# On chordal and perfect plane near-triangulations 

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#### Abstract

A plane near-triangulation $G$ can be decomposed into a collection of induced subgraphs, described here as the W-components of $G$, such that $G$ is perfect (respectively, chordal) if and only if each of its W-components is perfect (respectively, chordal). Each W-component is a 2 -connected plane near-triangulation, free of edge separators and separating triangles. Graphs satisfying these conditions will be called W-near-triangulations. A linear time decomposition of $G$ into its W -components is achievable using known techniques from the literature.

W-near-triangulations have the property that the open neighbourhood of every internal vertex induces a cycle. It follows that a W-near-triangulation $H$ of at least five vertices is non-chordal if and only if it contains an internal vertex. This yields a local structural characterization that a plane near-triangulation $G$ is chordal if and only if it does not contain an induced wheel of at least five vertices.

For W-near-triangulations that are free of induced wheels of five vertices, we derive a similar local criteria, that depends only on the neighbourhoods of individual vertices and faces, for checking perfectness. We show that a W-near-triangulation $H$ that is free of any induced wheel of five vertices is perfect if and only if there exists neither an internal vertex $x$, nor a face $f$ such that, the neighbours of $x$ or $f$ induces an odd hole. The above characterization leads to a linear time algorithm for determining perfectness of this class of graphs.


Keywords: Plane near-triangulated graphs, Plane triangulated graphs, Chordal graphs, Perfect graphs.

## 1. Introduction

A plane embedding of a (planar) graph is called a plane near-triangulation if the boundary of every face, except possibly the outer face, is a cycle of length three. We try to derive local characterizations for checking whether a plane near-triangulation is chordal or perfect. Here, a local characterization refers to a condition that can be checked by inspecting the neighbourhood of individual vertices, edges or faces of the graph. A graph is chordal if and only if it is free of induced cycles of length exceeding three [13]. A graph is perfect if and only if it is free of induced odd cycles of length exceeding three (or odd holes) [12.

Investigation of the structural properties of plane triangulations and some of their subfamilies like Apollonian networks have been elaborately undertaken in the literature [7, 2, 6, 3, owing to their rich and interesting geometric structure. Here we investigate local structural characterizations for chordal and perfect plane near-triangulations.

A plane near-triangulation $G$ can be decomposed in linear time, into a set of induced component subgraphs, which we call the $W$-components of $G$ (see Section 3). Each W-component $H$ of $G$ is essentially a 2-connected plane near-triangulation that is free of edge separators and separating triangles. Graphs satisfying these conditions are referred to as $W$-near-triangulations. The neighbourhood of every internal vertex of any W-near-triangulation induces a wheel.

The problem of determining whether a plane near-triangulation $G$ is chordal (respectively, perfect) can be reduced to the problem of checking whether each of its W -components is chordal (respectively, perfect).

[^0]In Section 3 we describe a linear time procedure to extract the W -components of $G$, by adapting a method known in the literature [5] for identifying 4-connected blocks in a plane triangulation.

In Section 4 it is shown that a W-near-triangulation $G$ that is not $K_{4}$ is chordal if and only if it does not contain an internal vertex. Consequently, we derive a local structural characterization that a plane near-triangulation $G$ is chordal if and only if it does not contain an induced wheel of at least five vertices.

In Section 6, we show that perfect W -near-triangulations that do not contain any induced wheel of five vertices admit a simple local characterization. It is shown that a W -near-triangulation $H$ that does not contain any induced wheel of five vertices is perfect if and only if there exists neither an internal vertex $x$, nor a face $f$ in $H$ such that, the neighbours of $x$ or $f$ induces an odd hole. This local structural characterization results in a linear time algorithm for determining whether a W-near-triangulation, that is free of any induced wheel of five vertices, is perfect. No sub-quadratic time algorithm appears to be known for recognizing perfect plane near-triangulations or perfect plane triangulations.

## 2. Preliminaries

Given a plane near-triangulation $G$, we call the vertices on the boundary of the external face of $G$ as the external vertices of $G$, denoted by $\operatorname{Ext}(G)$ and the remaining vertices as the internal vertices of $G$, denoted by $\operatorname{Int}(G)$. The notation $C_{n}$ will be used to denote a cycle of $n$ vertices. If $S \subseteq V(G)$, then $N_{G}(S)$ (respectively, $N_{G}[S]$ ) denotes the open (respectively, closed) neighbourhood of the set $S$. In the case when $S=\{u\}$ for a vertex $u \in V(G)$, we write $N_{G}(u)$ (respectively, $N_{G}[u]$ ) for the open (respectively, closed) neighbourhood of $u$. The suffix will be dropped when the underlying graph $G$ is clear from the context.

Definition 2.1 (Wheel). A wheel on $n(n \geq 4)$ vertices, $W_{n}$, is the graph obtained by adding a new vertex $v$ to a cycle $C_{n-1}$ and making it adjacent to all vertices in $C_{n-1}$. The cycle $C_{n-1}$ is called the rim of the wheel, the vertex $v$ is called the centre of the wheel and the added edges joining $v$ and vertices in $C_{n-1}$ are called spokes of the wheel.

A wheel $W_{n}$ is called an even wheel (respectively, odd wheel) if $n$ is even (respectively, odd). Note that the rim of an even wheel contains an odd number of vertices and the rim of an odd wheel has an even number of vertices. Any induced cycle of length at least four in a graph is called a hole. A hole with odd number of vertices is known as an odd hole. A separator in a connected graph is a set of vertices, the removal of which disconnects the graph. A clique in a graph is a set of pairwise adjacent vertices. A clique separator is a separator which is a clique. A clique separator of size two (respectively, three) is called an edge separator (respectively, a separating triangle).

Definition 2.2 (W-near-triangulation). A plane near-triangulation $G$ is called $a W$-near-triangulation if $G$ is two connected and, either $G$ is isomorphic to $K_{4}$ or $G$ contains neither a separating triangle, nor an edge separator. $A W$-near-triangulation $G$ is called an even $W$-near-triangulation if the degree of every vertex in $\operatorname{Int}(G)$ is even.

Note that a W-near-triangulation need not be 4-connected (for example, a wheel on five vertices is a W-near-triangulation, but contains a 3 -separator).

In the next section, we show that the study of chordality (respectively, perfectness) of plane-neartriangulations reduces to the study of chordality (respectively, perfectness) of W-near-triangulations.

## 3. W-decomposition

In this section we describe a method to decompose any plane near-triangulation $G$ into a collection of induced subgraphs, $G_{1}, G_{2}, \ldots, G_{k}$ (for some $k \geq 1$ ) in linear time, where each $G_{i}, i \in\{1,2 \ldots, k\}$ is a W-near-triangulation and $G$ is chordal (respectively, perfect) if and only if all of $G_{1}, G_{2}, \ldots G_{k}$ is chordal (respectively, perfect). The method described here is a combination of known techniques for handling plane triangulations, drawn from various sources. We sketch the details briefly here for the sake of completeness.

Let $G$ be a plane near-triangulation. $G$ is chordal (respectively, perfect) if and only if each of its 2connected blocks is chordal (respectively, perfect). Since we can identify the 2 -connected blocks of $G$ in linear time, we assume hereafter that $G$ is 2 -connected.

Let $u v$ be an edge separator in $G$. Let $H_{1}$ and $H_{2}$ be the two components of $G \backslash\{u, v\}$. It is easy to see that $G$ is chordal (respectively, perfect) if and only if the subgraphs $G_{1}$ and $G_{2}$ induced by $V\left(H_{1}\right) \cup\{u, v\}$ and $V\left(H_{2}\right) \cup\{u, v\}$ are chordal (respectively, perfect). Similarly, let $u v w$ is a separating triangle in $G$ and, let $H_{1}$ and $H_{2}$ be the two components of $G \backslash\{u, v, w\}$. It is easy to see that $G$ is chordal (respectively, perfect) if and only if the subgraphs $G_{1}$ and $G_{2}$ induced by $V\left(H_{1}\right) \cup\{u, v, w\}$ and $V\left(H_{2}\right) \cup\{u, v, w\}$ are chordal (respectively, perfect). We can recursively find edge separators and separating triangles in the components till we are left with a collection of induced subgraphs $G_{1}, G_{2}, \ldots G_{k}$ of $G$ such that none of them contains an edge separator or a separating triangle. That is, we have a decomposition of $G$ into a collection of maximal W-near-triangulated subgraphs of $G$ such that, $G$ is chordal (respectively, perfect) if and only if each of the subgraphs is chordal (respectively, perfect). We call each maximal W-near-triangulated subgraph of $G$ a $W$-component of $G$. This decomposition is a special case of the clique decomposition described by Tarjan [11. We need to show that the decomposition can be done in linear time.

The problem of finding edge separators in a 2-connected plane near-triangulation is reducible to finding separating triangles, using a folklore algorithmic trick. Given a plane near-triangulation $G$ that is not already a triangulation, we can triangulate $G$ by artificially adding a new vertex, say $p$, on the external face of $G$ and making all vertices in the external face of $G$ adjacent to $p$. Let the new graph be denoted by $G_{p}$. It is easy to see that any edge separator $u v$ in $G$ must be a chord connecting two vertices in the external face of $G$ and hence puv must be a separating triangle in $G_{p}$. Conversely, for any separating triangle puv in $G_{p}$ containing the newly added vertex $p, u v$ must be an edge separator in $G$.

To construct $G_{p}$ from $G$ in linear time, we need to find the vertices on the external face of $G$ from the adjacency list of $G$. Here is one possible way to do this. We first embed $G$ in an $n \times n$ grid in linear time using the algorithm by Schnyder [10]. Now start from a the vertex, say $v_{1}$ in $G$ whose $x$ coordinate is the smallest. This vertex must be on the external face of $G$. Traverse the adjacency list of $v_{1}$ to find the vertex $v_{2}$ such that the edge $v_{1} v_{2}$ has the largest slope among edges incident on $v_{1}$. Clearly, $v_{1} v_{2}$ must be an edge on the external face of $G$. By traversing the adjacency list of $v_{2}$ once and finding the angles between $v_{1} v_{2}$ and $v_{2} w$ for each neighbour $w$ of $v_{2}$, we can identify the edge, say $v_{2} v_{3}$ that appears next to $v_{1} v_{2}$, in the clockwise ordering of edges around the vertex $v_{2}$. It is not difficult to see that the edge $v_{2} v_{3}$ is on the external face of $G$. Continuing this way until we reach back $v_{1}$, we can find all the vertices on the boundary of $G$. Since the adjacency list of each vertex is traversed at most once in the process, the procedure takes only linear time. Adding the vertex $p$ to the adjacency list of every vertex on the boundary of the external face and adding the adjacency list of $p$ to $G$ can be done in linear time.

Thus, starting from a 2 connected plane near-triangulation $G$, we can construct a plane triangulation $G_{p}$ in linear time such that separating triangles in $G_{p}$ correspond to either edge separators or separating triangles in $G$. It is well known that a plane triangulation is 4 -connected if and only if it is free of separating triangles. Thus, to find the maximal W-components of $G_{p}$, it suffices to find the 4-connected blocks of $G_{p}$, which can be done in linear time using the algorithm by Kant [5]. It is not hard to see that, by removing the vertex $p$ from each W -component of $G_{p}$ (whenever $p$ is present), we can recover the W-components in $G$, which again requires only linear time. Hence we have:

Lemma 3.1. Given a plane near-triangulation $G$, we can find the maximal $W$-near-triangulated subgraphs ( $W$-components) of $G$ in linear time. Moreover, $G$ is chordal (respectively, perfect) if and only if each of the $W$-components is chordal (respectively, perfect).

Consequently, we study W-near-triangulations for the rest of the paper.

## 4. Chordal plane near-triangulations

The following Lemma describes the structure of W-near-triangulations.

Lemma 4.1. If $G$ is a $W$-near-triangulation with at least five vertices then for all $u \in \operatorname{Int}(G), N[u]$ induces a wheel $W_{k}$ for some $k \geq 5$.

Proof. Let $G$ be a W-near-triangulation with at least five vertices and $u \in \operatorname{Int}(G)$. As $u$ is an internal vertex, $|N(u)| \geq 3$. Let $N(u)=\left\{u_{0}, u_{1}, u_{2}, \ldots u_{t-1}\right\}$ for some $t \geq 3$ such that $u u_{0}, u u_{1}, \ldots, u u_{t-1}$ is the clockwise ordering of the edges incident with $u$. We claim that $u_{i} u_{i+1} \in E(G)$, where index $i \in\{0,1, \ldots, t-1\}$ is taken modulo $t$. Indeed, if $u_{i} u_{i+1}$ is not an edge, then $u u_{i}$ and $u u_{i+1}$ will be on the boundary of a face of length greater than three, contradicting that $G$ is a W-near-triangulation. Consequently, $u_{0}, u_{1}, \ldots u_{t-1} u_{0}$ is a cycle and as $G$ is free of separating triangles, we get $t \geq 4$. Now suppose that there exists an edge $u_{i} u_{j}$ with $j \notin\{i+1, i-1\}$. Then $\left\{u, u_{i}, u_{j}\right\}$ will be a separating triangle. Therefore $N[u]$ is the wheel $W_{t+1}$, containing at least five vertices.

The next observation is directly verifiable and form the base case of the inductive argument that follows.
Observation 4.2. Every plane near-triangulation with five or fewer vertices except $W_{5}$ is chordal. Every plane near-triangulation having no internal vertex is chordal.
Lemma 4.3. $A$ - $W$-near-triangulation except $K_{4}$ is chordal iff it does not contain any internal vertices.
Proof. Let $G$ be a W-near-triangulation. If $|V(G)| \leq 5$ then by Observation 4.2, $G$ is chordal iff $G$ is not $W_{5}$, which has an internal vertex. If $|V(G)|>5$ and there is no internal vertex in $G$, then by Observation 4.2 , $G$ is chordal. If $|V(G)|>5$ and $G$ contains at least one internal vertex say $u$, then by the Lemma $4.1, N[u]$ will induce a wheel say $W_{k}$ for $k \geq 5$. As the rim of $W_{k}$ is a chordless cycle of length $(k-1)>3, G$ is not chordal.

The following theorem gives a local structural characterization for chordal plane near-triangulations in terms of the closed neighbourhoods of internal vertices.

Theorem 4.4. A plane near-triangulated graph is not chordal iff it contains an induced wheel of at least five vertices.

Proof. Let $G$ be a plane near-triangulated graph. If $G$ contains an induced $W_{k}$ for some $k \geq 5$ then the $\operatorname{rim}$ of $W_{k}$ is a chordless cycle of length exceeding three and hence $G$ is not chordal. Conversely, if $G$ is not chordal, by Lemma 3.1, we can decompose $G$ into its W-components - say $G_{1}, G_{2}, \ldots, G_{t}$ for some $t>0$ such that $G$ is not chordal if and only if at least one $G_{i}, 1 \leq i \leq t$ is not chordal. Let $G_{i}$ be a non-chordal W-component of $G$. Since $G_{i}$ is a plane near-triangulation which is not chordal, by Lemma 4.3, $G_{i}$ contains at least one internal vertex, say $v$. By Lemma 4.1. $N_{G_{i}}[v]$ induces a wheel $W_{k}$ for some $k \geq 5$ in $G_{i}$. Since $G_{i}$ is an induced subgraph of $G$ (by definition), $N_{G_{i}}[v]$ induces a wheel of at least five vertices in $G$ as well.

Lemma 3.1 and Lemma 4.3 yield a linear time algorithm for recognizing chordal plane near-triangulations, different from the standard method based on perfect elimination ordering [8, as described below. Given a plane near-triangulation $G$, it suffices to decompose $G$ into its W-components in linear time and check whether any of the components contain an internal vertex. Checking whether a plane near-triangulation contains an internal vertex requires only linear time (for instance, find vertices on the boundary as described in the previous section and check whether the boundary includes every vertex or not). Thus, in linear time, chordal plane near-triangulations can be recognized.

## 5. Perfect plane near-triangulations

Our next objective is to investigate the problem of providing a local characterization for plane neartriangulated perfect graphs similar in spirit to Theorem 4.4. It is easy to see that the complement of cycle $C_{n}$ for $n \geq 7$ is not planar. Moreover, the complement of $C_{5}$ is isomorphic to $C_{5}$. Thus, it follows from the strong perfect graph theorem [4] that to prove a plane triangulated graph $G$ is perfect, it is enough to prove that $G$ does not contain an induced odd hole.

Let $G$ be a plane near-triangulated graph. If $G$ contains an induced wheel on an even number of vertices (even wheel) then clearly $G$ is not perfect. However the absence of an induced even wheel is not sufficient to guarantee the perfectness of a plane near-triangulation. For example, the graph shown in Figure 1 does not contain any induced even wheel. But the vertices on the boundary of external face induce an odd hole.

By Lemma 3.1, we know that the problem of characterizing perfect plane near-triangulations reduces to the problem of characterizing perfect W-near-triangulations. A local characterization that is simple enough to yield a linear time recognition procedure for arbitrary perfect W -near-triangulations appears hard to find. Instead, we characterize a subclass of W-near-triangulations that indeed admits a simple local characterization that leads to a linear time recognition procedure. we derive a simple local structural characterization for W-near-triangulations that do not contain any induced wheel of five vertices.


Figure 1: An even wheel-free non-perfect plane near-triangulation

## 6. $W_{5}$ free $W$-near-triangulations

In this section, we prove that any non-perfect $W_{5}$ free W-near-triangulation $G$ contains either an even wheel or contains three vertices forming an internal face such that the open neighbourhood of these vertices induces an odd hole. Throughout this section, we use the notation $W(u)$ to denote a wheel with vertex $u$ at the centre. We first establish some properties of W-near-triangulations that will be useful for deriving the characterization.

Lemma 6.1. If a $W$-near-triangulation $G$ contains e edges, $f$ internal faces and $t$ edges on the boundary of external face, then $f=t \bmod 2$. That is, $f$ is odd if and only if $t$ is odd.

Proof. Each internal face is bounded with exactly three edges and each edge except those in the boundary of the external face is shared by two faces. This implies $3 f=2 e-t$. Hence $t$ is odd if and only if $f$ is odd.

Definition 6.2 (Face intersecting wheels). Let $W(u)$ and $W(v)(u \neq v)$ be any two wheels in a $W$-neartriangulation. $W(u)$ and $W(v)$ are said to be face intersecting if they share at least one face.

Lemma 6.3. Let $G$ be a $W$-near-triangulation and $W(u)$ and $W(v)(u \neq v)$ be any two face-intersecting wheels in $G$, then $W(u)$ and $W(v)$ share exactly two faces. Further, the edge uv is on the boundary of these two faces.

Proof. Since $W(u)$ and $W(v)$ are face-intersecting and $u \neq v, u$ should be on the rim of $W(v)$. Similarly $v$ should lie on the rim of $W(u)$. Hence the edge $u v$ should be a spoke in both the wheels. As $u$ is on the $\operatorname{rim}$ of $W(v), u$ will have exactly two neighbours (say $x, y$ ) on the rim of $W(v)$. Similarly $v$ also have two neighbours (say $p, q$ ) on the rim of $W(u)$. If $p \neq x$ and $p \neq y$ then the edge $p u$ will be a chord on the wheel $W(v)$ and the vertices $u, p, v$ forms a separating triangle in $G$, which is a contradiction to the definition of W-near-triangulation. Hence $p=x$ or $p=y$ Similarly $q=y$ or $q=x$. This implies that either $p=x$ and $q=y$ or $p=y$ and $q=x$. So $x$ and $y$ are the only vertices in $N(u) \cap N(v)$ and the edges $u x$ and $u y$ on the $\operatorname{rim}$ of $W(v)$ are also spokes of $W(u)$ and $v x$ and $v y$ on the rim of $W(u)$ are also spokes of $W(v)$. That is, $\{u x, x v, v u)\}$ and $\{u y, y v, v u\}$ are the only two faces shared by $W(u)$ and $W(v)$.

Corollary 6.4. Let $W(x), W(y)$ and $W(z)$ (with $x \neq y \neq z$ ) be three (pair-wise) face-intersecting odd wheels in a $W$-near-triangulation $G$. Then they share exactly the common face $\{x y, y z, x z\}$.

Proof. Since $W(x)$ and $W(y)$ are face-intersecting, by Lemma 6.3, they share two faces (faces which has the edge $x y$ as one of its boundary). Similarly $W(y)$ and $W(z)$ share two faces (faces which has the edge $y z$ as one of its boundary) and $W(x)$ and $W(z)$ share two faces (faces which has the edge $x z$ as one of its boundary). This implies that $x y, y z$ and $x z$ forms either a separating triangle or a face which is shared by $W(x), W(y)$ and $W(z)$. But as $G$ is a W-near-triangulation, the edges $x y, y z$ and $x z$ can not form a separating triangle.

Definition $6.5\left(W_{\Delta}\right)$. Let $G$ be a $W$-near-triangulation and $W(x), W(y)$ and $W(z)$ be three face intersecting odd wheels in $G$. If $N[x] \cup N[y] \cup N[z] \backslash\{x, y, z\}$ induces an odd hole in $G$, then the subgraph induced by $N[x] \cup N[y] \cup N[z]$ is called a $W_{\Delta}$. The graph shown in Figure 1 is an example of $W_{\Delta}$.

If a W-near-triangulation $G$ with at least five vertices contains an internal vertex $u$ of odd degree exceeding 3 , then the rim of $W(u)$ induces an odd hole in $G$ and thus $G$ cannot be perfect. Since an internal vertex of degree 3 would induce a separating triangle, a W -near-triangulation with at least 5 vertices cannot contain an internal vertex whose degree is 3 . Consequently, the non-trivial case to handle is to characterize perfect W-near-triangulations whose internal vertices are all of even degree (even W-near-triangulations).

Lemma 6.6. Let $G$ be a $W_{5}$ free even $W$-near-triangulation and $W(x), W(y)$ and $W(z)$ be three face intersecting wheels in $G$. Then $N[x] \cup N[y] \cup N[z]$ induces $a W_{\Delta}$.
Proof. Let $x, y$ and $z$ be three vertices of $G$ such that $W(x), W(y)$ and $W(z)$ are face intersecting wheels in $G$. Let $G_{1}$ be the subgraph of $G$ induced by the vertices $x, y, z$ and their neighbours. That is, $G_{1}$ is a subgraph of $G$ induced by $N[x] \cup N[y] \cup N[z]$. Let $G_{2}$ be the subgraph of $G_{1}$ induced by $V\left(G_{1}\right) \backslash\{x, y, z\}$. If $G_{1}$ does not induce a $W_{\Delta}$ then there exists at least one chord in $G_{2}$. Without loss of generality we may assume that there exists two non consecutive vertices $p$ and $q$ on the rim of wheels $W(x)$ and $W(y)$ respectively such that $p q$ is a chord in $G_{2}$. We may further assume without loss of generality that there is no chord between the vertices of the clockwise boundary of $G_{1}$ from $p$ to $q$ (see Figure 22).

Let $P=p q_{1} q_{2} \ldots q_{r} q$ (where $r \geq 1$ ) be the path joining $p$ and $q$ in $G_{1}$ (see Figure 2). As $W(x)$ and $W(y)$ are face intersecting, there must be at least one vertex, say $\left.q_{i}, 1 \leq i \leq r\right)$ in $P$ that lies on the rim of both the wheels $W(x)$ and $W(y)$ (see Figure 22).

Let $s$ be the neighbour of $p$ on the rim of $W(x)$ in the anti clockwise direction and $t$ be the neighbour of $q$ on the rim of $W_{y}$ in the clockwise direction (see Figure 22 . Let $G_{3}$ be the subgraph of $G_{1}$ induced by the vertices $p, q_{1}, \ldots, q_{i}, \ldots, q_{r}, q$ and their neighbours except $x, y, s$ and $t$. That is, $G_{3}$ is the subgraph of $G_{1}$ induced by the vertices $\left(N[p] \cup N\left[q_{1}\right] \cup . . N\left[q_{i}\right] \cup N[q]\right) \backslash\{x, y, s, t\}$ (see figure 3). Let $e, f, n_{e}$ and $n_{i}$ be the number of edges, internal faces, external vertices and internal vertices in $G_{3}$ respectively. Let $n=n_{e}+n_{i}$ be the total number of vertices in $G_{3}$. Since $G_{3}$ is internally triangulated, by Lemma 6.1 we have,

$$
\begin{equation*}
3 f=2 e-n_{e} \tag{1}
\end{equation*}
$$

Using Euler's formula [13] we get:

$$
\begin{equation*}
3 n_{e}+3 n_{i}+3 f=3 e+3 \tag{2}
\end{equation*}
$$



Figure 2: $G_{1}$


Figure 3: $G_{3}$
from (1) and (2) we get:

$$
\begin{equation*}
e=2 n_{e}+3 n_{i}-3 \tag{3}
\end{equation*}
$$

Let $V_{i}$ and $V_{e}$ be the set of internal and external vertices in $G_{3}$ respectively. As $G_{3}$ is an induced subgraph of $G_{1}$ and $p q$ a chord in $G_{1}$, all vertices except $p$ and $q$ in $V_{e}$ are internal vertices of $G_{1}$ (see Figure 2 and Figure 3). Also every vertex in $V_{e}$ except $q_{i}$ must either be on the rim of $W_{x}$ or on the rim of $W_{y}$, but not on both. That is, for all external vertices in $G_{3}$ except $p, q$ and $q_{i}$, all but one of their neighbours in $G_{1}$ must be in the graph $G_{3}$. It follows that the degree of all vertices on the external face of $G_{3}$ except $p, q$ and $q_{i}$ will be at least five. This is true because we have assumed that $G_{1}$ is a $W_{5}$ free even W-near-triangulation and hence has no internal vertex of degree below six.

The vertices $p, q$ and $q_{i}$ have two neighbours on the cycle $p, q_{1}, \ldots, q_{i}, \ldots, q_{r}, q$ and as $G_{3}$ is plane neartriangulated, they must have at least one neighbour in $V_{i}$. So the degree of $p, q$ and $q_{i}$ will be at least three in $G_{3}$. Since every neighbour (in $G_{1}$ ) of vertices in $V_{i}$ is also present in $G_{3}$, degree of all vertices in $V_{i}$ must be at least six in $G_{3}$. Counting the degree of vertices, we get:

$$
\begin{equation*}
2 e \geq 6 n_{i}+5\left(n_{e}-3\right)+9 \tag{4}
\end{equation*}
$$

Substituting (3) we get,

$$
\begin{equation*}
4 n_{e}+6 n_{i}-6 \geq 6 n_{i}+5 n_{e}-6 \Longrightarrow 0 \geq 2 n_{e} \Longrightarrow 0 \geq n_{e} \tag{5}
\end{equation*}
$$

which is a contradiction.

The following lemma shows that Lemma 6.6 characterizes all non-perfect $W_{5}$ free even W -near-triangulations.
Lemma 6.7. Every $W_{5}$ free even $W$-near-triangulation $G$ that contains an induced odd hole must contain an induced $W_{\Delta}$.

Proof. Let $C$ be an induced odd hole in $G$. As $G$ is a plane near-triangulation, there must exist at least one vertex in $\operatorname{Int}(C)$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the subgraph induced by the vertices in $\operatorname{Int}(C)$. If $\left|V^{\prime}\right|=1$ then $V^{\prime} \cup V(C)$ will have to induce an odd wheel which is impossible as $G$ is an even W-near-triangulation. The case $\left|V^{\prime}\right|=2$ is also not possible as two face intersecting odd wheels will not induce an odd hole (See Lemma 6.1. Thus we may assume that $\left|V^{\prime}\right| \geq 3$. Let $G^{\prime \prime}$ be the subgraph of $G$ induced by the vertices $V^{\prime} \cup V(C)$. We have to consider the following cases.

1. $G^{\prime}$ contains an induced triangle $\triangle=(x y z)$ : In this case, $W(x), W(y)$ and $W(z)$ are face intersecting odd wheels and by Lemma 6.6. $N(x) \cup N(Y) \cup N(z)$ induces an odd hole, proving the lemma.
2. $G^{\prime}$ does not contain any induced triangles: In this case, as $G$ is a plane near-triangulation, the only possibility is that $V^{\prime}$ induces a tree $T$ of at least 3 vertices. (See Figure 4a, $T$ could possibly a path as in Figure 4b, Let $v_{0}, v_{1}, \ldots, v_{r}$ for some $r \geq 2$ be the vertices in $T$ ordered in such a way that $v_{0}$ is the root of the tree and each node $v_{j}$ for $j>0$ is a child of some unique $v_{i}, i<j$ in $T$. Note that every neighbour of a vertex $v_{i}$ in $T$ except its children and its parent in the tree $T$ must be a vertex in the odd hole $C$. Let $W\left(v_{i}\right)$ be the wheel induced by $N\left[v_{i}\right]$. Since $G$ is an even $W_{5}$ free plane near-triangulation, $v_{i}$ must have even degree (greater than 4) for each $i \in\{0,1, \ldots, r\}$. Further, each edge incident on $v_{i}(0 \leq i \leq r)$ is shared by exactly two internal faces in $G^{\prime \prime}$ (See Figure 4a). Hence, for each $i, j \in\{0,1, \ldots, r\}$, if $v_{i}$ is the parent of $v_{j}$ in the tree $T$, the wheels $W\left(v_{i}\right)$ and $W\left(v_{j}\right)$ must be face intersecting odd wheels sharing exactly two faces. Using this observation, we count the the total number of internal faces in $G^{\prime \prime}$ to be $f=\sum_{i=0}^{r} \operatorname{deg}\left(v_{i}\right)-2(r-1)$. As the degree of every internal vertex in $G^{\prime \prime}$ is even, $f$ must be even. Then by Lemma 6.1, the number of external vertices of $G^{\prime \prime}$ should be even. That is, $|V(C)|$ must be even. However, this contradicts the assumption that $C$ is an odd hole.


Figure 4a: $V^{\prime}$ induces a tree

The proof of Lemma 6.7 shows that if a $W_{5}$ free even W-near-triangulation $G$ contains an odd hole, then the interior of the odd hole cannot be a tree, and hence must contain a facial triangle $u v w$. On the other hand, if three internal vertices $u, v$ and $w$ forms a facial triangle in a $W_{5}$ free even W -near-triangulation $G$, by Lemma 6.6, the neighbours of the facial triangle uvw must induce an odd hole. Hence, we have the following computationally useful corollary.


Figure 4b: $V^{\prime}$ induces a path

Corollary 6.8. Let $G$ be a $W_{5}$ free even $W$-near-triangulation. The following conditions are equivalent.

1. $G$ is not perfect.
2. The subgraph induced by vertices of $\operatorname{Int}(G)$ contains a facial triangle.
3. The subgraph induced by vertices of $\operatorname{Int}(G)$ is not a tree.

Lemma 6.6 and Lemma 6.7 yields the following local characterization for perfect W-near-triangulations.
Theorem 6.9. $A W_{5}$ free plane triangulated $W$-near-triangulation $G$ other than a $K_{4}$ is perfect if and only if the following conditions hold

- $G$ does not contain an even wheel
- $G$ does not contain an induced $W_{\Delta}$

A simple linear time algorithm for checking whether a given $W_{5}$ free W-near-triangulation $G$ is perfect follows from Corollary 6.8 and Theorem 6.9, as explained below. We can find the vertices on the external face of $G$ in linear time using the method described in Section 3 and create an array whose $i^{t h}$ entry indicates whether the $i^{t h}$ vertex is in $\operatorname{Int}(G)$ or not, in linear time. Now we can check whether any vertex in $\operatorname{Int}(G)$ has odd degree, in which case, we immediately conclude that $G$ is not perfect. Otherwise, we perform a breadth first search on the subgraph induced by the vertices of $\operatorname{Int}(G)$ (the indicator array serves to ensures that the search never enters a vertex on the boundary of $G$ ) in linear time to decide whether the subgraph induced by vertices of $\operatorname{Int}(G)$ is a tree (Corollary 6.8).

## 7. Discussion and Conclusion

Investigation into the structure of perfect plane-triangulations or plane near-triangulations has been reported in two unpublished manuscripts in the literature. In a work done prior to this paper by Benchetrit and Bruhn [1], a structural characterization for perfect plane triangulations that is not a local characterization is reported. The characterization does not appear to yield any direct algorithmic consequences. In a work done subsequent to this paper by Salam et al. [9, a local characterization for W-near-triangulations has been derived using a different proof technique, but the characterization is more complex than the one derived here for $W_{5}$ free W-near-triangulations and consequently yields only a quadratic time recognition algorithm for perfectness. It is interesting to check whether the approach presented here can be extended to all W-near-triangulations in a way to yield a linear time recognition algorithm for plane perfect neartriangulations.

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