A note on the packing chromatic number of lexicographic products^{*}

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Abstract

The packing chromatic number $\chi_{\rho}(G)$ of a graph G is the smallest integer k such that there exists a k-vertex coloring of G in which any two vertices receiving color i are at distance at least i + 1. In this short note we present upper and lower bound for the packing chromatic number of the lexicographic product $G \circ H$ of graphs G and H. Both bounds coincide in many cases. In particular this happens if $|V(H)| - \alpha(H) \ge \operatorname{diam}(G) - 1$, where $\alpha(G)$ denotes the independence number of G.

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1 Introduction and preliminaries

Let G be a simple graph. To shorten the notation we use |G| instead of |V(G)| for the order of G. The distance $d_G(u, v)$ between vertices u and v of G is the length of a shortest path between u and v in G. The diameter of G is denoted by diam(G) and is the maximum length of a shortest path between any two vertices of G.

Let t be a positive integer. A set $X \subseteq V(G)$ is a t-packing if any two different vertices from X are at distance more than t. The t-packing number of G, denoted by $\rho_t(G)$, is the maximum cardinality of a t-packing of G. Notice, that if t = 1, then the 1-packing number equals to the independence number $\alpha(G)$ and we use the later more common notation for it. An independent set of cardinality $\alpha(G)$ is called $\alpha(G)$ -set. The packing chromatic number $\chi_{\rho}(G)$ of G is the smallest integer k such that V(G) can be partitioned into subsets X_1, \ldots, X_k , where X_i induces an *i*-packing for every $1 \leq i \leq k$. Another approach is from a k-packing coloring of G, which is a function $c: V(G) \to [k]$, where $[k] = \{1, \ldots, k\}$, such that if c(u) = c(v) = i, then $d_G(u, v) > i$. Clearly, $\chi_{\rho}(G)$ is the minimum integer k for which a k-packing coloring of G exists.

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The concept of packing chromatic number was introduced by Goddard et al. in [6] under the name broadcast chromatic number. The problem of determining the packing chromatic number of a graph is a very difficult problem and is NP-complete even for trees as shown in [2]. The attention was fast drawn to Cartesian product and infinite latices like hexagonal, triangular and similar. In [1] it was shown that the packing chromatic number of an infinite hexagonal lattice lies between 6 and 8. Upper bound was later improved to 7 in [3] and finally settled to 7 in [9]. For infinite triangular lattice and three-dimensional integer lattice \mathbb{Z}^3 the packing chromatic number is infinite as shown in [4]. The packing chromatic number of the Cartesian product was already considered in [1] where the general upper and lower bound were set. The lower bound was later improved in [8]. Several exact values and bounds for special families of Cartesian product graphs can be found in [8, 9].

In this note we switch from Cartesian to lexicographic product and prove an upper and a lower bound for the packing chromatic number of lexicographic product. It turns out that these two bounds coincide in many cases. In particular, if diam $(G) \leq 2$, if diam(G) = 3 and $H \ncong \overline{K}_n$ and if $|V(H)| - \alpha(H) \geq \operatorname{diam}(G) - 1$ and $H \ncong \overline{K}_n$.

The *lexicographic product* of graphs G and H is the graph $G \circ H$ (also sometimes denoted with G[H]) with the vertex set $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Set $G^h = \{(g, h) : g \in V(G)\}$ is called a *G*-layer through h and $H^g = \{(g, h) : h \in V(H)\}$ is called an *H*-layer through h. Clearly, subgraphs of $G \circ H$ induced by G^h and H^g are isomorphic to G and H, respectively. The distance between two vertices in lexicographic product is given by

$$d_G((g_1, h_1), (g_2, h_2)) = \begin{cases} d_G(g_1, g_2) & : & g_1 \neq g_2\\ \min\{2, d_H(h_1, h_2)\} & : & g_1 = g_2 \end{cases}$$
(1)

and depends heavily on the distance between projections of both vertices to G. For the independence number it is well known that

$$\alpha(G \circ H) = \alpha(G)\alpha(H), \tag{2}$$

see Theorem 1 in [5]. Lexicographic product $G \circ H$ is connected if and only if G is connected. For more properties of the lexicographic product see [7].

2 Results

In this section we present a lower and an upper bound for the packing chromatic number of lexicographic product of graphs. We start with the lower bound and we use the following notation

$$d(G) = \begin{cases} 1 & : \quad G \cong K_n \\ \operatorname{diam}(G) - 1 & : \quad \text{otherwise} \end{cases}$$

Theorem 2.1. If G and H are graphs, then

$$\chi_{\rho}(G \circ H) \ge |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\operatorname{diam}(G)-1} \rho_i(G) + d(G).$$

Proof. Denote $\ell = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\dim(G)-1} \rho_i(G) + d(G)$. Let X_1, \ldots, X_k be a partition of $V(G \circ H)$ that yields a k-packing coloring of $G \circ H$. We have at most $\alpha(G)\alpha(H)$ vertices in X_1 by (2). Denote by R_i a $\rho_i(G \circ H)$ -set for $2 \le i \le \dim(G) - 1$. By (1) we have $|H^g \cap R_i| \le 1$ for every

 $g \in V(G)$. So there are at most $\rho_i(G)$ vertices in X_i for $2 \leq i \leq \operatorname{diam}(G) - 1$. For $i \geq \operatorname{diam}(G)$ there can only be one vertex in X_i since all the vertices are at distance at most $\operatorname{diam}(G)$ from vertex in X_i . So we have at most $\alpha(G)\alpha(H)$ vertices colored with color 1, at most $\rho_i(G)$ vertices colored with color *i* for every $2 \leq i \leq \operatorname{diam}(G) - 1$, and we need one color for each one of the remaining vertices and there are $|G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\operatorname{diam}(G)-1} \rho_i(G)$ of them. Meaning that $\chi_{\rho}(G \circ H) \geq \ell$ because we have exactly d(G) color classes which possibly have more than one vertex.

We continue with an upper bound that has a similar structure as the lower bound from Theorem 2.1.

Theorem 2.2. Let G and H be graphs and $k = |H| - \alpha(H)$. If $H \ncong \overline{K}_n$, then

$$\chi_{\rho}(G \circ H) \le |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{k+1} \rho_i(G) + k + 1.$$

Proof. Denote $\ell = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{k+1} \rho_i(G) + k + 1$. We know that $\alpha(G \circ H) = \alpha(G)\alpha(H)$ and it is easy to see that $\alpha(G \circ H)$ -set can be written as $A_G \times A_H$ where A_G is an $\alpha(G)$ -set and A_H is an $\alpha(H)$ -set. We color all the vertices from $A_G \times A_H$ with color 1. Let $k = |H| - \alpha(H)$. There remain k G-layers with no colored vertices. In each of those layers we color $\rho_i(G)$ vertices with color $i, 2 \leq i \leq k+1$ (one color i is used in one layer). Each of the remaining uncolored vertices is colored with its own color. So we have $\alpha(G)\alpha(H)$ vertices colored with color 1, $\rho_i(G)$ vertices colored with color i for every $2 \leq i \leq k+1$, and we need one color for each one of the remaining uncolored vertices. Clearly, there are $|G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{k+1} \rho_i(G)$ vertices colored with its own color. Meaning that $\chi_{\rho}(G \circ H) \leq \ell$ because we have k + 1 color classes which possibly have more than one vertex.

Notice that if diam $(G) \leq 2$, then also diam $(G \circ H) \leq 2$ (see 1) and only color 1 can appear more then once in any packing coloring. Therefore, if diam $(G) \leq 2$, Theorem 2.2 also holds for $H \cong \overline{K}_n$. Next we show that if the number of vertices of H without its $\alpha(H)$ -set is comparable with diam(G), then both bounds coincide.

Corollary 2.3. Let G and H be graphs and $|H| - \alpha(H) \ge \operatorname{diam}(G) - 1$. If $H \ncong \overline{K}_n$, then

$$\chi_{\rho}(G \circ H) = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\operatorname{diam}(G)-1} \rho_i(G) + \operatorname{diam}(G) - 1.$$

Proof. Let first $G \cong K_n$. By Theorem 2.1 it holds that $\chi_{\rho}(G \circ H) \ge n|H| - \alpha(H) - \sum_{i=2}^{0} \rho_i(G) + 1 = n|H| - \alpha(H) + 1$ since d(G) = 1. On the other hand let $k = |H| - \alpha(H)$ and we have $\chi_{\rho}(G \circ H) \le n|H| - \alpha(H) - (k+1-2+1) + k + 1 = n|H| - \alpha(H) + 1$ by Theorem 2.2 since $\rho_i(G) = 1$ for every $2 \le i \le k+1$. Hence, the equality follows when $G \cong K_n$.

Otherwise $G \not\cong K_n$ and $d(G) = \operatorname{diam}(G) - 1$. So by Theorem 2.1 it holds that $\chi_{\rho}(G \circ H) \geq |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\operatorname{diam}(G)-1} \rho_i(G) + \operatorname{diam}(G) - 1$. Since $k \geq \operatorname{diam}(G) - 1$ and $\rho_i(G) = 1$ for

every diam $(G) \leq i \leq k+1$, by Theorem 2.2 it holds that

$$\chi_{\rho}(G \circ H) \leq |G| \cdot |H| - \alpha(G)\alpha(H) - \left(\sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G) + \sum_{i=\operatorname{diam}(G)}^{k+1} \rho_{i}(G)\right) + k + 1 = \\ = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G) - (k + 1 - \operatorname{diam}(G) + 1) + k + 1 =$$
(3)
$$= |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\operatorname{diam}(G)-1} \rho_{i}(G) + \operatorname{diam}(G) - 1.$$

We can expect that the condition of Corollary 2.3 will be fulfilled more frequently when diam(G) is small. In particular, for diam(G) = 1 the condition is always satisfied and we have

$$\chi_{\rho}(G \circ H) = |G| \cdot |H| - \alpha(H) + 1$$

as seen in the proof of the previous corollary. Notice that in the case of diam(G) = 2 the sum in the lower bound of Theorem 2.1 does not exist and that d(G) = 1. Also $\rho_i(G) = 1$ for every $2 \le i \le k$ since diam(G) = 2. Therefore we have $-\sum_{i=2}^{k+1} \rho_i(G) + k + 1 = -(k+1-2+1) + k + 1 = 1$ in the upper bound of Theorem 2.2. Hence both bounds coincide and we have the following corollary.

Corollary 2.4. Let G and H be graphs. If diam(G) = 2, then

$$\chi_{\rho}(G \circ H) = |G| \cdot |H| - \alpha(G)\alpha(H) + 1.$$

Similar holds also when diam(G) = 3. Namely in this case diam $(G \circ H) = 3$ by (1) and only two color classes $(X_1 \text{ and } X_2)$ can have more than one representative. Therefore bounds from Theorems 2.2 and 2.1 coincide again under condition that there is at least one *G*-layer without vertices from X_1 . This always occurs if $H \ncong \overline{K}_n$ and the following corollary holds.

Corollary 2.5. Let G and H be graphs. If diam(G) = 3 and $H \ncong \overline{K}_n$, then

$$\chi_{\rho}(G \circ H) = |G| \cdot |H| - \alpha(G)\alpha(H) - \rho_2(G) + 2$$

Continuing in this manner things get more complicated. Therefore we finish with an approach from the different side and concentrate on a family of graphs with big diameter, namely the case when $G \cong P_n$. For this we first improve the upper bound from Theorem 2.2.

Theorem 2.6. Let H a graph and n a positive integer. If $k = |H| - \alpha(H)$, then

$$\chi_{\rho}(P_n \circ H) \le n|H| - \left\lceil \frac{n}{2} \right\rceil \alpha(H) - \sum_{i=2}^{k+1} \left\lceil \frac{n}{i+1} \right\rceil - \sum_{j=k+2}^{|H|+1} \left(\left\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{\lfloor \frac{j}{2} \rfloor + 1} \right\rfloor + 1 \right) + |H| + 1.$$

Proof. Let $P_n = v_1 \dots v_n$ and A_H be an $\alpha(H)$ -set. Clearly, $A_{P_n} = \{v_{2i-1} : i \in \lfloor \lfloor \frac{n}{2} \rfloor\}$ is an $\alpha(P_n)$ -set and $A = A_{P_n} \times A_H$ is an $\alpha(P_n \circ H)$ -set. Firstly, we color vertices with k + 1 colors as in the proof of Theorem 2.2. For this we use

$$\ell = n|H| - \left\lceil \frac{n}{2} \right\rceil \alpha(H) - \sum_{i=2}^{k+1} \left\lceil \frac{n}{i+1} \right\rceil + k + 1$$

colors because $\rho_i(P_n) = \left\lceil \frac{n}{i+1} \right\rceil$.

In each G^h -layer, $h \in A_H$, there exist $\lfloor \frac{n}{2} \rfloor$ still not colored vertices with an even distance between any two of them. We denote them by $B^h = (V(P_n) - A_{P_n}) \times \{h\}$. Additionally we will color with color $j, k + 2 \leq j \leq |H| + 1$, some vertices of exactly one G^h -layer, $h \in A_H$. Denote by G_j^h the G^h -layer, $h \in A_H$, containing vertices of color $j, k + 2 \leq j \leq |H| + 1$. The biggest distance between two vertices from B^h equals $2 \lfloor \frac{n}{2} \rfloor - 2$. Notice that two vertices of G_j^h colored with j must be at least $p_j = 2 \lfloor \frac{j}{2} \rfloor + 2$ apart because every second vertex in G_j^h -layer, $h \in A_H, k + 2 \leq j \leq |H| + 1$, is already colored (with color 1). Therefore, we can color with j vertices from set

$$\{(v_{2+sp_j}, h) : 0 \le s \le \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor - 1}{\left\lfloor \frac{j}{2} \right\rfloor + 1} \right\rfloor\}.$$

Meaning that $t_j = \left(\left\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{\lfloor \frac{j}{2} \rfloor + 1} \right\rfloor + 1 \right)$ vertices can be colored with color $j, k + 2 \le j \le |H| + 1$ in G_j^h -layer, $h \in A_H$.

By Theorem 2.2 we use at most ℓ colors for coloring $P_n \circ H$. In addition t_j vertices of G_j^h are colored with $j, k+2 \leq j \leq |H|+1$. Meaning that

$$\chi_{\rho}(P_n \circ H) \le \ell - \sum_{j=k+2}^{|H|+1} t_j + |H| - k$$

which completes the proof.

For $H \cong K_m$ we have $\alpha(H) = 1$ and k = m - 1. The second sum of Theorem 2.6 has only one term and that is in the case of j = |H| + 1 so we immediately obtain the following.

Corollary 2.7. For positive integers n and m we have

$$\chi_{\rho}(P_n \circ K_m) \le nm - \left\lceil \frac{n}{2} \right\rceil - \sum_{i=2}^m \left\lceil \frac{n}{i+1} \right\rceil - \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor - 1}{\left\lfloor \frac{m+1}{2} \right\rfloor + 1} \right\rfloor + m.$$

The upper bound from Theorem 2.6 is not the best possible in the general case which we can see in the example of coloring $P_8 \circ P_6$. Using the coloring described in the proof of that theorem we use 32 colors to color $P_8 \circ P_6$, see left part of Figure 1. But the same graph can be colored with 31 colors, so $\chi_{\rho}(P_8 \circ P_6) \leq 31$, see right part of Figure 1.



Figure 1: Packing coloring for $P_8 \circ P_6$ using 32 colors according to Theorem 2.6 (a) and 31 colors (b) (not all edges of a graph are drawn).

Another example can be constructed as follows. Let $n_t = 1 + lcm(2, 3, ..., t + 1)$, $H \ncong \overline{K}_m$ a graph and $k = |H| - \alpha(H)$. Notice that n_t is chosen in such a way that every $\rho_i(P_{n_t})$ -set, $1 \le i \le t$, contains the first and the last vertex of P_{n_t} . If t - 1 > k, then we cannot obtain $\rho_i(P_{n_t})$ vertices of color i in $P_{n_t} \circ H$ for some $2 \le i \le t$ and the upper bound of Theorem 2.6 is not exact.

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