# A note on the packing chromatic number of lexicographic products* 

Dragana Božović ${ }^{(1)}$ and Iztok Peterin ${ }^{(1,2)}$<br>(1) Faculty of Electrical Engineering and Computer Science University of Maribor, Koroška cesta 46, 2000 Maribor, Slovenia.<br>${ }^{(2)}$ Institute of Mathematics, Physics and Mechanics<br>Jadranska ulica 19, 1000 Ljubljana, Slovenia.<br>e-mails:dragana.bozovic@um.si and iztok.peterin@um.si


#### Abstract

The packing chromatic number $\chi_{\rho}(G)$ of a graph $G$ is the smallest integer $k$ such that there exists a $k$-vertex coloring of $G$ in which any two vertices receiving color $i$ are at distance at least $i+1$. In this short note we present upper and lower bound for the packing chromatic number of the lexicographic product $G \circ H$ of graphs $G$ and $H$. Both bounds coincide in many cases. In particular this happens if $|V(H)|-\alpha(H) \geq \operatorname{diam}(G)-1$, where $\alpha(G)$ denotes the independence number of $G$.


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## 1 Introduction and preliminaries

Let $G$ be a simple graph. To shorten the notation we use $|G|$ instead of $|V(G)|$ for the order of $G$. The distance $d_{G}(u, v)$ between vertices $u$ and $v$ of $G$ is the length of a shortest path between $u$ and $v$ in $G$. The diameter of $G$ is denoted by $\operatorname{diam}(G)$ and is the maximum length of a shortest path between any two vertices of $G$.

Let $t$ be a positive integer. A set $X \subseteq V(G)$ is a $t$-packing if any two different vertices from $X$ are at distance more than $t$. The $t$-packing number of $G$, denoted by $\rho_{t}(G)$, is the maximum cardinality of a $t$-packing of $G$. Notice, that if $t=1$, then the 1 -packing number equals to the independence number $\alpha(G)$ and we use the later more common notation for it. An independent set of cardinality $\alpha(G)$ is called $\alpha(G)$-set. The packing chromatic number $\chi_{\rho}(G)$ of $G$ is the smallest integer $k$ such that $V(G)$ can be partitioned into subsets $X_{1}, \ldots, X_{k}$, where $X_{i}$ induces an $i$-packing for every $1 \leq i \leq k$. Another approach is from a $k$-packing coloring of $G$, which is a function $c: V(G) \rightarrow[k]$, where $[k]=\{1, \ldots, k\}$, such that if $c(u)=c(v)=i$, then $d_{G}(u, v)>i$. Clearly, $\chi_{\rho}(G)$ is the minimum integer $k$ for which a $k$-packing coloring of $G$ exists.

[^0]The concept of packing chromatic number was introduced by Goddard et al. in 6] under the name broadcast chromatic number. The problem of determining the packing chromatic number of a graph is a very difficult problem and is NP-complete even for trees as shown in [2]. The attention was fast drawn to Cartesian product and infinite latices like hexagonal, triangular and similar. In [1] it was shown that the packing chromatic number of an infinite hexagonal lattice lies between 6 and 8. Upper bound was later improved to 7 in [3] and finally settled to 7 in [9]. For infinite triangular lattice and three-dimensional integer lattice $\mathbb{Z}^{3}$ the packing chromatic number is infinite as shown in [4]. The packing chromatic number of the Cartesian product was already considered in [1] where the general upper and lower bound were set. The lower bound was later improved in [8]. Several exact values and bounds for special families of Cartesian product graphs can be found in [8, 9].

In this note we switch from Cartesian to lexicographic product and prove an upper and a lower bound for the packing chromatic number of lexicographic product. It turns out that these two bounds coincide in many cases. In particular, if $\operatorname{diam}(G) \leq 2$, if $\operatorname{diam}(G)=3$ and $H \not \equiv \bar{K}_{n}$ and if $|V(H)|-\alpha(H) \geq \operatorname{diam}(G)-1$ and $H \not \equiv \bar{K}_{n}$.

The lexicographic product of graphs $G$ and $H$ is the graph $G \circ H$ (also sometimes denoted with $G[H])$ with the vertex set $V(G) \times V(H)$. Two vertices $\left(g_{1}, h_{1}\right)$ and $\left(g_{2}, h_{2}\right)$ are adjacent if either $g_{1} g_{2} \in E(G)$ or $g_{1}=g_{2}$ and $h_{1} h_{2} \in E(H)$. Set $G^{h}=\{(g, h): g \in V(G)\}$ is called a $G$-layer through $h$ and $H^{g}=\{(g, h): h \in V(H)\}$ is called an $H$-layer through $h$. Clearly, subgraphs of $G \circ H$ induced by $G^{h}$ and $H^{g}$ are isomorphic to $G$ and $H$, respectively. The distance between two vertices in lexicographic product is given by

$$
d_{G}\left(\left(g_{1}, h_{1}\right),\left(g_{2}, h_{2}\right)\right)=\left\{\begin{array}{cl}
d_{G}\left(g_{1}, g_{2}\right) & : g_{1} \neq g_{2}  \tag{1}\\
\min \left\{2, d_{H}\left(h_{1}, h_{2}\right)\right\} & : g_{1}=g_{2}
\end{array}\right.
$$

and depends heavily on the distance between projections of both vertices to $G$. For the independence number it is well known that

$$
\begin{equation*}
\alpha(G \circ H)=\alpha(G) \alpha(H), \tag{2}
\end{equation*}
$$

see Theorem 1 in [5]. Lexicographic product $G \circ H$ is connected if and only if $G$ is connected. For more properties of the lexicographic product see [7].

## 2 Results

In this section we present a lower and an upper bound for the packing chromatic number of lexicographic product of graphs. We start with the lower bound and we use the following notation

$$
d(G)=\left\{\begin{array}{cl}
1 & : \quad G \cong K_{n} \\
\operatorname{diam}(G)-1 & : \quad \text { otherwise }
\end{array} .\right.
$$

Theorem 2.1. If $G$ and $H$ are graphs, then

$$
\chi_{\rho}(G \circ H) \geq|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)+d(G)
$$

Proof. Denote $\ell=|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)+d(G)$. Let $X_{1}, \ldots, X_{k}$ be a partition of $V(G \circ H)$ that yields a $k$-packing coloring of $G \circ H$. We have at most $\alpha(G) \alpha(H)$ vertices in $X_{1}$ by (22). Denote by $R_{i}$ a $\rho_{i}(G \circ H)$-set for $2 \leq i \leq \operatorname{diam}(G)-1$. By (1) we have $\left|H^{g} \cap R_{i}\right| \leq 1$ for every
$g \in V(G)$. So there are at most $\rho_{i}(G)$ vertices in $X_{i}$ for $2 \leq i \leq \operatorname{diam}(G)-1$. For $i \geq \operatorname{diam}(G)$ there can only be one vertex in $X_{i}$ since all the vertices are at distance at most $\operatorname{diam}(G)$ from vertex in $X_{i}$. So we have at most $\alpha(G) \alpha(H)$ vertices colored with color 1 , at most $\rho_{i}(G)$ vertices colored with color $i$ for every $2 \leq i \leq \operatorname{diam}(G)-1$, and we need one color for each one of the remaining vertices and there are $|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)$ of them. Meaning that $\chi_{\rho}(G \circ H) \geq \ell$ because we have exactly $d(G)$ color classes which possibly have more than one vertex.

We continue with an upper bound that has a similar structure as the lower bound from Theorem 2.1.

Theorem 2.2. Let $G$ and $H$ be graphs and $k=|H|-\alpha(H)$. If $H \not \equiv \bar{K}_{n}$, then

$$
\chi_{\rho}(G \circ H) \leq|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{k+1} \rho_{i}(G)+k+1
$$

Proof. Denote $\ell=|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{k+1} \rho_{i}(G)+k+1$. We know that $\alpha(G \circ H)=\alpha(G) \alpha(H)$ and it is easy to see that $\alpha(G \circ H)$-set can be written as $A_{G} \times A_{H}$ where $A_{G}$ is an $\alpha(G)$-set and $A_{H}$ is an $\alpha(H)$-set. We color all the vertices from $A_{G} \times A_{H}$ with color 1. Let $k=|H|-\alpha(H)$. There remain $k G$-layers with no colored vertices. In each of those layers we color $\rho_{i}(G)$ vertices with color $i, 2 \leq i \leq k+1$ (one color $i$ is used in one layer). Each of the remaining uncolored vertices is colored with its own color. So we have $\alpha(G) \alpha(H)$ vertices colored with color $1, \rho_{i}(G)$ vertices colored with color $i$ for every $2 \leq i \leq k+1$, and we need one color for each one of the remaining uncolored vertices. Clearly, there are $|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{k+1} \rho_{i}(G)$ vertices colored with its own color. Meaning that $\chi_{\rho}(G \circ H) \leq \ell$ because we have $k+1$ color classes which possibly have more than one vertex.

Notice that if $\operatorname{diam}(G) \leq 2$, then also $\operatorname{diam}(G \circ H) \leq 2$ (see $\mathbb{1}$ ) and only color 1 can appear more then once in any packing coloring. Therefore, if $\operatorname{diam}(G) \leq 2$, Theorem 2.2 also holds for $H \cong \bar{K}_{n}$. Next we show that if the number of vertices of $H$ without its $\alpha(H)$-set is comparable with $\operatorname{diam}(G)$, then both bounds coincide.

Corollary 2.3. Let $G$ and $H$ be graphs and $|H|-\alpha(H) \geq \operatorname{diam}(G)-1$. If $H \not \not \bar{K}_{n}$, then

$$
\chi_{\rho}(G \circ H)=|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)+\operatorname{diam}(G)-1
$$

Proof. Let first $G \cong K_{n}$. By Theorem 2.1] it holds that $\chi_{\rho}(G \circ H) \geq n|H|-\alpha(H)-\sum_{i=2}^{0} \rho_{i}(G)+$ $1=n|H|-\alpha(H)+1$ since $d(G)=1$. On the other hand let $k=|H|-\alpha(H)$ and we have $\chi_{\rho}(G \circ H) \leq n|H|-\alpha(H)-(k+1-2+1)+k+1=n|H|-\alpha(H)+1$ by Theorem 2.2 since $\rho_{i}(G)=1$ for every $2 \leq i \leq k+1$. Hence, the equality follows when $G \cong K_{n}$.
Otherwise $G \neq K_{n}$ and $d(G)=\operatorname{diam}(G)-1$. So by Theorem 2.1 it holds that $\chi_{\rho}(G \circ H) \geq$ $|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)+\operatorname{diam}(G)-1$. Since $k \geq \operatorname{diam}(G)-1$ and $\rho_{i}(G)=1$ for
every $\operatorname{diam}(G) \leq i \leq k+1$, by Theorem 2.2 it holds that

$$
\begin{align*}
\chi_{\rho}(G \circ H) & \leq|G| \cdot|H|-\alpha(G) \alpha(H)-\left(\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)+\sum_{i=\operatorname{diam}(G)}^{k+1} \rho_{i}(G)\right)+k+1= \\
& =|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)-(k+1-\operatorname{diam}(G)+1)+k+1=  \tag{3}\\
& =|G| \cdot|H|-\alpha(G) \alpha(H)-\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G)+\operatorname{diam}(G)-1 .
\end{align*}
$$

We can expect that the condition of Corollary 2.3 will be fulfilled more frequently when $\operatorname{diam}(G)$ is small. In particular, for $\operatorname{diam}(G)=1$ the condition is always satisfied and we have

$$
\chi_{\rho}(G \circ H)=|G| \cdot|H|-\alpha(H)+1
$$

as seen in the proof of the previous corollary. Notice that in the case of $\operatorname{diam}(G)=2$ the sum in the lower bound of Theorem 2.1 does not exist and that $d(G)=1$. Also $\rho_{i}(G)=1$ for every $2 \leq i \leq k$ since $\operatorname{diam}(G)=2$. Therefore we have $-\sum_{i=2}^{k+1} \rho_{i}(G)+k+1=-(k+1-2+1)+k+1=1$ in the upper bound of Theorem 2.2. Hence both bounds coincide and we have the following corollary.

Corollary 2.4. Let $G$ and $H$ be graphs. If $\operatorname{diam}(G)=2$, then

$$
\chi_{\rho}(G \circ H)=|G| \cdot|H|-\alpha(G) \alpha(H)+1 .
$$

Similar holds also when $\operatorname{diam}(G)=3$. Namely in this case $\operatorname{diam}(G \circ H)=3$ by (1) and only two color classes ( $X_{1}$ and $X_{2}$ ) can have more than one representative. Therefore bounds from Theorems 2.2 and 2.1 coincide again under condition that there is at least one $G$-layer without vertices from $X_{1}$. This always occurs if $H \not \equiv \bar{K}_{n}$ and the following corollary holds.

Corollary 2.5. Let $G$ and $H$ be graphs. If $\operatorname{diam}(G)=3$ and $H \nsubseteq \bar{K}_{n}$, then

$$
\chi_{\rho}(G \circ H)=|G| \cdot|H|-\alpha(G) \alpha(H)-\rho_{2}(G)+2
$$

Continuing in this manner things get more complicated. Therefore we finish with an approach from the different side and concentrate on a family of graphs with big diameter, namely the case when $G \cong P_{n}$. For this we first improve the upper bound from Theorem 2.2.,

Theorem 2.6. Let $H$ a graph and $n$ a positive integer. If $k=|H|-\alpha(H)$, then

$$
\chi_{\rho}\left(P_{n} \circ H\right) \leq n|H|-\left\lceil\frac{n}{2}\right\rceil \alpha(H)-\sum_{i=2}^{k+1}\left\lceil\frac{n}{i+1}\right\rceil-\sum_{j=k+2}^{|H|+1}\left(\left\lfloor\frac{\left\lfloor\frac{n}{2}\right\rfloor-1}{\left\lfloor\frac{j}{2}\right\rfloor+1}\right\rfloor+1\right)+|H|+1 .
$$

Proof. Let $P_{n}=v_{1} \ldots v_{n}$ and $A_{H}$ be an $\alpha(H)$-set. Clearly, $A_{P_{n}}=\left\{v_{2 i-1}: i \in\left[\left\lceil\frac{n}{2}\right\rceil\right]\right\}$ is an $\alpha\left(P_{n}\right)$-set and $A=A_{P_{n}} \times A_{H}$ is an $\alpha\left(P_{n} \circ H\right)$-set. Firstly, we color vertices with $k+1$ colors as in the proof of Theorem [2.2. For this we use

$$
\ell=n|H|-\left\lceil\frac{n}{2}\right\rceil \alpha(H)-\sum_{i=2}^{k+1}\left\lceil\frac{n}{i+1}\right\rceil+k+1
$$

colors because $\rho_{i}\left(P_{n}\right)=\left\lceil\frac{n}{i+1}\right\rceil$.
In each $G^{h}$-layer, $h \in A_{H}$, there exist $\left\lfloor\frac{n}{2}\right\rfloor$ still not colored vertices with an even distance between any two of them. We denote them by $B^{h}=\left(V\left(P_{n}\right)-A_{P_{n}}\right) \times\{h\}$. Additionally we will color with color $j, k+2 \leq j \leq|H|+1$, some vertices of exactly one $G^{h}$-layer, $h \in A_{H}$. Denote by $G_{j}^{h}$ the $G^{h}$-layer, $h \in A_{H}$, containing vertices of color $j, k+2 \leq j \leq|H|+1$. The biggest distance between two vertices from $B^{h}$ equals $2\left\lfloor\frac{n}{2}\right\rfloor-2$. Notice that two vertices of $G_{j}^{h}$ colored with $j$ must be at least $p_{j}=2\left\lfloor\frac{j}{2}\right\rfloor+2$ apart because every second vertex in $G_{j}^{h}$-layer, $h \in A_{H}, k+2 \leq j \leq|H|+1$, is already colored (with color 1). Therefore, we can color with $j$ vertices from set

$$
\left\{\left(v_{2+s p_{j}}, h\right): 0 \leq s \leq\left\lfloor\frac{\left\lfloor\frac{n}{2}\right\rfloor-1}{\left\lfloor\frac{j}{2}\right\rfloor+1}\right\rfloor\right\} .
$$

Meaning that $t_{j}=\left(\left\lfloor\frac{\left\lfloor\frac{n}{2}\right\rfloor-1}{\left\lfloor\frac{j}{2}\right\rfloor+1}\right\rfloor+1\right)$ vertices can be colored with color $j, k+2 \leq j \leq|H|+1$ in $G_{j}^{h}$-layer, $h \in A_{H}$.

By Theorem 2.2 we use at most $\ell$ colors for coloring $P_{n} \circ H$. In addition $t_{j}$ vertices of $G_{j}^{h}$ are colored with $j, k+2 \leq j \leq|H|+1$. Meaning that

$$
\chi_{\rho}\left(P_{n} \circ H\right) \leq \ell-\sum_{j=k+2}^{|H|+1} t_{j}+|H|-k
$$

which completes the proof.
For $H \cong K_{m}$ we have $\alpha(H)=1$ and $k=m-1$. The second sum of Theorem 2.6 has only one term and that is in the case of $j=|H|+1$ so we immediately obtain the following.
Corollary 2.7. For positive integers $n$ and $m$ we have

$$
\chi_{\rho}\left(P_{n} \circ K_{m}\right) \leq n m-\left\lceil\frac{n}{2}\right\rceil-\sum_{i=2}^{m}\left\lceil\frac{n}{i+1}\right\rceil-\left\lfloor\frac{\left\lfloor\frac{n}{2}\right\rfloor-1}{\left\lfloor\frac{m+1}{2}\right\rfloor+1}\right\rfloor+m .
$$

The upper bound from Theorem [2.6 is not the best possible in the general case which we can see in the example of coloring $P_{8} \circ P_{6}$. Using the coloring described in the proof of that theorem we use 32 colors to color $P_{8} \circ P_{6}$, see left part of Figure 1. But the same graph can be colored with 31 colors, so $\chi_{\rho}\left(P_{8} \circ P_{6}\right) \leq 31$, see right part of Figure 1.


Figure 1: Packing coloring for $P_{8} \circ P_{6}$ using 32 colors according to Theorem 2.6 (a) and 31 colors (b) (not all edges of a graph are drawn).

Another example can be constructed as follows. Let $n_{t}=1+\operatorname{lcm}(2,3, \ldots, t+1), H \nsubseteq \bar{K}_{m}$ a graph and $k=|H|-\alpha(H)$. Notice that $n_{t}$ is chosen in such a way that every $\rho_{i}\left(P_{n_{t}}\right)$-set, $1 \leq i \leq t$, contains the first and the last vertex of $P_{n_{t}}$. If $t-1>k$, then we cannot obtain $\rho_{i}\left(P_{n_{t}}\right)$ vertices of color $i$ in $P_{n_{t}} \circ H$ for some $2 \leq i \leq t$ and the upper bound of Theorem [2.6 is not exact.

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