

Global Defensive Alliances in the Lexicographic Product of Paths and Cycles

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Abstract

A set S of vertices of graph G is a *defensive alliance* of G if for every $v \in S$, it holds $|N[v] \cap S| \geq |N[v] - S|$. An alliance S is called *global* if it is also a dominating set. In this paper, we determine the exact values of the global defensive alliance number of lexicographic products of path and cycles.

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1 Introduction

We consider only finite, simple, and undirected graphs. Given a graph $G = (V, E)$, *open neighborhood* and the *closed neighborhood* of a $v \in V$ are denoted by $N(v), N[v]$, respectively. Given a set $S \subseteq V$, the subgraph of G induced by S is denoted by $G[S]$. If $v \in S$ and $|N[v] \cap S| \geq |N[v] - S|$, then v is said to be *defended in S* . We say that S is a *defensive alliance* if all vertices of S are defended. Note that if v is defended in S , then $|S \cap N(v)| \geq \lfloor \frac{d(v)}{2} \rfloor$. The set S is a *dominating set of G* if every vertex of G belongs to S or has a neighbor in S . A defensive alliance is *global* (GDA) if it is also a dominating set of the graph. The minimum cardinality of a global defensive alliance of G is its *global defensive alliance number* and is denoted by $\gamma_a(G)$.

The *lexicographic product* of graphs $G^1 = (V_1, E_1)$ and $G^2 = (V_2, E_2)$ is the graph $G = (V, E) = G^1 \circ G^2$ such that $V = V_1 \times V_2$ and $E = \{(u_1, u_2)(v_1, v_2) : (u_1 v_1 \in E_1) \text{ or } (u_1 = v_1$

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and $u_2v_2 \in E_2$). Given a graph $F = G^1 \circ G^2$ where the orders of G^1 and G^2 are n and m , respectively, it is clear that F contains n disjoint copies of G^2 , which will be denoted by $G_1^2, G_2^2, \dots, G_n^2$. Furthermore, for a set $S \subseteq V(G)$, we will denote by S_i the set $S \cap V(G_i^2)$ for $i \in \{1, \dots, n\}$ and $s_i = |S_i|$.

In this work, we present formulas that allow one determining the global defensive alliance number of a graph $F = G^1 \circ G^2$ where G^1 and G^2 are cycles or paths within a constant number of operations. In Section 2, we present a general characterization of $\gamma_a(G^1 \circ G^2)$ for $G^1 \in \{P_n, C_n\}$, $n \geq 3$, and any graph $G^2 \not\cong K_m$. Such characterization will be useful for the proposed solution presented in next sections. Section 3 contains useful properties of minimum global defensive alliances of the lexicographic product of paths and cycles. In Section 4.1, we present the formulas for $n \leq 7$ while the solution for $n \geq 8$ is given in Section 4.2. In Section 5, we explore the homogenous behavior of $\gamma_a(G^1 \circ G^2)$, when the orders of G_1 and G_2 change, for obtaining more structural results. The conclusions are in Section 6. We finish this section presenting related works.

The definition of alliances in graphs first appeared in [9]. Since then many variations appeared. The most extensively studied are defensive alliances [9, 8, 13, 19, 21], offensive alliances [7, 12, 18] and powerful or dual alliances [2, 3, 22]. A more generalized concept of alliance is represented by k -alliances [1, 15, 16, 17, 19], and Dourado et al. presented a new definition of alliances, namely, (f, g) -alliances [6], that generalizes previous concepts. In [23], Yero and Rodríguez-Velázquez published a summary of the major results obtained concerning defensive alliances up through 2013.

Since the decision problems of computing the minimum cardinality of these concepts for general graphs are NP-complete [4, 11, 14], several studies of alliances in graphs have been developed in graph classes and product of graphs; these advances are described in detail in [23, 10].

Haynes, Hedetniemi and Henning [9] determined the cardinality of the minimum set that can constitute a global defensive alliance for several classes of graphs and presented some limits on the minimum GDA in cubic, bipartite graphs and trees.

The initial studies of defensive alliances in Cartesian products were done by Brigham, Dutton and Hedetniemi in [2], and several parameters were also presented in [1, 19, 20] for Cartesian products of graphs for k -alliances. Following this trend, there is also the work by Chang *et al.* in 2012 [5], which presented some upper bounds for Cartesian products between paths and cycles. In 2013, Yero and Rodríguez-Velázquez [24] obtained closed formulas for the GDA number for several classes of Cartesian products of graphs.

2 Characterization of $\gamma_a(G^1 \circ G^2)$ for $G^1 \in \{C_n, P_n\}$

In this section, we present a characterization of $\gamma_a(G^1 \circ G^2)$ for $G^1 \in \{P_n, C_n\}$, $n \geq 3$, and any graph $G^2 \not\cong K_m$. Let S be a GDA of $F = G^1 \circ G^2$. The *spectrum of S in F* , $spe(S, F)$, is a sequence obtained in the following way. If $G^2 = C_m$ and there is $S_i = 0$, we assume that

$S_n = 0$: if $S_i \neq 0$ for $2 \leq i \leq n-1$, then $\text{spe}(S, F) = (n)$; otherwise let $i \geq 3$ be the minimum number such that $S_i = 0$. If $S_{i+1} = 0$, then $k_1 = i$; otherwise $k_1 = i-1$. In both cases, $\text{spe}(S, F) = (k_1)\text{spe}(S', F')$ where $S' = S \cap V(F')$ and $F' = F - (V(G_1^2) \cup \dots \cup V(G_{k_1}^2))$. When there is no doubt about which is the graph F , we can use $\text{spe}(S)$ to represent the spectrum of S in F .

We say that a sequence $w = (k_1, \dots, k_t)$ is *feasible* for $G^1 \circ G^2$ for $G^1 \in \{C_n, P_n\}$ and $G^2 \not\cong K_m$ if there is a GDA S of $G^1 \circ G^2$ whose spectrum is w . We denote $\max(w) = \max_{i \in t} \{k_i\}$ and say that w is an n -sequence if $\sum_{1 \leq i \leq t} k_i = n$. Observe that we can see w as a t -partition (V_1, \dots, V_t) of $V(F)$ where each part is associated with an element k_i and $F[V_i] \simeq P_{k_i} \circ G^2$. We call each such subgraph by a *section* of F . If $G^1 \simeq P_n$ and F_i is a section for $i \in \{1, t\}$, then we say that F_i is an *external section*; otherwise it is an *internal section*.

Given elements k_i and k_j of a sequence $w = (k_1, \dots, k_t)$ for $j > i$, the sequence formed by the elements that are between k_i and k_j in w will be denoted by $w_{i+1, j-1} = (k_{i+1}, \dots, k_{j-1})$, the sequence formed by the elements that precede k_i by $w_{1, i-1} = (k_1, \dots, k_{i-1})$, and the sequence formed by the elements that succeed k_j by $w_{j+1, t} = (k_{j+1}, \dots, k_t)$. The concatenation of sequences w and w' will be denoted by ww' . If all elements of w are equal, then we can write $w = ([t]k_1)$. This definition allows one to write $w = ([t_1]k_1, \dots, [t_p]k_p)$, which means that, for $1 \leq i \leq p$, there are t_i consecutive occurrences of k_i and $\sum_{1 \leq i \leq p} t_i k_i = n$. The feasible sequences are characterized in the following result.

Proposition 2.1 *For $G^1 \in \{P_n, C_n\}$ and $G^2 \not\cong K_m$, a sequence $w = (k_1, \dots, k_t)$ is feasible for $G^1 \circ G^2$ if and only if $k_1 \geq 2$, $k_i \geq 3$ for $i \in \{2, \dots, t\}$, and $\sum_{1 \leq i \leq t} k_i = n$.*

Proof. Let S be a GDA of F such that $\text{spe}(S) = w$. By the construction of a spectrum, it is clear that $\sum_{1 \leq i \leq t} k_i = n$. Since every vertex of S has a neighbour outside the copy of G^2 that it belongs, every $k_i \geq 2$. Since the definition of spectrum guarantees that for every k_i , for $i \geq 2$, the first copy of the section associated with it has no vertex of S , $k_i \geq 3$ for $i \geq 3$.

Conversely, let $F_i = F[V(G_j^2) \cup \dots \cup V(G_{j+k_i-1}^2)]$ be the section associated with k_i . If $k_i \geq 4$, add $V(G_{j+1}^2) \cup \dots \cup V(G_{j+k_i-2}^2)$ to S . If $k_i = 3$, add $V(G_{j+1}^2) \cup V(G_{j+2}^2)$ to S . If $k_1 = 2$, add $V(G_1^2) \cup V(G_2^2)$ to S . It is clear S is a dominating set, every vertex of S is defended, and that $\text{spe}(S) = w$, then w is feasible. \square

For the characterization of minimum GDA in terms of feasible sequences, we need some definitions. Given a positive integer k , we define

- for $k \geq 4$, $\text{val}_i(k, G^2)$ as the cardinality of a minimum GDA S of $P_k \circ G^2$ such that $s_1 = s_k = 0$;
- for $k \in \{2, 3\}$, $\text{val}_i(k, G^2) = \text{val}_i(4, G^2)$;
- for $k \geq 3$, $\text{val}_e(k, G^2)$ as the minimum GDA S of $P_k \circ G^2$ such that $s_1 = 0$;

- $val_e(2, G^2) = val_e(3, G^2)$.

For a sequence $w = (k_1, \dots, k_t)$, we define

- $val(P_n, G^2, w) = val_e(k_1, G^2) + \sum_{2 \leq i \leq t-1} val_i(k_i, G^2) + val_e(k_t, G^2);$
- $val(C_n, G^2, w) = \sum_{1 \leq i \leq t} val_i(k_i, G^2).$

Proposition 2.2 *If S is a GDA of $F = G^1 \circ G^2$ for $G^1 \in \{C_n, P_n\}$ and $G^2 \not\cong K_m$, then $|S| \geq val(G^1, G^2, spe(S))$ for $G \in \{P, C\}$. Furthermore, there is GDA of F of size $val(G^1, G^2, spe(S))$.*

Proof. Write $w = (k_1, \dots, k_t)$ and let $F_i = F[V(G_j^2) \cup \dots \cup V(G_{j+k_i-1}^2)]$ be the section associated with k_i and $S' = S \cap F_i$. First, consider $G = C$. For $k_i \geq 4$, since $s_{j+k_i} = 0$, it holds $|S'| \geq val_i(k_i, G^2)$. For $k_i = 3$, since $s_{j+3} = 0$, it holds $|S'| \geq val_i(4, G^2)$. Finally for $k_1 = 2$, since $s_3 = s_n = 0$, it holds $|S'| \geq val_i(4, G^2)$. Then, the result holds for $G = C$. Now consider $G = P$. For $k_i \geq 3$, since $s_j = 0$ or $s_{j+k_i} = 0$, it holds $|S'| \geq val_e(k_i, G^2)$. Finally for $k_1 = 2$, since $s_3 = 0$, it holds $|S'| \geq val_e(3, G^2)$, completing the proof. \square

Corollary 2.3 *Let $F = G^1 \circ G^2$ for $G^1 \in \{C_n, P_n\}$ and $G^2 \not\cong K_m$. Then $\gamma_a(F) = \min\{val(w)\}$ where w is a feasible sequence for G^2 .*

As a byproduct, we have that, for $G^1 \in \{C_n, P_n\}$ and $G^2 \not\cong K_m$, if the number of feasible sequences w that can reach the minimum GDA of $G^1 \circ G^2$ is bounded by a polynomial on n and m , one can determine them efficiently, and the values $val_i(k, G^2)$ and $val_e(k, G^2)$ are known for every $k \leq \max(w)$, one can find $\gamma_a(G^1 \circ G^2)$ efficiently. We show in next sections that this does hold for $G^2 \in \{P_m, C_m\}$. The last result of this section deals with the external sections.

Corollary 2.4 *If w is a feasible sequence of $F = P_n \circ G^2$ for $G^2 \not\cong K_m$ such that $val(P_n, G^2, w) = \gamma_a(F)$, then the following hold:*

- if 3 occurs in w , we can assume that $k_t = 3$;
- if $k_1 \neq 2$ and there are two occurrences of 3 in w , we can assume that $k_1 = k_t = 3$.

3 Properties of global defensive alliances

In this section, we present some bounds and properties of GDAs that will be useful in the remaining sections.

Proposition 3.1 *Let S be a GDA of $G^1 \circ G^2$ for $G^1 \in \{P_n, C_n\}$, $G^2 \in \{P_m, C_m\}$, $n \geq 3, m \geq 3$, and i be an integer such that $2 \leq i \leq n - 1$. Then, the following sentences hold*

- (i) *If $G^2 \simeq \{C_m, P_3\}$, then $s_{i-1} + s_i + s_{i+1} \geq m + 2$;*

- (ii) If $G^2 \simeq P_m$ for $m \geq 4$, then $s_{i-1} + s_i + s_{i+1} \geq m + 1$;
- (iii) If $s_i \geq 1$, then $s_{i-1} + s_{i+1} \geq m - 1$; and
- (iv) If $1 \leq s_i < m$, then $s_{i-1} + s_{i+1} \geq m$.

Proof. Let $v \in S_i$ and d be the number of neighbors of v in S_i .

- (i) Since $d(v) = 2m + 2$, S must contain at least $m + 1$ neighbors of v . This means that $s_{i-1} + s_i + s_{i+1} \geq m + 2$ because $N[v] \subseteq V(G_{i-1}^2) \cup V(G_i^2) \cup V(G_{i+1}^2)$.
- (ii) Since $d(v) = 2m + 1$, S must contain at least m neighbors of v . This means that $s_{i-1} + s_i + s_{i+1} \geq m + 1$ because $N[v] \subseteq V(G_{i-1}^2) \cup V(G_i^2) \cup V(G_{i+1}^2)$.
- (iii) Consequence of (i), (ii), and $1 \leq |N(v) \cap V(G_i^2)| \leq 2$.
- (iv) Consequence of (iii) and the fact that v can be chosen as a vertex having a neighbor in $V(G_i^2) \setminus S$. \square

Proposition 3.2 *Let S be a GDA of $G^1 \circ G^2$ for $G^1 \in \{P_n, C_n\}$, $G^2 \in \{P_m, C_m\}$, $n \equiv r \pmod{4}$, and $m \geq 3$ such that $s_i \geq 1$ for every $2 \leq i \leq n - 1$. Then the following hold.*

- (i) If $r = 0$, then $|S| \geq (2m - 1)\frac{n}{4}$
- (ii) If $r \in \{1, 2, 3\}$ and $n \geq 8$, then $|S| \geq (2m - 1)\lfloor \frac{n}{4} \rfloor + t$, where
$$t = \begin{cases} m + 1 & , \text{ if } r = 3 \text{ and } G^2 \simeq P_m \\ m + 2 & , \text{ if } r = 3 \text{ and } G^2 \simeq C_m \\ r & , \text{ if } r \in \{1, 2\} \end{cases}$$
- (iii) If $n \geq 6$ and $m = 3$, then $|S| \geq 6\lfloor \frac{n}{4} \rfloor + t$, where $t = \begin{cases} r + 2 & , \text{ if } r \in \{1, 2, 3\} \\ 0 & , \text{ if } r = 0 \end{cases}$
- (iv) If $n \geq 9$ and $m = 4$, then $|S| \geq 2n$ for $G^2 \simeq C_m$ and $|S| \geq 2n - 2$ for $G^2 \simeq P_m$

Proof. (i) By Proposition 3.1, $s_1 + s_3 \geq m - 1$ and $s_2 + s_4 \geq m - 1$. If $s_1 + s_3 = s_2 + s_4 = m - 1$, then S is not a GDA because some vertex of S_2 is not defended in S . Then $s_1 + s_2 + s_3 + s_4 \geq 2m - 1$. In fact, we can conclude that $s_i + s_{i+1} + s_{i+2} + s_{i+3} \geq 2m - 1$ for every $i \in \{1, \dots, n - 3\}$.

(ii) Since $n \geq 8$, $n - 4 - r = 4k$ for some positive integer k . It suffices to show that for $T = (V(G_5^2) \cup \dots \cup V(G_{5+r-1}^2)) \cap S$ it holds $|T| \geq t$. If $r \leq 2$, then $|T| \geq r$ because $s_i \geq 1$ for every $i \in \{2, \dots, n - 1\}$. If $r = 3$, then $|T| \geq m + 1$ if $G^2 \simeq P_m$ and $|T| \geq m + 2$ if $G^2 \simeq C_m$ due Proposition 3.1.

(iii) By Proposition 3.1 (i), it holds $s_i + s_{i+1} + s_{i+2} + s_{i+3} \geq 6$ for every $i \in \{1, \dots, n-3\}$. Then, the result is clear for $r = 0$. Case $r = 1$ is consequence of the fact that $s_1 + \dots + s_9 \geq 15$, case $r = 2$ because $s_1 + \dots + s_6 \geq 10$, and case $r = 3$ because $s_1 + \dots + s_7 \geq 11$.

(iv) It suffices to prove for $G^2 \simeq P_m$. We prove that if $s_i = 1$, then $s_{i+1} \geq 3$ or $s_{i+1} + s_{i+2} \geq 5$ or $s_{i+1} + s_{i+2} + s_{i+3} \geq 7$. If $s_{i+1} \geq 3$, we are done. If $s_{i+1} = 1$, then $s_{i+2} \geq 3$. If then $s_{i+2} = 3$, then there is a vertex of degree 2 in S_{i+2} having a neighbor in $V(G_2^2) \setminus S$, therefore $s_{i+3} \geq 3$. Then consider $s_{i+3} = 4$ and $s_{i+1} + s_{i+2} \geq 5$. Then consider $s_{i+1} = 2$. This means that there is a vertex of degree 2 in S_{i+1} having a neighbor in $V(G_1^2) \setminus S$, therefore $s_{i+2} \geq 3$ and $s_{i+1} + s_{i+2} \geq 5$.

Now, it remains to recall that $s_1 + s_2 + s_3 \geq 6$ and $s_{n-2} + s_{n-1} + s_n \geq 6$ for $G^2 \simeq C_4$ and $s_1 + s_2 + s_3 \geq 5$ and $s_{n-2} + s_{n-1} + s_n \geq 5$ for $G^2 \simeq P_4$. \square

Proposition 3.3 *If G is a spanning subgraph of G' and S is a minimum GDA of G such that no vertex of S is incident to any edge of $E(G') \setminus E(G)$, then S is also a minimum GDA of G' .*

Proof. Consequence of the fact that the neighborhood of each vertex of S is the same in G and in G' . \square

4 Determining γ_a for paths and cycles

For $n \geq 3$ and $m \geq 2$, we show in this section that $\gamma_a(G_1, G_2)$ for $G^1, G^2 \in \{C_n, P_m\}$ is the minimum among at most four values. Since these values are easily evaluated, one can determine $\gamma_a(G^1, G^2)$ within a constant number of operations. We consider first the case where G^1 has order at most 7.

4.1 Case $n \leq 7$

Let $F = G^1 \circ G^2$, for $G^1 \simeq P_n$, $G^2 \simeq C_m$, $n \in \{2, \dots, 7\}$, and $m \geq 3$. We define $X_{n,m} \subseteq V(F)$ as follows:

- $X_{2,3} = V(G_1^2)$. For $m \geq 4$, define $X_{2,m} = T_1 \cup T_2$, where T_1 and T_2 are the vertex sets of paths of order $\lfloor \frac{m}{2} \rfloor$ of G_1^2 and G_2^2 , respectively.
- $X_{3,m} = V(G_3^2) \cup T_2$, where T_2 is the vertex set of a path of order x of G_2^2 where $x = \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$.
- $X_{4,m} = V(G_3^2) \cup V(G_2^2) - \{u\}$, for some vertex $u \in V(G_2^2)$.
- $X_{5,m} = V(G_3^2) \cup T_2 \cup T_4$ where T_2 contains two adjacent vertices of G_2^2 and T_4 contains the vertices of a path of G_4^2 of size $\max\{2, m-3\}$.

- $X_{6,3} = V(G_1^2) \cup V(G_5^2) \cup T_4$, where T_4 is a pair of vertices; for $m \geq 4$, define $X_{6,m} = V(G_3^2) \cup V(G_4^2) \cup T_2 \cup T_5$, where T_2 and T_5 are two adjacent vertices of G_2^2 and G_5^2 , respectively.
- $X_{7,m} = V(G_3^2) \cup V(G_5^2) \cup T_2 \cup T_4 \cup T_6$, where T_2 and T_6 are two adjacent vertices of G_2^2 and G_6^2 , respectively, and T_4 is the vertex set of a path of order $m - 3$ of G_4^2 .

n	$P_n \circ C_3$	$P_n \circ C_m, m \geq 4$
2	3	$2\lfloor \frac{m}{2} \rfloor$
3	5	$m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$
4	5	$2m - 1$
5	7	$\max\{m + 4, 2m - 1\}$
6	8	$2m + 4$
7	10	$3m + 1$

Table 1: $P_n \circ C_m, 2 \leq n \leq 7$ and $m \geq 3$.

n	$C_n \circ C_3$	$C_n \circ C_m, m \geq 4$
3	5	$3\lceil \frac{m}{2} \rceil$
4	5	$2m - 1$
5	7	$\max\{m + 4, 2m - 1\}$
6	10	$2m + 4$
7	10	$3m + 1$

Table 2: $C_n \circ C_m, 3 \leq n \leq 7$ and $m \geq 3$.

Let $F = G^1 \circ G^2$, for $G^1 \simeq P_n, G^2 \simeq P_m, n \in \{2, \dots, 7\}$, and $m \geq 3$. We define $Y_{n,m} \subseteq V(F)$ as follows:

- $Y_{2,3} = V(G_1^2) \setminus \{v_1\} \cup \{v_2\}$. For $m \geq 4$, define $X_{2,m} = T_1 \cup T_2$, where T_1 and T_2 are the vertex sets of paths of order $\lfloor \frac{m}{2} \rfloor$ of G_1^2 and G_2^2 , respectively.
- For $m \in \{3, 4\}$, $Y_{3,m} = V(G_3^2) \cup \{v_2\}$ for $v_2 \in V(G_2^2)$; $Y_{3,5} = V(G_3^2) \setminus \{u\} \cup T_2$ where T_2 contains two adjacent vertices of G_2^2 ; for $m \geq 6$, $Y_{3,m} = V(G_3^2) \cup T_2$, where T_2 is the vertex set of a path of G_2^2 of order $\lfloor \frac{m-2}{2} \rfloor$.
- $Y_{4,m} = X_{4,m}$.
- $Y_{5,3} = V(G_3^2) \cup \{v_1\} \cup \{v_4\}$; $Y_{5,4} = V(G_3^2) \cup \{v_2\} \cup \{v_4, v_4'\}$; for $m \geq 5$, $Y_{5,m} = X_{5,m}$.
- $Y_{6,3} = V(G_3^2) \cup V(G_4^2) \cup \{v_2\} \cup \{v_5\}$, where $d(v_2) = d(v_5) = 7$; for $m \geq 4$, define $X_{6,m} = V(G_3^2) \cup V(G_4^2) \cup \{v_2\} \cup \{v_5\}$.
- $Y_{7,m} = V(G_3^2) \cup V(G_5^2) \cup \{v_2\} \cup T_4 \cup \{v_6\}$, where T_4 is the vertex set of a path of order $m - 2$ of G_4^2 .

Denote $x_{i,m} = |X_{i,m}|$ and $y_{i,m} = |Y_{i,m}|$.

n	$P_n \circ P_3$	$P_n \circ P_m, m \geq 4$
2	3	$2\lfloor \frac{m}{2} \rfloor$
3	4	$m + \lfloor \frac{m-2}{2} \rfloor$
4	5	$2m - 1$
5	5	$2m - 1$
6	8	$2m + 2$
7	9	$3m$

Table 3: $P_n \circ P_m, 2 \leq n \leq 7$ and $m \geq 3$.

n	$C_n \circ P_3$	$C_n \circ P_m, m \geq 4$
3	5	$3\lfloor \frac{m}{2} \rfloor$
4	5	$2m - 1$
5	5	$2m - 1$
6	8	$2m + 2$
7	9	$3m$

Table 4: $C_n \circ P_m, 3 \leq n \leq 7$ and $m \geq 3$.

Proposition 4.1 For $n \in \{2, \dots, 7\}$, $\gamma_a(P_n \circ C_m)$ is given in Table 1 and $\gamma_a(P_n \circ P_m)$ is given in Table 3.

Proof. It is easy to check that X_i^m is a GDA of $F = P_i \circ G^2$ for $i \in \{2, \dots, 7\}$ and $G^2 \in \{C_m, P_m\}$. For the converse, let S be a minimum GDA of F .

Case $i = 2$. For $m = 3$, it is easy to check that there is no GDA of size 2 and that $V(G_1^2)$ is a GDA of the graph. Then assume $m \geq 4$. Since $V(G_1^2)$ is not a GDA, $(V(G_1^2) \setminus \{v_1\}) \cup \{v_2\}$ is not a a GDA for $v_1 \in V(G_2^2)$ and $v_2 \in V(G_2^2)$, and $x_2 \leq m$, it holds $2 \leq s_1 < m$ and $2 \leq s_2 < m$ for a minimum GDA S . Then, we can assume that there is a vertex $v \in S \cap V(G_1^2)$ such that $d(v) = m + 2$ and having a neighbor in $V(G_1^2) \setminus S$. Therefore, we have $s_2 \geq \lfloor \frac{m+2}{2} \rfloor - 1 = \lfloor \frac{m}{2} \rfloor$. Since the same does hold for s_1 , the result is true.

Case $i = 3$. Suppose that $|S| < m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} = x_3^m < 2m$. If $s_2 = 0$, then we can assume that $v \in V(G_1^2)$ has at most one neighbor in S , then v is not defended in S . Hence $s_2 \geq 1$. First, consider $m \in \{3, 4, 5\}$ and $G_2 \simeq C_m$. Since $d(v) = 2m + 2$ for $v \in V(G_2^2)$, we have $|S \cap N(v)| < \frac{d(v)}{2}$, a contradiction. Case $m \in \{3, 4, 5\}$ and $G_2 \simeq P_m$ is direct from Proposition 3.1 (ii).

Now, consider $m \geq 6$. We can now write $|S| < m + \lfloor \frac{m-2}{2} \rfloor$. By Proposition 3.1, $s_1 + s_3 \geq m - 1$. Since $m - 1 > \lfloor \frac{m-2}{2} \rfloor$ for $m \geq 4$, we have $s_2 < m$. Consequently, by Proposition 3.1 again, we have $m \leq s_1 + s_3 < 2m$. Then, without loss of generality, there is a vertex in $V(G_1^2)$ having at most one neighbor in $S \cap V(G_1^2)$. Therefore $s_2 \geq \lfloor \frac{m+2}{2} \rfloor - 1 = \lfloor \frac{m}{2} \rfloor$, which means that $|S| \geq m + \lfloor \frac{m}{2} \rfloor > x_3$, a contradiction.

Case $i = 4$. First, consider $s_2 = 0$ and $s_3 = 0$. Then $m = 3$, $s_1 = s_4 = m$, and $|S| = 6 > x_4^3 = 5$. Next, consider $s_2 = 0$ and $s_3 \geq 1$. Then $m = 3$, $s_1 = m$ and, using Proposition 3.1, $s_2 + s_3 + s_4 \geq m + 2 = 5$. Which means $|S| \geq 8 > x_4^3 = 5$. The case $s_2 \geq 1$ and $s_3 = 0$ is

analogue. Then $s_2 \geq 1$ and $s_3 \geq 1$. By Proposition 3.2, $|S| \geq 2m - 1$, completing the proof for $i = 4$.

Case $i = 5$. Since $x_5^m = 2m - 1$ for $m \geq 5$, and $\gamma_a(P_5 \circ G^2) \geq \gamma_a(P_4 \circ G^2)$ for $G^2 \in \{C_m, P_m\}$, case $i = 4$ implies, for $m \geq 5$, $|S| \geq x_5^m$. The same argument holds for $m \leq 4$ and $G^2 \simeq P_m$.

For $m \leq 4$ and $G^2 \simeq C_m$, we have $x_5^m = m + 4$. If $s_2 = 0$, then $m = 3$ and $s_1 = 3$. Since s_3 and s_4 are both not equal to 0, then $s_2 + s_3 + s_4 + s_5 \geq m + 2$, which means $|S| > m + 4$. Then $s_2 \geq 1$ and $s_4 \geq 1$. Suppose that $s_3 = m - k$ for $k \geq 1$. This implies that $s_1 + s_2 \geq 2 + k$ and $s_4 + s_5 \geq 2 + k$. Then $|S| \geq m - k + 2 + k + 2 + k = m + 4 + k$, a contradiction. Therefore $s_3 = m$. Since every vertex of $S_2 \cup S_4$ has degree $2m + 2$, $s_1 + s_2 \geq 2$ and $s_4 + s_5 \geq 2$, which implies that $|S| \geq m + 4$.

Case $i = 6$. First, consider $m = 3$ and $G^2 \simeq C_m$. If $s_2 \geq 1$ and $s_5 \geq 1$, then $s_1 + s_2 + s_3 \geq 5$ and $s_4 + s_5 + s_6 \geq 5$, which means $|S| > x_6^3 = 8$. If $s_2 = 0$ and $s_5 = 0$, then $s_1 = s_6 = 3$, furthermore $s_3 + s_4 \geq 5$, which means $|S| > x_6^3$. Then, without loss of generality, we can assume $s_2 = 0$ and $s_5 \geq 1$. This implies $s_1 = 3$ and $s_4 + s_5 + s_6 \geq 5$, which means $|S| \geq x_6^3 = 8$.

Next, consider $m = 3$ and $G^2 \simeq P_m$. Since $V(G_1^2)$ is not a GDA of $P_2 \circ P_3$, then $s_2 \geq 1$ and $s_5 \geq 1$. Observe that a vertex $v \in S_2$ needs at least four neighbors in S because $d(v) = 8$. Then $s_1 + s_2 + s_3 \geq 5$ and $s_4 + s_5 + s_6 \geq 5$.

Now, consider $m \geq 4$. It is clear that $s_2 \geq 1$ and $s_5 \geq 1$. By Proposition 3.1, $s_1 + s_2 + s_3 \geq m + 2$ for $G^2 \simeq C_m$ and $s_1 + s_2 + s_3 \geq m + 1$ for $G^2 \simeq P_m$. By symmetry, $s_4 + s_5 + s_6 \geq m + 2$ for $G^2 \simeq C_m$ and $s_4 + s_5 + s_6 \geq m + 2$ for $G^2 \simeq P_m$. Therefore $|S| \geq 2m + 4$ for $G^2 \simeq C_m$ and $|S| \geq 2m + 2$ for $G^2 \simeq P_m$.

Case $i = 7$. If $s_2 = 0$, then $m = 3$, $G^2 \simeq C_m$, and $s_1 = 3$. If $s_3 \geq 1$, then $s_3 + s_4 \geq 5$. Since $s_5 + s_6 + s_7 \geq 3$, we have $|S| > x_7^3$. Then $s_3 = 0$. This implies that $s_4 \geq 1$, which means $s_4 + s_5 \geq 5$. Therefore $|S| < x_7^3$ if $s_6 + s_7 = 1$. But the vertex of $S_6 \cup S_7$ is not defendend in S .

Then, consider $s_2 \geq 1$ and $s_6 \geq 1$. If $s'_3 = 0$, then cases $i = 3$ and $i = 5$ imply that $s_1 + s_2 + s_3 \geq m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$ and $s_3 + s_4 + s_5 + s_6 + s_7 \geq \max\{m + 4, 2m - 1\}$ which is at least x_7^m for any $m \geq 3$. Then, we can consider $s_3 \geq 1$ and $s_5 \geq 1$. Since $s_1 + s_3$ and $s_2 + s_4$ cannot both be equal to $m - 1$ and for $G^2 \simeq C_m$, we have $s_5 + s_6 + s_7 \geq m + 2$, for $G^2 \simeq P_m$, we have $s_5 + s_6 + s_7 \geq m + 1$, we have $|S| \geq m - 1 + m + m + 2 = 3m + 1$ for $G^2 \simeq C_m$, and we have $|S| \geq m - 1 + m + m + 2 = 3m$ for $G^2 \simeq P_m$. \square

Corollary 4.2 For $n \in \{3, \dots, 7\}$, $\gamma_a(C_n \circ C_m)$ is given in Table 2 and $\gamma_a(C_n \circ P_m)$ is given in Table 4.

Proof. Proposition 3.2 implies that $\gamma_a(C_3 \circ C_3) \geq 5$ and $\gamma_a(C_3 \circ P_3) \geq 4$. It is easy to check that $C_3 \circ P_3$ has no GDA with less than 5 vertices, then since $X_{3,3}$ and $Y_{3,3}$ are GDAs of $C_3 \circ C_3$ and $C_3 \circ P_3$, respectively, $\gamma_a(C_3 \circ C_3) = \gamma_a(C_3 \circ P_3) = 5$.

Now, consider $n = 3$, $m \geq 4$, and let S be a minimum GDA of $C_n \circ G^2$ for $G^2 \in \{C_m, P_m\}$. We can assume that $v_2 \in S_2$ and $v_3 \in S_3$. Since $d(v_2) = d(v_3) \in \{2m + 1, 2m + 2\}$, $s_1 + s_3 \geq m - 1$

and $s_1 + s_2 \geq m - 1$. If $s_1 = 0$, then $|S| \geq 2m - 2$. Therefore, we can assume $s_1 \neq 1$, which implies, by the symmetry of the graph, that $s_1 = s_2 = s_3 = k$. If $d(v_2) = 2m + 1$ and $k < \lfloor \frac{m}{2} \rfloor$, then S is not a GDA. Therefore $\gamma_a(C_3 \circ P_n) = 3 \lfloor \frac{m}{2} \rfloor$ for $m \geq 4$. If $d(v_2) = 2m + 2$ and $k < \lceil \frac{m}{2} \rceil$, then S is not a GDA. Therefore $\gamma_a(C_3 \circ C_n) = 3 \lceil \frac{m}{2} \rceil$ for $m \geq 4$.

The cases $n \in \{4, 5, 6, 7\}$ are consequence of Propositons 3.3 and 4.1. \square

Corollary 4.3 *For $k \in \{3, \dots, 7\}$, $G^1 \in \{C_n, P_n\}$, and $G^2 \in \{C_m, P_m\}$, there is a minimum GDA S of $G^1 \circ G^2$ such that $\max(\text{spe}(S)) \leq k$.*

4.2 Case $n \geq 8$

We begin this section presenting a hierarchy of $\gamma_a(G^1 \circ G^2)$ which depends of the operands and is consequence of the previous results.

Corollary 4.4 *For $n \geq 2$ and $m \geq 3$, it holds $\gamma_a(P_n \circ P_m) \leq \gamma_a(C_n \circ P_m) \leq \gamma_a(C_n \circ C_m)$ and $\gamma_a(P_n \circ P_m) \leq \gamma_a(P_n \circ C_m) \leq \gamma_a(C_n \circ C_m)$.*

Proof. $\gamma_a(P_n \circ P_m) \leq \gamma_a(C_n \circ P_m)$ and $\gamma_a(P_n \circ C_m) \leq \gamma_a(C_n \circ C_m)$ are consequences of Corollary 2.3, while $\gamma_a(C_n \circ P_m) \leq \gamma_a(C_n \circ C_m)$ and $\gamma_a(P_n \circ P_m) \leq \gamma_a(P_n \circ C_m)$ are consequence of Corollaries 2.3, 4.2, and Proposition 4.1. \square

Now, we consider the case where G^1 has order at least 8. We divide the study into two cases, $m = 3$ and $m \geq 4$.

4.2.1 Case $m = 3$

Proposition 4.5 *For $n \geq 8$, $G^1 \in \{P_n, C_n\}$, and $G^2 \in \{P_3, C_3\}$, there is minimum GDA S of $G^1 \circ G^2$ such that $\max(\text{spe}(S)) \leq 6$.*

Proof. Write $F = G^1 \circ G^2$ and let $w = (k_1, \dots, k_t)$ be the spectrum of a minimum GDA S of F , $k_i \geq 7$ for some $i \in [t]$, and $r \equiv k_i \pmod{4}$.

By Proposition 3.2 (iii), $\text{val}_e(k_i, C_3) \geq 6 \lfloor \frac{k_i}{4} \rfloor + t$ where $t = \begin{cases} r + 2 & , \text{ if } r \in \{2, 3\} \\ 2 & , \text{ if } r = 1 \\ 0 & , \text{ if } r = 0 \end{cases}$

For each value of r , we present a k_i -sequence $w' = (\ell_1, \dots, \ell_p)$ such that $\max(w') \leq 6$ and $\text{val}(P_n, C_3, w') \leq 6 \lfloor \frac{k_i}{4} \rfloor + t$. Since $y_{t,3} \leq x_{t,3}$ for $3 \leq t \leq 6$ and the bound of Proposition 3.2 (iii) holds for $G^2 \simeq P_3$ and $G^2 \simeq C_3$, we only need to consider $G^2 \simeq C_3$.

For $r = 0$, define $w' = (\lfloor \frac{k_i}{4} \rfloor 4)$ containing $\frac{k_i}{4}$. Since $x_{4,3} = 5$, it holds $\text{val}(P_n, C_3, w') = 5 \frac{k_i}{4} \leq 6 \frac{k_i}{4} \leq \text{val}_e(k_i, C_3)$. For $r = 1$, consider $w' = (\lfloor \frac{k_i-5}{4} \rfloor 4, 5)$. Since $x_{5,3} = 5$, it holds $\text{val}(P_n, C_3, w') = 5 \frac{k_i-5}{4} + 5 \leq 6 \lfloor \frac{k_i}{4} \rfloor + 2 \leq \text{val}_e(k_i, C_3)$. For $r = 2$, define $w' = (\lfloor \frac{k_i-6}{4} \rfloor 4, 6)$. Since $x_{6,3} = 8$, it holds $\text{val}(P_n, C_3, w') = 5 \frac{k_i-6}{4} + 8 \leq 6 \lfloor \frac{k_i}{4} \rfloor + 4 \leq \text{val}_e(k_i, C_3)$. For $r = 3$, define $w' = (3, \lfloor \frac{k_i-3}{4} \rfloor 4)$. Since $\text{val}'(3) = x_{4,3} = 5$, it holds $\text{val}(P_n, C_3, w') = 5 \frac{k_i-3}{4} + 5 \leq 6 \lfloor \frac{k_i}{4} \rfloor + 5 \leq \text{val}_e(k_i, C_3)$.

Now, it remains to observe that $w'' = w_{1,i-1}w'w_{i+1,t}$ is a feasible n -sequence and $val(G^1, C_3, w'') \leq val(G^1, C_3, w)$. \square

Proposition 4.6 For $n \geq 8$, $G^1 \in \{C_n, P_n\}$, and $G^2 \in \{C_3, P_3\}$, there is a minimum GDA S of $G^1 \circ G^2$ such that

- (i) if $G^2 \simeq C_3$, then $spe(S)$ has at most one element in the set $\{3, 5, 6\}$ and no one is equal to 2;
- (ii) if $G^2 \simeq P_3$, then $spe(S)$ has at most one element in the set $\{2, 3, 4, 6\}$.

Proof. By Proposition 4.5, there is a minimum GDA S of $F = G^1 \circ C_3$ whose $max(spe(S)) \leq 6$. Suppose that k_i and k_j are values of $spe(S)$ and of $\{2, 3, 5, 6\}$. For each possible case, we present in Table 5 a sequence $w' = (\ell_1, \dots, \ell_t)$ for $t \leq 3$ such that $val(P_n, C_3, w') \leq val_e(k_i, C_3) + val_e(k_j, C_3)$, w' does not contain the number 2, and contains at most one element of the set $\{3, 5, 6\}$. The third column of the table is a lower bound of $val_e(k_i, C_3) + val_e(k_j, C_3)$, which is consequence of Proposition 4.1.

k_i	k_j	$val_e(k_i, C_3) + val_e(k_j, C_3)$	ℓ_1	ℓ_2	ℓ_3	$val(P_n, C_3, w')$
2	3	$5 + 5 = 10$	5			7
2	4	$5 + 5 = 10$	6			8
2	5	$5 + 7 = 12$	4	3		10
2	6	$5 + 8 = 13$	4	4		10
3	3	$5 + 5 = 10$	6			8
3	5	$5 + 7 = 12$	4	4		10
3	6	$5 + 8 = 13$	5	4		12
5	5	$7 + 7 = 14$	6	4		13
5	6	$7 + 8 = 15$	4	4	3	15
6	6	$8 + 8 = 16$	4	4	4	15

Table 5: Case $G^2 \simeq C_3$.

It is clear that the sequence $w'' = (k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_{j-1}, k_{j+1}, \dots, k_t)w'$ is feasible and $val(G^1, C_3, w'') \leq val(G^1, C_3, w)$. Since one can repeat this process until a sequence with the required properties be obtained, the result does hold.

The proof of (ii) is essentially the same of (i) by considering $G^2 \simeq P_3$ and Table 6. \square

$$f(n, 3) = \begin{cases} 5\frac{n}{4} & , \text{ if } n \equiv 0 \pmod{4} \\ 5\frac{n-5}{4} + 7 & , \text{ if } n \equiv 1 \pmod{4} \\ 5\frac{n-6}{4} + 8 & , \text{ if } n \equiv 2 \pmod{4} \\ 5\frac{n-3}{4} + 5 & , \text{ if } n \equiv 3 \pmod{4} \end{cases}$$

k_i	k_j	$val_e(k_i, P_3) + val_e(k_j, P_3)$	ℓ_1	ℓ_2	ℓ_3	$val(P_n, P_3, w')$
2	3	$4 + 4 = 8$	5			5
2	4	$4 + 5 = 9$	6			8
2	6	$4 + 8 = 12$	5	3		10
3	3	$4 + 4 = 8$	6			8
3	4	$4 + 5 = 9$	5	2		9
3	6	$4 + 8 = 12$	5	4		10
4	4	$5 + 5 = 10$	5	3		10
4	6	$5 + 8 = 13$	5	5		10
6	6	$8 + 8 = 16$	5	5	2	14

Table 6: Case $G^2 \simeq P_3$.

$$f'(n, 3) = \begin{cases} 5\frac{n}{5} & , \text{ if } n \equiv 0 \pmod{5} \\ 5\frac{n-6}{5} + 8 & , \text{ if } n \equiv 1 \pmod{5} \\ 5\frac{n-r}{5} + 4 & , \text{ if } n \equiv r \pmod{5} \text{ for } r \in \{2, 3\} \\ 5\frac{n-4}{5} + 5 & , \text{ if } n \equiv 4 \pmod{5} \end{cases}$$

Theorem 4.7 For $n \geq 8$ and $G^1 \in \{C_n, P_n\}$, $\gamma_a(G^1 \circ C_3) = f(n, 3)$ and $\gamma_a(G^1 \circ P_3) = f'(n, 3)$.

Proof. Corollary 2.4 and Propositions 4.5 and 4.6 (i) imply that, for $p = \lfloor \frac{n}{4} \rfloor$ and $r = n \pmod{4}$, it holds that a sequence w such that $\gamma_a(G^1 \circ C_3) = val(G^1, C_3, w)$ is

$$w = \begin{cases} ([p]4) & , \text{ if } r = 0, \\ ([p-1]4, 5) & , \text{ if } r = 1, \\ ([p-1]4, 6) & , \text{ if } r = 2, \\ ([p]4, 3) & , \text{ if } r = 3. \end{cases}$$

Using Proposition 4.1, we have $\gamma_a(G^1 \circ C_3) = f(n, 3)$. Now, Corollary 2.4 and Propositions 4.5 and 4.6 (ii) imply that, for $p = \lfloor \frac{n}{4} \rfloor$ and $r = n \pmod{5}$, it holds that a sequence w such that $\gamma_a(G^1 \circ P_3) = val(G^1, P_3, w)$ is

$$w = \begin{cases} ([p-1]5) & , \text{ if } r = 0, \\ ([p-1]5, 6) & , \text{ if } r = 1, \\ (r, [p]5) & , \text{ if } r \in \{2, 3, 4\}. \end{cases}$$

Using Proposition 4.1, we have $\gamma_a(G^1 \circ P_3) = f'(n, 3)$. □

4.2.2 Case $m \geq 4$

Proposition 4.8 For $n \geq 8, m \geq 4$, $G^1 \in \{P_n, C_n\}$, and $G^2 \in \{P_m, C_m\}$, there is a minimum GDA S of $G^1 \circ G^2$ such that $\max(\text{spe}(S)) \leq 7$.

Proof. Let $w = (k_1, \dots, k_t)$ be the spectrum of a minimum GDA S of $F = G^1 \circ G^2$ such that $k_i \geq 8$ for some $i \in [t]$. Let F' be the section of F associated with k_i and set $S' = |V(F') \cap S|$. For each case, we present a k_i -sequence $w' = (\ell_1, \dots, \ell_p)$ such that $\max(w') \leq 7$ and $\text{val}(P_n, G^2, w') \leq |S'|$.

For $k_i = 8$, consider $r \equiv k_i \pmod{4}$. Proposition 3.2 (i) implies $|S'| \geq 4m - 2$. If $m = 4$, let $w' = (4, 4)$. Since $y_{4,4} = x_{4,4} = 7$, $\text{val}(P_n, G^2, w') = 14 \leq |S'|$. If $m \geq 5$, let $w' = (5, 3)$. Since $y_{5,m} = x_{5,m} = 2m - 1$ and $y_{3,m} \leq x_{3,m} = m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$, it holds $\text{val}(P_n, G^2, w') \leq 3m - 1 + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} \leq 4m - 2 \leq |S'|$, which means that the result also holds for $k_i = 8$.

For $k_i \geq 9$, consider $r \equiv k_i \pmod{5}$. Let w' as follows

$$w' = \begin{cases} (\lfloor \frac{k_i}{5} \rfloor 5) & , \text{ if } r = 0 \\ (\lfloor \frac{k_i-6}{5} \rfloor 5, 6) & , \text{ if } r = 1 \\ (r, \lfloor \frac{k_i-r}{5} \rfloor 5) & , \text{ if } r \in \{2, 3, 4\} \end{cases}$$

Consider first $m = 4$. If $G^2 \simeq C_4$, Proposition 3.2 (iv) implies $|S'| \geq 2k_i$. By Proposition 4.1, it holds that $\text{val}(P_n, C_4, w')$ is $8\frac{k_i}{5}$ for $r = 0$, is $8\frac{k_i-6}{5} + 12$ for $r = 1$, is $8\frac{k_i-r}{5} + 6$ for $r \in \{2, 3\}$, is $8\frac{k_i-4}{5} + 7$ for $r = 4$. Since $\text{val}(P_n, C_4, w') \leq 2k_i \leq |S'|$ in all cases, the result follows for $G^2 \simeq C_4$. If $G^2 \simeq P_4$, Proposition 3.2 (iv) implies $|S'| \geq 2k_i - 2$. By Proposition 4.1, it holds that $\text{val}(P_n, P_4, w')$ is $7\frac{k_i}{5}$ for $r = 0$, is $7\frac{k_i-6}{5} + 10$ for $r = 1$, is $7\frac{k_i-r}{5} + 5$ for $r \in \{2, 3\}$, is $7\frac{k_i-4}{5} + 7$ for $r = 4$. Since $\text{val}(P_n, P_4, w') \leq 2k_i - 2 \leq |S'|$ in all cases, the result follows for $G^2 \simeq P_4$.

Consider now $m \geq 5$. Proposition 3.2 (i) and (ii) imply $|S'| \geq \lfloor \frac{k}{4} \rfloor (2m - 1) + t$ where

$$t = \begin{cases} m + 1 & , \text{ if } r = 3 \\ r & , \text{ if } r \in \{0, 1, 2\} \end{cases}$$

for $G^2 \in \{P_m, C_m\}$. By Proposition 4.1, $\text{val}(P_n, G^2, w')$ is $(2m - 1)\frac{k_i}{5}$ for $r = 0$, is at most $(2m - 1)\frac{k_i-6}{5} + 2m + 4$ for $r = 1$, is at most $(2m - 1)\frac{k_i-r}{5} + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$ for $r \in \{2, 3\}$, is $(2m - 1)\frac{k_i-4}{5} + 2m - 1$ for $r = 4$. Since $\text{val}(P_n, G^2, w') \leq |S'|$ in all cases, the proof is complete. \square

Proposition 4.9 *For $n \geq 8, m \geq 4, G^1 \in \{P_n, C_n\}$, and $G^2 \in \{P_m, C_m\}$, there is a minimum GDA S of $G^1 \circ G^2$ whose spectrum contains at most one element of the set $\{2, 3, 4, 7\}$.*

Proof. By Proposition 4.8, there is a minimum GDA S of $F = G^1 \circ G^2$ such that $\max(\text{spe}(S)) \leq 7$. Suppose that k_i and k_j are values of $\text{spe}(S)$ and of $\{2, 3, 4, 7\}$. For each possible case, we present in Tables 7 and 8 a sequence $w' = (\ell_1, \dots, \ell_t)$ for $t \leq 3$ such that $\text{val}(P_n, G^2, w') \leq \text{val}_e(k_i, G^2) + \text{val}_e(k_j, G^2)$ such that w' contains at most one element of the set $\{2, 3, 4, 7\}$. Table 7 contains the cases for $G^2 \simeq C_m$ and Table 8 for $G^2 \simeq P_m$. The third column of each table contains a lower bound of $\text{val}_e(k_i, G^2) + \text{val}_e(k_j, G^2)$, which is consequence of Proposition 4.1.

It is clear that the sequence $w'' = (k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_{j-1}, k_{j+1}, \dots, k_t,)w'$ is feasible and $\text{val}(G^1, G^2, w'') \leq \text{val}(G^1, G^2, w)$ for $G \in \{P, C\}$. Since one can repeat this process until a sequence with the required properties be obtained, the result does hold. \square

k_i	k_j	$val_e(k_i, C_m) + val_e(k_j, C_m)$	ℓ_1	ℓ_2	ℓ_3	$val(P_n, C_m, w')$
2	3	$2(m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\})$	5			$\max\{m + 4, 2m - 1\}$
3	3	$2(m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\})$	6			$2m + 4$
2	4	$2m - 1 + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$	6			$2m + 4$
3	4	$2m - 1 + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$	7			$3m + 1$
4	4	$2(2m - 1)$	5	3		$\max\{m + 4, 2m - 1\} + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$
2	7	$3m + 1 + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$	5	4		$\max\{m + 4, 2m - 1\} + 2m - 1$
3	7	$m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} + 3m + 1$	5	5		$2(\max\{m + 4, 2m - 1\})$
4	7	$3m + 1 + 2m - 1$	6	5		$2m + 4 + \max\{m + 4, 2m - 1\}$
7	7	$2(3m + 1)$	5	5	4	$2(\max\{m + 4, 2m - 1\}) + 2m - 1$

Table 7: Case $G^2 \simeq C_m$.

k_i	k_j	$val_e(k_i, P_m) + val_e(k_j, P_m)$	ℓ_1	ℓ_2	ℓ_3	$val(P_n, G^2, []P_m)w'$
2	3	$2(m + \lfloor \frac{m-2}{2} \rfloor)$	5			$2m - 1$
3	3	$2(m + \lfloor \frac{m-2}{2} \rfloor)$	6			$2m + 2$
2	4	$2m - 1 + m + \lfloor \frac{m-2}{2} \rfloor$	6			$2m + 2$
3	4	$2m - 1 + m + \lfloor \frac{m-2}{2} \rfloor$	7			$3m$
4	4	$2(2m - 1)$	5	3		$2m - 1 + m + \lfloor \frac{m-2}{2} \rfloor$
2	7	$3m + m + \lfloor \frac{m-2}{2} \rfloor$	5	4		$2m - 1 + 2m - 1$
3	7	$m + \lfloor \frac{m-2}{2} \rfloor + 3m$	5	5		$2(2m - 1)$
4	7	$3m + 2m - 1$	6	5		$2m + 2 + 2m - 1$
7	7	$2(3m)$	5	5	4	$2(2m - 1) + 2m - 1$

Table 8: Case $G^2 \simeq P_m$.

Proposition 4.10 *If $n \geq 8, m \geq 4, G^1 \in \{P_n, C_n\}, G^2 \in \{P_m, C_m\}$, and $w = (k_1, \dots, k_t)$ is the spectrum of a minimum GDA of $G^1 \circ G^2$ containing three numbers that are pairwise different, then we can assume that $w \in \{(3, [p-1]5, 6), (3, 5, [q-1]6)\}$ where $p = \frac{n-9}{5}$ and $q = \frac{n-8}{6}$.*

Proof. Suppose that, for $i, j, r \in [t]$, k_i, k_j , and k_r are pairwise different. By Proposition 4.9, we can assume that $k_j = 5$ and $k_r = 6$. In Tables 9 and 10, we show that if $k_i \neq 3$, then there is a k_i -sequence $w' = (\ell_1, \dots, \ell_{t'})$ for $t' \leq 4$ such that w' contains only numbers 3, 5, and 6, and $val(P_n, G^2, w') \leq val_e(k_i, G^2) + val_e(k_j, G^2) + val_e(k_r, G^2)$.

It remains to prove that $w \neq (3, [p]5), [q]6$ for $p, q \geq 2$. First, we consider $G^2 \simeq C_m$. We can assume that $w = (3, [2]5, [2]6, [p-2]5, [q-2]6)$. We know that $val(P_n, C_m, (3, [2]5, [2]6)) = 2(2m+4) + 2(\max\{m+4, 2m-1\}) + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} = 9m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} + 6$. For $m \leq 17$, $val(P_n, C_m, ([5]5)) = 5(\max\{m+4, 2m-1\}) = 10m - 5$ and for $m \geq 18$, $val(P_n, C_m, ([3]6, 7)) = 3(2m+4) + 3m + 1 = 9m + 13$, which means that w is not the spectrum of a minimum GDA of $G^1 \circ C_m$.

Finally consider $G^2 \simeq P_m$. We know that $val(P_n, P_m, (3, [2]5, [2]6)) = 2(2m+2) + 2(2m-1) + m + \lfloor \frac{m-2}{2} \rfloor = 9m + 3 + \lfloor \frac{m-2}{2} \rfloor$. For $m \leq 11$, $val(P_n, P_m, ([5]5)) = 5(2m-1) = 10m - 5$ and for $m \geq 12$, $val(P_n, P_m, ([3]6, 7)) = 3(2m+2) + 3m = 9m + 6$, which means that w is not the spectrum of a minimum GDA of $G^1 \circ P_m$. \square

The above results reduce the number of sequences that can reach $\gamma_a(F)$ for $G^1 \circ G^2, G^1 \in$

k_i	$val(i, j, r)$	ℓ_1	ℓ_2	ℓ_3	ℓ_4	$val(P_n, C_m, w')$
2	$m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} + 2m + 4 +$ $\max\{m + 4, 2m - 1\} =$ $5m + 3 + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$	5	5	3		$2(\max\{m + 4, 2m - 1\}) +$ $m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} =$ $5m - 2 + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$
4	$2m - 1 + \max\{m + 4, 2m - 1\} +$ $2m + 4 \geq 6m + 3$	5	5	5		$3(\max\{m + 4, 2m - 1\}) =$ $6m - 3$
7	$\max\{m + 4, 2m - 1\} + 2m + 4 +$ $3m + 1 = 7m + 4$	5	5	5	3	For $m \leq 10$, $3(\max\{m + 4, 2m - 1\}) +$ $m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} =$ $7m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} - 3$ For $m \geq 11$, $3(2m + 4) =$ $6m + 12$

Table 9: Case $G^2 \simeq C_m$, where $val(i, j, r) = val_e(k_i, C_m) + val_e(k_j, C_m) + val_e(k_r, C_m)$

k_i	$val(i, j, r)$	ℓ_1	ℓ_2	ℓ_3	ℓ_4	$val(P_n, P_m, w')$
2	$m + \lfloor \frac{m-2}{2} \rfloor + 2m + 2 +$ $2m - 1 = 5m + 1 + \lfloor \frac{m-2}{2} \rfloor$	5	5	3		$2(2m - 1) + m + \lfloor \frac{m-2}{2} \rfloor =$ $5m - 2 + \lfloor \frac{m-2}{2} \rfloor$
4	$2m - 1 + 2m - 1 + 2m + 2 =$ $6m$	5	5	5		$3(2m - 1) = 6m - 3$
7	$2m - 1 + 2m + 2 +$ $3m \geq 7m + 1$	5	5	5	3	For $m \leq 10$, $3(2m - 1) + m + \lfloor \frac{m-2}{2} \rfloor =$ $7m + \lfloor \frac{m-2}{2} \rfloor - 3$ For $m \geq 11$, $3(2m + 2) = 6m + 6$

Table 10: Case $G^2 \simeq P_m$, where $val(i, j, r) = val_e(k_i, P_m) + val_e(k_j, P_m) + val_e(k_r, P_m)$

$\{C_n, P_n\}$, $G^2 \in \{C_m, P_m\}$, $n \geq 8$, and $m \geq 4$. In fact, we will show that, for a given F , $\gamma_a(F)$ can be determined considering at most four sequences, the ones defined in the sequel.

$$f_{1,n} = \begin{cases} ([p]5) & , \text{ if } n \equiv 0 \pmod{5} \\ (r, [p]5) & , \text{ if } n \equiv r \pmod{5} \text{ for } r \in \{2, 3, 4\} \\ ([p]5, 6) & , \text{ if } n \equiv 6 \pmod{5} \end{cases}$$

$$f_{2,n} = \begin{cases} ([p]5, [q]6) \text{ for maximum } p & , \text{ if } n \neq 19 \\ (3, [2]5, 6) & , \text{ if } n = 19 \end{cases}$$

$$f_{3,n} = ([p]5, [q]6) \text{ for maximum } q$$

$$f_{4,n} = \begin{cases} ([q]6) & , \text{ if } n \equiv 0 \pmod{6} \\ (s, [q]6) & , \text{ if } n \equiv s \pmod{6} \text{ for } s \in \{3, 5\} \\ ([q]6, 7) & , \text{ if } n \equiv 1 \pmod{6} \\ (3, 5, [q]6) & , \text{ if } n \equiv 2 \pmod{6} \\ ([2]5, [q]6) & , \text{ if } n \equiv 4 \pmod{6} \end{cases}$$

For $i \in [4]$, $f_{i,n}$ is an infinite set of sequences, which is associated with at most one sequence if we fix the value of n . Therefore, when we can handle $f_{i,n}$ as a set.

Theorem 4.11 For $n \geq 8, m \geq 4, G^1 \in \{P_n, C_n\}$, and $G^2 \in \{P_m, C_m\}$, it holds $\gamma_a(G^1 \circ G^2) = \min\{val(G^1, G^2, f_{1,n}), val(G^1, G^2, f_{2,n}), val(G^1, G^2, f_{3,n}), val(G^1, G^2, f_{4,n})\}$.

Proof. Write $F = G^1 \circ G^2$. Corollary 2.4 and Propositions 4.8, 4.9, 4.10 imply that there is a sequence w such that $val(G^1, G^2, w) = \gamma_a(F)$ and w is a sequence of one of the following 8 sets of sequences for $G = C$ if $G^1 \simeq C_n$, $G = P$ if $G^1 \simeq P_n$, $p = \lfloor \frac{n}{5} \rfloor$, $q = \lfloor \frac{n}{6} \rfloor$, $r = n \pmod{5}$, and $s = n \pmod{6}$:

$$T_1 = \{([p]5)\},$$

$$T_2 = \{([p]5, r) \text{ for } r \in \{2, 3, 4\}\},$$

$$T_3 = \{([q]6)\},$$

$$T_4 = \{([q]6, s) \text{ for } s \in \{2, 3, 4, 5\}\},$$

$$T_5 = \{([q-1]6, 7)\},$$

$$T_6 = \{(3, 5, [q-1]6)\},$$

$$T_7 = \{(3, [p-1]5, 6)\},$$

$$T_8 = \{([p']5, [q']6), \text{ for all positive integers } p' \text{ and } q' \text{ such that } 5p' + 6q' = n\}.$$

We note that there are values of n such that some of these sets are empty. Therefore, we need to show that, if w belongs to some T_i for $i \in [8]$ and $val(G^1, G^2, w) = \gamma_a(F)$, then w appears in some $f_{i,n}$, for $i \in [4]$.

- The sequences of T_1, T_2, T_3, T_5 , and T_6 appear in $f_{1,n}, f_{1,n}, f_{4,n}, f_{4,n}$, and $f_{4,n}$, respectively, so there is nothing to do for these cases.
- The sequences of T_4 appear in $f_{4,n}$ for $s \in \{3, 5\}$. We will show: (i) for $s \in \{2, 4\}$, $val(G^1, G^2, ([q]6, s)) \leq val(G^1, G^2, w)$ for some w that appears in $f_{j,n}$ for some $j \in [4]$.
- The 19-sequence of T_7 appears in $f_{2,n}$. We will show: (ii) for $n \geq 22$, $val(G^1, G^2, (3, [p-1]5, 6)) \leq val(G^1, G^2, w)$ for some w that appears in $f_{j,n}$ for some $j \in [4]$.
- Only two sequences of T_8 are considered, one in $f_{2,n}$ and the other in $f_{3,n}$. We will show: (iii) only these two sequences of T_8 can reach the minimum.

Hence, to complete the proof it suffices to prove (i), (ii), and (iii).

(i) We show that $val(G^1, G^2, w) \leq val(G^1, G^2, w')$ for $w \in T_6$ and $w' \in T_4$ with $s = 2$. First, consider $G^2 \simeq C_m$. For $G^1 \simeq P_n$, suppose that $qx_{6,m} + x_{3,m} < (q-1)x_{6,m} + x_{5,m} + x_{3,m}$ for some n . We have $q(2m+4) + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} < (q-1)(2m+4) + \max\{m+4, 2m-1\} + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$. Then $2m+4 < \max\{m+4, 2m-1\}$, a contradiction. For $G^1 \simeq C_n$, suppose that $qx_{6,m} + x_{4,m} < (q-1)x_{6,m} + x_{5,m} + x_{4,m}$ for some n . We have $q(2m+4) < (q-1)(2m+4) + \max\{m+4, 2m-1\} \Rightarrow 2m+4 < \max\{m+4, 2m-1\}$, a contradiction.

Now, consider $G^2 \simeq P_m$. For $G^1 \simeq P_n$, suppose that $qy_{6,m} + y_{3,m} < (q-1)y_{6,m} + y_{5,m} + y_{3,m}$ for some n . We have $q(2m+2) + m + \lfloor \frac{m-2}{2} \rfloor < (q-1)(2m+2) + 2m - 1 + m + \lfloor \frac{m-2}{2} \rfloor$. Then $2m+2 < 2m-1$, a contradiction. For $G^1 \simeq C_n$, suppose that $qy_{6,m} + y_{4,m} < (q-1)y_{6,m} + y_{5,m} + y_{4,m}$ for some n . We have $q(2m+2) < (q-1)(2m+2) + 2m - 1 \Rightarrow 2m+2 < 2m-1$, a contradiction.

Next, we show that $val(G^1, G^2, ([2]5, [q-1]6)) \leq val(G^1, G^2, ([q]6, 4))$. Suppose that $qx_{6,m} + x_{4,m} < (q-1)x_{6,m} + 2x_{5,m}$ for some n . We have $q(2m+4) + 2m - 1 < (q-1)(2m+4) + 2(\max\{m+4, 2m-1\})$. Then $4m+3 < 2(\max\{m+4, 2m-1\})$, For $m=3$, $15 < 14$; $m=4$, $19 < 16$; $m=5$, $23 < 18$, a contradiction.

(ii) We show that $val(G^1, G^2, w_7) \geq \min\{val(G^1, G^2, w_2), val(G^1, G^2, w_8)\}$ where $w_7 = (3, [p-1]5, 6) \in T_7$, $w_2 = ([p]5, 4) \in T_2$, and $w_8 = ([p']5, [q']6) \in T_8$ for $n \geq 24$. Since the 3 sequences have a 24-subsequence, we do the analysis comparing the corresponding 24-subsequences w'_7 , w'_2 , and w'_8 . First consider $G^1 \simeq P_n$ and $G^2 \simeq C_m$. If $m \leq 5$, $4x_{5,m} + x_{4,m} = 6m + 15$ while $3x_{5,m} + x_{6,m} + x_{3,m} = 3(m+4) + 2m + 4 + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} = 6m + 16 + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$. Then for $m=4$, $val(P_n, C_m, w'_2) = 39 < val(P_n, C_m, w'_7) = 42$; and for $m=5$, $val(P_n, C_m, w'_2) = 45 \leq val(P_n, C_m, w'_7) = 48$. If $m \geq 6$, $3x_{5,m} + x_{6,m} + x_{3,m} = 6m - 3 + 2m + 4 + m + \lfloor \frac{m-2}{2} \rfloor = 9m + 1 + \lfloor \frac{m-2}{2} \rfloor$ and $4x_{6,m} = 4(2m+4) = 8m + 8$, which means that $val(P_n, C_m, w'_8) = 8m + 8 < 9m + 1 + \lfloor \frac{m-2}{2} \rfloor = val(P_n, C_m, w'_7)$ for $m \geq 6$. The result for $G^1 \simeq C_n$ and $G^2 \simeq C_m$ is consequence of the fact that $val(C_n, C_m, w_7) \geq val(P_n, C_m, w_7)$ is true due Corollary 4.4. It remains to consider $G^1 \in \{C_n, P_n\}$ and $G^2 \simeq P_m$. Now, it suffices to observe that $val(G^1, P_m, w'_2) = 10m - 5 < val(G^1, P_m, w'_7) = 10m - 2$ for every $m \geq 4$.

(iii) Suppose that the minimum one is achieved by $([p']5, [q']6)$ for $p' < p$ and $q' < q''$. (We consider p maximum and q'' maximum). This means that we can change either $([6]5)$ for $([5]6)$ or vice-versa obtaining a smaller GDA, a contradiction. \square

Theorems 4.7 and 4.11 lead to a constant-time algorithm for computing $\gamma_a(G^1 \circ G^2)$ for $G^1 \in \{C_n, P_n\}$, $G^2 \in \{C_m, P_m\}$, $n \geq 8$ and $m \geq 3$. It consists in computing at most four values and choosing the minimum one. In the next section, we show that functions $f_{k,n}$ have an homogeneous behavior, which allows one to characterize, for each pair $\{n, m\}$, which function gives the global defensive alliance number of $G^1 \circ G^2$.

5 Deepening the results

It is easy to verify that if $n \geq 8$ is such that $f_{i,n}$ and $f_{i+1,n}$ are defined, then there is an integer m_0 such that $val(G^1, G^2, f_{i,n}) \geq val(G^1, G^2, f_{i,n})$ for $G^2 \in \{C_m, P_m\}$ and $m \geq m_0$. The minimum m_0 with this property is the *threshold between $f_{i,n}$ and $f_{i+1,n}$* and will be denoted by $t_{n,i}$. If one of the functions is not defined or if $val(G^1, G^2, f_{i,n}) = val(G^1, G^2, f_{i,n})$ for every m that both functions are defined, we will say that $t_{n,i}$ is undefined.

Proposition 5.1 *If $t_{n,2}$ is defined for n , then $t_{n,2}^{CC} = t_{n,2}^{PC} = 13$ and $t_{n,2}^{CP} = t_{n,2}^{PP} = 8$.*

Proof. Let $w_2 = ([p]5, [q]6) \in f_{2,n}$ and $w_3 = ([p']5, [q']6) \in f_{3,n}$. If $val(G^1, G^2, w_2) \neq val(G^1, G^2, w_3)$, then $p > p'$. Furthermore, $p = 6k + p'$ and $q' = 5k + q$ for $k \geq 1$.

Since $val_i(k, G^2) = val_e(k, G^2)$ for $G^2 \in C_m$ and $k \in \{5, 6\}$, we have $val(G^1, G^2, w_2) = p \times val_i(5, G^2) + q \times val_i(6, G^2)$ and $val(G^1, G^2, w_3) = p' \times val_i(5, G^2) + q' \times val_i(6, G^2)$. Replacing, we have $val(G^1, G^2, w_2) = (6k + p')val_i(5, G^2) + (q' - 5k)val_i(6, G^2) = 6k \times val_i(5, G^2) - 5k \times val_i(6, G^2) + val(G^1, G^2, w_3)$.

For $G^2 \simeq C_m$, we have $val(G^1, C_m, w_2) = 6k(2m - 1) - 5k(2m + 4) + val(G^1, C_m, w_3) = k(2m - 26) + val(G^1, C_m, w_3)$, which meanst that $t_{n,2}^{PC} = t_{n,2}^{CC} = 13$.

For $G^2 \simeq P_m$, we have $val(G^1, P_m, w_2) = 6k(2m - 1) - 5k(2m + 2) + val(G^1, P_m, w_3) = k(2m - 16) + val(G^1, P_m, w_3)$, which meanst that $t_{n,2}^{PP} = t_{n,2}^{CP} = 8$. \square

Proposition 5.2 For every n and $m \geq 4$, $t_{n,1}$ is given in Table 11.

$n \pmod 5$	$t_{n,1}^{PC}$	$t_{n,1}^{CC}$	$t_{n,1}^{PP}$	$t_{n,1}^{CP}$
0	13	13	8	8
1	*	*	*	*
2	8	6	5	4
3	9	8	7	5
4, $n \neq 19$	11	11	7	7
$n = 19$	9	*	5	*

Table 11: $t_{n,1}$.

Proof. **Case 1** ($n \equiv 1 \pmod 5$, $f_{1,n} = f_{2,n}$)

Case 2 ($n \equiv 2 \pmod 5$, $f_{1,n} = (2, [p]5)$, and $f_{2,n} = ([p-2]5, [2]6)$)

For $P_n \circ C_m$, using Corollary 2.3, $val(P_n, C_m, f_{1,n}) = px_{5,m} + x_{3,m} \geq val(P_n, C_m, f_{2,n}) = (p-2)x_{5,m} + 2x_{6,m}$.

$$2(2m-1) + m + \lfloor \frac{m-2}{2} \rfloor \geq 2(2m+4)$$

$$5m-2 + \lfloor \frac{m-2}{2} \rfloor \geq 4m+8 \Rightarrow m + \lfloor \frac{m-2}{2} \rfloor \geq 10$$

which is true for $m \geq 8$.

For $C_n \circ C_m$, using Corollary 2.3, $val(C_n, C_m, f_{1,n}) = px_{5,m} + x_{4,m} \geq val(C_n, C_m, f_{2,n}) = (p-2)x_{5,m} + 2x_{6,m}$

$$p(2m-1) + 2m-1 \geq (p-2)(2m-1) + 2(2m+4)$$

$$3(2m-1) \geq 2(2m+4) \Rightarrow 6m-3 \geq 4m+8 \Rightarrow 2m \geq 11$$

that is true for $m \geq 6$.

For $P_n \circ P_m$, using Corollary 2.3, $val(P_n, P_m, f_{1,n}) = py_{5,m} + y_{3,m} \geq val(P_n, P_m, f_{2,n}) = (p-2)y_{5,m} + 2y_{6,m}$.

$$2(2m-1) + m + \lfloor \frac{m-2}{2} \rfloor \geq 2(2m+2)$$

$$m + \lfloor \frac{m-2}{2} \rfloor \geq 6$$

which is true for $m \geq 5$.

For $C_n \circ P_m$, using Corollary 2.3, $val(C_n, P_m, f_{1,n}) = py_{5,m} + y_{4,m} \geq val(C_n, P_m, f_{2,n}) = (p-2)y_{5,m} + 2y_{6,m}$.

$$2(2m-1) + 2m - 1 \geq 2(2m+2) \Rightarrow 2m \geq 7$$

which is true for $m \geq 4$.

Case 3 ($n \equiv 3 \pmod{5}$, $f_{1,n} = (3, [p]5)$, and $f_{2,n} = ([p-3]5, [3]6)$)

For $P_n \circ C_m$, $val(P_n, C_m, f_{1,n}) = px_{5,m} + x_{3,m} \geq val(P_n, C_m, f_{2,n}) = (p-3)x_{5,m} + 3x_{6,m}$.

$$3(\max\{m+4, 2m-1\}) + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\} \geq 3(2m+4)$$

which is true for $m \geq 9$.

For $C_n \circ C_m$, $val(C_n, C_m, f_{1,n}) = px_{5,m} + x_{4,m} \geq val(C_n, C_m, f_{2,n}) = (p-3)x_{5,m} + 3x_{6,m}$.

$$3(\max\{m+4, 2m-1\}) + 2m - 1 \geq 3(2m+4)$$

which is valid for $m \geq 8$.

For $P_n \circ P_m$, $val(P_n, P_m, f_{1,n}) = py_{5,m} + y_{3,m} \geq val(P_n, P_m, f_{2,n}) = (p-3)y_{5,m} + 3y_{6,m}$.

$$3(2m-1) + m + \lfloor \frac{m-2}{2} \rfloor \geq 3(2m+2)$$

$$m + \lfloor \frac{m-2}{2} \rfloor \geq 9$$

which is true for $m \geq 7$.

For $C_n \circ P_m$, $val(C_n, P_m, f_{1,n}) = py_{5,m} + y_{4,m} \geq val(C_n, P_m, f_{2,n}) = (p-3)y_{5,m} + 3y_{6,m}$.

$$3(2m-1) + 2m - 1 \geq 3(2m+2)$$

$$2m \geq 10$$

which is true for $m \geq 5$.

Case 4 ($n \equiv 4 \pmod{5}$, $n \neq 19$, $f_{1,n} = (4, [p]5)$, and $f_{2,n} = ([p-4]5, [4]6)$)

For $P_n \circ C_m$ and $C_n \circ C_m$, we have $4x_{5,m} + x_{4,m} \geq 4x_{6,m}$

$$4(\max\{m + 4, 2m - 1\}) + 2m - 1 \geq 4(2m + 4)$$

which is true for $m \geq 11$.

For $P_n \circ P_m$ and $C_n \circ P_m$, we have $4y_{5,m} + y_{4,m} \geq 4y_{6,m}$

$$4(2m - 1) + 2m - 1 \geq 4(2m + 2) \Rightarrow 2m \geq 13$$

which is true for $m \geq 7$.

Case 5 ($n \equiv 0 \pmod{5}$, $f_{1,n} = ([p]5)$, and $f_{2,n} = ([p - 6]5, [5]6)$)

For $P_n \circ C_m$ and $C_n \circ C_m$, we have

$$6x_{5,m} \geq 5x_{6,m} \Rightarrow 6(2m - 1) \geq 5(2m + 4) \Rightarrow 12m - 6 \geq 10m + 20 \Rightarrow 2m \geq 26$$

which is true for $m \geq 13$.

For $P_n \circ P_m$ and $C_n \circ P_m$, we have

$$6y_{5,m} \geq 5y_{6,m} \Rightarrow 6(2m - 1) \geq 5(2m + 2) \Rightarrow 12m - 6 \geq 10m + 10$$

which is true for $m \geq 8$.

Case 6 ($n = 19$, $f_{1,n} = (4, [3]5)$, and $f_{2,n} = (3, [2]5, 6)$)

For $P_n \circ C_m$,

$$3(\max\{m + 4, 2m - 1\}) + 2m - 1 \geq 2(2m - 1) + 2m + 4 + m + \lfloor \frac{m - 2}{2} \rfloor$$

$$8m - 4 \geq 7m + 2 + \lfloor \frac{m - 2}{2} \rfloor \Rightarrow m \geq 6 + \lfloor \frac{m - 2}{2} \rfloor$$

that is true for $m \geq 9$.

For $C_n \circ C_m$, $val(C_n, C_m, f_{1,n}) = 3x_{5,m} + x_{4,m} \geq val(C_n, C_m, f_{2,n}) = 2x_{5,m} + x_{6,m} + x_{4,m}$.

$$\max\{m + 4, 2m - 1\} \geq 2m + 4$$

since there is no m satisfying the above inequality, $t_{19,1}^{CP}$ is not defined.

For $P_n \circ P_m$,

$$3(2m - 1) + 2m - 1 \geq 2(2m - 1) + 2m + 2 + m + \lfloor \frac{m - 2}{2} \rfloor$$

$$8m - 4 \geq 7m + \lfloor \frac{m - 2}{2} \rfloor \Rightarrow m - \lfloor \frac{m - 2}{2} \rfloor \geq 4$$

that is true for $m \geq 5$.

For $C_n \circ P_m$,

$$3(2m - 1) + 2m - 1 \geq 2(2m - 1) + 2m + 2 + 2m - 1$$

$$8m - 4 \geq 8m - 1$$

since there is no m satisfying the above inequality, $t_{19,1}^{CC}$ is not defined. □

Proposition 5.3 For $n \geq 8$ and $m \geq 4$, $t_{n,3}$ is given in Table 12.

$n \bmod 6$	$t_{n,3}^{PC}$	$t_{n,3}^{CC}$	$t_{n,3}^{PP}$	$t_{n,3}^{CP}$
0	*	*	*	*
1	18	18	11	11
2	19	*	6	*
3	19	*	6	*
4	*	*	*	*
5	*	*	*	*

Table 12: $t_{n,3}$.

Proof. Case 1 ($n \equiv 1 \pmod{6}$, $f_{3,n} = ([5]5, [q-4]6)$, and $f_{4,n} = ([q-1]6, 7)$)

For $P_n \circ C_m$ and $C_n \circ C_m$, $5x_{5,m} \geq 3x_{6,m} + x_{7,m}$.

$$5(\max\{m+4, 2m-1\}) \geq 3(2m+4) + 3m+1$$

From, $10m-5 \geq 9m+13$, we have that $t_{n,3} = 18$.

For $P_n \circ P_m$ and $C_n \circ P_m$, we can write $5y_{5,m} \geq 3y_{6,m} + y_{7,m}$. Thus

$$5(2m-1) \geq 3(2m+2) + 3m$$

From, $10m-5 \geq 9m+6$, we have that $t_{n,3} = 11$.

Case 2 ($n \equiv 2 \pmod{6}$, $f_{3,n} = ([4]5, [q-3]6)$, and $f_{4,n} = (3, 5, [q-1]6)$)

For $P_n \circ C_m$, we have $4x_{5,m} \geq 2x_{6,m} + x_{5,m} + x_{3,m}$.

$3(\max\{m+4, 2m-1\}) \geq 2(2m+4) + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$. Thus

$$6m-3 \geq 5m+8 + \lfloor \frac{m-2}{2} \rfloor \Rightarrow m \geq 11 + \lfloor \frac{m-2}{2} \rfloor$$

is true for $m \geq 19$.

For $C_n \circ C_m$, we have $4x_{5,m} \geq 2x_{6,m} + x_{5,m} + x_{4,m}$.

Since there is no positive m satisfying $3(2m-1) \geq 2m-1 + 2m-1$, $t_{n,3}$ is undefined for this case.

For $P_n \circ P_m$, we have $4y_{5,m} \geq 2y_{6,m} + y_{5,m} + y_{3,m}$.

$3(2m-1) \geq 2(2m+2) + m + \lfloor \frac{m-2}{2} \rfloor$. Then

$$6m-3 \geq 5m+4 + \lfloor \frac{m-2}{2} \rfloor \Rightarrow m \geq 7 + \lfloor \frac{m-2}{2} \rfloor$$

is true for $m \geq 6$.

For $C_n \circ P_m$, we have $4y_{5,m} \geq 2y_{6,m} + y_{5,m} + y_{4,m}$.

$3(2m - 1) \geq 2(2m + 2) + 2m - 1$. Since $6m - 3 \geq 6m + 3$ is not true for any positive m , $t_{n,3}$ is undefined for this case.

Case 3 ($n \equiv 3 \pmod{6}$, $f_{3,n} = ([3]5, [q - 2]6)$, and $f_{4,n} = (3, [q]6)$)

For $P_n \circ C_m$, one has $3x_{5,m} \geq 2x_{6,m} + x_{3,m} \Rightarrow 3(\max\{m + 4, 2m - 1\}) \geq 2(2m + 4) + m + \max\{2, \lfloor \frac{m-2}{2} \rfloor\}$

$$6m - 3 \geq 4m + 8 + m + \lfloor \frac{m-2}{2} \rfloor \Rightarrow m \geq 11 + \lfloor \frac{m-2}{2} \rfloor$$

that is true for $m \geq 19$.

For $C_n \circ C_m$, one has $3x_{5,m} \geq 2x_{6,m} + x_4 \Rightarrow 3(\max\{m + 4, 2m - 1\}) \geq 2(2m + 4) + 2m - 1$.

From $6m - 3 \geq 6m + 7$, we conclude that $t_{n,3}$ is undefined for this case.

For $P_n \circ P_m$, one has $3y_{5,m} \geq 2y_{6,m} + y_{3,m} \Rightarrow 3(2m - 1) \geq 2(2m + 2) + m + \lfloor \frac{m-2}{2} \rfloor$.

$$6m - 3 \geq 5m + 4 + \lfloor \frac{m-2}{2} \rfloor \Rightarrow m + \lfloor \frac{m-2}{2} \rfloor \geq 7$$

which is true for $m \geq 6$.

For $C_n \circ P_m$, one has $3y_{5,m} \geq 2y_{6,m} + y_{4,m} \Rightarrow 3(2m - 1) \geq 2(2m + 2) + 2m - 1$.

Since $6m - 3 \geq 6m + 3$ is not true for any positive m , $t_{n,3}$ is undefined for this case.

Case 4 ($n \equiv 4 \pmod{6}$, $f_{3,n} = ([2]5, [q - 1]6)$, and $f_{4,n} = ([2]5, [q - 1]6)$)

Since $f_{n,3} = f_{n,4}$, $t_{n,3}$ is undefined for this case.

Case 5 ($n \equiv 5 \pmod{6}$, $f_{3,n} = (5, [q]6)$, and $f_{4,n} = (5, [q]6)$)

Since $f_{n,3} = f_{n,4}$, $t_{n,3}$ is undefined for this case.

Case 6 ($n \equiv 0 \pmod{6}$, $f_{3,n} = ([q]6)$, and $f_{4,n} = ([q]6)$).

Since $f_{n,3} = f_{n,4}$, $t_{n,3}$ is undefined for this case. □

Corollary 5.4 For $n \geq 8$, $m \geq 4$, $G^1 \in \{C_n, P_n\}$, and $G^2 \in \{C_m, P_m\}$, it holds

$$\gamma_a(G^1 \circ G^2) = \begin{cases} \text{val}(G^1, G^2, f_{1,n}) & , \text{ if } m < \min\{t_{n,1}, t_{n,2}\} \\ \text{val}(G^1, G^2, f_{2,n}) & , \text{ if } t_{n,1} \text{ is defined and } t_{n,1} \leq m \leq t_{n,2} \\ \text{val}(G^1, G^2, f_{3,n}) & , \text{ if } t_{n,3} \text{ is defined and } t_{n,2} \leq m < t_{n,3} \\ \text{val}(G^1, G^2, f_{4,n}) & , \text{ if } m \geq \max\{t_{n,3}, t_{n,2}\} \end{cases}$$

Proof. For $m \geq 4$, the result is consequence of Theorem 4.11 and Propositions 5.1 to 5.3. □

6 Conclusion

One can determine the global defensive alliance number of a graph $F = G^1 \circ G^2$ for $G^1 \in \{C_n, P_n\}$ and $G^2 \in \{C_m, P_m\}$ within a constant number of arithmetic operations.

For $n \leq 7$, the answer is obtained directly from Tables 1 to 4. For instance, $\gamma_a(P_5 \circ C_3) = 7$ due Proposition 4.1 and $\gamma_a(C_5 \circ P_3) = 5$ due Proposition 4.2.

For $n \geq 8$, consider as an example $P_{20} \circ C_{15}$. Since $t_{2,3}^{PC} = 13$ (Proposition 5.1) and $t_{2,3}^{PC} = 19$ (Proposition 5.3), Corollary 5.4, implies that $\gamma_a(P_{20} \circ C_{15}) = f_{3,20} = \text{val}(P_n, C_{15}, ([4]5)) = 4x_{5,15} = 116$. As another example, consider the graph $C_{20} \circ P_{15}$. Since $t_{2,3}^{CP} = 8$ (Proposition 5.1) and $t_{2,3}^{CP}$ is undefined (Proposition 5.3), Corollary 5.4, implies that $\gamma_a(C_{20} \circ P_{15}) = f_{4,20} = \text{val}(C_n, P_{15}, (3, 5, [2]6)) = y_{4,15} + y_{5,15} + 2y_{6,15} = 29 + 29 + 2 * 32 = 122$.

For concluding, we remark that the four examples presented in this section show that the only relation not contained in Corollary 4.4 indeed cannot be established because $\gamma_a(P_5 \circ C_3) = 7 > 5 = \gamma_a(C_5 \circ P_3)$ and $\gamma_a(P_{20} \circ C_{15}) = 116 < 122 = \gamma_a(C_{20} \circ P_{15})$.

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