# On selecting a fraction of leaves with disjoint neighborhoods in a plane tree* 

Kolja Junginger, Ioannis Mantas, and Evanthia Papadopoulou ${ }^{\dagger}$
Faculty of Informatics, USI Università della Svizzera italiana, Lugano, Switzerland


#### Abstract

We present a generalization of a combinatorial result by Aggarwal, Guibas, Saxe and Shor [Discrete E Computational Geometry, 1989] on a linear-time algorithm that selects a constant fraction of leaves, with pairwise disjoint neighborhoods, from a binary tree embedded in the plane. This result of Aggarwal et al. is essential to the lineartime framework, which they also introduced, that computes certain Voronoi diagrams of points with a tree structure in linear time. An example is the diagram computed while updating the Voronoi diagram of points after deletion of one site. Our generalization allows that only a fraction of the tree leaves is considered, and it is motivated by lineartime Voronoi constructions for non-point sites. We are given a plane tree $T$ of $n$ leaves, $m$ of which have been marked, and each marked leaf is associated with a neighborhood (a subtree of $T$ ) such that any two topologically consecutive marked leaves have disjoint neighborhoods. We show how to select in linear time a constant fraction of the marked leaves having pairwise disjoint neighborhoods.


## 1 Introduction

In 1987, Aggarwal, Guibas, Saxe and Shor [1] introduced a linear-time technique to compute the Voronoi diagram of points in convex position, which

[^0]

Figure 1: The divide \& conquer algorithm of $[1] .(\longrightarrow)$ indicates the two different divide phases; $(\longrightarrow)$ indicates the recursive constructions; $(\longrightarrow)$ indicates site insertion; $(\longrightarrow)$ indicates merging.
can also be used to compute other Voronoi diagrams of point-sites with a tree structure such as: (1) updating a nearest-neighbor Voronoi diagram of points after deletion of one site; (2) computing the farthest-point Voronoi diagram, after the convex hull of the points is known; (3) computing an order- $k$ Voronoi diagram of points, given its order- $(k-1)$ counterpart. Since then, this framework has been used (and extended) in various ways to tackle various linear-time Voronoi constructions, including the medial axis of a simple polygon by Chin et al. 4], the Hamiltonian abstract Voronoi diagram by Klein and Lingas [8], and some forest-like abstract Voronoi diagrams by Bohler et al. [2]. The linear-time construction for problem (3) improves by a logarithmic factor the standard iterative construction by Lee 9] to compute the order-k Voronoi diagram of point-sites, which is in turn used in different scenarios; for example, algorithms for coverage problems in wireless networks by So and Ye [10]. A much simpler randomized linear-time approach for problems (1)-(3) was introduced by Chew [3].

The linear-time technique of Aggarwal et al. [1] is a doubly-recursive divide-and-conquer scheme operating on an ordered set of points $S$ whose Voronoi diagram is a tree with connected Voronoi regions. At a high level it can be described as follows, see Figure 1. In an initial divide phase, the set $S$ is split in two sets $\mathcal{S}_{R}$ (red) and $\mathcal{S}_{B}$ (blue) of roughly equal size, with the property that every two consecutive red sites in $\mathcal{S}_{R}$ have disjoint Voronoi regions. In a second divide phase, the set $\mathcal{S}_{R}$ is split further in sets $\mathcal{S}_{C}$ (crimson) and $\mathcal{S}_{G}$ (garnet), so that any two sites in $\mathcal{S}_{C}$ have pairwise disjoint regions in the Voronoi diagram of $\mathcal{S}_{B} \cup \mathcal{S}_{C}$, and the cardinality of $\mathcal{S}_{C}$ is a constant fraction of the cardinality of $\mathcal{S}_{R}$. In the merge phase, the sites of
$\mathcal{S}_{C}$ are inserted one by one in the recursively computed Voronoi diagram of $\mathcal{S}_{B}$, deriving the Voronoi diagram of $\mathcal{S}_{C} \cup \mathcal{S}_{B}$, and the result is merged with the recursively computed diagram of $\mathcal{S}_{G}$.

The key factor in obtaining the linear-time complexity is that the cardinality of the set $\mathcal{S}_{C}$ is a constant fraction of $\mathcal{S}_{R}$, which is $\Theta(|\mathcal{S}|)$, and that $\mathcal{S}_{C}$ can be obtained in linear time. This is possible due to the following combinatorial result of [1] on a geometric binary tree embedded in the plane. This result is, thus, inherently used by any algorithm that is based on the lineartime framework of Aggarwal et al. A binary tree that contains no nodes of degree 2 is called proper.
Theorem 1 ([1]). Let $\mathcal{T}$ be an unrooted (proper) binary tree embedded in the plane. Each leaf of $\mathcal{T}$ is associated with a neighborhood, which is a (proper) subtree of $\mathcal{T}$ rooted at that leaf; consecutive leaves in the topological ordering of $\mathcal{T}$ have disjoint neighborhoods. Then, there exists a fixed fraction of the leaves whose neighborhoods are pairwise disjoint, they have a constant size, and no tree edge has its endpoints in two different neighborhoods. Such a set of leaves can be found in linear time.

Overall, the time complexity of the algorithm is described by the following recursive equation and can be proved to be $\Theta(n)$, where $n=|\mathcal{S}|$.

$$
\begin{aligned}
T(n) & =T\left(\left|\mathcal{S}_{B}\right|\right)+T\left(\left|\mathcal{S}_{G}\right|\right)+\Theta\left(\left|\mathcal{S}_{R}\right|\right)+\left|\mathcal{S}_{C}\right| \cdot \Theta(1)+\Theta(n) \\
& \left.=T\left(\left|\mathcal{S}_{B}\right|\right)+T\left(\left|\mathcal{S}_{G}\right|\right)+\Theta(n) \quad \text { (Because }\left|\mathcal{S}_{C}\right|=\Theta(n)\right) \\
& =\Theta(n) \quad
\end{aligned}
$$

It is worth understanding what Theorem 1 represents, in order to have a spherical perspective of its connection to Voronoi diagrams. An embedded tree corresponds to the graph structure of a Voronoi diagram, and leaves are the endpoints of unbounded Voronoi edges "at infinity"; see Figure 2a. The neighborhood of a leaf corresponds to the part of the diagram (of $\mathcal{S}_{B}$ ) that gets deleted if a point-site is inserted there; see Figure 2b, Hence, Theorem 1 aims to select leaves with pairwise disjoint neighborhoods $\left(\mathcal{S}_{C}\right)$, as they can easily, and independently from one another, be inserted in the diagram.

For generalized sites, other than points in the plane, or for abstract Voronoi diagrams, deterministic linear-time algorithms for the counterparts of problems (1)-(3) have not been known so far. This includes the diagrams of very simple geometric sites such as line segments and circles in the Euclidean plane. A major complication over points is that the underlying diagrams have disconnected Voronoi regions. Recently Papadopoulou et al. [6, 7]

(a) The Voronoi diagram of 5 points $(■)$; the neighborhood of $\ell_{2}$ is shaded.

(b) The dashed part of the diagram will get deleted if point ( $\square$ ) is inserted.

(c) The topological ordering of the leaves in $\mathcal{T}$.

Figure 2: An embedded binary tree $\mathcal{T}$ in the setting of Aggarwal et al. [1.
presented a randomized linear-time technique for these problems, based on a relaxed Voronoi structure, called a Voronoi-like diagram [6, 7]. Whether this structure can be used within the framework of Aggarwal et al., leading to deterministic linear-time constructions, remains still an open problem. Towards resolving this problem we need a generalized version of Theorem 1 .

The problem is formulated as follows. We have an unrooted binary tree $\mathcal{T}$ embedded in the plane, which corresponds to a Voronoi-like structure. Not all leaves of $\mathcal{T}$ are eligible for inclusion in the set $S_{C}$ of the linear-time framework. As in the original problem, each of the eligible leaves is associated with a neighborhood, which is a subtree of $\mathcal{T}$ rooted at that leaf, and adjacent leaves in the topological ordering of $\mathcal{T}$ have disjoint neighborhoods. In linear time, we need to compute a constant fraction of the eligible leaves such that their neighborhoods are pairwise disjoint. The non-eligible leaves spread arbitrarily along the topological ordering of the tree leaves. This paper addresses this problem by proving the following generalization of Theorem 1 .

Theorem 2. Let $\mathcal{T}$ be an unrooted (proper) binary tree embedded in the plane having $n$ leaves, $m$ of which have been marked. Each marked leaf of $\mathcal{T}$ is associated with a neighborhood, which is a proper subtree of $\mathcal{T}$ rooted at this leaf, and any two consecutive marked leaves in the topological ordering of $\mathcal{T}$ have disjoint neighborhoods. Then, there exist at least $\frac{1}{10} m$ marked leaves whose neighborhoods are pairwise disjoint and no tree edge has its endpoints in two of these neighborhoods. Further, we can select at least a fraction p of these $\frac{1}{10} m$ marked leaves in time $O\left(\frac{1}{1-p} n\right)$, for any $p \in(0,1)$.

(a) Neighborhoods of the marked leaves are shown shaded. The intersection of neighborhoods is highlighted with red.

(b) A marked tree $\mathcal{T}$, which serves as an example instance for illustrating the notions of Section 2.

Figure 3: Two marked trees, where marked leaves are shown with $(\bullet)$ and unmarked leaves are shown with ( $\quad$ ).

The algorithm of Theorem 2 allows for a trade-off between the number of the returned marked leaves and its time complexity, using a parameter $p \in(0,1)$. If $p$ is constant then the algorithm returns a constant fraction of the marked leaves in $O(n)$ time. Theorem 2 is a combinatorial result on an embedded tree, and thus, we expect it to find applications in different contexts as well.

## 2 Preliminaries

Throughout this work, we consider an unrooted binary tree $\mathcal{T}$ of $n$ leaves that is embedded in the plane. The tree $\mathcal{T}$ contains no nodes of degree 2 and has the following additional properties:

- $m$ out of the $n$ leaves of $\mathcal{T}$ have been marked, and the remaining $r=n-m$ leaves are unmarked (see Figure 3b).
- Every marked leaf $\ell$ is associated with a neighborhood, denoted $n h(\ell)$, which is a subtree of $\mathcal{T}$ rooted at $\ell$ (see Figure 3a).
- Every two consecutive marked leaves in the topological ordering of $\mathcal{T}$ have disjoint neighborhoods (see Figure 3a).


Figure 4: Illustration of Definition 1 applied to the tree $\mathcal{T}$ of Figure 3b.

We call a binary tree $\mathcal{T}$ that follows these properties, a marked tree. Given a marked tree $\mathcal{T}$, let $\mathcal{T}_{u}$ denote the unmarked tree obtained by deleting all the unmarked leaves of $\mathcal{T}$ and contracting the resulting degree- 2 nodes, see Figure 4a. We apply to $\mathcal{T}_{u}$ the following definition, which is extracted from the proof of Theorem 1 in [1], see Figure 4.

Definition 1. Let $T$ be a proper binary tree and let $T^{*}$ be the tree obtained from $T$ after deleting all its leaves. A node $u$ in $T^{*}$ is called:
a) Leaf or $L$-node if $\operatorname{deg}(u)=1$ in $T^{*}$, i.e., $u$ neighbors two leaves in $T$.
b) Comb or $C$-node if $\operatorname{deg}(u)=2$ in $T^{*}$, i.e., $u$ neighbors one leaf in $T$.
c) Junction or $J$-node if $\operatorname{deg}(u)=3$ in $T^{*}$, i.e., $u$ neighbors no leaves in $T$.

A spine is a maximal sequence of consecutive $C$-nodes, which is delimited by $J$ - or $L$-nodes. Each spine has two sides and marked leaves may lie in either side of a spine.

Let $\mathcal{T}_{u}^{*}$, be the tree obtained by applying Definition 1 to the unmarked tree $\mathcal{T}_{u}$. The nodes $\mathcal{T}_{u}^{*}$ are labeled as $L-, C$ - and $J$-nodes, see, e.g., Figure 4b. The labeling of nodes in $\mathcal{T}_{u}^{*}$ is then carried back to their corresponding nodes in the original marked tree $\mathcal{T}$ obtaining a marked tree $\mathcal{T}$ with labels, see Figure 6 . Some nodes in $\mathcal{T}$ remain unlabeled, see, e.g., node $u$ in Figure 6.


Figure 5: The marked tree $\mathcal{T}$ of Figure 3b with labels $(\mathrm{O}, \square, \square)$. Node $u$ is not labeled.


Figure 6: The components of $\mathcal{T}$ shown shaded. The dashed parts do not belong to any component.

Definition 2. Given a marked tree $\mathcal{T}$ with labels we define the following two types of components:
a) $L$-component: an $L$-node $\lambda$ defines an $L$-component that consists of $\lambda$ union the two subtrees of $\mathcal{T}$ that are incident to $\lambda$ and contain no labeled node, see, e.g., $K_{2}$ in Figure 6. The $L$-component contains exactly the two marked leaves that labeled $\lambda$.
b) 5-component: a group of five successive $C$-nodes $c_{i}, \ldots, c_{i+4}$ on a spine defines a 5 -component that consists of the path $P_{c_{i}: c_{i+4}}$ from $c_{i}$ to $c_{i+4}$ (which may contain unlabeled nodes) union the subtrees of $\mathcal{T}$, which are incident to the nodes of $P_{c_{i}: c_{i+4}}$ and contain no labeled node, see, e.g., $K_{1}$ in Figure 6. Nodes $c_{i}$ and $c_{i+4}$ are referred to as the extreme nodes of $K$. The 5 -component contains exactly the five marked leaves, which labeled the five $C$-nodes.

Each spine is partitioned into consecutive groups of 5 -components and at most four remaining ungrouped $C$-nodes.

Figure 5 and Figure 6 illustrates these definitions. The tree $\mathcal{T}$ has three $L$-components and two 5 -components which are indicated shaded in Figure 6 . The 5-component $K_{1}$ contains the path $P_{c_{1}: c_{5}}$ from $c_{1}$ to $c_{5}$, which is shown in thick black lines, and contains one unlabeled node. Node $c_{6}$ is an ungrouped $C$-node. Figure 6 also illustrates a spine consisting of the $C$-nodes
$c_{1}, c_{2}, c_{3}, c_{4}, c_{5}, c_{6}$. The spine is delimited by the $L$-node $\lambda^{\prime}$ and the $J$-node $\iota$; it has five marked leaves from one side and one marked leaf from the other.

Observation 1. The components of $\mathcal{T}$ are pairwise vertex disjoint. Every L-component contains exactly two marked leaves and every 5 -component contains exactly five marked leaves.

Among the components of $\mathcal{T}$ there may be subtrees of $\mathcal{T}$ consisting of unlabeled nodes and unmarked leaves that may be arbitrarily large. These subtrees hang off any unlabeled nodes and ungrouped $C$-nodes. For example, in Figure 6, node $u^{\prime}$ is unlabeled and the gray dotted subtree incident to it consists solely of unmarked leaves and unlabeled nodes that do not belong to any component.

## 3 Existence of leaves with pairwise disjoint neighborhoods

Aggarwal et al. [1 showed that for every eight ungrouped $C$-nodes in $\mathcal{T}_{u}$ there exists at least one $L$-node. Their argument holds for the marked tree $\mathcal{T}$ as well, which is described in the following lemma for completeness.

Lemma 1. For every eight ungrouped $C$-nodes in $\mathcal{T}$ there exists at least one L-component.

Proof. We count the $L$-nodes of $\mathcal{T}$ using the tree $\mathcal{T}_{u}^{*}$ following the argument of [1]. Let $k$ be the number of leaves in $\mathcal{T}_{u}^{*}$, which also equals the number of $L$-nodes in $\mathcal{T}$. Contracting all degree- 2 vertices in $\mathcal{T}_{u}^{*}$ yields a binary tree $\mathcal{T}_{b}^{*}$, which has the same leaves as $\mathcal{T}_{u}^{*}$. Since $\mathcal{T}_{b}^{*}$ is an unrooted binary tree with $k$ leaves, it has $2 k-2$ nodes and $2 k-3$ edges. Every edge in $\mathcal{T}_{b}^{*}$ corresponds to at most one spine in $\mathcal{T}_{u}^{*}$ and in every spine there are at most four ungrouped $C$-nodes. Thus,

$$
\mid \text { ungrouped } C \text {-nodes }|\leqslant 4| \text { spines }|\leqslant 4 \cdot(2 k-3)<8| L \text {-nodes } \mid,
$$

where $|\cdot|$ denotes cardinality. So, there exists at least one $L$-node for every eight ungrouped $C$-nodes, and an $L$-node corresponds to exactly one L-component.


Figure 7: Marked leaves with their neighborhoods shaded. The neighborhood $n h\left(\ell_{i}\right)$ is confined to the component in both cases.

The following lemmata establish that there exists a constant fraction of the marked leaves, which have pairwise disjoint neighborhoods. The counting arguments follow those in [1] while they are further enhanced to account for the unmarked leaves, which are arbitrarily distributed among the marked leaves. We say that the neighborhood $n h(\ell)$ of a marked leaf $\ell$ is confined to a component $K$ if it is a subtree of $K$.

Lemma 2. In every component $K$, there exists a marked leaf $\ell \in K$ whose neighborhood is confined to $K$. This neighborhood may contain no L-node and no extreme $C$-node.

Proof. Let $K$ be an $L$-component and let $s$ be the $L$-node that defines $K$. Let $\ell_{i}$ and $\ell_{i+1}$ be the two marked leaves of $K$. Since the neighborhoods $n h\left(\ell_{i}\right)$ and $n h\left(\ell_{i+1}\right)$ are disjoint, at least one of them cannot contain $s$. This neighborhood is, thus, entirely contained in the relevant subtree rooted at $s$, see Figure 7a, and contains no labeled node.

Let $K$ be a 5 -component. Since a 5 -component has two sides, at least three out of the five marked leaves of the component must lie on the same side of $K$, call them $\ell_{i-1}, \ell_{i}$ and $\ell_{i+1}$. Let $q, s$, and $t$ be their corresponding $C$-nodes, i.e., the first $C$-nodes in $K$ reachable from $\ell_{i-1}, \ell_{i}$, and $\ell_{i+1}$, respectively, see Figure 7b. There are three cases. If $t \in n h\left(\ell_{i}\right)$, then $t \notin n h\left(\ell_{i+1}\right)$ (since the two neighborhoods are disjoint), and thus, $n h\left(\ell_{i+1}\right)$ is confined to the subtree of $t$ that contains $\ell_{i+1}$. Similarly, if $q \in n h\left(\ell_{i}\right)$, then $q \notin n h\left(\ell_{i-1}\right)$, so $n h\left(\ell_{i-1}\right)$ is confined to the subtree of $q$ containing $\ell_{i-1}$. If neither $q$ nor $t$ are in $n h\left(\ell_{i}\right)$, then clearly $n h\left(\ell_{i}\right)$ is confined to $K$. In all cases the confined neighborhood cannot contain neither $q$ nor $t$. So, at least one of the five
marked leaves must have a neighborhood confined to $K$ and this neighborhood cannot contain the extreme $C$-nodes in $K$.

Lemma 3. Let $\mathcal{T}$ be a marked tree with $m$ marked leaves. At least $\frac{1}{10} m$ marked leaves must have pairwise disjoint neighborhoods such that no tree edge may have its endpoints in two different neighborhoods.

Proof. Every spine of $\mathcal{T}$ has up to four ungrouped $C$-nodes. By Lemma 1, there exists at least one $L$-component for every eight ungrouped $C$-nodes. By Lemma 2, every component of $\mathcal{T}$ has at least one marked leaf whose neighborhood is confined to the component. So, overall, at least $\frac{1}{5}$ of the marked leaves from each 5 -component and at least $\frac{1}{10}$ marked leaves of the remaining nodes, which label ungrouped $C$-nodes or $L$-nodes, have a confined neighborhood. The components are pairwise disjoint, so at least $\frac{1}{10}$ marked leaves have pairwise disjoint neighborhoods. Furthermore, confined neighborhoods do not contain any $L$-node or extreme $C$-node, as shown in Lemma 2. Thus, no tree edge may have its endpoints in two different neighborhoods.

We remark that the neighborhoods implied by Lemma 3 may not contain any $L$-node nor any extreme $C$-node. We also remark that these neighborhoods need not be of constant complexity as their counterparts in [1] are. These neighborhoods may have complexity $\Theta(r)$, where $r=n-m$ is the number of unmarked leaves. Since $r$ may be $\Theta(n)$, this poses a challenge on how we can select these leaves efficiently.

## 4 Selecting leaves with pairwise disjoint neighborhoods

Given a marked tree $\mathcal{T}$ with $m$ marked leaves, we have already established the existence of $\frac{1}{10} m$ marked leaves that have pairwise disjoint neighborhoods. In this section, we present an algorithm to select a fraction $p$ of these leaves, i.e., $\frac{p}{10} m$ marked leaves with pairwise disjoint neighborhoods, in time $O\left(\frac{1}{1-p} n\right)$, where $0<p<1$.

The main challenge over the algorithm of [1] is that the $r$ unmarked leaves are arbitrarily distributed among the $m$ marked leaves, and thus, the components of $\mathcal{T}$ and the neighborhoods of the marked leaves may have complexity $\Theta(r)$. If for each component we spend time proportional to its
size, then the time complexity of the algorithm will be $\Theta(m r)$, i.e., $\Theta\left(n^{2}\right)$ if $r, m \in \Theta(n)$.

To keep the complexity of the algorithm linear, we spend time up to a predefined number of steps in each component depending on the ratio $c=\left\lceil\frac{r}{m}\right\rceil$ and the trade-off parameter $p \in(0,1)$. Our algorithm guarantees to find at least a fraction $p$ of the possible $\frac{1}{10} m$ marked leaves in time $O\left(\frac{1}{1-p} n\right)$. We first present a series of results necessary to establish the correctness of the approach and then describe the algorithm.

Let $\ell_{1}, \ldots, \ell_{m}$ be the marked leaves in $\mathcal{T}$ ordered in a counterclockwise topological ordering. Let the interval $\left(\ell_{i}, \ell_{i+1}\right)$ denote the set of unmarked leaves between $\ell_{i}$ and $\ell_{i+1}$ in the same order. The interval tree of $\left(\ell_{i}, \ell_{i+1}\right)$, denoted $T_{\left(\ell_{i}, \ell_{i+1}\right)}$, is the minimal subtree of $\mathcal{T}$ that contains the marked leaves $\ell_{i}$ and $\ell_{i+1}$, including the unmarked leaves in $\left(\ell_{i}, \ell_{i+1}\right)$, see Figure 8 b . We show the following pigeonhole lemma involving unmarked leaves and intervals.

Lemma 4. Suppose that $r$ items (unmarked leaves) are distributed in $k \geqslant m$ containers (intervals), and let $c=\left\lceil\frac{r}{m}\right\rceil$. For any natural number $x \leqslant r$, let $k_{x}$ denote the number of containers that contain more than $x$ items. Then $k_{x} \leqslant \frac{c m}{x+1}$.
Proof. Each of the $k_{x}$ containers contains at least $x+1$ items. Thus,

$$
\begin{align*}
& k_{x}(x+1) \leqslant r \Rightarrow k_{x} \leqslant \frac{r}{x+1} .  \tag{1}\\
& c=\left\lceil\frac{r}{m}\right\rceil \Rightarrow c \geqslant \frac{r}{m} \Rightarrow r \leqslant c m  \tag{2}\\
& 11 \xlongequal{|2|} k_{x} \leqslant \frac{c m}{x+1} \tag{3}
\end{align*}
$$

For a component $K$, let $\delta_{K}$ denote the maximum number of topologically consecutive unmarked leaves in $K$. The unmarked leaves counted in $\delta_{K}$ belong to some interval $\left(\ell_{i}, \ell_{i+1}\right)$.

Lemma 5. Let $K$ be a component of $\mathcal{T}$ and let $\ell_{i}$ be a marked leaf whose neighborhood $n h\left(\ell_{i}\right)$ is confined to $K$.
a) If $K$ is an $L$-component, then $n h\left(\ell_{i}\right)$ has at most $4 \delta_{K}$ nodes.
b) If $K$ is a 5-component, then $n h\left(\ell_{i}\right)$ has at most $10 \delta_{K}$ nodes.

Proof. Let $K$ be an $L$-component whose $L$-node is $s$, see Figure 8a. Since $n h\left(\ell_{i}\right)$ is confined to $K$ then $s \notin n h\left(\ell_{i}\right)$. Thus, $s$ disconnects $n h\left(\ell_{i}\right)$ from the


Figure 8: Illustration of a component $K$ in different settings for the proof of Lemma 5. The neighborhood $n h\left(\ell_{i}\right)$ is shaded gray. Marked leaves of $K$ are indicated with $(\bullet)$ and the other marked leaves with (■).
rest of $\mathcal{T}$, making $n h\left(\ell_{i}\right)$ disjoint from any interval tree, other than $T_{\left(\ell_{i-1}, \ell_{i}\right)}$ and $T_{\left(\ell_{i}, \ell_{i+1}\right)}$. Hence, $n h\left(\ell_{i}\right)$ contains at most $2 \delta_{K}+1$ leaves, and since it is a proper binary tree, it can have at most $4 \delta_{K}$ nodes in total.

Suppose $K$ is a 5 -component. Since $K$ contains exacly five marked leaves, there can be at most seven interval trees that may share a node with $K$. Let $a$ and $b$ be the two extreme $C$-nodes of $K$ and let $\ell_{a}$ and $\ell_{b}$ be their corresponding marked leaves, which labeled $a$ and $b$ as $C$-nodes. Let $\ell_{a}^{*}$ (resp. $\left.\ell_{b}^{*}\right)$ be the neighboring marked leaf of $\ell_{a}\left(\right.$ resp. $\left.\ell_{b}\right)$ in the topological ordering of the marked leaves, which does not belong to $K$. Refer to Figure 8b and Figure 8c. Neighborhood $n h\left(\ell_{i}\right)$ is confined to $K$, thus, $a, b \notin n h\left(\ell_{i}\right)$. If $\ell_{i}=\ell_{a}$ (resp. $\ell_{i}=\ell_{b}$ ) the $C$-node $a$ (resp. $b$ ), disconnects $n h\left(\ell_{i}\right)$ from the rest of $\mathcal{T}$. Thus, $n h\left(\ell_{i}\right)$ has a node in common with only two interval trees, $T_{\left(\ell_{i-1}, \ell_{i}\right)}$ and $T_{\left(\ell_{i}, \ell_{i+1}\right)}$, see Figure 8b. If $\ell_{i} \notin\left\{\ell_{a}, \ell_{b}\right\}$, then nodes $a$ and $b$ disconnect $n h\left(\ell_{i}\right)$ from the rest of $\mathcal{T}$, thus, $n h\left(\ell_{i}\right)$ is disjoint from both $T_{\left(\ell_{a}, \ell_{a}^{*}\right)}$ and $T_{\left(\ell_{b}, \ell_{b}^{*}\right)}$, see Figure 8c. Then $n h\left(\ell_{i}\right)$ may have a node in common with at most five out of the seven interval trees that could be related to $K$. Concluding, $n h\left(\ell_{i}\right)$ has at most $5 \delta_{K}+1$ leaves, and since it is a proper binary tree, it has at most $10 \delta_{K}$ nodes overall.

For each component $K$ we define a so-called representative leaf and at
most two delimiting nodes. These are used by our algorithm to identify a confined neighborhood within the component.

Definition 3. For a component $K$, we define its representative leaf and delimiting nodes as follows:
a) If $K$ is an $L$-component, there is one delimiting node, which is its $L$-node. The representative leaf is the first marked leaf of $K$ in the topological ordering of leaves. In Figure 7a, $\ell_{i}$ is the representative leaf and $s$ is the delimiting node.
b) If $K$ is a 5 -component, consider the side of $K$ containing at least three marked leaves. The representative leaf is the second leaf among these three leaves in the topological ordering. The delimiting nodes are the $C$ nodes defined by the other two leaves in the same side. In Figure 7b, $\ell_{i}$ is the representative leaf and $q, t$ are the delimiting nodes.

Our algorithm takes as input a marked tree $\mathcal{T}$ and a parameter $p \in(0,1)$, and returns $\frac{p}{10} m$ marked leaves that have pairwise disjoint neighborhoods. A pseudocode description is given in Algorithm 1. The algorithm iterates over all the components of $\mathcal{T}$, and selects at most one marked leaf for each component.

For each component $K$, the algorithm first identifies its representative leaf and delimiting nodes (lines 6,13 ), and then traverses the neighborhood of the representative leaf performing a depth-first-search in the component up to a predefined number of steps (lines 7,14). If, while traversing the neighborhood, a delimiting node is detected (lines $8,15,17$ ), then a marked leaf is selected (lines 9,16,18), following the case analysis of Lemma 2. If the entire neighborhood is traversed within the allowed number of steps without detecting a delimiting node (lines 10,19), then the representative leaf is selected (lines 11,20). Otherwise, $K$ is abandoned and the algorithm proceeds to the next component.

Lemma 6. Algorithm 1 returns at least $\frac{p}{10} m$ marked leaves with pairwise disjoint neighborhoods such that no tree edge has its endpoints in two different neighborhoods.

Proof. Let $K$ be a component. The algorithm traverses the neighborhood of the representative leaf $\ell_{i}$ and takes a decision after at most $4 z$, or $10 z$, steps. In Lemma 5, we proved that if $n h\left(\ell_{i}\right)$ is confined, $n h\left(\ell_{i}\right)$ has at most

```
Algorithm 1: Selecting leaves with pairwise disjoint neighborhoods.
    Input : A marked tree \(\mathcal{T}\) with \(n=r+m\) leaves and a parameter
                \(p \in(0,1)\).
    Output: A set sol of marked leaves.
    Obtain the labeling of \(\mathcal{T}\);
    Partition \(\mathcal{T}\) into components as indicated in Definition 2;
    sol \(\leftarrow \varnothing ; \quad c \leftarrow\left\lceil\frac{r}{m}\right\rceil ; \quad z \leftarrow\left\lceil\frac{10 c}{1-p}\right\rceil-1 ;\)
    for each component \(K\) of \(\mathcal{T}\) do
        if \(K\) is an \(L\)-component then
            \(\ell_{i} \leftarrow\) representative leaf; \(\quad s \leftarrow\) delimiting node;
            for at most \(4 z\) steps traverse \(n h\left(\ell_{i}\right)\)
                if \(s\) is visited then
                    sol \(\leftarrow\) sol \(\cup\left\{\ell_{i+1}\right\} ; \quad\) break;
            if \(n h\left(\ell_{i}\right)\) is traversed and \(s\) is not visited then
                sol \(\leftarrow\) sol \(\cup\left\{\ell_{i}\right\} ;\)
            else if \(K\) is a 5 -component then
            \(\ell_{i} \leftarrow\) representative leaf, \(\quad q, t \leftarrow\) delimiting nodes;
            for at most \(10 z\) steps traverse \(n h\left(\ell_{i}\right)\)
                if \(q\) is visited then
                    sol \(\leftarrow\) sol \(\cup\left\{\ell_{i-1}\right\} ; \quad\) break;
                if \(t\) is visited then
                    sol \(\leftarrow\) sol \(\cup\left\{\ell_{i+1}\right\} ; \quad\) break;
            if \(n h\left(\ell_{i}\right)\) is traversed and \(q, t\) are not visited then
                    sol \(\leftarrow \operatorname{sol} \cup\left\{\ell_{i}\right\} ;\)
    return sol;
```

$4 \delta_{K}$, or $10 \delta_{K}$, nodes. Hence, if $\delta_{K} \leqslant z$, the algorithm will succeed to select a marked leaf from $K$, because either $n h\left(\ell_{i}\right)$ is confined to $K$, and thus, the entire $n h\left(\ell_{i}\right)$ is traversed (lines 10-11,19-20), or else a delimiting node gets visited, and thus, the corresponding marked leaf is selected (lines 8-9,15-18). In all cases, we follow the proof of Lemma 2 and the neighborhood of the selected leaf is confined to $K$. Thus the selected leaf is among those counted in Lemma 3.

If on the other hand $\delta_{k}>z$, then the algorithm may fail to identify a marked leaf of $K$. We use the pigeonhole Lemma 4 to bound the number of


Figure 9: An interval $\left(\ell_{i}, \ell_{i+1}\right)$ related to three components $K_{1}, K_{2}$ and $K_{3}$. Interval $\left(\ell_{i}, \ell_{i+1}\right)$ is further subdivided into three intervals $\left(\ell_{i}, \ell_{i+1}\right)_{K_{1}},\left(\ell_{i}, \ell_{i+1}\right)_{K_{2}}$ and $\left(\ell_{i}, \ell_{i+1}\right)_{K_{3}}$.
these components. To this aim, we consider the set $I$ of all intervals induced by the marked leaves and the component of $\mathcal{T}$. For an interval $\left(\ell_{i}, \ell_{i+1}\right)$, which is not disjoint from $K$, let $\left(\ell_{i}, \ell_{i+1}\right)_{K}:=\left(\ell_{i}, \ell_{i+1}\right) \cap K$ denote its subinterval of unmarked leaves that belong to $K$, see an example in Figure 9. Let $I_{z}$ be the intervals in $I$ that contain more than $z$ unmarked leaves. Then the algorithm may fail in at most $\left|I_{z}\right|$ components.

To bound $\left|I_{z}\right|$, we use Lemma 4 for $x=z=\left\lceil\frac{10 c}{1-p}\right\rceil-1$. Then,

$$
\begin{equation*}
\left|I_{z}\right| \stackrel{(3)}{\leqslant} \frac{c m}{z+1}=\frac{c m}{\left\lceil\left.\frac{10 c}{1-p} \right\rvert\,-1+1\right.} \leqslant \frac{c m}{\frac{10 c}{1-p}}=\frac{1-p}{10} m \tag{4}
\end{equation*}
$$

Thus, the algorithm may fail for at most $\frac{1-p}{10} m$ components. By Lemma 1 . there exist at least $\frac{1}{10} m$ components in $\mathcal{T}$, thus, the algorithm will succeed in selecting a marked leaf from at least

$$
\begin{equation*}
\frac{1}{10} m-\left|I_{z}\right| \stackrel{\sqrt{4}}{\Rightarrow} \frac{1}{10} m-\frac{1-p}{10} m=m \frac{p}{10} \tag{5}
\end{equation*}
$$

components, concluding the proof.
Lemma 7. Algorithm 1 has time complexity $O\left(\frac{1}{1-p} n\right)$.
Proof. Labeling and partitioning the tree $\mathcal{T}$ into components can be done in $\Theta(n)$ time. Then, for each component the algorithm traverses a neighborhood performing at most $10 z=\Theta\left(\frac{c}{1-p}\right)$ steps. There are $\Theta(m)$ components, so we have $O\left(\frac{c}{1-p} \cdot m\right)$ time complexity. Recall that $c=\left\lceil\frac{r}{m}\right\rceil$. If $m=\Theta(n)$, then
$c=\Theta(1)$, so $c m=\Theta(n)$. Else if $m=o(n)$, then $c m=\Theta(r)=\Theta(n)$. In all cases, the time complexity of the algorithm is $O\left(\frac{1}{1-p} n\right)$.

By combining Lemmas 3, 6 and 7 we establish (and re-state) Theorem 2.
Theorem 2. Let $\mathcal{T}$ be a marked tree of $n$ total leaves and $m$ marked leaves. Then there exist at least $\frac{1}{10} m$ leaves in $\mathcal{T}$ with pairwise disjoint neighborhoods such that no tree edge has its endpoints in two different neighborhoods. We can select at least a fraction $p$ of these $\frac{1}{10} m$ marked leaves in time $O\left(\frac{1}{1-p} n\right)$, for any $p \in(0,1)$.

If the parameter $p \in(0,1)$ is a constant, then the algorithm returns a constant fraction of the marked leaves and the time complexity of the algorithm is $O(n)$.

## References

[1] A. Aggarwal, L. Guibas, J. Saxe, and P. Shor. A linear-time algorithm for computing the Voronoi diagram of a convex polygon. Discrete $\mathcal{B}$ Computational Geometry, 4:591-604, 1989.
[2] C. Bohler, R. Klein, A. Lingas, and C.-H. Liu. Forest-like abstract Voronoi diagrams in linear time. Computational Geometry, 68:134-145, 2018.
[3] L. P. Chew. Building Voronoi diagrams for convex polygons in linear expected time. Technical report, Dartmouth College, Hanover, USA, 1990.
[4] F. Chin, J. Snoeyink, and C. A. Wang. Finding the medial axis of a simple polygon in linear time. Discrete $\mathfrak{E}$ Computational Geometry, 21(3):405-420, 1999.
[5] K. Junginger, I. Mantas, and E. Papadopoulou. On selecting a fraction of leaves with disjoint neighborhoods in a plane tree. Discrete Applied Mathematics, 319:141-148, 2022.
[6] K. Junginger and E. Papadopoulou. Deletion in abstract Voronoi diagrams in expected linear time and related problems. arXiv preprint arXiv:1803.05372, 2018.
[7] E. Khramtcova and E. Papadopoulou. An expected linear-time algorithm for the farthest-segment Voronoi diagram. arXiv preprint arXiv:1411.2816, 2017.
[8] R. Klein and A. Lingas. Hamiltonian abstract Voronoi diagrams in linear time. In Proceedings of the 5th International Symposium on Algorithms and Computation (ISAAC 1994), pages 11-19. Springer, 1994.
[9] D.-T. Lee. On k-nearest neighbor Voronoi diagrams in the plane. IEEE transactions on computers, 100(6):478-487, 1982.
[10] A. M.-C. So and Y. Ye. On solving coverage problems in a wireless sensor network using Voronoi diagrams. In Proceedings of the 1st Workshop on Internet and Network Economics (WINE 2005), pages 584-593. Springer, 2005.


[^0]:    *This work was supported in part by the Swiss National Science Foundation, project SNF 200021E-154387. It appears in Discrete Applied Mathematics 5.
    ${ }^{\dagger}$ Corresponding author: evanthia.papadopoulou@usi.ch

