Decomposing Claw-free Subcubic Graphs and 4-Chordal Subcubic Graphs

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Abstract

Hoffmann-Ostenhof's conjecture states that the edge set of every connected cubic graph can be decomposed into a spanning tree, a matching and a 2-regular subgraph. In this paper, we show that the conjecture holds for claw-free subcubic graphs and 4-chordal subcubic graphs.

Keywords: Subcubic graph, Hoffmann-Ostenhofs conjecture, Claw-free graph, 4-Chordal subcubic graph, Planar graph
2010 MSC: 05C70

1. Introduction

In this paper, all graphs are assumed to be finite without loops or multiple edges. Let G be a finite graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V(G)$, the degree of v in G and the maximum degree of G are denoted by $d_G(v)$ and $\triangle(G)$, respectively. Here $N_G(v)$ denotes the set of all neighbours of v. The complete graph of order n is denoted by K_n. The complete bipartite graph with partite sets of sizes m and n is denoted by $K_{m,n}$. A graph is called *cubic* if the degree of every vertex is 3 and it is called a *subcubic* graph if its maximum degree is at most 3. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. A cycle is called *chordless* if it has no

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chord. A graph G is called *chordal* if every cycle of G of length greater than 3 has a chord and a graph is 4-*chordal* if it has no induced cycle of length greater than 4. A *cut-edge* of a connected graph G is an edge $e \in E(G)$ such that $G \setminus e$ is disconnected. A subdivision of a graph G is a graph obtained from G by replacing some of the edges of G by internally vertex-disjoint paths. An edge decomposition of a graph G is called a 3-*decomposition*, if the edges of G can be decomposed into a spanning tree, a matching and a 2-regular subgraph (the matching or the 2-regular subgraph may be empty).

Hoffmann-Ostenhof proposed the following conjecture in his thesis [5], this conjecture also was appeared as a problem of BCC22 [4].

Hoffmann-Ostenhof's conjecture. Every connected cubic graph admits a 3-decomposition.

Hoffmann-Ostenhof's conjecture is known to be true for some families of cubic graphs. Kostochka [7] showed that the Petersen graph, the prism over cycles, and many other graphs have 3-decompositions. Bachstein [3] proved that every 3-connected cubic graph embedded in torus or Klein-bottle has a 3-decomposition. Furthermore, Ozeki and Ye [8] proved that 3-connected plane cubic graphs have 3-decompositions. Akbari et. al. [2] showed that Hamiltonian cubic graphs have 3-decompositions. Also, it has been proved that the traceable cubic graphs have 3-decompositions [1]. In 2017, Hoffmann-Ostenhof et. al. [6] proved that planar cubic graphs have 3-decompositions. In this paper it is shown that the connected claw-free subcubic graphs and the connected 4-chordal subcubic graphs have 3-decompositions.

2. Connected Claw-free Subcubic Graphs Have 3-Decompositions

In this section, we show that Hoffmann-Ostenhof's conjecture holds even for claw-free subcubic graphs.

Theorem 1. *If G is a connected claw-free subcubic graph, then G has a 3-decomposition.*

PROOF. We apply by induction on $n = |V(G)|$. If $|V(G)| \leq 3$, then the assertion is trivial. Now, we consider the following two cases:

Case 1. Assume that the graph G has a cut-edge e. Indeed we would like to prove that a connected subcubic graph G (not necessarily claw-free) with a cut-edge e has a 3-decomposition if and only if each component of $G \setminus e$ has a 3-decomposition. Let H and K be the connected components of $G \setminus e$. By induction hypothesis, both H and K have 3-decompositions. Let T_i, $i = 1, 2$, be the spanning trees in the 3-decompositions of H and K. Add e to $T_1 \cup T_2$ and consider this tree as the spanning tree in a 3-decomposition of G. Note that we take the union of cycles and matchings obtained in two components H and K as the 2-regular subgraph and the matching in the 3-decomposition of G. Now, if G has a 3-decomposition, then e is contained in the spanning tree T. Since $T \setminus e$ is union of two trees which one of them is a spanning tree of H and another is a spanning tree of K, so we are done.

Case 2. By Case 1 we may assume that G is 2-edge connected. If G is triangle-free, then since G is claw-free, we conclude that $\triangle(G) \leq 2$. Thus G is a cycle and hence in this case the assertion is trivial.

Now, let $xyzx$ be a triangle in G. If $d_G(x) = d_G(y) = 2$ and $d_G(z) = 3$, then G has a cut-edge, a contradiction.

Assume that $d_G(x) = d_G(y) = d_G(z) = 3$. If there is a vertex incident to all x, y and z, then $G = K_4$ which satisfied in the conjecture. Since G is 2-edge connected, $H = G \setminus \{x, y, z\}$ is connected. By induction hypothesis, H admits a 3-decomposition. Let e, f and g be the three edges with one end-point in $\{x, y, z\}$, and another end-point in $V(H)$. Add e, f and g to T_1 to obtain a spanning tree for G, where T_1 is the spanning tree in the 3-decomposition of H. Now, consider $xyzx$ as a cycle in the 3-decomposition of G, as desired.

Finally assume that $d_G(x) = 2$ and $d_G(y) = d_G(z) = 3$. Assume that y and z have a common neighbour, say $b \neq x$. Clearly, if $d_G(b) = 2$, then G has a 3-decomposition. Now, if $d_G(b) = 3$, then G has a cut-edge, a contradiction. Now, assume that $N_G(y) \cap N_G(z) = \{x\}$. Identify x, y and z. Call the new vertex a and denote the resulting graph by H'. Clearly, H' is a claw-free subcubic graph.

By induction hypothesis H' has a 3-decomposition. Let T_1 be the spanning tree in the 3-decomposition of H'. If $d_{T_1}(a) = 2$, then let T be the spanning tree of the 3-decomposition of G formed by $T_1 \cup \{xy, yz\}$. Also consider xz as an edge of the matching in the 3-decomposition of G. Finally, if $d_{T_1}(a) = 1$, then the edge incident with z is contained in T_1. Let T be a spanning tree of G formed by $T_1 \cup \{xy, yz\}$. Note that xz is contained in the matching in the 3-decomposition of G. The proof is complete. □

3. Connected 4-Chordal Subcubic Graphs Have 3-Decompositions

In this section we show that every connected 4-chordal subcubic graph has a 3-decomposition.

Theorem 2. *If G is a connected 4-chordal subcubic graph, then G has a 3-decomposition.*

Proof. We show that if G does not have a 3-decomposition, then G is planar and so by Corollary 13 of [6], G has a 3-decomposition, as desired. To the contrary, suppose that G is not planar. By Kuratowski's Theorem [9, p.310], G either contains a subdivision of K_5 or a subdivision of $K_{3,3}$. Since every vertex of K_5 has degree 4, G cannot contain a subdivision of K_5. Now, suppose that G contains a subdivision of $K_{3,3}$.

First, we introduce some notation which we need for the rest of the proof.

Let $K^*_{3,3}$ be a subgraph of G which is a subdivision of $K_{3,3}$. For every edge $e \in E(K_{3,3})$, let $L(e)$ be the set of all new added vertices on the edge e in G. For every $v \in V(K_{3,3})$, denote the set of all edges incident with v by $\{e_i(v) \mid 1 \leq i \leq 3\}$. If for every $e \in E(K_{3,3})$, $L(e) = \varnothing$, then $G = K_{3,3}$ and so G has a 3-decomposition, as desired. The rest of the proof is based on the following three claims:

Claim 1. *For every $e \in E(K_{3,3})$, $|L(e)| \geq 2$.*

Proof of Claim. By contradiction, if there exists an edge $e \in E(K_{3,3})$ such that $L(e) = \{p\}$ and e is incident with the vertex a in $K^*_{3,3}$, then let C be a

4-cycle of $K_{3,3}$ containing e. Let C^* be the subdivision of C in $K^*_{3,3}$. Assume that $\langle C^* \rangle$ denotes the induced subgraph of G on $V(C^*)$. If $\langle C^* \rangle$ is chordless, then G has an induced cycle of length at least 5, a contradiction. Now, let wx be a chord of $\langle C^* \rangle$ in G such that the cycle $a \dots wx \dots upa$ is chordless in G (note that there might exist some vertices between a and w and also between x and u), see Figure 1. So G contains an induced cycle of length at least 5, a contradiction. If $x = p$, then consider the path between p and w on C^* which contains a and remove all of its vertices except p and w. Similarly, one can see that the remaining graph has an induced cycle of length at least 5, a contradiction. So the claim is proved. □

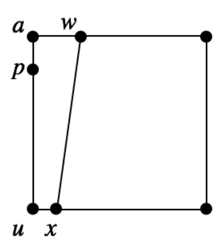

Figure 1: The subdivision of a 4-cycle in $K^*_{3,3}$

For every $v \in V(K_{3,3})$, by Claim 1, let $M(e_i(v))$ be the set of two vertices in $L(e_i(v))$ which have the smallest distances from v. In particular, for $a \in V(K_{3,3})$, let $M(e_1(a)) = \{p, q\}$, $M(e_2(a)) = \{r, s\}$ and $M(e_3(a)) = \{t, z\}$, see Figure 2.

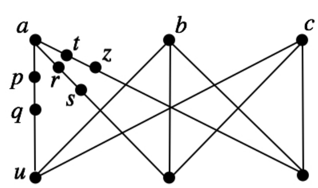

Figure 2: The subdivision of $K_{3,3}$ in G

Claim 2. *If there exists $v \in \{a, b, c\}$ (a, b, c are the vertices of one part of $K_{3,3}$)*

and $y \in M(e_i(v))$, for some i, $1 \leq i \leq 3$, such that $N_G(y) \cap M(e_j(v)) = \varnothing$ for every $j \in \{1,2,3\} \setminus \{i\}$, $1 \leq j \leq 3$, then G has a 3-decomposition.

Proof of Claim. Assume that $v = a$ and $y = p$. Since $d_G(q) \leq 3$, there exists j, $j \in \{2,3\}$, such that $N_G(q) \cap M(e_j(a)) = \varnothing$. Now, consider a 4-cycle in $K_{3,3}$ containing $e_1(a)$ and $e_j(a)$ and call it by C. Without loss of generality, let $j = 3$. Assume that C^* is the subdivision of C in $K^*_{3,3}$. If $\langle C^* \rangle$ is chordless, then by Claim 1, C^* is an induced cycle of length at least 12, a contradiction. Now, suppose that the cycle $\langle C^* \rangle$ has at least one chord. Let wx be that chord in $\langle C^* \rangle$ such that the cycle $atz \ldots wx \ldots qpa$ is chordless (note that $w \in \{t, z\}$ is possible). This implies that G has an induced cycle of length at least 5, see Figure 3, a contradiction. The same conclusion can be deduced for the case $v = a$ and $y = q$. □

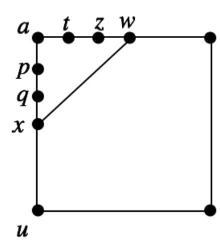

Figure 3: The subdivision of a 4-cycle in $K^*_{3,3}$

Claim 3. *If for every $v \in \{a, b, c\}$ (a, b, c are the vertices of one part of $K_{3,3}$) and any $y \in M(e_i(v))$, $i = 1, 2, 3$,*

$$N_G(y) \cap (\cup_{j \in \{1,2,3\} \setminus \{i\}} M(e_j(v)) \neq \varnothing,$$

then G has a 3-decomposition.

Proof of Claim. Without loss of generality assume that $v = a$ and $N_G(p) \cap M(e_2(a)) = \{r\}$. If $N_G(q) \cap M(e_3(a)) = \{z\}$, then G contains the induced 5-cycle $apqzta$, a contradiction. Now, assume that $N_G(q) \cap M(e_3(a)) = \{t\}$. Note that s and z are adjacent. In this case by considering the induced 5-cycle $arszta$, we obtain a contradiction to the fact that G is 4-chordal. □

Thus G is planar and by Corollary 13 of [6], G has a 3-decomposition. $\square$

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