# On the packing chromatic number of Moore graphs

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#### Abstract

The packing chromatic number  $\chi_{\rho}(G)$  of a graph G is the smallest integer k for which there exists a vertex coloring  $\Gamma: V(G) \to \{1, 2, ..., k\}$  such that any two vertices of color i are at distance at least i + 1. For  $g \in \{6, 8, 12\}, (q + 1, g)$ -Moore graphs are (q + 1)-regular graphs with girth g which are the incidence graphs of a symmetric generalized g/2-gons of order q. In this paper we study the packing chromatic number of a (q + 1, g)-Moore graph G. For g = 6 we present the exact value of  $\chi_{\rho}(G)$ . For g = 8, we determine  $\chi_{\rho}(G)$  in terms of the intersection of certain structures in generalized quadrangles. For g = 12, we present lower and upper bounds for this invariant when  $q \geq 9$  an odd prime power.

Keywords: Packing chromatic number, Moore graphs, Cages, Ovoids

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### 1. Introduction and definitions

The concept of packing coloring was introduced, under the name broadcast coloring, by Goddard et al. [16] to solve problems related to the assignment of broadcast frequencies to radio stations. The idea behind packing colorings is that in a network, the signals of two stations using the same frequency will interfere unless they are located sufficiently far apart. The term broadcast coloring was renamed as packing chromatic by Brešar et al. [8]. A packing k-coloring of a graph G is a function  $\Gamma: V(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two vertices of color *i* are at distance at least i + 1. The packing chromatic number of G,  $\chi_p(G)$ , is the smallest integer k for which G has a packing k-coloring. The packing chromatic number has been studied for several families of graphs: lattices and grids [13, 18], cubic graphs [4], subcubic outerplanar graphs [15], distance graphs [11], sierpinsky graphs [9], hypercubes [21], cartesian products and trees [8], among others. Kim et al. [17] showed that determining the packing chromatic number of graphs with diameter at least 3 is NP-complete.

All graphs considered in this work are finite, simple and undirected. We follow the book of Bondy and Murty [7] for terminology and notations not defined here. The *distance* between two vertices u and

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v is denoted by d(u, v). The *m*-neighborhood of a vertex set S, denoted by  $N^m(S)$  is the set of vertices at distance m from S. If  $S = \{v\}$  we denote it by  $N^m(v)$  and if m = 1 we omit the superindex m.

Given two integers  $k \ge 2$  and  $g \ge 3$  a (k, g)-graph is a k-regular graph of girth g. If G is a (k, g)-graph, then

$$|V(G)| \ge n_0(k,g) = \begin{cases} \frac{2(k-1)^{g/2} - 2}{k-2} & \text{if } g \text{ is even;} \\ \frac{k(k-1)^{(g-1)/2} - 2}{k-2} & \text{if } g \text{ is odd.} \end{cases}$$

The number  $n_0(k,g)$  is called the *Moore bound*. A (k,g)-graph of order  $n_0(k,g)$  is called a (k,g)-Moore graph and they are almost completely characterized [5, 10]. By definition, a (k,g)-Moore graph is a (k,g)-graph of minimum order. In general, the (k,g)-graphs of minimum order are called (k,g)-cages and have received a lot of attention. For more information on Moore graphs and cages see [12, 19].

In this paper we study the packing chromatic number of (k, g)-Moore graphs. In Section 2, we summarize some known results of the packing chromatic number and use them to obtain this invariant for Moore graphs with either k = 2 or  $g \leq 5$  and  $k \neq 57$ . In Section 3, we obtain the exact value of  $\chi_{\rho}(G)$  of Moore graphs with girth 6. For girth 8, we determine the packing chromatic number of Moore graphs in terms of the intersection of certain structures in generalized quadrangles. For (q+1, 12)-Moore graphs, we present lower and upper bounds for this invariant when  $q \geq 9$  is an odd prime power and we use the unofficial GAP [14] package YAGS [24] to obtain an upper bound for  $q \in \{2, 3\}$ .

### 2. Previous results

We begin this section with the definition of a symmetric generalized polygon (or *n*-gon). Let  $\mathcal{P}$  (the set of points) and  $\mathcal{L}$  (the set of lines) be disjoint non-empty sets, and let I be the point-line incidence relation. Let  $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ , and let  $G = G[\mathcal{P}, \mathcal{L}]$  be the bipartite incidence graph on  $\mathcal{P} \cup \mathcal{L}$  with edges joining the points from  $\mathcal{P}$  to their incident lines in  $\mathcal{L}$ . Following notation of [12], the ordered triple  $\mathcal{I}$  is a symmetric generalized *n*-gon of order *q* subject to the following regularity conditions.

- GP1: There exists an integer  $q \ge 1$  such that every line is incident to exactly q+1 points and every point is incident to exactly q+1 lines.
- GP2: Any two distinct lines intersect in at most one point and there is at most one line through any two distinct points.
- GP3: The incidence graph  $G = G[\mathcal{P}, \mathcal{L}]$  has diameter n and girth 2n.

For more information on symmetric generalized g/2-gons see [20, 22]. The following theorem summarize the partial characterization of Moore graphs. **Theorem 2.1.** [5, 10] There exists a Moore graph of degree k and girth g if and only if any of the following conditions hold

- 1. k = 2 and  $g \ge 3$  (cycles);
- 2. g = 3 and  $k \ge 2$  (complete graphs);
- 3. g = 4 and  $k \ge 2$  (complete bipartite graphs);
- 4. g = 5, and k = 3 (Petersen graph) or k = 7 (Hoffman-Singleton graph), or possibly k = 57;
- 5.  $g \in \{6, 8, 12\}$  and there exists a symmetric generalized g/2-gon of order k 1 (incidence graphs of symmetric generalized g/2-gons).

The packing chromatic number has already been determined for some Moore graphs [16] such as complete bipartite graphs, complete graphs and cycles. Let  $\beta(G)$  be the independence number of a graph G. We rewrite a result of Goddard et al. in terms of the independence number.

**Proposition 2.1.** [16] For every graph G of order n and diameter two,  $\chi_{\rho}(G) = n - \beta(G) + 1$ .

Let P denote the Petersen graph and let HS denote the Hoffmann-Singleton graph, recall that diam(P) = diam(HS) = 2. Using Proposition 2.1 and results obtained by Goddard et al. [16], we obtain the following remark.

**Remark 2.1.** Let  $n \ge 1$  and  $m \ge 3$  be integers. Then:  $\chi_{\rho}(K_n) = n$ ,  $\chi_{\rho}(K_{n,n}) = n + 1$ ,  $\chi_{\rho}(C_m) = 3$  if  $m \equiv 0 \pmod{4}$  and  $\chi_{\rho}(C_m) = 4$  if  $m \not\equiv 0 \pmod{4}$ ,  $\chi_{\rho}(P) = 7$  and  $\chi_{\rho}(HS) = 36$ .

Hence, the for items 1., 2., 3. and 4. of Theorem 2.1 the packing chromatic number is known except for the possibly existing (57, 5)-Moore graph.

## 3. The packing chromatic number of (q+1,g)-Moore graphs for $g \in \{6,8,12\}$

In this section we study the packing chromatic number for graphs considered in item 5. of Theorem 2.1. For  $g \in \{6, 8, 12\}$ , (q + 1, g)-Moore graphs are incidence graphs of symmetric generalized g/2-gons and thus  $\beta(G) = n/2$ . We use the following proposition.

**Proposition 3.1.** [16] If G is a bipartite graph of order n and diameter 3, then

$$n - \beta(G) \le \chi_{\rho}(G) \le n - \beta(G) + 1.$$

The following lemma is straightforward, but for sake of completeness we include the proof.

**Lemma 3.1.** Let G be a (q + 1, g)-Moore graph of n vertices with vertex partition  $(\mathcal{P}, \mathcal{L})$ . If S is a maximum independent set, then either  $S = \mathcal{P}$  or  $S = \mathcal{L}$ .

**Proof** Let  $P_S = S \cap \mathcal{P}$  and  $L_S = S \cap \mathcal{L}$ . Suppose for a contradiction, that  $L_S \neq \emptyset \neq P_S$ . Let  $|L_S| = r$ , let  $|P_S| = n/2 - r$  and let G' be the bipartite subgraph induced by  $L_S \cup (\mathcal{P} \setminus P_S)$  of order 2r. Since S is an maximum independent set,  $N_G(L_S) \subset (\mathcal{P} \setminus P_S)$ , G' has size r(q+1) and it is (q+1)-regular. Therefore, G' is a connected component of G, contradicting that G is connected.

**Theorem 3.1.** If G is a (q+1, 6)-Moore graph, then

$$\chi_{\rho}(G) = q^2 + q + 2.$$

**Proof** Let G be a (q + 1, 6)-Moore graph of order  $n = n_0(q + 1, 6) = 2(q^2 + q + 1)$ . Since G is the incidence graph of a symmetric generalized 3-gon (projective plane), it follows that the diameter of G is 3,  $\beta(G) = n/2$  and there is a bipartition  $(\mathcal{P}, \mathcal{L})$  of V(G) such that  $|\mathcal{P}| = |\mathcal{L}| = n/2$ . By Proposition 3.1,  $n/2 \leq \chi_{\rho}(G) \leq n/2 + 1$ . Suppose, for the sake of contradiction, that  $\chi_{\rho}(G) = n/2$ . Let  $\Gamma: V(G) \rightarrow \{1, 2, \dots, n/2\}$  be an optimal packing coloring of G. Since diameter of G is 3, it follows that  $\Gamma^{-1}(i)$  is a singular class for  $3 \leq i \leq n/2$ ,  $\Gamma$  has at least n/2 - 2 singular classes and exactly n/2 + 2vertices of color 1 or 2. If  $\Gamma^{-1}(2)$  is a singular class, then  $|\Gamma^{-1}(1)| = n/2 + 1 > \beta(G)$  contradicting the fact that the vertices of color 1 induce an independent set. Hence,  $\Gamma^{-1}(2)$  is not a singular class. Let  $u, v \in \Gamma^{-1}(2)$ , then d(u, v) = 3, implying that u and v belong to different parts of the bipartition. Therefore,  $|\Gamma^{-1}(2)| = 2$ ,  $|\Gamma^{-1}(1)| = n/2$  and  $\Gamma^{-1}(1) \cap \mathcal{P} \neq \emptyset \neq \Gamma^{-1}(1) \cap \mathcal{P}$  which contradicts Lemma 3.1. Thus,  $\chi_{\rho}(G) = n/2 + 1$  and the result follows.

Two points or two lines of a symmetric generalized g/2-gon of order q are called *opposite* if they are at distance g/2. An *ovoid*  $\mathcal{O}$  (resp. *spread*  $\mathcal{S}$ ) in a generalized g/2-gon of order q is a set of  $q^{g/4} + 1$  mutually opposite points (lines), such that every element x of the generalized g/2-gon is at distance at most g/4 from at least one point p of  $\mathcal{O}$  (resp. one line l of  $\mathcal{S}$ ). Ovoids and spreads have similar properties and have been widely studied, for more information see [22]. In graph terminology, an ovoid of a (q + 1, g)-Moore graph for  $g \in \{6, 8, 12\}$  is a set  $q^{g/4} + 1$  of vertices which are mutually at distance g/2.

**Proposition 3.2.** [2] A (q+1,g)-Moore graph with q a prime power and g = 8, or q an odd prime power different from 5 and 7 and g = 12, contain exactly  $q^{g/4} + 1$  vertices which are mutually at distance g/2.

We use the following coordinatization of a (q+1, 8)-Moore graph, where  $V_0 = \mathcal{P}$  and  $V_1 = \mathcal{L}$ .

**Definition 3.1.** [1] Let  $\mathbb{F}_q$  be a finite field with  $q \ge 2$  a prime power and  $\rho$  a symbol not belonging to  $\mathbb{F}_q$ . Let G be the incidence graph of a generalized quadrangle of order q. Let  $(V_0, V_1)$  be the bipartition of G with  $V_i = \mathbb{F}_q^3 \cup \{(\rho, b, c)_i, (\rho, \rho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_i\}, i \in \{0, 1\}$  and edge set defined as follows:

$$\begin{aligned} & \text{For all } a \in \mathbb{F}_q \cup \{\rho\} \text{ and for all } b, c \in \mathbb{F}_q : \\ & N_G((a, b, c)_1) = \begin{cases} & \{(w, aw + b, a^2w + 2ab + c)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ & \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, c)_0\} & \text{if } a = \rho. \end{cases} \\ & N_G((\rho, \rho, c)_1) = \{(\rho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_0\} \\ & N_G((\rho, \rho, \rho)_1) = \{(\rho, \rho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_0\}. \end{cases} \\ & \text{Or equivalently, for all } i \in \mathbb{F}_q \cup \{\rho\} \text{ and for all } j, k \in \mathbb{F}_q : \\ & N_G((i, j, k)_0) = \begin{cases} & \{(w, j - wi, w^{2i} - 2wj + k)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, j, i)_1\} & \text{if } i \in \mathbb{F}_q; \\ & \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, j)_1\} & \text{if } i = \rho. \end{cases} \\ & N_G((\rho, \rho, k)_0) = \{(\rho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_1\}; \end{cases} \\ & N_G((\rho, \rho, \rho)_0) = \{(\rho, \rho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_1\}. \end{aligned}$$

Recall that a (q + 1, 8)-Moore graph G has order  $2(q + 1)(q^2 + 1)$ , diameter 4, an ovoid of G has cardinality  $q^2 + 1$  and a (q + 1, 8)-Moore graph exist when q is a prime power.

**Theorem 3.2.** Let q be an odd prime power. If G is a (q+1, 8)-Moore graph, then

$$(q^{2}+1)(q-1)+3 \le \chi_{\rho}(G) \le (q^{2}+1)(q-1)+4.$$

**Proof** Let  $G = G[V_0, V_1]$  be a (q + 1, 8)-Moore graph. For the upper bound, we use the cordinatization in Definition 3.1 to construct a packing coloring of G with  $(q^2 + 1)(q - 1) + 4$  colors. We prove that the sets  $O = \{(\rho, \rho, \rho)_0\} \cup \{(i, j, 0)_0 \mid i, j \in \mathbb{F}_q\}$  and  $O' = \{(\rho, \rho, \rho)_0\} \cup \{(i, j, 1)_0 \mid i, j \in \mathbb{F}_q\}$  are ovoids in G. Let  $x, y \in O$  be two distinct vertices. Since  $x, y \in V_0$ , the distance between them is even. We prove that d(x, y) = 4. By Definition 3.1,  $d((\rho, \rho, \rho)_0, (i, j, 0)_0) = 4$ . Suppose by contradiction that there exist  $x, y \in O$  such that d(x, y) = 2. Let  $x = (i, j, 0)_0$ , let  $y = (i', j', 0)_0$  and let  $L = (a, b, c)_1$  be the unique vertex adjacent to x and y. Then, by Definition 3.1,

$$(i, j, 0)_0 = (i, ai + b, a^2i + 2ab + c)_0$$
 and  $(i', j', 0)_0 = (i', ai' + b, a^2i' + 2ab + c)_0$ 

Since  $a^2i + 2ab + c = 0 = a^2i' + 2ab + c$ , it follows that i = i', j = ai + b = ai' + b = j' and x = y, a contradiction. Hence, every pair of vertices in O are at distance 4. Furthermore,  $i, j \in \mathbb{F}_q$  hence  $|O| = q^2 + 1$  and O is an ovoid. Analogously O' is an ovoid and by Definition 3.1,  $O \cap O' = \{(\rho, \rho, \rho)_0\}$ . We define  $\Gamma : V(G) \to \{1, 2, \dots, (q^2 + 1)(q - 1) + 4\}$  as follows:  $\Gamma^{-1}(1) = V_1$ ,  $\Gamma^{-1}(2) = O$ ,  $\Gamma^{-1}(3) = O' \setminus \{(\rho, \rho, \rho)_0\}$ , and color each vertex of the remaining vertices with a different color. Thus,  $\Gamma$  is a packing coloring that uses  $2(q^2 + 1)(q + 1) - (q^2 + 1)(q + 1) - (q^2 + 1)(q - 1) + 4$  colors and

$$\chi_{\rho}(G) \le (q^2 + 1)(q - 1) + 4.$$

For the lower bound, we use geometric properties of the generalized quadrangles. Recall that  $V_0 = \mathcal{P}$ and  $V_1 = \mathcal{L}$ . Observe that  $\Gamma^{-1}(i)$  is a singular class for  $i \ge 4$ , and  $\Gamma^{-1}(3)$  is contained in the same part of G, because vertices of color 3 are at distance 4 and diam(G) = 4. Let  $\beta_4$  be the maximal order of a set of vertices at distance at least 4. By Proposition 3.2,  $\beta_4 = q^2 + 1$  and thus  $|\Gamma^{-1}(3)| \leq q^2 + 1$ . Since  $\beta(G) = (q^2 + 1)(q+1)$ , it follows that  $|\Gamma^{-1}(1)| \leq (q^2 + 1)(q+1)$ . Hence, it suffices to prove that  $|\Gamma^{-1}(2)| \leq q^2 + 1$ . If  $\Gamma^{-1}(2) \subset \mathcal{P}$ , then  $\Gamma^{-1}(2)$  is a set of vertices at distance 4, and by Proposition 3.2,  $|\Gamma^{-1}(2)| \leq q^2 + 1$  and the result holds. The case  $\Gamma^{-1}(2) \subset \mathcal{L}$  is analogous. Assume that  $\Gamma^{-1}(2) \cap \mathcal{P} \neq \emptyset \neq \Gamma^{-1}(2) \cap \mathcal{L}$ . Let  $\mathcal{P}_2 = \Gamma^{-1}(2) \cap \mathcal{P}$  and  $\mathcal{L}_2 = \Gamma^{-1}(2) \cap \mathcal{L}$ . If  $|\mathcal{P}_2| = q^2$ , then  $\mathcal{P}_2$  can be extended to an ovoid (Theorem 2.7.1 [20]). Let  $v \in \mathcal{P}$  such that  $\mathcal{P}_2 \cup \{v\}$  is an ovoid. By the definition of ovoids, every vertex in  $\mathcal{L}$  is adjacent to vertices in  $\mathcal{P}_2$ , thus every vertex in  $\mathcal{L}_2$  is adjacent to v. Since vertices in  $\Gamma^{-1}(2)$  are pairwise at distance at least 3, therefore  $|\mathcal{L}_2| = 1$  and  $|\Gamma^{-1}(2)| = q^2 + 1$ .

Assume that  $|\mathcal{P}_2| \leq q^2 - 1$ . Let  $|\mathcal{P}_2| = q^2 - \delta$  with  $\delta \geq 1$ . Since vertices in  $\Gamma^{-1}(2)$  are pairwise at distance at least 3, the vertices of  $\mathcal{P}_2$  do not have neighbours in common, therefore  $|N(\mathcal{P}_2)| = (q^2 - \delta)(q + 1)$ . Let  $S = \mathcal{L} \setminus N(\mathcal{P}_2)$ . Observe that  $\mathcal{L}_2$  is contained in S. Since  $|\mathcal{L}| = (q^2 + 1)(q + 1)$ , it follows that  $|S| = (q + 1)(\delta + 1)$ . Let  $l \in S$  and let  $N(l) = \{y_0, y_1, \ldots, y_q\}$ . Notice that  $y_i \notin \mathcal{P}_2$  for  $i \in \{0, 1, \ldots, q\}$ . Let  $z_i \in N(y_i)$  and let  $z_j \in N(y_j)$  with  $z_i \neq l \neq z_j$ . Then,  $z_i \neq z_j$ , otherwise  $(l, y_i, z_i, y_j, l)$  is a 4-cycle, a contradiction. Moreover,  $z_i$  and  $z_j$  do not share neighbours in common, the neighbours of  $\mathcal{P}_2$  induce a partition of  $\mathcal{L} - S$  into  $q^2 - \delta$  classes,  $U_v$  for  $v \in \mathcal{P}_2$ . In order to prove that each class  $U_v$  has at most one neighbour in N(l), we assume for a contradiction, that there is a class  $U_\alpha$  such that  $N(U_\alpha)$  contains at least two vertices  $y_i, y_j \in N(l)$ . In this case, either  $(x, y_i, l, y_j, x)$  is a 4-cycle for some  $x \in U_\alpha$  or  $(\alpha, x_1, y_i, l, y_j, x_2, \alpha)$  is a 6-cycle for some  $x_1, x_2 \in U_\alpha$ , a contradiction. Hence, each class  $U_v$  has at most one neighbour in N(l), see Figure 1.

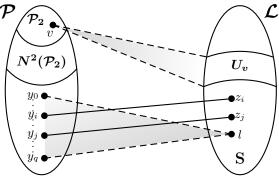


Figure 1: Case  $|\mathcal{P}_2| \leq q^2 - 1$  and  $\mathcal{L}_2 \neq \emptyset$ .

Let  $t_i = d_{S-l}(y_i)$ . Since g = 8,  $N(u) \cap N(v) = \{l\}$  for any pair of vertices  $u, v \in N(l)$ . For each  $v \in \mathcal{P}_2$ ,  $U_v$  has at most one neighbour in N(l), implying that  $|N_{S-l}^2(l)| = \sum_{i=0}^q t_i \ge (q+1)q - (q^2 - \delta) = q + \delta$ . Since l has at least  $q + \delta$  vertices in S - l at distance 2 and  $\mathcal{L}_2 \subset S$ , it follows that if  $|\mathcal{L}_2| = k$ , then  $k + k(q + \delta) \le |\mathcal{L}_2| + |N_{S-l}^2(\mathcal{L}_2)| \le |S| = (q + 1)(\delta + 1).$  Therefore,

$$k \le \frac{(q+1)(\delta+1)}{q+\delta+1}$$

Hence,

$$|\Gamma^{-1}(2)| = k + q^2 - \delta \le q^2 - \delta + \frac{(q+1)(\delta+1)}{q+\delta+1} = q^2 + \frac{q+1-\delta^2}{q+1+\delta} < q^2 + 1.$$

Thus,  $|\Gamma^{-1}(2)| \leq q^2 + 1$  which implies  $(q^2 + 1)(q - 1) + 3 \leq \chi_{\rho}(G)$  and the result holds.

**Corollary 3.1.** Let Q be a generalized quadrangle of order q and let G be the incidence graph of Q. Then Q contains two disjoint ovoids or two disjoint spreads if and only if  $\chi_{\rho}(G) = (q^2 + 1)(q - 1) + 3$ .

**Proof** Let  $G = G[\mathcal{P}, \mathcal{L}]$  be the incidence graph of a generalized quadrangle Q. If Q contains two disjoint ovoids (resp. spreads), then we can color the vertices of  $\mathcal{L}$  (resp.  $\mathcal{P}$ ) with color 1, the vertices of each ovoid (resp. spread) with colors 2 and 3, respectively, and the  $(q^2 + 1)(q - 1)$  remaining vertices with different colors. Therefore,  $\chi_{\rho}(G) \leq (q^2 + 1)(q - 1) + 3$  and the equality is attained.

If  $\chi_{\rho}(G) = (q^2 + 1)(q - 1) + 3$ , then let  $\Gamma$  be a packing coloring attaining this number. Since diam(G) = 4,  $|\Gamma^{-1}(i)| = 1$  for  $i \ge 4$ . Also,  $|\Gamma^{-1}(1)| \le q^3 + q^2 + q + 1$ ,  $|\Gamma^{-1}(2)| \le q^2 + 1$  and  $|\Gamma^{-1}(3)| \le q^2 + 1$ . Since we have  $(q^2 + 1)(q - 1)$  singular classes, to achieve  $\chi_{\rho}(G)$  the equality must hold in the previous inequalities. By Lemma 3.1,  $\Gamma^{-1}(1)$  is either  $\mathcal{P}$  or  $\mathcal{L}$ . Therefore  $\Gamma^{-1}(2)$  and  $\Gamma^{-1}(3)$  must be disjoint ovoids or two disjoint spreads.

For q an odd prime power, every pair of ovoids intersects [3, 6]. For q an even prime power, it is conjectured that every pair of ovoids intersects, but as far as the authors know, a proof is only known for  $2 \le q \le 64$ .

**Corollary 3.2.** If G is a (q + 1, 8)-Moore graph with q an odd prime power or  $q \le 64$  an even prime power, then

$$\chi_{\rho}(G) = (q^2 + 1)(q - 1) + 4.$$

We continue studying the packing chromatic number of (q+1, 12)-Moore graphs which are incidence graphs of generalized hexagons. First we establish upper bounds for some chromatic classes.

**Lemma 3.2.** Let  $\Gamma$  be a packing coloring of a (q+1, 12)-Moore graph with q an odd prime power different from 5 and 7. Then  $|\Gamma^{-1}(4)| \leq 2q^3 - 2q^2 + 2q$ .

**Proof** Let  $\mathcal{H}$  be a generalized hexagon of order q. Let  $G = G[\mathcal{P}, \mathcal{L}]$  be the incidence graph of  $\mathcal{H}$ . For every  $u, v \in \Gamma^{-1}(4), d(u, v) \geq 5$ , then by Proposition 3.2,  $|\Gamma^{-1}(4) \cap \mathcal{L}|, |\Gamma^{-1}(4) \cap \mathcal{P}| \leq q^3 + 1$ . If  $|\Gamma^{-1}(4) \cap \mathcal{P}| = q^3 + 1$ , then  $\Gamma^{-1}(4) \cap \mathcal{P}$  is an ovoid and  $|\Gamma^{-1}(4) \cap \mathcal{L}| = 0$ . Therefore  $|\Gamma^{-1}(4)| = q^3 + 1$  and the result follows. Similarly if  $|\Gamma^{-1}(4) \cap \mathcal{L}| = q^3 + 1$ , then  $\Gamma^{-1}(4) \cap \mathcal{L}$  is a spread, and the result holds. Let  $P_4 = \Gamma^{-1}(4) \cap \mathcal{P}$  and let  $L_4 = \Gamma^{-1}(4) \cap \mathcal{L}$ . Assume that  $|P_4| \ge |L_4| \ge 1$ , and let  $r \ge 0$  be an integer such that  $|P_4| = q^3 - r$ . For  $u, v \in P_4$ , d(u, v) = 6 and  $N^2(u) \cap N^2(v) = \emptyset$ . Hence

$$|N^{2}(P_{4})| = (q+1)(q^{3}-r)q = q^{5} + q^{4} - r(q^{2}+q).$$

Let  $P' = \mathcal{P} \setminus (P_4 \cup N^2(P_4))$ . Hence  $|P'| = (r+1)(q^2+q+1)$ . Since d(u,v) = 5, for every  $u \in P_4$  and  $v \in L_4$ , it follows that  $N(L_4) \subset P'$  and  $(q+1)|L_4| \leq (r+1)(q^2+q+1)$ . Therefore

$$|L_4| \le (r+1)q + \frac{r+1}{q+1}$$

Since  $|L_4| \le |P_4| = q^3 - r$ , then

$$|L_4| \le \min\left\{q^3 - r, (r+1)q + \frac{r+1}{q+1}\right\}.$$

Let  $f(q) = (q^4 + q^3 - q^2 - q - 1)/(q^2 + 2q + 2)$ . Observe that if  $r \ge f(q)$ , then  $|L_4| \le q^3 - r$  and  $|\Gamma^{-1}(4)| = |P_4| + |L_4| \le 2q^3 - 2f(q)$ . If r < f(q), then

$$|L_4| < r\left(q + \frac{1}{q+1}\right) + q + \frac{1}{q+1}$$

and

$$\begin{aligned} |\Gamma^{-1}(4)| &= |P_4| + |L_4| < q^3 - r + r\left(q + \frac{1}{q+1}\right) + q + \frac{1}{q+1} \\ &= q^3 + q + \frac{1}{q+1} + r\left(q - 1 + \frac{1}{q+1}\right) \\ &= 2q^3 - 2\frac{q^4 - q^3 + q^2 - q - q}{q^4 + 2q^2 + 2} \\ &= 2q^3 - 2f(q). \end{aligned}$$

In both cases  $|\Gamma^{-1}|(4)| \le 2q^3 - 2f(q)$ . Since  $f(q) \le q^2 - q$  for  $q \ge 2$ , the result follows. The case when  $|L_4| \ge |P_4|$  is proved similarly.

A distance-2 ovoid of a generalized n-gon is a subset of the point set with the property that every line contains exactly one point of that subset. Dual notion is that of a distance-2 spread. From this definition it follows that a distance-2 ovoid or distance-2 spread of a generalized hexagon has  $q^4 + q^2 + 1$  elements.

**Lemma 3.3.** Let  $\Gamma$  be a packing coloring of a (q+1, 12)-Moore graph. Then  $|\Gamma^{-1}(3)| \leq q^4 + q^2 + 1$  and the equality holds if  $\Gamma^{-1}(3)$  is a distance-2 ovoid or a distance-2 spread.

**Proof** Let  $\mathcal{H}$  be a generalized hexagon of order q. Let  $G = G[\mathcal{P}, \mathcal{L}]$  be the incidence graph of  $\mathcal{H}$ . For every  $u, v \in \Gamma^{-1}(3)$ ,  $d(u, v) \geq 4$ . Let  $P_3 = \Gamma^{-1}(3) \cap \mathcal{P}$  and let  $L_3 = \Gamma^{-1}(3) \cap \mathcal{L}$ . Observe that  $|P_3| + |N^2(P_3)| + |N(L_3)| \leq |\mathcal{P}|$  and  $|L_3| + |N^2(L_3)| + |N(P_3)| \leq |\mathcal{L}|$ . Therefore

$$|P_3| + |L_3| + |N^2(P_3)| + |N^2(L_3)| + |N(P_3)| + |N(L_3)| \le |\mathcal{P}| + |\mathcal{L}| = |V(G)|.$$
(1)

Let B be the bipartite graph constructed as follows:  $V(B) = P_3 \cup N^2(P_3)$  and for  $x \in P_3$  and  $y \in N^2(P_3)$ ,  $xy \in E(B)$  if and only if  $y \in N^2(x)$ . Observe that for every  $x \in P_3$ ,  $d_B(x) = q(q+1)$ . On the other hand, let  $y \in N^2(P_3)$ . If  $d_B(y) \ge q+2$ , since  $d_G(y) = q+1$  there exists at least two vertices  $x_1, x_2 \in P_3 \cap N^2(y)$  such that  $d(x_1, x_2) = 2$ , a contradiction. Hence  $d_B(y) \le q+1$  and

$$|E(B)| = |P_3|q(q+1) = \sum_{y \in N^2(P_3)} d(y) \le |N^2(P_3)|(q+1),$$

implying that  $|N^2(P_3)| \ge |P_3|q$ . Analogously,  $|N^2(L_3)| \ge |L_3|q$ . Since G is (q+1)-regular,  $|N(P_3)| = |P_3|(q+1)$  and  $|N(L_3)| = |L_3|(q+1)$ . By (1),

$$|V(G)| \geq |P_3| + |L_3| + |N^2(P_3)| + |N^2(L_3)| + |N(P_3)| + |N(L_3)|$$
  
$$\geq |P_3| + |L_3| + (|P_3| + |L_3|)q + (|P_3| + |L_3|)(q+1)$$
  
$$= (|P_3| + |L_3|)(2q+2).$$

Therefore  $|P_3| + |L_3| \le q^4 + q^2 + 1$ .

**Lemma 3.4.** Let  $\Gamma$  be a packing coloring of a (q+1, 12)-Moore graph. Then  $|\Gamma^{-1}(2)| \le 2\frac{q+1}{q+2}(q^4+q^2+1)$ .

**Proof** Let  $\mathcal{H}$  be a generalized hexagon of order q. Let  $G = G[\mathcal{P}, \mathcal{L}]$  be the incidence graph of  $\mathcal{H}$ . For every  $u, v \in \Gamma^{-1}(2), d(u, v) \geq 3$ . Let  $P_2 = \Gamma^{-1}(2) \cap \mathcal{P}$  and let  $L_2 = \Gamma^{-1}(2) \cap \mathcal{L}$ . Observe that  $|P_2| + |N(L_2)| \leq |\mathcal{P}|$  and  $|L_2| + |N(P_2)| \leq |\mathcal{L}|$ . Therefore

$$|P_2| + |L_2| + |N(P_3)| + |N(L_3)| \le |\mathcal{P}| + |\mathcal{L}| = |V(G)|.$$
(2)

Since G is (q + 1)-regular,  $|N(P_2)| = |P_2|(q + 1)$  and  $|N(L_2)| = |L_2|(q + 1)$ . By (2),

$$\begin{aligned} |V(G)| &\geq |P_2| + |L_2| + |N(P_2)| + |N(L_2)| \\ &\geq |P_2| + |L_2| + (|P_2| + |L_2|)(q+1) \\ &= (|P_3| + |L_3|)(q+2). \end{aligned}$$

Therefore,

$$|\Gamma^{-1}(2)| = |P_2| + |L_2| \le 2\frac{q+1}{q+2}(q^4+q^2+1).$$

**Theorem 3.3.** Let G be a (q+1, 12)-Moore graph with q an odd prime power  $q \ge 9$ , then

$$q^{5} - 2q^{4} - 4q^{2} + 9q - 18 + \frac{42}{q+2} \le \chi_{\rho}(G) \le q^{5} + q^{4} - 2q^{3} - q^{2} + 4$$

**Proof** Let  $q \ge 9$  be an odd prime power, let G be a (q + 1, 12)-Moore graph and let  $\Gamma$  be a packing coloring of G. Since diam(G) = 6,  $\Gamma^{-1}(i)$  is singular, for  $i \ge 6$ . For the lower bound observe that by Proposition 3.2,  $\Gamma^{-1}(5) \le q^3 + 1$ . Let S be the set of singular classes of  $\Gamma$ , then

$$|S| \ge |V(G)| - \sum_{i=1}^{5} |\Gamma^{-1}(i)|.$$

Using that  $\Gamma^{-1}(1)$  is an independent set and Lemmas 3.2, 3.3 and 3.4, it follows that

$$|S| \ge q^5 - 2q^4 - 4q^2 + 9q - 23 - \frac{42}{q+2}.$$

Therefore,  $\chi_{\rho}(G) \ge |S| + 5$  and the lower bound holds.

For the upper bound we give a packing chromatic coloring of a (q + 1, 12)-Moore graph. Consider an ovoid  $\mathcal{O}$  of G (which exist by Proposition 3.2) and assign color 5 to the vertices of  $\mathcal{O}$ . Let  $x \in \mathcal{O}$  and let  $y \in N(x)$ . Let T be the spanning tree such that every vertex of T is at distance at most (g - 2)/2 from the edge xy, where x and y are labeled by  $(0,0)_0$  and  $(0,0)_1$ , respectively. The vertex set of T is labeled by  $(i,j)_k$  where  $k \in \mathbb{Z}_2$  and the subindices are taken in  $\mathbb{Z}_2$ , j denotes the distance between the edge xy and the vertex  $(i,j)_k$  and  $0 \le i \le q^j - 1$ . The adjacencies of  $(0,0)_k$  in T are  $(i,1)_{k+1}$  for  $0 \le i < q$  and  $(0,0)_{k+1}$ . The adjacencies of  $(i,j)_k$  in T are the vertices labeled by  $(iq + r, j + 1)_{k+1}$  where  $0 \le r < q$  and the vertex labelled by  $(i', j - 1)_{k+1}$  where i = i'q + r and  $0 \le r < q$ . For  $i \in \{0, 1, \ldots, q^{\alpha} - 1\}$ , let  $B_i^{\alpha} = \{(iq + r, \alpha)_0 : 0 \le r < q\}$ . Observe that  $\mathcal{O} \subseteq (0,0)_0 \bigcup \left( \bigcup_{i=0}^{q^4-1} B_1^5 \right)$ . A general scheme of such labeled tree can be seen in Figure 2.

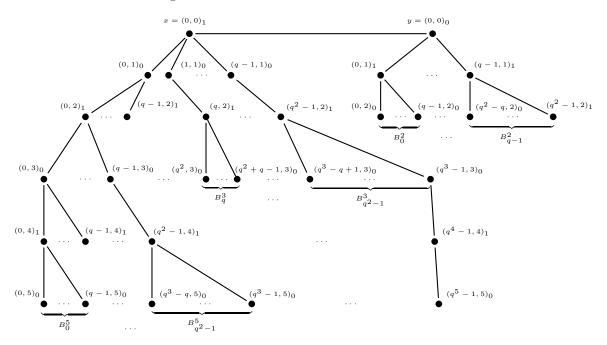


Figure 2: The labeled spanning tree of a (q + 1, 12)-Moore graph.

The vertices of color 1 are those labelled by  $(i, j)_1$ . Assign color 2 to vertex  $(0, 1)_0$ . For every  $i \in \{0, \ldots, q^2 - 1\}$  the set  $B_i^5$  contains at least  $q - 1 \ge 3$  vertices that are not in  $\mathcal{O}$ . Hence, we can choose

two vertices of  $B_i^5$  and color them with colors 2 and 3. For  $a, b \in \{0, \ldots, q^3 - 1\}$  with  $|b - a| \ge q$ , it follows that  $(a, 5)_0 \in B_i^5$  and  $(b, 5)_0 \in B_j^5$  where  $i \ne j$ . Therefore, the distance in T between  $(a, 5)_0$  and  $(b, 5)_0$  is 4 or 6. And their distance in G is at least 4.

Similarly, for every  $q \leq i \leq q^2 - 1$  we can choose two vertices of  $B_i^3$  and color them with colors 2 and 3. For  $a, b \in \{q^2, \ldots, q^3 - 1\}$  with  $|b - a| \geq q$ , it follows that  $(a, 3)_0 \in B_i^3$  and  $(b, 3)_0 \in B_j^3$  where  $i \neq j$ . Therefore the distance in T between  $(a, 3)_0$  and  $(b, 3)_0$  is 4 or 6. And their distance in G is at least 4. Moreover, the distance in T between  $(a, 3)_0$  and  $(b, 5)_0$ , for every  $q^2 \leq a \leq q^3 - 1$  and  $0 \leq b \leq q^3 - 1$  is 8, and the distance between them in G is at least 4. Observe that for  $0 \leq i \leq q - 1$ ,  $B_i^2 \cap \mathcal{O} = \emptyset$ . For every  $0 \leq i \leq q - 1$  we can choose two vertices of  $B_i^2$  and color them with colors 2 and 3. Observe that for  $a, b \in \{0, \ldots, q^2 - 1\}$  such that  $|b - a| \geq q$ ,  $(a, 2)_0 \in B_i^2$  and  $(b, 2)_0 \in B_j^2$  with  $i \neq j$ . Therefore the distance in T between  $(a, 2)_0$  and  $(b, 2)_0$  is 4. And their distance in G is 4. Moreover, the distance in Tbetween  $(a, 2)_0$  and  $(b, i)_0$ , for  $i \in \{1, 3, 5\}$  is 4,6 or 8, implying that they are at least at distance 4 in G.

For every set  $B_{iq}^5$  with  $0 \le i < q$ , we can choose a vertex and assign it color 4. Observe that the distance between these vertices is 6.

Hence, for this coloring we have  $|\Gamma^{-1}(1)| = q^5 + q^4 + q^3 + q^2 + q + 1$ ,  $|\Gamma^{-1}(2)| = q^3 + q^2 + 1$ ,  $|\Gamma^{-1}(3)| = q^3 + q^2$ ,  $|\Gamma^{-1}(4)| = q$ ,  $|\Gamma^{-1}(5)| = q^3 + 1$  and the upper bound holds.

Finally, for the (3, 12)-Moore graph and the (4, 12)-Moore graph, we computed an upper bound for the packing chromatic number using the backtracking function in the YAGS [24] package for GAP. For (4, 12)-Moore graph we used the Adjacency matrix from [23]. For (3, 12)-Moore graph we used the following definition in YAGS

g:=GraphByAdjacencies([ [64, 65, 78], [65, 66, 99], [66, 67, 84], [67, 68, 74], [68, 69, 95], [69, 70, 80], [70, 71, 76], [71, 72, 106], [64, 72, 73], [73, 74, 87], [74, 75, 108], [75, 76, 93], [76, 77, 83], [77, 78, 104], [78, 79, 89], [79, 80, 85], [80, 81, 115], [73, 81, 82], [82, 83, 96], [83, 84, 117], [84, 85, 102], [85, 86, 92], [86, 87, 113], [87, 88, 98], [88, 89, 94], [89, 90, 124], [82, 90, 91], [91, 92, 105], [92, 93, 126], [93, 94, 111], [94, 95, 101], [95, 96, 122], [96, 97, 107], [97, 98, 103], [70, 98, 99], [91, 99, 100], [100, 101, 114], [72, 101, 102], [102, 103, 120], [103, 104, 110], [68, 104, 105], [105, 106, 116], [106, 107, 112], [79, 107, 108], [100, 108, 109], [109, 110, 123], [81, 110, 111], [66, 111, 112], [12, 113, 119], [77, 113, 114], [114, 115, 125], [115, 116, 121], [88, 116, 117], [109, 117, 118], [69, 118, 119], [90, 119, 120], [75, 120, 121], [65, 121, 122], [86, 122, 123], [71, 123, 124], [67, 124, 125], [97, 125, 126], [64, 118, 126], [1, 9, 63], [1, 2, 58], [2, 3, 48], [3, 4, 61], [4, 5, 41], [5, 6, 55], [6, 7, 35], [7, 8, 60], [8, 9, 38], [9, 10, 18], [4, 10, 11], [11, 12, 57], [7, 12, 13], [13, 14, 50], [1, 14, 15], [15, 16, 44], [6, 16, 17], [17, 18, 47], [18, 19, 27], [13, 19, 20], [3, 20, 21], [16, 21, 22], [22, 23, 59], [10, 23, 24], [24, 25, 53], [15, 25, 26], [26, 27, 56], [27, 28, 36], [24, 39, 40], [14, 40, 41], [28, 41, 42], [8, 42, 43], [33, 43, 44], [11, 44, 45], [45, 46, 54], [40, 46, 47], [30, 47, 48], [43, 48, 49], [23, 49, 50], [34, 39, 40], [14, 40, 41], [28, 41, 42], [8, 42, 43], [33, 43, 44], [11, 44, 45], [45, 46, 54], [40, 46, 47], [30, 47, 48], [43, 48, 49], [23, 49, 50], [37, 50, 51], [17, 51, 52], [42, 52, 53], [20, 53, 54], [54, 55, 63], [49, 55, 56], [39, 56, 57], [52, 57, 58], [32, 58, 59], [46, 59, 60], [26, 60, 61], [51, 61, 62], [29, 62, 63]]);

and the following function for backtracking algorithm

```
L:=[];;
chk:=function(L,g)
local x,y;
if L=[] then return true; fi;
x:=Length(L);
for y in [1..x-1] do
if Distance(g,x,y)<=L[x] and L[x]=L[y] then
return false;
fi;
od;
return true;
end;
Backtrack(L,[1..26],chk,Order(g),g);
```

implying that for the (3, 12)-Moore graph G,  $\chi_{\rho}(G) \leq 26$ . Similarly, for the (4, 12)-Moore graph G, we obtained a packing coloring using 219 colors. Thus, for the (4, 12)-Moore graph G,  $\chi_{\rho}(G) \leq 219$ .

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