On selecting a fraction of leaves with disjoint neighborhoods in a plane tree^{*}

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Abstract

We present a generalization of a combinatorial result by Aggarwal, Guibas, Saxe and Shor Discrete & Computational Geometry, 1989 on a linear-time algorithm that selects a constant fraction of leaves, with pairwise disjoint neighborhoods, from a binary tree embedded in the plane. This result of Aggarwal et al. is essential to the lineartime framework, which they also introduced, that computes certain Voronoi diagrams of points with a tree structure in linear time. An example is the diagram computed while updating the Voronoi diagram of points after deletion of one site. Our generalization allows that only a fraction of the tree leaves is considered, and it is motivated by lineartime Voronoi constructions for non-point sites. We are given a plane tree T of n leaves, m of which have been marked, and each marked leaf is associated with a *neighborhood* (a subtree of T) such that any two topologically consecutive marked leaves have disjoint neighborhoods. We show how to select in linear time a constant fraction of the marked leaves having pairwise disjoint neighborhoods.

1 Introduction

In 1987, Aggarwal, Guibas, Saxe and Shor [1] introduced a linear-time technique to compute the Voronoi diagram of points in convex position, which

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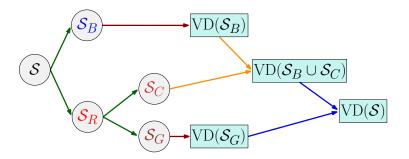


Figure 1: The divide & conquer algorithm of [1]. (\longrightarrow) indicates the two different divide phases; (\longrightarrow) indicates the recursive constructions; (\longrightarrow) indicates site insertion; (\longrightarrow) indicates merging.

can also be used to compute other Voronoi diagrams of point-sites with a tree structure such as: (1) updating a nearest-neighbor Voronoi diagram of points after deletion of one site; (2) computing the farthest-point Voronoi diagram, after the convex hull of the points is known; (3) computing an order-k Voronoi diagram of points, given its order-(k-1) counterpart. Since then, this framework has been used (and extended) in various ways to tackle various linear-time Voronoi constructions, including the medial axis of a simple polygon by Chin et al. [4], the Hamiltonian abstract Voronoi diagram by Klein and Lingas [8], and some forest-like abstract Voronoi diagrams by Bohler et al. [2]. The linear-time construction for problem (3) improves by a logarithmic factor the standard iterative construction by Lee [9] to compute the order-k Voronoi diagram of point-sites, which is in turn used in different scenarios; for example, algorithms for coverage problems in wireless networks by So and Ye [10]. A much simpler randomized linear-time approach for problems (1)-(3) was introduced by Chew [3].

The linear-time technique of Aggarwal et al. [1] is a doubly-recursive divide-and-conquer scheme operating on an ordered set of points S whose Voronoi diagram is a tree with connected Voronoi regions. At a high level it can be described as follows, see Figure 1. In an initial divide phase, the set S is split in two sets S_R (red) and S_B (blue) of roughly equal size, with the property that every two consecutive red sites in S_R have disjoint Voronoi regions. In a second divide phase, the set S_R is split further in sets S_C (crimson) and S_G (garnet), so that any two sites in S_C have pairwise disjoint regions in the Voronoi diagram of $S_B \cup S_C$, and the cardinality of S_C is a constant fraction of the cardinality of S_R . In the merge phase, the sites of S_C are inserted one by one in the recursively computed Voronoi diagram of S_B , deriving the Voronoi diagram of $S_C \cup S_B$, and the result is merged with the recursively computed diagram of S_G .

The key factor in obtaining the linear-time complexity is that the cardinality of the set S_C is a constant fraction of S_R , which is $\Theta(|S|)$, and that S_C can be obtained in linear time. This is possible due to the following combinatorial result of [1] on a geometric binary tree embedded in the plane. This result is, thus, inherently used by any algorithm that is based on the lineartime framework of Aggarwal et al. A binary tree that contains no nodes of degree 2 is called *proper*.

Theorem 1 ([1]). Let \mathcal{T} be an unrooted (proper) binary tree embedded in the plane. Each leaf of \mathcal{T} is associated with a neighborhood, which is a (proper) subtree of \mathcal{T} rooted at that leaf; consecutive leaves in the topological ordering of \mathcal{T} have disjoint neighborhoods. Then, there exists a fixed fraction of the leaves whose neighborhoods are pairwise disjoint, they have a constant size, and no tree edge has its endpoints in two different neighborhoods. Such a set of leaves can be found in linear time.

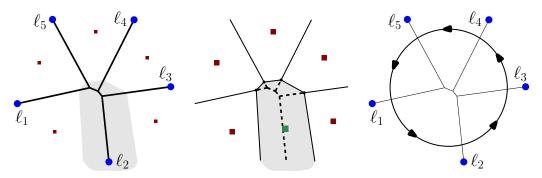
Overall, the time complexity of the algorithm is described by the following recursive equation and can be proved to be $\Theta(n)$, where $n = |\mathcal{S}|$.

$$T(n) = T(|\mathcal{S}_B|) + T(|\mathcal{S}_G|) + \Theta(|\mathcal{S}_R|) + |\mathcal{S}_C| \cdot \Theta(1) + \Theta(n)$$

= $T(|\mathcal{S}_B|) + T(|\mathcal{S}_G|) + \Theta(n)$
= $\Theta(n)$ (Because $|\mathcal{S}_C| = \Theta(n)$)

It is worth understanding what Theorem 1 represents, in order to have a spherical perspective of its connection to Voronoi diagrams. An embedded tree corresponds to the graph structure of a Voronoi diagram, and leaves are the endpoints of unbounded Voronoi edges "at infinity"; see Figure 2a. The neighborhood of a leaf corresponds to the part of the diagram (of S_B) that gets deleted if a point-site is inserted there; see Figure 2b. Hence, Theorem 1 aims to select leaves with pairwise disjoint neighborhoods (S_C), as they can easily, and independently from one another, be inserted in the diagram.

For generalized sites, other than points in the plane, or for abstract Voronoi diagrams, deterministic linear-time algorithms for the counterparts of problems (1)-(3) have not been known so far. This includes the diagrams of very simple geometric sites such as line segments and circles in the Euclidean plane. A major complication over points is that the underlying diagrams have disconnected Voronoi regions. Recently Papadopoulou et al. [6, 7]



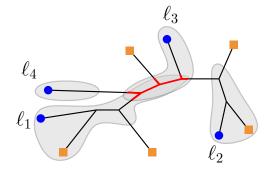
(a) The Voronoi diagram (b) The dashed part of the (c) The topological orderof 5 points (\blacksquare); the neighborhood of ℓ_2 is shaded. point (\blacksquare) is inserted.

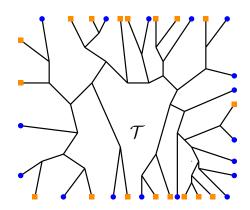
Figure 2: An embedded binary tree \mathcal{T} in the setting of Aggarwal et al. [1].

presented a randomized linear-time technique for these problems, based on a relaxed Voronoi structure, called a *Voronoi-like diagram* [6, 7]. Whether this structure can be used within the framework of Aggarwal et al., leading to deterministic linear-time constructions, remains still an open problem. Towards resolving this problem we need a generalized version of Theorem 1.

The problem is formulated as follows. We have an unrooted binary tree \mathcal{T} embedded in the plane, which corresponds to a Voronoi-like structure. Not all leaves of \mathcal{T} are eligible for inclusion in the set S_C of the linear-time framework. As in the original problem, each of the eligible leaves is associated with a neighborhood, which is a subtree of \mathcal{T} rooted at that leaf, and adjacent leaves in the topological ordering of \mathcal{T} have disjoint neighborhoods. In linear time, we need to compute a constant fraction of the eligible leaves such that their neighborhoods are pairwise disjoint. The non-eligible leaves spread arbitrarily along the topological ordering of the tree leaves. This paper addresses this problem by proving the following generalization of Theorem 1.

Theorem 2. Let \mathcal{T} be an unrooted (proper) binary tree embedded in the plane having n leaves, m of which have been marked. Each marked leaf of \mathcal{T} is associated with a neighborhood, which is a proper subtree of \mathcal{T} rooted at this leaf, and any two consecutive marked leaves in the topological ordering of \mathcal{T} have disjoint neighborhoods. Then, there exist at least $\frac{1}{10}m$ marked leaves whose neighborhoods are pairwise disjoint and no tree edge has its endpoints in two of these neighborhoods. Further, we can select at least a fraction p of these $\frac{1}{10}m$ marked leaves in time $O(\frac{1}{1-p}n)$, for any $p \in (0,1)$.





(a) Neighborhoods of the marked leaves are shown shaded. The intersection of neighborhoods is highlighted with red.

(b) A marked tree \mathcal{T} , which serves as an example instance for illustrating the notions of Section 2.

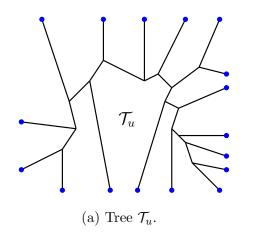
Figure 3: Two marked trees, where marked leaves are shown with (\bullet) and unmarked leaves are shown with (\bullet) .

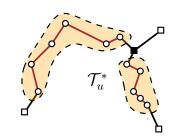
The algorithm of Theorem 2 allows for a trade-off between the number of the returned marked leaves and its time complexity, using a parameter $p \in (0, 1)$. If p is constant then the algorithm returns a constant fraction of the marked leaves in O(n) time. Theorem 2 is a combinatorial result on an embedded tree, and thus, we expect it to find applications in different contexts as well.

2 Preliminaries

Throughout this work, we consider an unrooted binary tree \mathcal{T} of n leaves that is embedded in the plane. The tree \mathcal{T} contains no nodes of degree 2 and has the following additional properties:

- m out of the n leaves of \mathcal{T} have been marked, and the remaining r = n-m leaves are unmarked (see Figure 3b).
- Every marked leaf ℓ is associated with a *neighborhood*, denoted $nh(\ell)$, which is a subtree of \mathcal{T} rooted at ℓ (see Figure 3a).
- Every two consecutive marked leaves in the topological ordering of \mathcal{T} have disjoint neighborhoods (see Figure 3a).





(b) Tree \mathcal{T}_u^* : *L*-nodes are shown with (\Box), *C*-nodes with (\circ), *J*-nodes with (\bullet), and spines are highlighted.

Figure 4: Illustration of Definition 1 applied to the tree \mathcal{T} of Figure 3b.

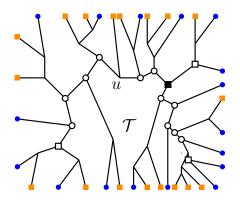
We call a binary tree \mathcal{T} that follows these properties, a marked tree. Given a marked tree \mathcal{T} , let \mathcal{T}_u denote the unmarked tree obtained by deleting all the unmarked leaves of \mathcal{T} and contracting the resulting degree-2 nodes, see Figure 4a. We apply to \mathcal{T}_u the following definition, which is extracted from the proof of Theorem 1 in [1], see Figure 4.

Definition 1. Let T be a proper binary tree and let T^* be the tree obtained from T after deleting all its leaves. A node u in T^* is called:

- a) Leaf or L-node if deg(u) = 1 in T^* , i.e., u neighbors two leaves in T.
- b) Comb or C-node if deg(u) = 2 in T^* , i.e., u neighbors one leaf in T.
- c) Junction or J-node if deg(u) = 3 in T^* , i.e., u neighbors no leaves in T.

A *spine* is a maximal sequence of consecutive C-nodes, which is delimited by J- or L-nodes. Each spine has two *sides* and marked leaves may lie in either side of a spine.

Let \mathcal{T}_u^* , be the tree obtained by applying Definition 1 to the unmarked tree \mathcal{T}_u . The nodes \mathcal{T}_u^* are labeled as *L*-, *C*- and *J*-nodes, see, e.g., Figure 4b. The labeling of nodes in \mathcal{T}_u^* is then carried back to their corresponding nodes in the original marked tree \mathcal{T} obtaining a marked tree \mathcal{T} with labels, see Figure 6. Some nodes in \mathcal{T} remain unlabeled, see, e.g., node *u* in Figure 6.



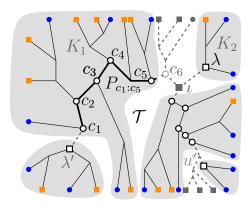


Figure 5: The marked tree \mathcal{T} of Figure 3b with labels (\circ, \Box, \bullet) . Node u is not labeled.

Figure 6: The components of \mathcal{T} shown shaded. The dashed parts do not belong to any component.

Definition 2. Given a marked tree \mathcal{T} with labels we define the following two types of *components*:

- a) *L-component*: an *L*-node λ defines an *L*-component that consists of λ union the two subtrees of \mathcal{T} that are incident to λ and contain no labeled node, see, e.g., K_2 in Figure 6. The *L*-component contains exactly the two marked leaves that labeled λ .
- b) 5-component: a group of five successive C-nodes c_i, \ldots, c_{i+4} on a spine defines a 5-component that consists of the path $P_{c_i:c_{i+4}}$ from c_i to c_{i+4} (which may contain unlabeled nodes) union the subtrees of \mathcal{T} , which are incident to the nodes of $P_{c_i:c_{i+4}}$ and contain no labeled node, see, e.g., K_1 in Figure 6. Nodes c_i and c_{i+4} are referred to as the *extreme nodes* of K. The 5-component contains exactly the five marked leaves, which labeled the five C-nodes.

Each spine is partitioned into consecutive groups of 5-components and at most four remaining *ungrouped* C-nodes.

Figure 5 and Figure 6 illustrates these definitions. The tree \mathcal{T} has three *L*-components and two 5-components which are indicated shaded in Figure 6. The 5-component K_1 contains the path $P_{c_1:c_5}$ from c_1 to c_5 , which is shown in thick black lines, and contains one unlabeled node. Node c_6 is an ungrouped *C*-node. Figure 6 also illustrates a spine consisting of the *C*-nodes $c_1, c_2, c_3, c_4, c_5, c_6$. The spine is delimited by the *L*-node λ' and the *J*-node ι ; it has five marked leaves from one side and one marked leaf from the other.

Observation 1. The components of \mathcal{T} are pairwise vertex disjoint. Every L-component contains exactly two marked leaves and every 5-component contains exactly five marked leaves.

Among the components of \mathcal{T} there may be subtrees of \mathcal{T} consisting of unlabeled nodes and unmarked leaves that may be arbitrarily large. These subtrees hang off any unlabeled nodes and ungrouped *C*-nodes. For example, in Figure 6, node u' is unlabeled and the gray dotted subtree incident to it consists solely of unmarked leaves and unlabeled nodes that do not belong to any component.

3 Existence of leaves with pairwise disjoint neighborhoods

Aggarwal et al. [1] showed that for every eight ungrouped C-nodes in \mathcal{T}_u there exists at least one L-node. Their argument holds for the marked tree \mathcal{T} as well, which is described in the following lemma for completeness.

Lemma 1. For every eight ungrouped C-nodes in \mathcal{T} there exists at least one L-component.

Proof. We count the *L*-nodes of \mathcal{T} using the tree \mathcal{T}_u^* following the argument of [1]. Let *k* be the number of leaves in \mathcal{T}_u^* , which also equals the number of *L*-nodes in \mathcal{T} . Contracting all degree-2 vertices in \mathcal{T}_u^* yields a binary tree \mathcal{T}_b^* , which has the same leaves as \mathcal{T}_u^* . Since \mathcal{T}_b^* is an unrooted binary tree with *k* leaves, it has 2k-2 nodes and 2k-3 edges. Every edge in \mathcal{T}_b^* corresponds to at most one spine in \mathcal{T}_u^* and in every spine there are at most four ungrouped *C*-nodes. Thus,

 $|\text{ungrouped } C\text{-nodes}| \leq 4|\text{spines}| \leq 4 \cdot (2k-3) < 8|L\text{-nodes}|,$

where $|\cdot|$ denotes cardinality. So, there exists at least one *L*-node for every eight ungrouped *C*-nodes, and an *L*-node corresponds to exactly one *L*-component.

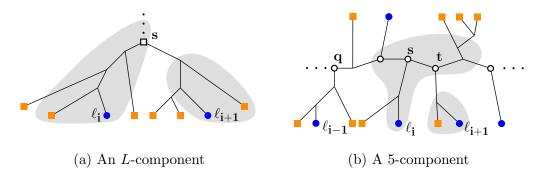


Figure 7: Marked leaves with their neighborhoods shaded. The neighborhood $nh(\ell_i)$ is confined to the component in both cases.

The following lemmata establish that there exists a constant fraction of the marked leaves, which have pairwise disjoint neighborhoods. The counting arguments follow those in [1] while they are further enhanced to account for the unmarked leaves, which are arbitrarily distributed among the marked leaves. We say that the neighborhood $nh(\ell)$ of a marked leaf ℓ is confined to a component K if it is a subtree of K.

Lemma 2. In every component K, there exists a marked leaf $\ell \in K$ whose neighborhood is confined to K. This neighborhood may contain no L-node and no extreme C-node.

Proof. Let K be an L-component and let s be the L-node that defines K. Let ℓ_i and ℓ_{i+1} be the two marked leaves of K. Since the neighborhoods $nh(\ell_i)$ and $nh(\ell_{i+1})$ are disjoint, at least one of them cannot contain s. This neighborhood is, thus, entirely contained in the relevant subtree rooted at s, see Figure 7a, and contains no labeled node.

Let K be a 5-component. Since a 5-component has two sides, at least three out of the five marked leaves of the component must lie on the same side of K, call them ℓ_{i-1}, ℓ_i and ℓ_{i+1} . Let q, s, and t be their corresponding C-nodes, i.e., the first C-nodes in K reachable from ℓ_{i-1}, ℓ_i , and ℓ_{i+1} , respectively, see Figure 7b. There are three cases. If $t \in nh(\ell_i)$, then $t \notin nh(\ell_{i+1})$ (since the two neighborhoods are disjoint), and thus, $nh(\ell_{i+1})$ is confined to the subtree of t that contains ℓ_{i+1} . Similarly, if $q \in nh(\ell_i)$, then $q \notin nh(\ell_{i-1})$, so $nh(\ell_{i-1})$ is confined to the subtree of q containing ℓ_{i-1} . If neither q nor t are in $nh(\ell_i)$, then clearly $nh(\ell_i)$ is confined to K. In all cases the confined neighborhood cannot contain neither q nor t. So, at least one of the five marked leaves must have a neighborhood confined to K and this neighborhood cannot contain the extreme C-nodes in K.

Lemma 3. Let \mathcal{T} be a marked tree with m marked leaves. At least $\frac{1}{10}m$ marked leaves must have pairwise disjoint neighborhoods such that no tree edge may have its endpoints in two different neighborhoods.

Proof. Every spine of \mathcal{T} has up to four ungrouped *C*-nodes. By Lemma 1, there exists at least one *L*-component for every eight ungrouped *C*-nodes. By Lemma 2, every component of \mathcal{T} has at least one marked leaf whose neighborhood is confined to the component. So, overall, at least $\frac{1}{5}$ of the marked leaves from each 5-component and at least $\frac{1}{10}$ marked leaves of the remaining nodes, which label ungrouped *C*-nodes or *L*-nodes, have a confined neighborhood. The components are pairwise disjoint, so at least $\frac{1}{10}$ marked leaves have pairwise disjoint neighborhoods. Furthermore, confined neighborhoods do not contain any *L*-node or extreme *C*-node, as shown in Lemma 2. Thus, no tree edge may have its endpoints in two different neighborhoods.

We remark that the neighborhoods implied by Lemma 3 may not contain any *L*-node nor any extreme *C*-node. We also remark that these neighborhoods need not be of constant complexity as their counterparts in [1] are. These neighborhoods may have complexity $\Theta(r)$, where r = n - m is the number of unmarked leaves. Since r may be $\Theta(n)$, this poses a challenge on how we can select these leaves efficiently.

4 Selecting leaves with pairwise disjoint neighborhoods

Given a marked tree \mathcal{T} with m marked leaves, we have already established the existence of $\frac{1}{10}m$ marked leaves that have pairwise disjoint neighborhoods. In this section, we present an algorithm to select a fraction p of these leaves, i.e., $\frac{p}{10}m$ marked leaves with pairwise disjoint neighborhoods, in time $O(\frac{1}{1-p}n)$, where 0 .

The main challenge over the algorithm of [1] is that the r unmarked leaves are arbitrarily distributed among the m marked leaves, and thus, the components of \mathcal{T} and the neighborhoods of the marked leaves may have complexity $\Theta(r)$. If for each component we spend time proportional to its size, then the time complexity of the algorithm will be $\Theta(mr)$, i.e., $\Theta(n^2)$ if $r, m \in \Theta(n)$.

To keep the complexity of the algorithm linear, we spend time up to a predefined number of steps in each component depending on the ratio $c = \left[\frac{r}{m}\right]$ and the trade-off parameter $p \in (0, 1)$. Our algorithm guarantees to find at least a fraction p of the possible $\frac{1}{10}m$ marked leaves in time $O(\frac{1}{1-p}n)$. We first present a series of results necessary to establish the correctness of the approach and then describe the algorithm.

Let ℓ_1, \ldots, ℓ_m be the marked leaves in \mathcal{T} ordered in a counterclockwise topological ordering. Let the *interval* (ℓ_i, ℓ_{i+1}) denote the set of unmarked leaves between ℓ_i and ℓ_{i+1} in the same order. The *interval tree* of (ℓ_i, ℓ_{i+1}) , denoted $T_{(\ell_i, \ell_{i+1})}$, is the minimal subtree of \mathcal{T} that contains the marked leaves ℓ_i and ℓ_{i+1} , including the unmarked leaves in (ℓ_i, ℓ_{i+1}) , see Figure 8b. We show the following *pigeonhole lemma* involving unmarked leaves and intervals.

Lemma 4. Suppose that r items (unmarked leaves) are distributed in $k \ge m$ containers (intervals), and let $c = \left\lceil \frac{r}{m} \right\rceil$. For any natural number $x \le r$, let k_x denote the number of containers that contain more than x items. Then $k_x \le \frac{cm}{x+1}$.

Proof. Each of the k_x containers contains at least x + 1 items. Thus,

$$k_x(x+1) \leqslant r \implies k_x \leqslant \frac{r}{x+1}.$$
 (1)

$$c = \left| \frac{r}{m} \right| \Rightarrow c \ge \frac{r}{m} \Rightarrow r \le cm \tag{2}$$

$$(1) \stackrel{(2)}{\Longrightarrow} k_x \leqslant \frac{cm}{x+1} \tag{3}$$

For a component K, let δ_K denote the maximum number of topologically consecutive unmarked leaves in K. The unmarked leaves counted in δ_K belong to some interval (ℓ_i, ℓ_{i+1}) .

Lemma 5. Let K be a component of \mathcal{T} and let ℓ_i be a marked leaf whose neighborhood $nh(\ell_i)$ is confined to K.

- a) If K is an L-component, then $nh(\ell_i)$ has at most $4\delta_K$ nodes.
- b) If K is a 5-component, then $nh(\ell_i)$ has at most $10\delta_K$ nodes.

Proof. Let K be an L-component whose L-node is s, see Figure 8a. Since $nh(\ell_i)$ is confined to K then $s \notin nh(\ell_i)$. Thus, s disconnects $nh(\ell_i)$ from the

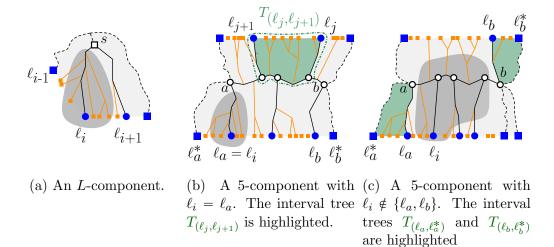


Figure 8: Illustration of a component K in different settings for the proof of Lemma 5. The neighborhood $nh(\ell_i)$ is shaded gray. Marked leaves of K are indicated with (•) and the other marked leaves with (•).

rest of \mathcal{T} , making $nh(\ell_i)$ disjoint from any interval tree, other than $T_{(\ell_{i-1},\ell_i)}$ and $T_{(\ell_i,\ell_{i+1})}$. Hence, $nh(\ell_i)$ contains at most $2\delta_K + 1$ leaves, and since it is a proper binary tree, it can have at most $4\delta_K$ nodes in total.

Suppose K is a 5-component. Since K contains exacly five marked leaves, there can be at most seven interval trees that may share a node with K. Let a and b be the two extreme C-nodes of K and let ℓ_a and ℓ_b be their corresponding marked leaves, which labeled a and b as C-nodes. Let ℓ_a^* (resp. ℓ_b^*) be the neighboring marked leaf of ℓ_a (resp. ℓ_b) in the topological ordering of the marked leaves, which does not belong to K. Refer to Figure 8b and Figure 8c. Neighborhood $nh(\ell_i)$ is confined to K, thus, $a, b \notin nh(\ell_i)$. If $\ell_i = \ell_a$ (resp. $\ell_i = \ell_b$) the C-node a (resp. b), disconnects $nh(\ell_i)$ from the rest of \mathcal{T} . Thus, $nh(\ell_i)$ has a node in common with only two interval trees, $T_{(\ell_{i-1},\ell_i)}$ and $T_{(\ell_i,\ell_{i+1})}$, see Figure 8b. If $\ell_i \notin {\ell_a,\ell_b}$, then nodes a and b disconnect $nh(\ell_i)$ from the rest of \mathcal{T} , thus, $nh(\ell_i)$ is disjoint from both $T_{(\ell_a,\ell_a^*)}$ and $T_{(\ell_b,\ell_b^*)}$, see Figure 8c. Then $nh(\ell_i)$ may have a node in common with at most five out of the seven interval trees that could be related to K. Concluding, $nh(\ell_i)$ has at most $5\delta_K + 1$ leaves, and since it is a proper binary tree, it has at most $10\delta_K$ nodes overall.

For each component K we define a so-called *representative leaf* and at

most two *delimiting nodes*. These are used by our algorithm to identify a confined neighborhood within the component.

Definition 3. For a component K, we define its *representative leaf* and *delimiting nodes* as follows:

- a) If K is an L-component, there is one *delimiting node*, which is its L-node. The *representative leaf* is the first marked leaf of K in the topological ordering of leaves. In Figure 7a, ℓ_i is the representative leaf and s is the delimiting node.
- b) If K is a 5-component, consider the side of K containing at least three marked leaves. The *representative leaf* is the second leaf among these three leaves in the topological ordering. The *delimiting nodes* are the C-nodes defined by the other two leaves in the same side. In Figure 7b, ℓ_i is the representative leaf and q, t are the delimiting nodes.

Our algorithm takes as input a marked tree \mathcal{T} and a parameter $p \in (0, 1)$, and returns $\frac{p}{10}m$ marked leaves that have pairwise disjoint neighborhoods. A pseudocode description is given in Algorithm 1. The algorithm iterates over all the components of \mathcal{T} , and selects at most one marked leaf for each component.

For each component K, the algorithm first identifies its representative leaf and delimiting nodes (lines 6,13), and then traverses the neighborhood of the representative leaf performing a depth-first-search in the component up to a predefined number of steps (lines 7,14). If, while traversing the neighborhood, a delimiting node is detected (lines 8,15,17), then a marked leaf is selected (lines 9,16,18), following the case analysis of Lemma 2. If the entire neighborhood is traversed within the allowed number of steps without detecting a delimiting node (lines 10,19), then the representative leaf is selected (lines 11,20). Otherwise, K is abandoned and the algorithm proceeds to the next component.

Lemma 6. Algorithm 1 returns at least $\frac{p}{10}m$ marked leaves with pairwise disjoint neighborhoods such that no tree edge has its endpoints in two different neighborhoods.

Proof. Let K be a component. The algorithm traverses the neighborhood of the representative leaf ℓ_i and takes a decision after at most 4z, or 10z, steps. In Lemma 5, we proved that if $nh(\ell_i)$ is confined, $nh(\ell_i)$ has at most

Algorithm 1: Selecting leaves with pairwise disjoint neighborhoods.

Input : A marked tree \mathcal{T} with n = r + m leaves and a parameter $p \in (0, 1).$ Output: A set *sol* of marked leaves. 1 Obtain the labeling of \mathcal{T} ; 2 Partition \mathcal{T} into components as indicated in Definition 2; **3** sol $\leftarrow \emptyset$; $c \leftarrow \left\lceil \frac{r}{m} \right\rceil$; $z \leftarrow \left\lceil \frac{10c}{1-p} \right\rceil - 1$; 4 for each component K of \mathcal{T} do if K is an L-component then $\mathbf{5}$ $\ell_i \leftarrow$ representative leaf; $s \leftarrow$ delimiting node; 6 for at most 4z steps traverse $nh(\ell_i)$ 7 if s is visited then 8 $sol \leftarrow sol \cup \{\ell_{i+1}\};$ break; 9 if $nh(\ell_i)$ is traversed and s is not visited then 10 $sol \leftarrow sol \cup \{\ell_i\};$ 11 else if K is a 5-component then 12 $q, t \leftarrow$ delimiting nodes; $\ell_i \leftarrow$ representative leaf, $\mathbf{13}$ for at most 10z steps traverse $nh(\ell_i)$ 14 if q is visited then $\mathbf{15}$ $sol \leftarrow sol \cup \{\ell_{i-1}\};$ break; 16 if t is visited then $\mathbf{17}$ $sol \leftarrow sol \cup \{\ell_{i+1}\};$ break; 18 if $nh(\ell_i)$ is traversed and q,t are not visited then 19 $sol \leftarrow sol \cup \{\ell_i\};$ $\mathbf{20}$ 21 return sol:

 $4\delta_K$, or $10\delta_K$, nodes. Hence, if $\delta_K \leq z$, the algorithm will succeed to select a marked leaf from K, because either $nh(\ell_i)$ is confined to K, and thus, the entire $nh(\ell_i)$ is traversed (lines 10-11,19-20), or else a delimiting node gets visited, and thus, the corresponding marked leaf is selected (lines 8-9,15-18). In all cases, we follow the proof of Lemma 2 and the neighborhood of the selected leaf is confined to K. Thus the selected leaf is among those counted in Lemma 3.

If on the other hand $\delta_k > z$, then the algorithm may fail to identify a marked leaf of K. We use the pigeonhole Lemma 4 to bound the number of

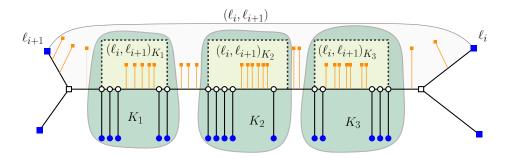


Figure 9: An interval (ℓ_i, ℓ_{i+1}) related to three components K_1, K_2 and K_3 . Interval (ℓ_i, ℓ_{i+1}) is further subdivided into three intervals $(\ell_i, \ell_{i+1})_{K_1}, (\ell_i, \ell_{i+1})_{K_2}$ and $(\ell_i, \ell_{i+1})_{K_3}$.

these components. To this aim, we consider the set I of all intervals induced by the marked leaves and the component of \mathcal{T} . For an interval (ℓ_i, ℓ_{i+1}) , which is not disjoint from K, let $(\ell_i, \ell_{i+1})_K := (\ell_i, \ell_{i+1}) \cap K$ denote its subinterval of unmarked leaves that belong to K, see an example in Figure 9. Let I_z be the intervals in I that contain more than z unmarked leaves. Then the algorithm may fail in at most $|I_z|$ components.

To bound $|I_z|$, we use Lemma 4 for $x = z = \left\lceil \frac{10c}{1-p} \right\rceil - 1$. Then,

$$|I_z| \stackrel{(3)}{\leqslant} \frac{cm}{z+1} = \frac{cm}{\left\lceil \frac{10c}{1-p} \right\rceil - 1 + 1} \leqslant \frac{cm}{\frac{10c}{1-p}} = \frac{1-p}{10}m$$
(4)

Thus, the algorithm may fail for at most $\frac{1-p}{10}m$ components. By Lemma 1, there exist at least $\frac{1}{10}m$ components in \mathcal{T} , thus, the algorithm will succeed in selecting a marked leaf from at least

$$\frac{1}{10}m - |I_z| \stackrel{(4)}{\geqslant} \frac{1}{10}m - \frac{1-p}{10}m = m\frac{p}{10}$$
(5)

components, concluding the proof.

Lemma 7. Algorithm 1 has time complexity $O(\frac{1}{1-n}n)$.

Proof. Labeling and partitioning the tree \mathcal{T} into components can be done in $\Theta(n)$ time. Then, for each component the algorithm traverses a neighborhood performing at most $10z = \Theta(\frac{c}{1-p})$ steps. There are $\Theta(m)$ components, so we have $O(\frac{c}{1-p} \cdot m)$ time complexity. Recall that $c = \lceil \frac{r}{m} \rceil$. If $m = \Theta(n)$, then

 $c = \Theta(1)$, so $cm = \Theta(n)$. Else if m = o(n), then $cm = \Theta(r) = \Theta(n)$. In all cases, the time complexity of the algorithm is $O(\frac{1}{1-p}n)$.

By combining Lemmas 3, 6 and 7 we establish (and re-state) Theorem 2.

Theorem 2. Let \mathcal{T} be a marked tree of n total leaves and m marked leaves. Then there exist at least $\frac{1}{10}m$ leaves in \mathcal{T} with pairwise disjoint neighborhoods such that no tree edge has its endpoints in two different neighborhoods. We can select at least a fraction p of these $\frac{1}{10}m$ marked leaves in time $O(\frac{1}{1-p}n)$, for any $p \in (0, 1)$.

If the parameter $p \in (0, 1)$ is a constant, then the algorithm returns a constant fraction of the marked leaves and the time complexity of the algorithm is O(n).

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