On the anti-Ramsey number of forests

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Abstract

We call a subgraph of an edge-colored graph rainbow subgraph, if all of its edges have different colors. The anti-Ramsey number of a graph G in a complete graph K_n , denoted by $ar(K_n, G)$, is the maximum number of colors in an edge-coloring of K_n with no rainbow subgraph copy of G. In this paper, we determine the exact value of the anti-Ramsey number for star forests and the approximate value of the anti-Ramsey number for linear forests. Furthermore, we compute the exact value of $ar(K_n, 2P_4)$ for $n \geq 8$ and $ar(K_n, S_{p,q})$ for large n, where $S_{p,q}$ is the double star with p + q leaves.

Keywords: Anti-Ramsey number, star forest, linear forest, double star.

1. Introduction

Let G be a simple undirected graph. For $x \in V(G)$, we denote the neighborhood (the set of neighbors of x) and the degree of x in G by $N_G(x)$ and $d_G(x)$, respectively. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For $\emptyset \neq X \subset V(G)$, G[X] is the subgraph of G induced by X and G-X is the subgraph of G induced by $V(G)\setminus X$. If $X = \{x\}$, then G-X will be denoted by G-x for short. Given a graph G = (V, E), for any (not necessarily disjoint) vertex sets $A, B \subset V$, let $E_G(A, B) := \{uv \in E(G) | u \neq v, u \in A, v \in B\}$. A star forest is a forest whose components are stars and a linear forest is a forest whose components are paths. We use G and G to denote G to denote G to denote G and G to denote by G to denote by G to G the G

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graphs G_1 and G_2 , that is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2 .

We call a subgraph of an edge-colored graph rainbow, if all of its edges have different colors. Let G be a graph. The anti-Ramsey number $ar(K_n, G)$ is the maximum number of colors in an edge-coloring of K_n which has no rainbow copy of G. The Tur'an number ex(n, G) is the maximum number of edges of a simple graph on n vertices without a copy of G. The anti-Ramsey number was first studied by Erdős, Simonovits and Sós [6]. They showed that the anti-Ramsey number is closely related to Tur\'an number. Since then, there are plentiful results in this field, including cycles [1, 17], cliques [16, 19], trees [12, 13] and so on. See Fujita, Magnant and Ozeki [7, 8] for an abundant survey. Among these results, almost the considered graphs are connected graphs, and a few unconnected graphs are considered including matchings [4, 11], vertex-disjoint cliques [21]. In this paper, we will consider the cases that G is a star forest or a linear forest.

We mention some of the results, which are relevant to our work.

Jiang [12] and Montellano-Ballesteros [18] independently found the anti-Ramsey number for stars.

Theorem 1. ([12],[18]) For $n \ge p + 2 \ge 3$,

$$\left\lfloor \frac{(p-1)n}{2} \right\rfloor + \left\lfloor \frac{n}{n-p+1} \right\rfloor \le ar(K_n, K_{1,p+1}) \le \left\lfloor \frac{(p-1)n}{2} + \frac{1}{2} \left\lfloor \frac{2n}{n-p+1} \right\rfloor \right\rfloor.$$

By Theorem 1, $ar(K_n, K_{1,p}) = \lfloor \frac{(p-2)n}{2} \rfloor + 1$ for $n \geq 3p + 4$ and $p \geq 2$.

Simonovits and Sós [20] considered the anti-Ramsey number for paths and obtained the following result.

Theorem 2. ([20]) Let P_{k+1} be a path of length $k \geq 2$ and $k \equiv r \pmod{2}, 0 \leq r \leq 1$. For large enough $n \pmod{2}$ $n \geq \frac{5}{4}k + C$ for some universal constant C),

$$ar(K_n, P_{k+1}) = \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) n - \left(\left\lfloor \frac{k}{2} \right\rfloor \right) + 1 + r.$$

Let Ω_k denote the family of graphs that contain k vertex-disjoint cycles. Jin and Li [14] computed the anti-Ramsey number for Ω_2 .

Theorem 3. ([14]) For any $n \ge 6$, $ar(K_n, \Omega_2) = \max\{2n - 2, 11\}$.

The Turán number of tK_2 was determined by Erdős and Gallai [5] as $ex(n, tK_2) = \max\{\binom{2t-1}{2}, \binom{t-1}{2} + (t-1)(n-t+1)\}$ for $n \geq 2t \geq 2$. The anti-Ramsey number of matchings was first considered by Schiermeyer [19].

Theorem 4.([19]) For $n \ge 3t + 3 \ge 8$, we have

$$ar(K_n, tK_2) = ex(n, (t-1)K_2) + 1 = (t-2)n - {t-1 \choose 2} + 1.$$

Later, Chen, Li and Tu [4] and independently Fujita, Kaneko, Schiermeyer and Suzuki [7] showed that $ar(K_n, tK_2) = ex(n, (t-1)K_2) + 1$ for $n \ge 2t + 1 \ge 5$. The values $ar(K_{2t}, tK_2) = ex(2t, (t-1)K_2) + 1$ for $1 \le t \le 6$ and $1 \le 6$ and

Gilboa and Roditty [9] considered the graphs with small connected components and proved inductive results of the form "if $ar(K_n, G \cup t_0 P_s) \leq f(n, t_0, G)$ for sufficiently large n, then $ar(K_n, G \cup tP_s) \leq f(n, t, G)$ for sufficiently large n and $t \geq t_0$, where s = 2 or 3." These results imply the following theorem.

Theorem 5. ([9]) For sufficiently large n,

(1)
$$ar(K_n, P_3 \cup tP_2) = (t-1)(n-\frac{t}{2}) + 1 \text{ for } t \ge 2;$$

(2)
$$ar(K_n, P_4 \cup tP_2) = t(n - \frac{t+1}{2}) + 1 \text{ for } t \ge 1;$$

(3)
$$ar(K_n, C_3 \cup tP_2) = t(n - \frac{t+1}{2}) + 1 \text{ for } t \ge 1;$$

(4)
$$ar(K_n, tP_3) = (t-1)(n-\frac{t}{2}) + 1 \text{ for } t \ge 1;$$

(5)
$$ar(K_n, P_{k+1} \cup tP_3) = (t + \lfloor \frac{k}{2} \rfloor - 1)(n - \frac{t + \lfloor k/2 \rfloor}{2}) + 1 + (k \mod 2) \text{ for } k \geq 3 \text{ and } t \geq 0;$$

(6)
$$ar(K_n, P_2 \cup tP_3) = (t-1)(n-\frac{t}{2}) + 2 \text{ for } t \ge 1;$$

(7)
$$ar(K_n, kP_2 \cup tP_3) = (t + k - 2)(n - \frac{t+k-1}{2}) + 1$$
 for $k \ge 2$ and $t \ge 2$.

The Turán number of star forests and linear forests are considered by Lidický, Liu and Palmer [15].

Theorem 6.([15]) Let $F = \bigcup_{i=1}^{t} K_{1,p_i}$ be a star forest and $p_1 \geq p_2 \geq \cdots \geq p_t \geq 1$. For a sufficiently large,

$$ex(n,F) = \max_{1 \le i \le t} \left\{ (i-1)n - {i \choose 2} + \left\lfloor \frac{p_i - 1}{2}(n-i+1) \right\rfloor \right\}.$$

Theorem 7.([15]) Let $F = \bigcup_{i=1}^k P_{p_i}$ be a linear forest, where $k \geq 2$ and $p_i \geq 2$ for $1 \leq i \leq k$. If at least one p_i is not 3, then for n sufficiently large,

$$ex(n,F) = \left(\sum_{i=1}^{k} \left\lfloor \frac{p_i}{2} \right\rfloor - 1\right) n - \left(\sum_{i=1}^{k} \left\lfloor \frac{p_i}{2} \right\rfloor\right) + c,$$

where c = 1 if all p_i are odd and c = 0 otherwise.

In Sections 2 and 3, we generalize Theorem 5 by considering the anti-Ramsey number of star forests and linear forests, respectively.

Theorem 8. Let $F = \bigcup_{i=1}^{t} K_{1,p_i}$ be a star forest, where $p_1 \geq 3$, $p_1 \geq p_2 \geq \cdots \geq p_t \geq 1$. Let $s = \max\{i : p_i \geq 2, 1 \leq i \leq t\}$. For $n \geq 3t^2(p_1 + 1)^2$, we have

$$ar(K_n, F) = \max \left\{ \max_{1 \le i \le s} \left\{ (i-1)n - {i \choose 2} + \left\lfloor \frac{p_i - 2}{2}(n-i+1) \right\rfloor + 1 \right\}, (t-2)n - {t-1 \choose 2} + r \right\},$$

where r = 1 if $p_{t-1} = 1$ and r = 2 otherwise.

Theorem 9. Let $F = \bigcup_{i=1}^k P_{p_i}$ be a linear forest, where $k \geq 2$ and $p_i \geq 2$ for $1 \leq i \leq k$. We have

$$ar(K_n, F) = \left(\sum_{i=1}^k \left\lfloor \frac{p_i}{2} \right\rfloor - \epsilon \right) n + O(1),$$

where $\epsilon = 1$ if all p_i are odd and $\epsilon = 2$ otherwise.

We get the approximate value of the anti-Ramsey number for linear forests by Theorem 9 and it would be interesting to determine the exact value. Bialostocki, Gilboa and Roditty [2] and independently Gorgol and Görlich [10] showed that $ar(K_n, 2P_3) = \max\{n, 7\}$ for $n \geq 6$. Gorgol and Görlich showed that $ar(K_n, 3P_3) = 2n - 2$ for $n \geq 13$. In Section 4, we will use Theorem 3 to compute the exact value of $ar(K_n, 2P_4)$ for $n \geq 8$.

Theorem 10. For any $n \ge 8$, $ar(K_n, 2P_4) = \max\{2n - 2, 16\}$.

Another motivation of this paper is the following conjecture of Gorgol and Görlich [10]:

Let G be a connected graph on $n_0 \geq 3$ vertices and $t \geq 1$, then for large n,

$$ar(K_n, tG) = (t-1)n - {t \choose 2} + ar(K_{n-t+1}, G)$$

if and only if G is a tree.

Statement (4) in Theorem 5(4) and Theorem 8 show that this conjecture is true for P_3 and $K_{1,p}$ ($p \ge 3$), respectively. However, from Theorem 9, some simple calculation shows that this conjecture fails for P_l , $l \ge 4$.

Actually, for an arbitrary tree T_k with k edges, it is difficult to determine the (approximate) value of $ar(K_n, T_k)$. Jiang and West [13] showed that for $n \geq 2k$,

$$\left| \frac{k-2}{2} \right| \frac{n}{2} + O(1) \le ar(K_n, T_k) \le (k-1)n.$$

The upper bound comes from the well-known bound of $ex(n, T_k) \leq (k-1)n$. Erdős and Sós gave the following conjecture.

Conjecture 1.

$$ex(n,T_k) \le \frac{k-1}{2}n.$$

If Conjecture 1 is true (Ajtai, Komlós, Simonovits, Szemerédi announced it for large k), then the upper bound of $ar(K_n, T_k)$ can also be reduced to $\frac{k-1}{2}n$. Also, Jiang and West [13] conjectured that:

Conjecture 2.

$$ar(n, T_k) \le \frac{k-2}{2}n + O(1).$$

Notice that if T_k is a star or a path of even length, then $ar(n, T_k) = \frac{k-2}{2}n + O(1)$.

The double star $S_{p,q}$, where $p \geq q \geq 1$, is the graph consisting of the union of two stars $K_{1,p}$ and $K_{1,q}$ together with an edge joining their centers. In Section 5, we compute the anti-Ramsey number of double stars.

Theorem 11. For $p \ge 2, 1 \le q \le p$ and $n \ge 6(p^2 + 2p)$, we have

$$ar(K_n, S_{p,q}) = \begin{cases} \lfloor \frac{(p-1)n}{2} \rfloor + 1, \ 1 \le q \le p-1; \\ \lfloor \frac{p(n-1)}{2} \rfloor + 1, \ q = p. \end{cases}$$

Notice that if we take $T_k = S_{p,p-1}$, then we have $ar(K_n, T_k) = \lfloor \frac{k-2}{2} \rfloor \frac{n}{2} + O(1)$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 8. The proof of Theorem 9 will be given in Section 3. The proof of Theorem 10 will be given in Section 4. The proof of Theorem 11 will be given in Section 5. Finally we will give a conjecture in Section 6.

Notation: Given an edge-coloring c of G, we denote the color of an edge uv by c(uv). We denote the number of colors by |c|. For any $v \in V(G)$, let $C(v) := \{c(vw) | w \in N_G(v)\}$ and $d_c(v) := |C(v)|$. Let H be a subgraph of G. Denote $C(H) = \{c(uv) | uv \in E(H)\}$. A color a is stared (at x) if all the edges with color a induce a star $K_{1,r}$ (centered at the vertex x). We let $d^c(v) = |\{a \in C(v) | a \text{ is stared at } v\}|$. A representing subgraph in an edge-coloring of K_n is a spanning subgraph containing exactly one edge of each color. In the rest of this paper, we will use V to denote the vertex set of K_n for short.

2. Star forests

In this Section, we use the idea of [9] to prove Theorem 8.

Theorem 8. Let $F = \bigcup_{i=1}^{t} K_{1,p_i}$ be a star forest, where $p_1 \geq 3, p_1 \geq p_2 \geq \cdots \geq p_t \geq 1$. Let $s = \max\{i : p_i \geq 2, 1 \leq i \leq t\}$. For $n \geq 3t^2(p_1 + 1)^2$,

$$ar(K_n, F) = \max \left\{ \max_{1 \le i \le s} \left\{ (i-1)n - \binom{i}{2} + \left| \frac{p_i - 2}{2}(n-i+1) \right| + 1 \right\}, (t-2)n - \binom{t-1}{2} + r \right\},$$

where r = 1 if $p_{t-1} = 1$ and r = 2 otherwise.

Proof. For $1 \leq i \leq s$, we color K_n as follows. We color $K_{i-1} + \overline{K}_{n-i+1}$ rainbow and color K_{n-i+1} with new $ar(K_{n-i+1}, K_{1,p_i})$ colors without producing a rainbow copy of K_{1,p_i} . In such way, we use exactly $(i-1)(n-i+1) + \binom{i-1}{2} + ar(K_{n-i+1}, K_{1,p_i}) = (i-1)n - \binom{i}{2} + \lfloor \frac{p_i-2}{2}(n-i+1) \rfloor + 1$ colors. Since any rainbow K_{1,p_j} $(1 \leq j \leq i)$ contains at least one vertex of $V(K_{i-1})$, we do not obtain any rainbow $\bigcup_{j=1}^{i} K_{1,p_j}$. Also, we do not obtain any rainbow F.

For another lower bound, it is enough to consider the case $t \geq 3$. If $p_{t-1} = 1$, then $F \supset tK_2$. By Theorem 4, we have

$$ar(K_n, F) \ge ar(K_n, tK_2) = ex(n, (t-1)K_2) + 1 = (t-2)n - {t-1 \choose 2} + 1.$$

If $p_{t-1} \geq 2$, we have $F \supset (t-1)K_{1,2} \cup P_2$. By Theorem 5(6), we have

$$ar(K_n, F) \ge ar(K_n, (t-1)K_{1,2} \cup K_2) = (t-2)n - {t-1 \choose 2} + 2.$$

Now we consider the upper bound. If t = 1, then the result holds obviously by Theorem 1. Suppose $t \ge 2$. Let

$$f(n,F) = \max \left\{ \max_{1 \le i \le s} \left\{ (i-1)n - \binom{i}{2} + \left\lfloor \frac{p_i - 2}{2}(n-i+1) \right\rfloor + 1 \right\}, (t-2)n - \binom{t-1}{2} + r \right\}.$$

By Theorems 4 and 5 (statements (1), (4), (6), (7)), we have

$$ar(K_n, kK_{1,2} \cup lK_{1,1}) = \begin{cases} (k+l-2)n - {k+l-1 \choose 2} + 1, & k \ge 0, l \ge 2; \\ (k-1)n - {k \choose 2} + 2, & k \ge 1, l = 1; \\ (k-1)n - {k \choose 2} + 1, & k \ge 1, l = 0. \end{cases}$$
(*)

Let c be any edge-coloring of K_n using f(n,F) + 1 colors. We will find a rainbow F by considering the following two cases.

Case 1. There is some vertex $v_0 \in V$ such that $d_c(v_0) \geq \sum_{i=1}^t p_i + t$.

Choose some color c_0 . Consider an edge-coloring c' of $K_n - v_0$: c'(e) = c(e) if $c(e) \notin C(v_0)$; else $c'(e) = c_0$, where $e \in E(K_n - v_0)$.

Claim 1. There is a rainbow copy of $F - K_{1,p_1}$ in $K_n - v_0$ with respect to c'.

Proof of Claim 1. We consider the following two subcases.

Subcase 1.1 $p_2 \le 2$.

In this subcase, we have $F - K_{1,p_1} = (s-1)K_{1,2} \cup (t-s)K_{1,1}$ and

$$f(n,F) = \max \left\{ \left\lfloor \frac{p_1 - 2}{2} n \right\rfloor + 1, (s - 1)n - \binom{s}{2} + 1, (t - 2)n - \binom{t - 1}{2} + r \right\}.$$

If $F - K_{1,p_1} = K_{1,1}$, then the result holds obviously. Suppose $F - K_{1,p_1} \neq K_{1,1}$. Then

$$|c'| \ge f(n,F) + 1 - (n-1)$$

$$\ge \max\left\{ (s-1)n - \binom{s}{2} + 1, (t-2)n - \binom{t-1}{2} + r \right\} + 1 - (n-1)$$

$$= \max\left\{ (s-2)(n-1) - \binom{s-1}{2} + 1, (t-3)(n-1) - \binom{t-2}{2} + r \right\} + 1$$

$$\ge ar(K_{n-1}, F - K_{1,p_1}) + 1,$$

by (*). Thus there is a rainbow $F - K_{1,p_1}$ in $K_n - v_0$ with respect to c'.

Subcase 1.2 $p_2 \ge 3$.

In this subcase, we have

$$|c'| \ge f(n,F) + 1 - (n-1) \ge f(n-1,F-K_{1,p_1}) + 1 = ar(K_{n-1},F-K_{1,p_1}) + 1.$$

By induction hypothesis, there is a rainbow $F - K_{1,p_1}$ in $K_n - v_0$ with respect to c'.

By Claim 1, there is a rainbow $F - K_{1,p_1}$ in $K_n - v_0$ with respect to c' (also respect to c). Since $d_c(v_0) \geq \sum_{i=1}^t p_i + t$, we are surely left with at least p_1 edges, say $v_0 w_1, v_0 w_2, \ldots, v_0 w_{p_1}$, such that $w_1, w_2, \ldots, w_{p_1} \notin V(F - K_{1,p_1})$ and $c(v_0 w_1), c(v_0 w_2), \ldots, c(v_0 w_{p_1}) \notin \{c(e) | e \in E(F - K_{1,p_1})\}$. By adding such p_1 edges to $F - K_{1,p_1}$, we get a rainbow F.

Case 2. $d_c(v) \leq \sum_{i=1}^t p_i + t - 1$ for all $v \in V$.

By induction hypothesis, K_n clearly contains a rainbow $F - K_{1,p_t}$. Assume, by contradiction, that K_n does not contain a rainbow F. Let G be a representing subgraph of K_n such that $E(F - K_{1,p_t}) \subseteq E(G)$. Then we have |E(G)| = f(n,F) + 1 and $d_G(v) \le \sum_{i=1}^t p_i + t - 1$ for all $v \in V$.

Let $W = V - V(F - K_{1,p_t})$. Since $d_G(v) \leq \sum_{i=1}^t p_i + t - 1$ for all $v \in V(F - K_{1,p_t})$ and G[W] does not contain a copy of K_{1,p_t} , we have

$$|E(G)| \leq |E_G(V(F - K_{1,p_t}), V)| + |E(G[W])|$$

$$\leq \left(\sum_{i=1}^{t-1} p_i + t - 1\right) \left(\sum_{i=1}^{t} p_i + t - 1\right) + \frac{(p_t - 1)[n - (\sum_{i=1}^{t-1} p_i + t - 1)]}{2}$$

$$\leq \left(\sum_{i=1}^{t} p_i + t - 1\right)^2 + \frac{p_t - 1}{2}n.$$

We will finish the proof by considering the following two subcases.

Subcase 2.1 $p_t = 1$.

Since $p_1 \geq 3$ and $p_t = 1$, we have

$$\left(\sum_{i=1}^{t} p_i + t - 1\right)^2 \ge |E(G)| = f(n, F) + 1 \ge \left\lfloor \frac{p_1 - 2}{2} n \right\rfloor + 2 \ge \frac{n}{2} + 1,$$

a contradiction with $n \ge 3t^2(p_1+1)^2$.

Subcase 2.2 $p_t \ge 2$.

In this subcase, we have s = t and

$$\left(\sum_{i=1}^{t} p_i + t - 1\right)^2 + \frac{p_t - 1}{2}n \ge |E(G)| = f(n, F) + 1$$

$$\ge (t - 1)n - {t \choose 2} + \left\lfloor \frac{p_t - 2}{2}(n - t + 1) \right\rfloor + 2$$

$$\ge \frac{(2t + p_t - 4)n}{2} - \frac{(t - 1)(p_t + t - 2)}{2}.$$

Thus we have $n \leq \frac{2}{2t-3}[(\sum_{i=1}^t p_i + t - 1)^2 + (t-1)(p_t + t - 2)]$, a contradiction with $n \geq 3t^2(p_1 + 1)^2$.

3. Linear forests

First, we have the lower bound of the anti-Ramsey number for linear forests.

Proposition 1. Let F be a linear forest with components of order p_1, p_2, \ldots, p_k , where $k \geq 2$ and $p_i \geq 2$ for $1 \leq i \leq k$. Let $n \geq \sum_{i=1}^k p_i$. Then we have

$$ar(K_n, F) \ge \max \left\{ \binom{\sum_{i=1}^k p_i - 2}{2} + 1, sn - \binom{s+1}{2} + r \right\},$$

where $s = \sum_{i=1}^{k} \lfloor \frac{p_i}{2} \rfloor - \epsilon$; $\epsilon = 1$ if all p_i are odd and $\epsilon = 2$ otherwise; r = 2 if exactly one p_i is even and r = 1 otherwise.

Proof. For the first lower bound, we choose a subgraph $K_{\sum_{i=1}^k p_i - 2}$ and color it rainbow. Then we use one extra color to color the remaining edges. In this way, we use exactly $\binom{\sum_{i=1}^k p_i - 2}{2} + 1$ colors and do not obtain a rainbow F.

For the second lower bound, we color $K_s + \overline{K}_{n-s}$ rainbow and color the edges of K_{n-s} with r new colors. Every copy of F in K_n have at least (r+1) edges in K_{n-s} . In this way we do not obtain a rainbow F and use exactly $s(n-s) + {s \choose 2} + r = sn - {s+1 \choose 2} + r$ colors.

If all the components of the linear forest are even paths or odd paths, we can get the following corollary from Theorems 2 and 7.

Corollary 1. Let F be a linear forest with components of order p_1, p_2, \ldots, p_k , where $k \geq 2$ and $p_i \geq 2$ for $1 \leq i \leq k$ and n sufficiently large. Let $s = \sum_{i=1}^k \lfloor \frac{p_i}{2} \rfloor - 2$. If all p_i are even, we have

$$sn - {s+1 \choose 2} + 1 \le ar(K_n, F) \le sn - {s+1 \choose 2} + 2.$$

If all p_i are odd, we have

$$ar(K_n, F) = (s+1)n - {s+2 \choose 2} + 1.$$

Proof. The lower bound is due to Proposition 1.

For the upper bound, when all p_i are even, by Theorem 2,

$$ar(K_n, F) \le ar(K_n, P_{\sum_{i=1}^k p_i}) = sn - \binom{s+1}{2} + 2.$$

When all p_i are odd, if $p_1 = p_2 = \cdots = p_k = 3$, by Theorem 5(4), $ar(K_n, F) = ar(K_n, kP_3) = (k-1)n - \binom{k}{2} + 1$. If at least one p_i is not 3, by Theorem 7,

$$ar(K_n, F) \le ex(n, F) = (s+1)n - \binom{s+2}{2} + 1.$$

It is enough to consider the linear forests with at least one even path.

Theorem 12. Let F be a linear forest with components of order p_1, p_2, \ldots, p_k , where $k \geq 1$, $p_i \geq 2$ for $1 \leq i \leq k$ and at least one p_i is even. Then

$$ar(K_n, F) = \left(\sum_{i=1}^k \lfloor \frac{p_i}{2} \rfloor - 2\right) n + O(1).$$

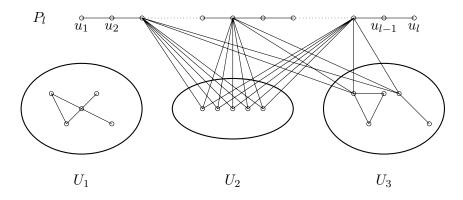
Proof. By Proposition 1, we just need to show the upper bound. We will use the idea of [20] to prove it. The following results of Erdős and Gallai [5] will be used in our proof.

(a)
$$ex(n, P_r) \le \frac{r-2}{2}n;$$

(b)
$$ex(n, \{C_{r+1}, C_{r+2}, \dots\}) \le \frac{r(n-1)}{2}$$
.

Since we can regard the union of even paths as the subgraph of one long even path (see Corollary 1), we just need to prove the upper bound is correct for linear forest with exact one even path and some odd paths. So we assume that $F = P_{2s} \cup P_{2t_1+1} \cup P_{2t_2+1} \cup \cdots \cup P_{2t_k+1}$, where $s \ge 1, k \ge 1$ and $t_k \ge t_{k-1} \ge \cdots \ge t_1 \ge 1$. Let $s + t_1 + t_2 + \cdots + t_k = m$. In this proof, we just consider the case $s \ge 6$. The case s < 6 can be proved by the similar arguments but need to distinguish more cases as in [20].

Consider an edge-coloring of K_n with $ar(K_n, F)$ colors such that there is no rainbow F. First we take a rainbow path $P_l = u_1 u_2 \dots u_l$ with maximum length. If $l \geq 2m + k$, we can get a rainbow F, a contradiction. By Proposition 1 and Theorem 2, $ar(K_n, F) \geq ar(K_n, P_{2m-1}) + 1$ for large n, which implies there is a rainbow P_{2m-1} . Hence we assume that $2m-1 \leq l \leq 2m+k-1$. Take a representing subgraph G of K_n such that $P_l \subset G$. Then $|E(G)| = ar(K_n, F)$. We would partition $V \setminus V(P_l)$ into three sets U_1, U_2 and U_3 as follows:



 U_1 is the subset of vertices of $V \setminus V(P_l)$ which are not jointed to P_l at all: neither by edges nor by paths;

 U_2 is the set of isolated vertices of $V \setminus V(P_l)$ which are jointed to P_l by edges;

$$U_3 = V \setminus (V(P_l) \cup U_1 \cup U_2).$$

Claim 1. $|E(G[U_1])| \leq (m-2)|U_1|$.

Proof of Claim 1 We first prove that there is an $P_{2s} \cup P_{2t_1+1} \cup ... \cup P_{2t_k+1}$ in $P_{2s+2t_1+...+2t_k-1} \cup P_{2s+2t_1+...+2t_k-2}$ by induction on k. The base case k=1 is correct since $P_{2s} \cup P_{2t_1+1} \subset P_{2s+2t-1} \cup P_{2s+2t_1-2}$. Suppose the statement holds for k-1. We divide $P_{2s+2t_1+...+2t_k-1}$ and $P_{2s+2t_1+...+2t_k-2}$ respectively into two parts $P_{2s+2t_1+...+2t_{k-1}-1}$, P_{2t_k} and $P_{2s+2t_1+...+2t_{k-1}-2}$, P_{2t_k} . We can find an $P_{2s} \cup P_{2t_1+1} \cup ... \cup P_{2t_{k-1}+1}$ in $P_{2s+2t_1+...+2t_{k-1}-1} \cup P_{2s+2t_1+...+2t_{k-1}-2}$ by induction hypothesis. Since $(2s+2t_1+...+2t_{k-1}-1)+(2s+2t_1+...+2t_{k-1}-1)+(2s+2t_1+...+2t_{k-1}-1)$ or $P_{2s+2t_1+...+2t_{k-1}-2}$ which is not used in $P_{2s} \cup P_{2t_1+1} \cup ... \cup P_{2t_{k-1}+1}$. Hence, we can find an $P_{2s} \cup P_{2t_1+1} \cup ... \cup P_{2t_{k+1}}$ in the original two long paths.

Since G contains no F, $G[U_1]$ contains no P_{2m-2} by the statement above. Thus $|E(G[U_1])| \le (m-2)|U_1|$ by (a).

Claim 2.
$$|\{v \in U_2 : d_G(v) \ge m-1\}| \le (m+k) {l-2 \choose m-1}.$$

Proof of Claim 2 It is obvious that $N_G(v) \subset V(P_l) \setminus \{u_1, u_l\}$ for all $v \in U_2$. Suppose $|\{v \in U_2 : d_G(v) \geq m-1\}| \geq (m+k) \binom{l-2}{m-1} + 1$. Note that there are $\binom{l-2}{m-1}$ subsets of order m-1 in $V(P_l) \setminus \{u_1, u_l\}$. By Pigeonhole Principle, there are $A \subseteq V(P_l) \setminus \{u_1, u_l\}$ with |A| = m-1 and $B \subseteq U_2$ with |B| = m+k+1 such that $A \subseteq N_G(v)$ for any $v \in B$. So there is a rainbow $K_{m-1,m+k+1}$. By adding one edge whose endpoints are in the large part of $K_{m-1,m+k+1}$ to $K_{m-1,m+k+1}$, we can find a rainbow F, a contradiction.

The following claim 3 is Lemma 1 in [20]. We include the proof for the sake of completeness.

Claim 3.
$$|E(G[U_3])| + |E_G(P_l, U_3)| \le (m-2)|U_3|$$
.

Proof of Claim 3 Take a component H of $G[U_3]$ and denote by r the length of the longest cycle of H. If H is a tree, let r=2. By (b), we have $|E(H)| \leq \frac{r(|V(H)|-1)}{2}$. For any $v \in V(H)$, we can find an P_r in H starting from it. Hence, u_1, \ldots, u_r and u_{l-r+1}, \ldots, u_l cannot be joined to v. Otherwise, there is a rainbow P_{l+1} , a contradiction. For any three consecutive vertices $\{u_i, u_{i+1}, u_{i+2}\}$, there is no two independent edges in $E_G(\{u_i, u_{i+1}, u_{i+2}\}, V(H))$ by the maximality of P_l . Hence, we have

$$|E(H)| + |E_G(P_l, H)| \le \frac{r(|V(H)| - 1)}{2} + \frac{l - 2r + 2}{3}|V(H)| \le \frac{l + 1}{3}|V(H)|.$$

Adding all the components of $G[U_3]$ up, we get $|E(G[U_3])| + |E_G(P_l, U_3)| \le \frac{l+1}{3}|U_3|$. Note that $l \le 2m + k - 1$. Since $s \ge 6$, $m \ge s + k \ge 6 + k$ which implies $|E(G[U_3])| + |E_G(P_l, U_3)| \le (m-2)|U_3|$

By Claims 1, 2 and 3, we have

$$ar(K_n, F) = |E(G)| = |E(G[P_l])| + |E(G[U_1])| + |E_G(U_2, P_l)| + |E(G[U_3])| + |E_G(U_3, P_l)|$$

$$\leq {l \choose 2} + (m-2)|U_1| + (l-2)(m+k){l-2 \choose m-1} + (m-2)|U_2| + (m-2)|U_3|$$

$$\leq {l \choose 2} + (l-2)(m+k){l-2 \choose m-1} + (m-2)(n-l)$$

$$= (m-2)n + O(1).$$

By Corollary 1 and Theorem 12, we can get Theorem 9.

4. The exact value of $ar(K_n, 2P_4)$

In this Section, we will prove Theorem 10. We denote the complete graph on n vertices minus one edge by K_n^- . The following fact is trivial.

Fact 1. Let $n \geq 8$. If there is an edge coloring of K_n using 17 colors such that there is a rainbow K_6 or K_6^- , then there is a rainbow $2P_4$.

We first prove the following lemma.

Lemma 1. $ar(K_8, 2P_4) = 16$.

Proof. By Proposition 1 in Section 3, we just need to show the upper bound. Consider an 17-edge-coloring c of K_8 . Suppose there is no rainbow $2P_4$ in K_8 . By Theorem 3, there must be a rainbow $C_k \cup C_l$. Assume that $k \leq l$. Then k = 3. Let $T_1 = C_3 = x_1x_2x_3x_1$ and $T_2 = C_l = y_1 \dots y_l y_l$. We choose a representing subgraph G such that $G \supset T_1 \cup T_2$. We just need to consider the following three cases.

Case 1. l = 5.

We claim that $c(xy) \in C(T_1 \cup T_2)$ for all $x \in V(T_1)$ and $y \in V(T_2)$. Otherwise, say $c(x_1y_1) \notin C(T_1 \cup T_2)$, then $x_2x_3x_1y_1 \cup y_2y_3y_4y_5$ is a rainbow $2P_4$, a contradiction. Then the total number of colors is at most $3 + \binom{5}{2} = 13 < 17$, a contradiction.

Case 2. l = 4.

Let the remaining vertex be z. We have $c(x_1z)=c(x_2x_3)$; otherwise there must be a rainbow $2P_4$ in $T_1 \cup T_2 \cup \{x_1z\}$. Similarly, we have $c(x_2z)=c(x_1x_3)$ and $c(x_3z)=c(x_1x_2)$. Now we have four rainbow C_3 's: $C_3^1=x_1x_2x_3x_1$, $C_3^2=zx_2x_3z$, $C_3^3=zx_1x_2z$ and $C_3^4=zx_1x_3x_1z$. If there are $u \in \{z, x_1, x_2, x_3\}$, say u=z, and $1 \le j \le 4$, say j=1, such that $c(zy_1) \ne c(zy_2)$ and $c(zy_1), c(zy_2) \notin C(C_3^1 \cup T_2)$ then we have a rainbow C_3^1 and a rainbow $C_5=y_1zy_2y_3y_4y_1$ and the situation is the same as Case 1. Hence we have $|E_G(u,T_2)| \le 2$ for any $u \in \{z, x_1, x_2, x_3\}$. Then $17 = |E(G)| \le 3 + 4 \times 2 + {4 \choose 2} = 17$, which means $G[V(T_2)] = K_4$ and $|E_G(u,T_2)| = 2$ for any $u \in \{z, x_1, x_2, x_3\}$. Thus we can find a rainbow $2P_4$, a contradiction.

Case 3. l = 3.

Let the remaining vertices be z_1, z_2 . By Case 2 and G containing no rainbow $2P_4$, we have $|E_G(\{z_1, z_2\}, T_1 \cup T_2)| \le 2$ and then $|E(G[V(T_1 \cup T_2)])| \ge 17 - 2 - 1 = 14$. Thus $G[V(T_1 \cup T_2)] \cong K_6$ or K_6^- and we can get a rainbow $2P_4$ by Fact 1, a contradiction.

Now we will complete the proof of Theorem 10.

Theorem 10. For any $n \ge 8$, $ar(K_n, 2P_4) = \max\{2n - 2, 16\}$.

Proof. By Proposition 1, we just need to show the upper bound. We will prove it by induction on n.

Consider an (2n-1)-edge-coloring of K_n for $n \geq 9$. Suppose there is no rainbow $2P_4$. By Theorem 3, there is a rainbow $C_k \cup C_l$. Assume that $k \leq l$. Then k = 3. Let $T_1 = C_3 = x_1x_2x_3x_1$ and $T_2 = C_l = y_1 \dots y_l y_l$. We finish the proof by considering the following four cases.

Case 1.
$$4 \le l \le n - 5$$
.

In this case, there are at least two vertices $z_1, z_2 \notin V(T_1 \cup T_2)$. Since there is no rainbow $2P_4$, we have $c(x_1z_i) = c(x_2x_3)$, $c(x_2z_i) = c(x_1x_3)$ and $c(x_3z_i) = c(x_1x_2)$ for i = 1, 2. But we can have a rainbow $2P_4$ whatever the color of z_1z_2 is, a contradiction.

Case 2. l = n - 4.

Let $V(K_n) \setminus V(T_1 \cup T_2) = \{z\}$. Since there is no rainbow $2P_4$, we have that $c(x_1z) = c(x_2x_3)$, $c(x_2z) = c(x_1x_3)$, $c(x_3z) = c(x_1x_2)$, $c(xy) \in C(T_1 \cup T_2)$ for all $x \in V(T_1) \cup \{z\}$ and $y \in V(T_2)$. By Case 1, we can assume that $c(y_iy_j) \in C(T_1 \cup T_2)$ for all $y_i, y_j \in V(T_2)$. Hence the total

number of colors is at most 3 + n - 4 = n - 1 < 2n - 1, a contradiction.

Case 3. l = n - 3.

Since there is no rainbow $2P_4$, we have that $c(xy) \in C(T_1 \cup T_2)$ for all $x \in V(T_1)$ and $y \in V(T_2)$. By Case 1 and Case 2, we can assume that $c(y_iy_j) \in C(T_1 \cup T_2)$ for all $y_i, y_j \in V(T_2)$. Hence the total number of colors is at most 3 + n - 3 = n < 2n - 1, a contradiction.

Case 4. l = 3.

In this case we will consider the following two subcases. We choose a representing subgraph G such that $G \supset T_1 \cup T_2$.

Subcase 4.1 n = 9.

If there is a vertex v_0 such that $d^c(v_0) = 0$, then $K_9 - v_0$ uses exactly 17 colors and there is a rainbow $2P_4$ in $K_9 - v_0$ by Lemma 1. Hence we have $d^c(v) \ge 1$, for all $v \in V(K_9)$. Thus $\delta(G) \ge 1$.

Let $Z = V(K_9) \setminus V(T_1 \cup T_2) = \{z_1, z_2, z_3\}$. We claim that $|E_G(z_j, T_s)| \le 1$ for any $1 \le j \le 3$ and $1 \le s \le 2$. Otherwise G contains an $C_3 \cup C_4$ and the situation is the same as Case 1. If there are $z_i \in Z$ and $s \in \{1, 2\}$ such that $|E_G(z_i, T_s)| = 1$, then $|E_G(z, T_t)| = 0$ for any $z \in Z \setminus \{z_i\}$ and $t \in \{1, 2\} \setminus \{s\}$ by G having no $2P_4$. If $E_G(Z, T_1 \cup T_2) = \emptyset$, then $|E(G[V(T_1 \cup T_2)])| = 17 - |E(G[Z])| \ge 17 - 3 = 14$ which implies $G[V(T_1 \cup T_2)] \cong K_6$ or K_6^- and there is a rainbow $2P_4$ by Fact 1, a contradiction. Hence we have $E_G(Z, T_1 \cup T_2) \ne \emptyset$. Assume $z_1x_1 \in E(G)$ and we will consider the following two subcases.

Subcase 4.1.1 $E_G(Z, T_2) = \emptyset$.

Recall $z_1x_1 \in E(G)$ and $|E_G(z_j, T_1)| \le 1$ for any $1 \le j \le 3$. If there is $z \in \{z_2, z_3\}$ such that $z_1z \in E(G)$, then $E_G(\{x_2, x_3\}, T_2) = \emptyset$ which implies $|E(G)| \le 15$, a contradiction. Hence we have $|E(G[Z])| \le 1$ and then $|E(G[V(T_1 \cup T_2)])| \ge 17 - 3 = 14$. Thus $G[V(T_1 \cup T_2)] \cong K_6$ or K_6^- and there is a rainbow $2P_4$ by Fact 1, a contradiction.

Subcase 4.1.2 $E_G(Z, T_2) \neq \emptyset$.

In this case, we have $|E_G(Z, T_1 \cup T_2)| = |E_G(z_1, T_1 \cup T_2)| = 2$ and assume $z_1y_1 \in E(G)$. If $E_G(z_1, \{z_2, z_3\}) \neq \emptyset$, say $z_1z_2 \in E(G)$, then $E_G(T_1, T_2) \subseteq \{x_1y_1\}$. Hence $|E(G)| \leq 12$, a contradiction. So we have $E_G(z_1, \{z_2, z_3\}) = \emptyset$, which implies $z_2z_3 \in E(G)$ by $\delta(G) \geq 1$. Thus $|E(G[V(T_1 \cup T_2)])| = 17 - 3 = 14$ and $G[V(T_1 \cup T_2)] \cong K_6^-$. We can get a rainbow $2P_4$ by Fact 1, a contradiction.

Subcase 4.2 $n \ge 10$.

If there is a vertex v_0 such that $d^c(v_0) \leq 2$, then $K_n - v_0$ uses $2n - 1 - d^c(v_0) \geq 2n - 1 - 2 = 2(n-1) - 1$ colors and we get that $K_n - v_0$ contains a rainbow $2P_4$ by induction hypothesis, a contradiction. So we assume that $d^c(v) \geq 3$ for all $v \in V(K_n)$. Thus $\delta(G) \geq 3$.

Let $Z = V(K_n) \setminus V(T_1 \cup T_2) = \{z_1, z_2, \dots, z_{n-6}\}$. Since G contains no rainbow $2P_4$, we have that if there are $z \in Z$ and $s \in \{1, 2\}$ such that $E_G(z, T_s) \neq \emptyset$, then $E_G(z', T_t) = \emptyset$ for any $z' \in Z \setminus \{z\}$ and $t \in \{1, 2\} \setminus \{s\}$.

Subcase 4.2.1 $E_G(Z, T_1) \neq \emptyset$ and $E_G(Z, T_2) \neq \emptyset$.

In this case, there is exactly one vertex in Z, say z_1 , such that $E_G(z_1, T_1)$, $E_G(z_1, T_2) \neq \emptyset$ and $E_G(Z \setminus \{z_1\}, T_1 \cup T_2) = \emptyset$. Since $d_G(v) \geq 3$, $G[T_1 \cup T_2 \cup \{z_1\}]$ contains a cycle with length at least 4 and $G[Z \setminus \{z_1\}]$ contains a cycle of length at least 3. Then we can have a contradiction by Cases 1 to 3.

Subcase 4.2.2 $E_G(Z, T_1) = \emptyset$ or $E_G(Z, T_2) = \emptyset$.

Assume that $E_G(Z, T_1) = \emptyset$. By Case 1, we can assume that $|E_G(z, T_2)| \le 1$ for any $z \in Z$. Then $\delta(G[Z]) \ge 2$ and there is a cycle in G[Z]. Since $d_G(x_i) \ge 3$, we have $|E_G(x_i, T_2)| \ge 1$ for any $1 \le i \le 3$ and there is a cycle with length at least 4 in $G[T_1 \cup T_2]$. Then we can have a contradiction by Cases 1 to 3.

5. Double stars

The following Lemma 2 is an extension of Theorem 1. The idea of the proof is the same as the idea used in [12]. We include the proof for the sake of completeness.

Lemma 2. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. For $1 \le p_1 \le p_2 \le \dots \le p_n \le \frac{n}{3}$, the maximum number of colors of an edge-coloring of K_n such that $d_c(v_i) \le p_i$ for any $1 \le i \le n$ is at most $\frac{\sum_{i=1}^n p_i - n}{2} + 1$.

Proof. Let c be an edge-coloring of K_n such that $d_c(v_i) \leq p_i$ for any $1 \leq i \leq n$. Let G be a representing subgraph of K_n . Then $d_G(v_i) \leq d_c(v_i) \leq p_i$ for $1 \leq i \leq n$ and |E(G)| = |c|. Write $V = V(G) = V(K_n)$. Let $S = \{v_i \in V : d_G(v_i) = p_i\}$. If $S = \emptyset$, then $d_G(v_i) \leq p_i - 1$ for $1 \leq i \leq n$ and hence

$$|c| = \frac{\sum_{i=1}^{n} d_G(v_i)}{2} \le \frac{\sum_{i=1}^{n} (p_i - 1)}{2} = \frac{\sum_{i=1}^{n} p_i - n}{2}$$

and we are done. Hence we may assume that $S \neq \emptyset$. For $v \in V$, let $C_G(v)$ be the set of colors used on the edges incident to v in G. Clearly, we have $C_G(v) \subset C(v)$ for all $v \in V$ and $C_G(v) = C(v)$ for $v \in S$. Particularly, $\{c(uv)\} = C(u) \cap C(v) = C_G(u) \cap C_G(v)$ for $v \in S$.

Claim 1. G[S] is a clique.

Proof of Claim 1. Let $u, v \in S$. Since $c(uv) \in C(u) \cap C(v) = C_G(u) \cap C_G(v)$, $uv \in E(G)$. Hence G[S] is a clique.

Claim 2. Let $u, v \in S$ and $w \in V - S$. If c(uw) = c(vw), then c(uw) = c(vw) = c(uv).

Proof of Claim 2. Since $c(uw) = c(vw) \in C(u) \cap C(v) = C_G(u) \cap C_G(v) = \{c(uv)\}$, we have c(uw) = c(vw) = c(uv).

Claim 3. Let $u \in S, v \notin S$ and $uv \notin E(G)$. Then $c(uv) \notin C_G(v)$.

Proof of Claim 3. Suppose $c(uv) \in C_G(v)$. Since $u \in S$, we have $c(uv) \in C_G(u)$. Hence $c(uv) \in C_G(u) \cap C_G(v)$ which implies $uv \in E(G)$, a contradiction.

Claim 4. For all $v_i \notin S$, $d_G(v_i) \leq p_i - \frac{|S \setminus N_G(v_i)|}{2}$.

Proof of Claim 4. Let $S \setminus N_G(v_i) = \{u_1, u_2, \dots, u_k\}$. By Claim 3, $c(u_j v_i) \notin C_G(v_i)$ for all $1 \leq j \leq k$. Furthermore, by Claim 2, if $c(u_j v_i) = c(u_l v_i)$, then $c(u_j v_i) = c(u_l v_i) = c(u_l v_i)$.

This implies that no three edges in the set $\{u_1v_i, u_2v_i, \dots, u_kv_i\}$ have the same color; otherwise, by Claim 1, G[S] would contain a monochromatic triangle, a contradiction. Hence at least $\frac{k}{2}$ distinct colors are used on the edges $v_iu_1, v_iu_2, \dots, v_iu_k$, and those colors are not in $C_G(v_i)$. So $|C(v_i)| \geq |C_G(v_i)| + \frac{k}{2} = d_G(v_i) + \frac{k}{2}$. Since $|C(v_i)| = d_c(v_i) \leq p_i$, we have $d_G(v_i) + \frac{k}{2} \leq p_i$, which yields $d_G(v_i) \leq p_i - \frac{k}{2} = p_i - \frac{|S \setminus N_G(v_i)|}{2}$.

By Claim 4, we have

$$\sum_{i=1}^{n} d_G(v_i) = \sum_{v_i \in S} d_G(v_i) + \sum_{v_i \notin S} d_G(v_i)$$

$$\leq \sum_{v_i \in S} p_i + \sum_{v_i \notin S} \left(p_i - \frac{|S \setminus N_G(v_i)|}{2} \right)$$

$$= \sum_{i=1}^{n} p_i - \sum_{v_i \notin S} \frac{|S \setminus N_G(v_i)|}{2}.$$

Notice that $\sum_{v_i \notin S} |S \setminus N_G(v_i)|$ counts exactly the number of non-edges in G between S and $V \setminus S$. We have $\sum_{v_i \notin S} |S \setminus N_G(v_i)| = \sum_{v_i \in S} (n-1-p_i)$. Hence,

$$\sum_{i=1}^{n} d_G(v_i) \le \sum_{i=1}^{n} p_i - n + n - \frac{\sum_{v_i \in S} (n - 1 - p_i)}{2}.$$

On the other hand, for $v_i \notin S$, $d_G(v_i) \leq p_i - 1$. Hence, we have

$$\sum_{i=1}^{n} d_G(v_i) \le \sum_{v_i \in S} p_i + \sum_{v_i \notin S} (p_i - 1) \le \sum_{i=1}^{n} p_i - n + |S|.$$

We have

$$\sum_{i=1}^{n} d_G(v_i) \le \sum_{i=1}^{n} p_i - n + \min\left\{ |S|, n - \frac{\sum_{v_i \in S} (n - 1 - p_i)}{2} \right\}.$$

Since $n \geq 3p_n$, we have $\min\{|S|, n - \frac{\sum_{v_i \in S}(n-1-p_i)}{2}\} \leq 2$. Therefore,

$$|c| = \frac{\sum_{i=1}^{n} d_G(v_i)}{2} \le \frac{\sum_{i=1}^{n} p_i - n}{2} + 1.$$

Now we will prove Theorem 11.

Theorem 11. For $p \ge 2, 1 \le q \le p$ and $n \ge 6(p^2 + 2p)$,

$$ar(K_n, S_{p,q}) = \begin{cases} \lfloor \frac{(p-1)n}{2} \rfloor + 1, \ 1 \le q \le p-1; \\ \lfloor \frac{p(n-1)}{2} \rfloor + 1, \ q = p. \end{cases}$$

Proof. If $1 \le q \le p-1$, we have

$$ar(K_n, S_{p,q}) \ge ar(K_n, K_{1,p+1}) = \left\lfloor \frac{(p-1)n}{2} \right\rfloor + 1.$$

When q = p, we color the edges of K_n as follows. We color $K_{1,n-1}$ rainbow and color K_{n-1} with new $ar(K_{n-1}, K_{1,p})$ colors without producing a rainbow $K_{1,p}$. In such way, we use exactly

$$n - 1 + ar(K_{n-1}, K_{1,p}) = n - 1 + \left\lfloor \frac{(p-2)(n-1)}{2} \right\rfloor + 1 = \left\lfloor \frac{p(n-1)}{2} \right\rfloor + 1$$

colors and do not obtain a rainbow $S_{p,p}$.

Now we consider the upper bound. Let

$$g(n, p, q) = \begin{cases} \lfloor \frac{(p-1)n}{2} \rfloor + 1, \ 1 \le q \le p - 1, \\ \lfloor \frac{p(n-1)}{2} \rfloor + 1, \ q = p. \end{cases}$$
 (1)

Let c be any edge-coloring of K_n using g(n, p, q) + 1 colors. We will find a rainbow $S_{p,q}$ by considering the following two cases.

Case 1. There is some vertex v_0 such that $d_c(v_0) \ge p + 2q + 1$.

Let $U = \{v \in V \setminus \{v_0\} : \text{the color } c(v_0v) \text{ is stared at } v_0\}$. Consider the induced edge-coloring c' of c on $K_n - v_0$. We have

$$|c'| = |C(K_n - v_0)| = g(n, p, q) + 1 - d^c(v_0) \ge g(n, p, q) + 1 - |U|.$$

Claim 1. $K_n - v_0$ contains a rainbow $K_{1,q}$ with center in U or a rainbow $K_{1,q+1}$ with center in $V(K_n - v_0) \setminus U$ with respect to c' (also c).

Proof of Claim 1. Suppose $d_{c'}(u) \leq q-1$ for all $u \in U$ and $d_{c'}(v) \leq q$ for all $v \in V(K_n-v_0) \setminus U$, by Lemma 2, we have

$$|c'| \le \frac{|U|(q-1) + (n-1-|U|)q - (n-1)}{2} + 1.$$

While $|c'| \ge g(n, p, q) + 1 - |U|$, we can get $|U| \ge n$, a contradiction.

By Claim 1, we get a rainbow $K_{1,q}$ in $K_n - v_0$ with respect to c whose center is u_0 such that the color $c(v_0u_0)$ does not present in $c(K_{1,q})$. The vertex v_0 is the endpoint of at least p + 2q + 1 edges with distinct colors (with respect to c). Since $d_c(v_0) \geq p + 2q + 1$, there are at least p edges, say $v_0v_1, v_0v_2, \ldots, v_0v_p$, such that $v_i \notin V(K_{1,q})$ and $c(v_0v_i) \notin C(K_{1,q}) \cup \{c(v_0u_0)\}$ for all $i = 1, \ldots, p$. So we can get a rainbow $S_{p,q}$.

Case 2. $d_c(v) \leq p + 2q$ for all $v \in V$.

Since $g(n, p, q) + 1 \ge ar(K_n, K_{1,p+1}) + 1$, we can find a rainbow $K_{1,p+1}$. Choose some color c_0 which is not in $C(K_{1,p+1})$. Consider an edge-coloring c'' of $K_n - V(K_{1,p+1})$: for $e \in E(K_n - V(K_{1,p+1}))$, if $c(e) \notin C(K_{1,p+1})$, then c''(e) = c(e); else $c''(e) = c_0$. Then

$$|c'| \ge g(n, p, q) + 1 - (p + 2q)(p + 2) \ge ar(K_{n-p-2}, K_{1,q+1}) + 1.$$

So there is a rainbow $K_{1,q+1}$ in $K_n - K_{1,p+1}$ with respect to c'' (also c). $K_{1,q+1}$ has at most one edge e with color c_0 . By joining the centers $K_{1,p+1}$ and $K_{1,q+1}$ and deleting one edge of $K_{1,p+1}$ and $K_{1,q+1}$ respectively, we can find a rainbow $S_{p,q}$.

6. Open problems

A *spider* is a tree with at most one vertex of degree more than 2, called the center of the spider (if no vertex of degree more than two, then any vertex can be the center). A *leg* of a spider is a path from the center to a vertex of degree 1. Thus, a star with p edges is a spider of p legs, each of length 1, and a path is a spider of 1 or 2 legs.

The number of edges in a maximum matching of a graph G is called the *matching number* of G and denoted by $\nu(G)$.

Observation 1. Let $p \geq 2$ and T be a spider of p legs, each of length at least 2. We have

$$\min\{\nu(T - e_1 - e_2) : e_1, e_2 \in E(T)\} \le \min\{\nu(T - e) : e \in E(T)\},\$$

the equality holds if and only if T has exactly one leg with length even.

Let $p \ge 2$ and T be a spider of p legs, each of length at least 2. Let $\beta(T) = \min\{\nu(T-e) : e \in E(T)\}.$

Proposition 2. Let $p \ge 2$ and T be a spider of p legs, each of length at least 2. We have

$$ar(K_n,T) \ge (\beta(T)-1)n - {\beta(T) \choose 2} + r,$$

where r = 2 if there is exactly one leg of T with length even and r = 1 otherwise.

Proof. Let $\beta = \beta(T)$. We take an $K_{\beta-1} + \overline{K}_{n-\beta+1}$ and color it rainbow, and use r extra colors for all the remaining edges. Suppose there is a rainbow T in this coloring. Then T - e contains a matching of size β for any $e \in E(T)$ (or $T - e_1 - e_2$ contains a matching of size β for any $e_1, e_2 \in E(T)$ if T has exactly one leg with length even). But $K_{\beta-1} + \overline{K}_{n-\beta+1}$ does not contain a matching of size β , a contradiction.

If we regard P_{k+1} as a spider of 2 legs, the lower bound of Proposition 2 is sharp for p=2 and large n by Theorem 2. We conjecture that the lower bound is sharp for $p \geq 3$ and large n.

Conjecture 3. Let $p \geq 3$ and T be a spider of p legs, each of length at least 2. For large n,

$$ar(K_n,T) = (\beta(T) - 1)n - {\beta(T) \choose 2} + r,$$

where r = 2 if there is exactly one leg of T with length even and r = 1 otherwise.

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