitarian distribution. The players of these egalitarian admissible coalitions are the *egalitarian claimants* and the corresponding payoffs in the egalitarian admissible coalitions are their *egalitarian claims*. In a next iteration, the egalitarian distribution allocates in each coalition the egalitarian claims to the members which are egalitarian claimants, and allocates weakly Pareto efficient egalitarian payoffs to the other members. The egalitarian admissible coalitions, the egalitarian claimants, and their corresponding egalitarian claims are determined in the same way in all further iterations. The egalitarian procedure is formally defined after an example.

Example 1 (*cf. Roth [16]*). Let $N=\{1,2,3\}$ and consider the game $V^p \in \mathrm{NTU}^N$ which is for all $p \in (0, \frac{1}{2})$ given by

$$V^p(\{1,2\}) = \left\{x \in \mathbb{R}^{\{1,2\}} \;\middle|\; (x_1,x_2) \leq (\tfrac{1}{2},\tfrac{1}{2})\right\};$$
$$V^p(\{i,3\}) = \left\{x \in \mathbb{R}^{\{i,3\}} \;\middle|\; (x_i,x_3) \leq (p, 1-p)\right\} \text{ for } i \in \{1,2\};$$
$$V^p(\{1,2,3\}) = \mathrm{comp}(\mathrm{conv}(\{(\tfrac{1}{2},\tfrac{1}{2},0),(p,0,1-p),(0,p,1-p)\})),$$

where $\mathrm{conv}(A)$ denotes the convex hull of the set $A \subseteq \mathbb{R}^N$. The utopia values, the maximal payoffs within the grand coalition, are given by $u^{V^p} = (\frac{1}{2},\frac{1}{2},1-p)$. The following table presents the egalitarian distribution of the egalitarian procedure.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V^p,1}(S)$	0	0	0	$(\frac{1}{2},\frac{1}{2},\cdot)$	$(p,\cdot,2p(1-p))$	$(\cdot,p,2p(1-p))$	$\lambda^{V^p,1}(N)u^{V^p}$
$\chi^{V^p,2}(S)$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	0	$(\mathbf{\frac{1}{2}},\mathbf{\frac{1}{2}},\cdot)$	$(\mathbf{\frac{1}{2}},\cdot,0)$	$(\cdot,\mathbf{\frac{1}{2}},0)$	$(\mathbf{\frac{1}{2}},\mathbf{\frac{1}{2}},0)$
$\chi^{V^p,3}(S)$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{0}$	$(\mathbf{\frac{1}{2}},\mathbf{\frac{1}{2}},\cdot)$	$(\mathbf{\frac{1}{2}},\cdot,\mathbf{0})$	$(\cdot,\mathbf{\frac{1}{2}},\mathbf{0})$	$(\mathbf{\frac{1}{2}},\mathbf{\frac{1}{2}},\mathbf{0})$
$\chi^{V^p,\cdots}(S)$	...	...	...	...	...	...	...

In the first iteration, the egalitarian distribution $\chi^{V^p,1}$ assigns to any coalition $S \in 2^N \setminus \{\emptyset\}$ the weakly Pareto efficient egalitarian allocation, i.e. $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ proportional to the utopia vector. The collection of egalitarian admissible coalitions $\mathcal{A}^{V^p,1} = \{\{1,2\}\}$ consists of the coalitions in which all members are allocated their highest payoff concerning the egalitarian distribution. The members of these coalitions $P^{V^p,1} = \{1,2\}$ are the egalitarian claimants and their claims $\gamma^{V^p,1} = (\frac{1}{2},\frac{1}{2},\cdot)$ equal their allocated payoffs in the egalitarian admissible coalitions. In the second iteration, the egalitarian distribution $\chi^{V^p,2}$ assigns in any coalition to all claimants $P^{V^p,1} = \{1,2\}$ their claims $\gamma^{V^p,1} = (\frac{1}{2},\frac{1}{2},\cdot)$ and assigns the weakly Pareto efficient egalitarian payoffs to the other members if possible, zero otherwise. In the egalitarian admissible coalitions $\mathcal{A}^{V^p,2} = \{\{3\},\{1,2\},\{1,2,3\}\}$, all members $P^{V^p,2} = N$ can obtain their highest allocated payoffs $\gamma^{V^p,2} = (\frac{1}{2},\frac{1}{2},0)$. In the third iteration, the egalitarian distribution $\chi^{V^p,3}$ assigns in any coalition to all claimants $P^{V^p,2} = N$ their claims $\gamma^{V^p,2} = (\frac{1}{2},\frac{1}{2},0)$. The egalitarian admissible coalitions, the egalitarian claimants, and their corresponding egalitarian claims are the same for all further iterations. △

Definition 1 (*Egalitarian Procedure*). Let $V \in \mathrm{NTU}^N$ be a nontransferable utility game. The vector of *utopia values* $u^V \in \mathbb{R}^N_{++}$ is given by $u^V_i = \max\{x_i \mid x \in V(N) \cap \mathbb{R}^N_+\}$ for all $i \in N$. The set of 0-*egalitarian claimants* is given by $P^{V,0} = \emptyset$. Let $k \in \mathbb{N}$. The *k-egalitarian distribution* $\chi^{V,k}$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ the payoff allocation $\chi^{V,k}(S) \in \mathbb{R}^S$ given by

$$\chi^{V,k}(S) = \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, \lambda^{V,k}(S)u^V_{S\setminus P^{V,k-1}}\right),$$

where $\lambda^{V,k}$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ for which $S \not\subseteq P^{V,k-1}$ the scalar $\lambda^{V,k}(S) \in \mathbb{R}$ given by

$$\lambda^{V,k}(S) = \begin{cases} \max\left\{t \in \mathbb{R} \;\middle|\; \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, tu^V_{S\setminus P^{V,k-1}}\right) \in V(S)\right\} & \text{if } \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}\right) \in V(S); \\ 0 & \text{if } \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}\right) \notin V(S). \end{cases}$$

The collection of *k-egalitarian admissible coalitions* is given by

$$\mathcal{A}^{V,k} = \left\{S \in 2^N \setminus \{\emptyset\} \;\middle|\; \chi^{V,k}(S) \in \mathrm{WP}(V(S)), \forall i \in S\ \forall T \in 2^N, i \in T : \chi^{V,k}_i(T) \leq \chi^{V,k}_i(S)\right\}.$$

The set of *k-egalitarian claimants* $P^{V,k} \in 2^N \setminus \{\emptyset\}$ is given by $P^{V,k} = \bigcup_{S\in\mathcal{A}^{V,k}} S$. The vector of *k-egalitarian claims* $\gamma^{V,k} \in \mathbb{R}^{P^{V,k}}_+$ is given by $\gamma^{V,k}_i = \chi^{V,k}_i(S)$ for all $i \in P^{V,k}$, where $S \in \mathcal{A}^{V,k}$ and $i \in S$.

Lemma 3.1. *Let $V \in \mathrm{NTU}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $\chi^{V,k}(S) \in \mathrm{WUC}(V(S))$ for all $k \in \mathbb{N}$.*

Proof. We show the statement by induction on k. Suppose that $\chi^{V,1}(S) \notin \mathrm{WUC}(V(S))$. Then there exists an $x \in V(S)$ for which $x > \chi^{V,1}(S)$. Since $V(S)$ is comprehensive, this means that there exists a $y \in V(S)$ with $y > \chi^{V,1}(S)$ for which $y = tu^V_S$ for some $t \in \mathbb{R}$. Using $P^{V,0} = \emptyset$, this means that $t > \lambda^{V,1}(S)$, which contradicts the definition of $\lambda^{V,1}(S)$. Hence, $\chi^{V,1}(S) \in \mathrm{WUC}(V(S))$.

Let $k \in \mathbb{N}$ and assume that $\chi^{V,k}(S) \in \mathrm{WUC}(V(S))$. If $S \subseteq P^{V,k}$, then $\chi^{V,k+1}(S) = \gamma^{V,k}_S \geq \chi^{V,k}(S)$, so $\chi^{V,k+1}(S) \in \mathrm{WUC}(V(S))$. Assume that $S \not\subseteq P^{V,k}$ and suppose that $\chi^{V,k+1}(S) \notin \mathrm{WUC}(V(S))$. Then there exists an $x \in V(S)$ for which $x > \chi^{V,k+1}(S)$. Since $V(S)$ is comprehensive, this means that there exists a $y \in V(S)$ with $y \geq \chi^{V,k+1}(S)$ and $y \neq \chi^{V,k+1}(S)$ for which $y = (\gamma^{V,k}_{S\cap P^{V,k}}, tu^V_{S\setminus P^{V,k}})$ for some $t \in \mathbb{R}$. This means that $t > \lambda^{V,k+1}(S)$, which contradicts the definition of $\lambda^{V,k+1}(S)$. Hence, $\chi^{V,k+1}(S) \in \mathrm{WUC}(V(S))$. □

Lemma 3.1 shows that the egalitarian distribution assigns an element of the weak upper contour set to each coalition. Only coalitions which are assigned an element of the weak Pareto set can be egalitarian admissible. This can only be achieved when it is possible to allocate to the members which are egalitarian claimants their corresponding egalitarian claims. Formally, for all $S \in 2^N \setminus \{\emptyset\}$ and any $k \in \mathbb{N}$, we have $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ if and only if $(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}) \in V(S)$. To an egalitarian admissible coalition, the egalitarian distribution assigns a weakly Pareto efficient allocation for which no member is allocated a higher payoff in any other coalition. This suggests that the payoff allocation is an element of the core. The *core* of any $V \in \mathrm{NTU}^N$ is given by

$$\mathcal{C}(V) = \left\{x \in V(N) \mid \forall S \in 2^N \setminus \{\emptyset\} : x_S \in \mathrm{WUC}(V(S))\right\}.$$

Indeed, for each egalitarian admissible coalition, the corresponding vector of egalitarian claims is a core element of the induced subgame. For any $V \in \mathrm{NTU}^N$, the *subgame* $V_S \in \mathrm{NTU}^S$ on $S \in 2^N \setminus \{\emptyset\}$ is given by $V_S(R) = V(R)$ for all $R \in 2^S \setminus \{\emptyset\}$.

Proposition 3.2. *Let $V \in \mathrm{NTU}^N$ and let $k \in \mathbb{N}$. Then $\gamma^{V,k}_S \in \mathcal{C}(V_S)$ for all $S \in \mathcal{A}^{V,k}$.*

Proof. Let $S \in \mathcal{A}^{V,k}$. By definition, we have $\gamma^{V,k}_S = \chi^{V,k}(S)$ and $\chi^{V,k}(S) \in V_S(S)$. Suppose that $\gamma^{V,k}_S \notin \mathcal{C}(V_S)$. Then there exists an $R \in 2^S \setminus \{\emptyset\}$ for which $\gamma^{V,k}_R \in V_S(R) \setminus \mathrm{WP}(V_S(R))$. We can write

$$\gamma^{V,k}_R = \chi^{V,k}_R(S) \geq \chi^{V,k}(R).$$

Since $V_S(R)$ is comprehensive, this means that $\chi^{V,k}(R) \in V_S(R) \setminus \mathrm{WP}(V_S(R))$. This contradicts Lemma 3.1. Hence, $\gamma^{V,k}_S \in \mathcal{C}(V_S)$. $\square$

The question arises whether egalitarian admissible coalitions and egalitarian claimants exist in every game. Are players always able to acquire an egalitarian claim? The answer turns out to be affirmative.

Lemma 3.3. *Let $V \in \mathrm{NTU}^N$ and let $k \in \mathbb{N}$. Then $\mathcal{A}^{V,k} \subseteq \mathcal{A}^{V,k+1}$. Moreover, if $P^{V,k-1} \neq N$, then $P^{V,k-1} \subset P^{V,k}$.*

Proof. Let $S \in \mathcal{A}^{V,k}$. Then we have $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ and $S \subseteq P^{V,k}$. We can write $\chi^{V,k+1}(S) = \gamma^{V,k}_S = \chi^{V,k}(S)$. This means that $\chi^{V,k+1}(S) \in \mathrm{WP}(V(S))$ and for all $i \in S$ we have $\chi^{V,k+1}_i(T) = \gamma^{V,k}_i \leq \chi^{V,k+1}_i(S)$ for all $T \in 2^N$ for which $i \in T$, so $S \in \mathcal{A}^{V,k+1}$. Hence, $\mathcal{A}^{V,k} \subseteq \mathcal{A}^{V,k+1}$.

Assume that $P^{V,k-1} \neq N$. Let $S \in 2^N$ with $S \not\subseteq P^{V,k-1}$ and $(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}) \in V(S)$ be a coalition such that $\lambda^{V,k}(S)$ equals the maximum $\lambda^{V,k}(R)$ over all coalitions $R \in 2^N$ with $R \not\subseteq P^{V,k-1}$. Then we have $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ and $\chi^{V,k}_i(T) \leq \chi^{V,k}_i(S)$ for all $i \in S$ and all $T \in 2^N$ for which $i \in T$. This means that $S \in \mathcal{A}^{V,k}$ and $S \subseteq P^{V,k}$. Hence, $P^{V,k-1} \subset P^{V,k}$. $\square$

Lemma 3.3 shows that the nonempty collection of egalitarian admissible coalitions weakly extends in each iteration and eventually covers all players. The structure of this collection depends on the structure of the underlying game. An NTU-game $V \in \mathrm{NTU}^N$ is

- *superadditive* if $V(S) \times V(T) \subseteq V(S \cup T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \cap T = \emptyset$;
- *ordinal convex* (cf. Vilkov [21]) if V is superadditive and for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \cap T \neq \emptyset$ and any $x \in \mathbb{R}^N$ for which $x_S \in V(S)$ and $x_T \in V(T)$, we have $x_{S\cup T} \in V(S \cup T)$ or $x_{S\cap T} \in V(S \cap T)$;
- *coalitional merge convex* (cf. Hendrickx et al. [10]) if V is superadditive and for all $R \in 2^N \setminus \{\emptyset\}$ and $S, T \in 2^{N\setminus R} \setminus \{\emptyset\}$ for which $S \subset T$, and any $s \in \mathrm{WP}(V(S))$, $t \in \mathrm{WP}(V(T))$, and $x \in V(S \cup R)$ for which $x_S \geq s$, there exists a $y \in V(T \cup R)$ for which $y_T \geq t$ and $y_R \geq x_R$;
- *balanced* (cf. Scarf [18]) if for all balanced collections $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$, we have $x \in V(N)$ if $x_S \in V(S)$ for all $S \in \mathcal{B}$. Here, a collection of coalitions $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is *balanced* if there exists a function $\delta : \mathcal{B} \to \mathbb{R}_{++}$ for which $\sum_{S\in\mathcal{B}:i\in S} \delta(S) = 1$ for all $i \in N$.

Interestingly, all these properties have implications for the relation of the collections of egalitarian admissible coalitions in two subsequent iterations.

Proposition 3.4. *Let $V \in \mathrm{NTU}^N$ and let $k \in \mathbb{N}$.*

(i) If V is superadditive, then $S \cup T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$ with $S \cap T = \emptyset$.
(ii) If V is ordinal convex, then $S \cup T \in \mathcal{A}^{V,k+1}$ or $S \cap T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$.
(iii) If V is coalitional merge convex, then $S \cup T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$.
(iv) If V is balanced, then $N \in \mathcal{A}^{V,k+1}$ if there exists a balanced collection $\mathcal{B} \subseteq \mathcal{A}^{V,k}$.

Proof. (i) Assume that V is superadditive. Let $S, T \in \mathcal{A}^{V,k}$ with $S \cap T = \emptyset$. Then we have $\gamma^{V,k}_S \in V(S)$ and $\gamma^{V,k}_T \in V(T)$. Since V is superadditive, this means that $\gamma^{V,k}_{S\cup T} \in V(S \cup T)$. From Lemma 3.1 we know that $\chi^{V,k+1}(S \cup T) \in \mathrm{WUC}(V(S \cup T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma^{V,k}_{S\cup T}$, this implies that $\chi^{V,k+1}(S \cup T) \in \mathrm{WP}(V(S \cup T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$.

(ii) Assume that V is ordinal convex. Let $S, T \in \mathcal{A}^{V,k}$ with $S \cap T \neq \emptyset$. Then we have $\gamma_S^{V,k} \in V(S)$ and $\gamma_T^{V,k} \in V(T)$. Since V is ordinal convex, this means that $\gamma_{S\cup T}^{V,k} \in V(S \cup T)$ or $\gamma_{S\cap T}^{V,k} \in V(S \cap T)$. From Lemma 3.1 we know that $\chi^{V,k+1}(S \cup T) \in \mathrm{WUC}(V(S \cup T))$ and $\chi^{V,k+1}(S \cap T) \in \mathrm{WUC}(V(S \cap T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma_{S\cup T}^{V,k}$ and $\chi^{V,k+1}(S \cap T) = \gamma_{S\cap T}^{V,k}$, this implies that $\chi^{V,k+1}(S \cup T) \in \mathrm{WP}(V(S \cup T))$ or $\chi^{V,k+1}(S \cap T) \in \mathrm{WP}(V(S \cap T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$ or $S \cap T \in \mathcal{A}^{V,k+1}$.

(iii) Assume that V is coalitional merge convex. Let $S, T \in \mathcal{A}^{V,k}$ with $S \cap T \neq \emptyset$, $S \not\subseteq T$ and $T \not\subseteq S$. Then we have $\gamma_S^{V,k} \in V(S)$ and $\gamma_T^{V,k} \in V(T)$. Since V is coalitional merge convex, there exists a $y \in V(S \cup T)$ for which $y_S \geq \gamma_S^{V,k}$ and $y_{T\setminus S} \geq \gamma_{T\setminus S}^{V,k}$, i.e. $y \geq \gamma_{S\cup T}^{V,k}$. Since $V(S \cup T)$ is comprehensive, this means that $\gamma_{S\cup T}^{V,k} \in V(S \cup T)$. From Lemma 3.1 we know that $\chi^{V,k+1}(S \cup T) \in \mathrm{WUC}(V(S \cup T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma_{S\cup T}^{V,k}$, this implies that $\chi^{V,k+1}(S \cup T) \in \mathrm{WP}(V(S \cup T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$.

(iv) Assume that V is balanced. Let $\mathcal{B} \subseteq \mathcal{A}^{V,k}$ be a balanced collection. Then we have $\gamma_S^{V,k} \in V(S)$ for all $S \in \mathcal{B}$. Since V is balanced, this means that $\gamma^{V,k} \in V(N)$. From Lemma 3.1 we know that $\chi^{V,k+1}(N) \in \mathrm{WUC}(V(N))$. Since $\chi^{V,k+1}(N) = \gamma^{V,k}$, this implies that $\chi^{V,k+1}(N) \in \mathrm{WP}(V(N))$. Hence, $N \in \mathcal{A}^{V,k+1}$. □

The egalitarian procedure reaches a steady state when all players have acquired an egalitarian claim. Lemma 3.3 shows that the number of iterations needed to converge to this steady state is bounded by the number of players. Players which are member of all inclusion-wise maximal egalitarian admissible coalitions are called strong claimants.

Definition 2. Let $V \in \mathrm{NTU}^N$ be a nontransferable utility game. The iteration $n^V \in \{1, \dots, |N|\}$ is given by $n^V = \min\{k \in \mathbb{N} \mid P^{V,k} = N\}$. The vector of *egalitarian claims* $\widehat{\gamma}^V \in \mathbb{R}_+^N$ is given by $\widehat{\gamma}^V = \gamma^{V,n^V}$. The collection $\widehat{\mathcal{A}}^V \subseteq 2^N \setminus \{\emptyset\}$ is given by

$$\widehat{\mathcal{A}}^V = \left\{S \in 2^N \setminus \{\emptyset\} \mid \widehat{\gamma}_S^V \in V(S), \forall T \in 2^N \setminus \{\emptyset\}, \widehat{\gamma}_T^V \in V(T) : S \not\subset T\right\}.$$

The set of *strong egalitarian claimants* $D^V \in 2^N$ is given by $D^V = \bigcap_{S \in \widehat{\mathcal{A}}^V} S$.

The egalitarian claims can be interpreted as aspiration levels for a payoff allocation for the grand coalition. The procedural egalitarian solution prioritizes the strong egalitarian claimants over the other players to prescribe such an allocation, i.e. it allocates the egalitarian claims to the corresponding players according to this priority. The possibly resulting infeasibility is modeled as a bankruptcy problem in which the egalitarian claims are adopted.

A *bankruptcy problem with nontransferable utility* (cf. Orshan et al. [15]) is a triple (N, E, c) in which $E \subseteq \mathbb{R}_+^N$ is a nonempty, closed, bounded, and comprehensive *estate* and $c \in \mathrm{WUC}(E)$ is a vector of *claims*. For convenience, an NTU-bankruptcy problem is denoted by (E, c). The *constrained relative equal awards rule* CREA (cf. Dietzenbacher et al. [5]) assigns to any bankruptcy problem (E, c) the payoff allocation

$$\mathrm{CREA}(E, c) = \left(\min\left\{c_i, \alpha^{E,c} u_i^E\right\}\right)_{i \in N},$$

where $u_i^E = \max\{x_i \mid x \in E\}$ and $\alpha^{E,c} = \max\{t \in [0, 1] \mid (\min\{c_i, t u_i^E\})_{i \in N} \in \mathrm{WP}(E)\}$.

To prescribe a payoff allocation for the grand coalition, the procedural egalitarian solution solves the possibly resulting infeasibility using the constrained relative equal awards rule, i.e. prioritizing the strong egalitarian claimants, it allocates payoffs as relatively equal as possible provided that players cannot get more than their egalitarian claims. In Section 5, we further comment on the choice of this specific bankruptcy rule.

Definition 3 (*Procedural Egalitarian Solution*)**.** The *procedural egalitarian solution* $\Gamma : \mathrm{NTU}^N \to \mathbb{R}_+^N$ assigns to any game $V \in \mathrm{NTU}^N$ the payoff allocation

$$\Gamma(V) = \begin{cases} \left(\widehat{\gamma}_{D^V}^V, \mathrm{CREA}\left(\left\{x \in \mathbb{R}_+^{N\setminus D^V} \;\middle|\; \left(\widehat{\gamma}_{D^V}^V, x\right) \in V(N)\right\}, \widehat{\gamma}_{N\setminus D^V}^V\right)\right) & \text{if } \left(\widehat{\gamma}_{D^V}^V, 0_{N\setminus D^V}\right) \in V(N); \\ \left(\mathrm{CREA}\left(\left\{x \in \mathbb{R}_+^{D^V} \;\middle|\; \left(x, 0_{N\setminus D^V}\right) \in V(N)\right\}, \widehat{\gamma}_{D^V}^V\right), 0_{N\setminus D^V}\right) & \text{if } \left(\widehat{\gamma}_{D^V}^V, 0_{N\setminus D^V}\right) \notin V(N). \end{cases}$$

This generalizes the procedural egalitarian solution for TU-games (cf. Dietzenbacher et al. [4]), which in turn generalizes the Dutta and Ray [6] solution for convex TU-games.

Example 2. Let $N = \{1, 2, 3\}$ and consider the game $V^p \in \mathrm{NTU}^N$ from Example 1. We have $n^{V^p} = 2$, $\widehat{\gamma}^{V^p} = (\frac{1}{2}, \frac{1}{2}, 0)$, $\widehat{\mathcal{A}}^{V^p} = \{N\}$, and $D^{V^p} = N$. Consequently, $\Gamma(V^p) = (\frac{1}{2}, \frac{1}{2}, 0)$. Besides, the Shapley [19] solution equals $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the Harsanyi [7] solution and the Kalai and Samet [12] solution both equal $(\frac{1}{2} - \frac{1}{3}p, \frac{1}{2} - \frac{1}{3}p, \frac{2}{3}p)$. Note that, contrary to the procedural egalitarian solution, these solutions do not belong to the core.

This example initiated an interesting and extensive discussion on the interpretation of solutions for nontransferable utility games. Roth [16] argues that the payoff allocation prescribed by the procedural egalitarian solution is the unique outcome of this game which is consistent with the hypothesis that the players are rational utility maximizers, since this payoff allocation is strictly preferred by both players 1 and 2, and it can be achieved without player 3. For more details, we refer to Harsanyi [8], Aumann [1], Hart [9], Roth [17], and Aumann [2]. △

The procedural egalitarian solution satisfies

- *weak Pareto efficiency*: it prescribes a weakly Pareto efficient allocation of the grand coalition;
- *symmetry*: it allocates equal payoffs to symmetric players;
- *scale covariance*: it is covariant under individual rescaling of utility;
- *individual rationality*: it allocates at least the maximal individually attainable payoffs;
- *restricted monotonicity*: it allocates nondecreasing payoffs when the set of attainable allocations of the grand coalition expands but the utopia values remain equal.

The procedural egalitarian solution does not generally satisfy *coalitional rationality*, i.e. it does not necessarily prescribe a core element. In order to better understand the trade-off between relative egalitarianism and coalitional rationality, the games in which these principles do not conflict are particularly interesting. This is for instance the case in Example 2, where the grand coalition is egalitarian admissible. Consequently, all players are strong egalitarian claimants, there is no infeasibility, and the procedural egalitarian solution allocates to all players their egalitarian claims. Such games are called egalitarian stable.

Definition 4 (*Egalitarian Stability*)**.** A game $V \in \mathrm{NTU}^N$ is *egalitarian stable* if $\widehat{\mathcal{A}}^V = \{N\}$.

Proposition 3.2 shows that the procedural egalitarian solution prescribes a core element for all egalitarian stable games, i.e. egalitarian stability is a sufficient condition for coalitional rationality. The following example shows that this condition is not necessary.

Example 3. Let $N = \{1, 2, 3\}$ and consider the game $V \in \mathrm{NTU}^N$ given by

$$V(\{1, i\}) = \left\{x \in \mathbb{R}^{\{1,i\}} \;\middle|\; (x_1, x_i) \leq (4, 4)\right\} \text{ for } i \in \{2, 3\};$$
$$V(\{2, 3\}) = \left\{x \in \mathbb{R}^{\{2,3\}} \;\middle|\; (x_2, x_3) \leq (0, 0)\right\};$$
$$V(\{1, 2, 3\}) = \left\{x \in \mathbb{R}^{\{1,2,3\}} \;\middle|\; x_1 + x_2 + x_3 \leq 6\right\}.$$

We have $u^V = (6, 6, 6)$. The following table illustrates the egalitarian distribution in the first iteration of the egalitarian procedure.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\chi^{V,1}(S)$	$(0, \cdot, \cdot)$	$(\cdot, 0, \cdot)$	$(\cdot, \cdot, 0)$	$(4, 4, \cdot)$	$(4, \cdot, 4)$	$(\cdot, 0, 0)$	$(2, 2, 2)$

We have $\mathcal{A}^{V,1} = \{\{1, 2\}, \{1, 3\}\}$, $P^{V,1} = N$, and $\gamma^{V,1} = (4, 4, 4)$. This means that $n^V = 1$, $\widehat{\gamma}^V = (4, 4, 4)$, $\widehat{\mathcal{A}}^V = \{\{1, 2\}, \{1, 3\}\}$, and $D^V = \{1\}$. Consequently,

$$\Gamma(V) = \left(4, \mathrm{CREA}\left(\left\{x \in \mathbb{R}^{\{2,3\}}_{+} \mid x_2 + x_3 \leq 2\right\}, (\cdot, 4, 4)\right)\right) = (4, 1, 1).$$

Note that $\Gamma(V) \in \mathcal{C}(V)$. △

In Example 3, the sets of payoff allocations $V(\{1, 2\})$ and $V(\{1, 3\})$ are not nonleveled. Note that for nontransferable utility games $V \in \mathrm{NTU}^N$ for which $V(S) \cap \mathbb{R}^N_+$ is nonleveled for all $S \in 2^N \setminus \{\emptyset\}$, egalitarian stability is a necessary and sufficient condition for coalitional rationality. The question arises which games are egalitarian stable. From Proposition 3.4 we know that coalitional merge convex games are egalitarian stable. In the next sections we show that bargaining games and bankruptcy games are egalitarian stable as well.

4. Bargaining games

In this section, we analyze the procedural egalitarian solution on the class of bargaining games. A *bargaining problem* (cf. Nash [14]) is a triple (N, F, d) in which $F \subseteq \mathbb{R}^N$ is a nonempty, closed, and comprehensive *feasible set* and $d \in \mathbb{R}^N$ is a *disagreement point* such that

- $\{x \in F \mid x \geq d\}$ is bounded;
- $d \notin \mathrm{WUC}(F)$.

Let BG^N denote the class of all bargaining problems (N, F, d) for which $d = 0_N$. For convenience, such a bargaining problem is denoted by $F \in \mathrm{BG}^N$.

Any bargaining problem $F \in \mathrm{BG}^N$ gives rise to the corresponding *bargaining game* $V^F \in \mathrm{NTU}^N$ which is for all $S \in 2^N \setminus \{\emptyset\}$ given by

$$V^F(S) = \begin{cases} F & \text{if } S = N; \\ \mathrm{comp}(\{0_S\}) & \text{if } S \in 2^N \setminus \{\emptyset, N\}. \end{cases}$$

The core of a bargaining game is given by $\mathcal{C}(V^F) = \mathrm{WP}(F) \cap \mathbb{R}^N_+$. Note that bargaining games are coalitional merge convex, which implies that bargaining games are egalitarian stable and that the procedural egalitarian solution prescribes a core element.

The Kalai and Smorodinsky [13] solution $\mathrm{KS} : \mathrm{BG}^N \to \mathbb{R}^N_+$ assigns to any bargaining problem $F \in \mathrm{BG}^N$ the payoff allocation

$$\mathrm{KS}(F) = \kappa^F u^{V^F},$$

where $\kappa^F = \max\{t \in [0, 1] \mid tu^{V^F} \in \mathrm{WP}(F)\}$.

Theorem 4.1. *The procedural egalitarian solution of a bargaining game coincides with the Kalai and Smorodinsky [13] solution of the underlying bargaining problem.*

Proof. Let $F \in \mathrm{BG}^N$ be a bargaining problem. In the first iteration of the egalitarian procedure, we have

$$\chi^{V^F,1}(S) = \begin{cases} \lambda^{V^F,1}(N)u^{V^F} & \text{if } S = N; \\ 0_S & \text{if } S \in 2^N \setminus \{\emptyset, N\}, \end{cases}$$

where $\lambda^{V^F,1}(N) \in \mathbb{R}_{++}$ is such that $\lambda^{V^F,1}(N)u^{V^F} \in \mathrm{WP}(F)$. This means that $N \in \mathcal{A}^{V^F,1}$, $P^{V^F,1} = N$, and $\gamma^{V^F,1} = \lambda^{V^F,1}(N)u^{V^F}$, which implies that $n^{V^F} = 1$, $\widehat{\gamma}^{V^F} = \lambda^{V^F,1}(N)u^{V^F}$, $\widehat{\mathcal{A}}^{V^F} = \{N\}$, and $D^{V^F} = N$. Consequently, $\Gamma(V^F) = \lambda^{V^F,1}(N)u^{V^F}$. Since $\Gamma(V^F) \in \mathrm{WP}(F)$ and $\mathrm{KS}(F, d) \in \mathrm{WP}(F)$, the assumptions on F imply that $\lambda^{V^F,1}(N) = \kappa^F$. Hence, $\Gamma(V^F) = \mathrm{KS}(F)$. □

5. Bankruptcy games

In this section, we analyze the procedural egalitarian solution on the class of bankruptcy games. Let BR^N denote the class of all bankruptcy problems (E, c) for which $E \neq \{0_N\}$ is nonleveled and $(c_{N\setminus\{i\}}, 0) \in \mathrm{WUC}(E)$ for all $i \in N$. Any bankruptcy problem $(E, c) \in \mathrm{BR}^N$ gives rise to the corresponding *bankruptcy game* $V^{E,c} \in \mathrm{NTU}^N$ (cf. Dietzenbacher [3]) which is for all $S \in 2^N \setminus \{\emptyset\}$ given by

$$V^{E,c}(S) = \begin{cases} \mathrm{comp}(\{x \in \mathbb{R}^S \mid (x, c_{N\setminus S}) \in E\}) & \text{if } (0_S, c_{N\setminus S}) \in E; \\ \mathrm{comp}(\{0_S\}) & \text{if } (0_S, c_{N\setminus S}) \notin E. \end{cases}$$

The core of a bankruptcy game is given by $\mathcal{C}(V^{E,c}) = \{x \in \mathrm{WP}(E) \mid x \leq c\}$.

In the next theorem, we show that bankruptcy games are egalitarian stable, which means that the procedural egalitarian solution allocates to all players their egalitarian claims without having to rely on the constrained relative equal awards rule in its definition. Interestingly, the procedural egalitarian solution for bankruptcy games corresponds to the constrained relative equal awards rule for bankruptcy problems. This illustrates the strong connection between the procedural egalitarian solution and the constrained relative equal awards rule, and justifies the use of the latter in the definition of the procedural egalitarian solution for games which are not egalitarian stable.

Theorem 5.1. *All bankruptcy games are egalitarian stable and the procedural egalitarian solution of a bankruptcy game coincides with the constrained relative equal awards rule of the underlying bankruptcy problem.*

Proof. Let $(E, c) \in \mathrm{BR}^N$. First, we show that $\widehat{\gamma}^{V^{E,c}} \leq c$. Suppose that there exists an $i \in N$ such that $\widehat{\gamma}_i^{V^{E,c}} > c_i$. Let $k \in \mathbb{N}$ be such that $i \in P^{V^{E,c},k} \setminus P^{V^{E,c},k-1}$ and let $S \in \mathcal{A}^{V^{E,c},k}$ be such that $i \in S$. Then $S \neq \{i\}$ since $\widehat{\gamma}_i^{V^{E,c}} \notin V^{E,c}(\{i\})$. We have $\chi^{V^{E,c},k}(S) \in \mathrm{WP}(V^{E,c}(S))$, i.e.

$$\left(\lambda^{V^{E,c},k}(S)u^{V^{E,c}}_{S\setminus P^{V^{E,c},k-1}}, \gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}\right) \in \mathrm{WP}(V^{E,c}(S)).$$

Since E is comprehensive and nonleveled,

$$\left(\lambda^{V^{E,c},k}(S)u^{V^{E,c}}_{S\setminus P^{V^{E,c},k-1}}, \gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, c_{N\setminus S}\right) \in \mathrm{SP}(E).$$

Since E is comprehensive,

$$\left(\lambda^{V^{E,c},k}(S)u^{V^{E,c}}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}, \gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, c_i, c_{N\setminus S}\right) \in E \setminus \mathrm{SP}(E).$$

Since E is nonleveled,

$$\left(\lambda^{V^{E,c},k}(S)u^{V^{E,c}}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}, \gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}\right) \in V^{E,c}(S \setminus \{i\}) \setminus \mathrm{WP}(V^{E,c}(S \setminus \{i\})).$$

Since we know from Lemma 3.1 that $\chi^{V^{E,c},k}(S \setminus \{i\}) \in \mathrm{WUC}(V^{E,c}(S \setminus \{i\}))$, we can write

$$\chi^{V^{E,c},k}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}(S \setminus \{i\}) = \lambda^{V^{E,c},k}(S \setminus \{i\})u^{V^{E,c}}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})} > \lambda^{V^{E,c},k}(S)u^{V^{E,c}}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})} = \chi^{V^{E,c},k}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}(S).$$

This contradicts that $S \in \mathcal{A}^{V^{E,c},k}$. Hence, $\widehat{\gamma}^{V^{E,c}} \leq c$.

Suppose that $c \in E$. Then we have $\chi^{V^{E,c},n^{V^{E,c}}}(N) \leq \gamma^{V^{E,c},n^{V^{E,c}}} = \widehat{\gamma}^{V^{E,c}} \leq c$. From Lemma 3.1 we know that $\chi^{V^{E,c},n^{V^{E,c}}}(N) \in \text{WUC}(E)$. Since $c \in \text{WP}(E)$ and E is nonleveled, this means that $\widehat{\gamma}^{V^{E,c}} = c$, $\widehat{\mathcal{A}}^{V^{E,c}} = \{N\}$, and $D^{V^{E,c}} = N$. Consequently, $V^{E,c}$ is egalitarian stable and $\Gamma(V^{E,c}) = c = \text{CREA}(E, c)$.

Now suppose that $c \notin E$. First, we show that $\chi^{V^{E,c},1}(S) \leq \alpha^{E,c} u_S^E$ for all $S \in 2^N \setminus \{\emptyset\}$. Suppose there exists an $S \in 2^N \setminus \{\emptyset\}$ such that $\chi_i^{V^{E,c},1}(S) > \alpha^{E,c} u_i^E$ for some $i \in S$. Then we have $\chi^{V^{E,c},1}(S) \in \text{WP}(V^{E,c}(S))$ and

$$\chi^{V^{E,c},1}(S) = \lambda^{V^{E,c},1}(S) u_S^{V^{E,c}} = \lambda^{V^{E,c},1}(S) u_S^E > \alpha^{E,c} u_S^E \geq \text{CREA}_S(E, c).$$

This means that $(\chi^{V^{E,c},1}(S), c_{N\setminus S}) \in \text{WP}(E)$. Moreover, $(\chi^{V^{E,c},1}(S), c_{N\setminus S}) \geq \text{CREA}(E, c)$ and $(\chi^{V^{E,c},1}(S), c_{N\setminus S}) \neq \text{CREA}(E, c)$. Since E is nonleveled, this contradicts $\text{CREA}(E, c) \in \text{WP}(E)$. Hence, $\chi^{V^{E,c},1}(S) \leq \alpha^{E,c} u_S^E$ for all $S \in 2^N \setminus \{\emptyset\}$.

Next, define $H^{E,c} \in 2^N \setminus \{\emptyset\}$ by

$$H^{E,c} = \left\{ i \in N \mid \text{CREA}_i(E, c) = \alpha^{E,c} u_i^E \right\}.$$

We have $\chi^{V^{E,c},1}(H^{E,c}) \in \text{WP}(V^{E,c}(H^{E,c}))$ and

$$\chi^{V^{E,c},1}(H^{E,c}) = \lambda^{V^{E,c},1}(H^{E,c}) u_{H^{E,c}}^{V^{E,c}} = \lambda^{V^{E,c},1}(H^{E,c}) u_{H^{E,c}}^E = \alpha^{E,c} u_{H^{E,c}}^E = \text{CREA}_{H^{E,c}}(E, c).$$

This means that $H^{E,c} \in \mathcal{A}^{V^{E,c},1}$, $H^{E,c} \subseteq P^{V^{E,c},1}$, and $\widehat{\gamma}_{H^{E,c}}^{V^{E,c}} = \gamma_{H^{E,c}}^{V^{E,c},1} = \text{CREA}_{H^{E,c}}(E, c)$. We have

$$\chi^{V^{E,c},n^{V^{E,c}}}(N) \leq \gamma^{V^{E,c},n^{V^{E,c}}} = \widehat{\gamma}^{V^{E,c}} \leq \left(\text{CREA}_{H^{E,c}}(E, c), c_{N\setminus H^{E,c}}\right) = \text{CREA}(E, c).$$

From Lemma 3.1 we know that $\chi^{V^{E,c},n^{V^{E,c}}}(N) \in \text{WUC}(E)$. Since $\text{CREA}(E, c) \in \text{WP}(E)$ and E is nonleveled, this means that $\widehat{\gamma}^{V^{E,c}} = \text{CREA}(E, c)$, $\widehat{\mathcal{A}}^{V^{E,c}} = \{N\}$, and $D^{V^{E,c}} = N$. Consequently, $V^{E,c}$ is egalitarian stable and $\Gamma(V^{E,c}) = \text{CREA}(E, c)$. □

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